Sec8.1. Arc-length. *LecNote6.*

Q1 Find the exact arc-length of $x = \frac{2}{3}(y^2+1)^{3/2}$ from $y = 0$ to $y = 2$.

$$x'(y) = \frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}\cdot(y^2+1)^{\frac{1}{2}}\cdot 2y = (y^2+1)^{\frac{1}{2}}\cdot 2y$$

$$\text{Arc-length} = \int_0^2 \sqrt{1+(x'(y))^2}\cdot dy$$

$$= \int_0^2 \sqrt{1+\left[(y^2+1)^{\frac{1}{2}}\cdot 2y\right]^2}\, dy$$

$$= \int_0^2 \sqrt{1+(y^2+1)\cdot 4y^2}\cdot dy$$

$$= \int_0^2 \sqrt{4y^4+4y^2+1}\cdot dy$$

$$= \int_0^2 \sqrt{(2y^2)^2+2\cdot 2y^2\cdot 1+1}\, dy \quad \text{complete the square}$$

$$= \int_0^2 \sqrt{(2y^2+1)^2}\, dy$$

$$= \int_0^2 2y^2+1\cdot dy$$

$$= 2\cdot\frac{1}{3}y^3+y\Big|_0^2 = 2\cdot\frac{1}{3}\cdot 2^3+2-0 = \frac{16}{3}+2 = \boxed{\frac{22}{3}}$$

Sec11.1. Sequences. LecNote7.

Q2(Limit of a sequence.) Find the limit if the sequence below converges or state why it diverges.

(a)

$$a_n = \frac{1}{n}\ln(\frac{1}{n})$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}\cdot\ln(\frac{1}{n}) = \lim_{n\to\infty} \frac{-\ln n}{n} \qquad \frac{\infty}{\infty}$$

$$\overset{\text{l'Hop}}{=} \lim_{n\to\infty} \frac{(-\ln n)'}{(n)'}$$

$$= \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} = 0 \text{ , conV.}$$

(b)

$$a_k = \frac{\sqrt{1+k^3}}{3k^2+7k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\sqrt{1+k^3}}{3k^2+7k} = \lim_{k\to\infty} \frac{\sqrt{k^3}}{3k^2} = \lim_{k\to\infty} \frac{k^{\frac{3}{2}}}{3k^2} = \lim_{k\to\infty} \frac{1}{3\cdot k^{\frac{1}{2}}}$$

$$= 0 \text{ , conV.}$$

(c)

$$a_n = n(e^{\frac{1}{n}} - 1)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\cdot(e^{\frac{1}{n}}-1) \qquad \infty\cdot(e^0-1) = \infty\cdot 0$$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} \qquad \frac{0}{0}$$

$$\overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{e^{\frac{1}{n}}\cdot(-\frac{1}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1 \text{ , conV.}$$

Q3 (n-th term test for divergence). Which statements (more than one option) are true about

$\lim_{n\to\infty} e^{\frac{2}{n}} = e^0 = 1 \neq 0$

$$a_n = e^{\frac{2}{n}}, \quad \text{and} \quad \sum_{n=1}^{\infty} e^{\frac{2}{n}}$$

A. The sequence a_n is convergent but the series is divergent.

B. The sequence a_n converges, therefore, the **nth term test** concludes that the series ~~converges~~.

C. The sequence a_n has limit zero, therefore, the **nth term test** concludes that the series converges.

nonzero

D. The **nth term test** concludes that the series diverges. since $\lim_{n\to\infty} a_n = 1 \neq 0$

E. If the series is convergent, then the sequence a_n should have zero limit, which is a contradiction. Therefore, the series can not be convergent.

Q4 (Geometric Sequence/Sum/Series). Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{9^{n/2}}{3(2^{2n+1})}$$

sln 1:

First term: $n=1$ $\quad \frac{9^{\frac{1}{2}}}{3\cdot 2^3} = \frac{3}{3\cdot 8} = \frac{1}{8}, \quad a = \frac{1}{8}$

Second term: $n=2$. $\quad \frac{9^{\frac{2}{2}}}{3\cdot 2^5} = \frac{9}{3\cdot 2^5} = \frac{3}{32}, \quad r = \frac{\text{2nd term}}{\text{1st term}} = \frac{\frac{3}{32}}{\frac{1}{8}} = \frac{3}{4}$

$$\sum_{n=1}^{\infty} \frac{9^{\frac{n}{2}}}{3(2^{2n+1})} = \frac{a}{1-r} = \frac{\frac{1}{8}}{1-\frac{3}{4}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \boxed{\frac{1}{2}}.$$

sln 2: $\quad \frac{9^{\frac{n}{2}}}{3\cdot 2^{2n+1}} = \frac{(\sqrt{9})^n}{3\cdot 2^{2n}\cdot 2} = \frac{3^n}{6\cdot 4^n} = \frac{1}{6}\cdot\left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right)^{n-1}$

$= \frac{1}{8}\cdot\left(\frac{3}{4}\right)^{n-1}$

$a = \frac{1}{8}, \quad r = \frac{3}{4}.$

Sec11.3. Integral Test and the p-Series. *LecNote7.*

Q5 (Integral Test) Test the following series for convergence or divergence by THE INTEGRAL TEST.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{1}{n}\right)$$

$f(x) = \frac{1}{x^2}\cdot \sin(\frac{1}{x})$ is positive, continuous and decreasing on $[1, +\infty)$

It is enough to test the improper integral:

$$\int_1^{\infty} \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx, \quad u = \frac{1}{x},\ du = \frac{-1}{x^2}dx$$

$$= \lim_{t\to\infty} \int \sin u \cdot (-du)$$

$$= \lim_{t\to\infty} -\cos u = \lim_{t\to\infty} -\cos\left(\frac{1}{x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} -\cos\left(\frac{1}{t}\right) + \cos 1 = -1 + \cos 1$$

According to the integral test, $\int_1^{\infty} \frac{1}{x^2}\sin(\frac{1}{x})dx$ conv implies $\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{1}{n})$ conv.

Q6 (p-series) Which statements (more than one option) are true

~~**A.**~~ The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $\cancel{|p| <} 1$. $p > 1$

~~**B.**~~ The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if and only if $\cancel{|p| > 1}$. $p \le 1$

✓ **C.** The series $\sum_{n=1}^{\infty} r^n$ is convergent if $|r| < 1$.

✓ **D.** The series $\sum_{n=0}^{\infty} r^n$ is divergent at $r = -1$.

$\frac{\sqrt{n}}{n^{1.5}} = \frac{1}{n}$, $p = 1$

~~**E.**~~ The series $\sum_{n=1}^{\infty} \frac{-3\sqrt{n}}{n^{1.5}}$ is convergent since it is a constant multiple of a p-series with $\cancel{p = 1.5} > 1$.

~~**F.**~~ The p-series $\sum_{n=1}^{\infty} n^{-2}$ is divergent since $p \cancel{= -2}$ and $|p| = 2 > 1$.

$n^{-2} = \frac{1}{n^2}$, $p = 2 > 1$.

Sec11.4. Comparison Test. *LecNote8.*

Q7 Determine whether the following series converge or diverge by (Direct/Limit) Comparison Test.

(a)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$$

S/n1: $\frac{\sqrt{n+1}}{n} > \frac{1}{n}$, $\sum \frac{1}{n}$ DIV $\Rightarrow \sum \frac{\sqrt{n+1}}{n}$ DIV.

according to Comparison Test

S/n2: $a_n = \frac{\sqrt{n+1}}{n}$, $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. $\sum b_n$ DIV $(p=\frac{1}{2})$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{n} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1$ $\Rightarrow \sum a_n$ DIV. (L.C.T.)

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{3n^3-7n}$$

$a_n = \frac{\sqrt{n^2+1}}{3n^3-7n}$, $b_n = \frac{\sqrt{n^2}}{3n^3} = \frac{n}{3n^3} = \frac{1}{3n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{3n^3-7n} \cdot 3n^2 \xlongequal{\text{leading term}} \lim_{n\to\infty} \frac{\sqrt{n^2} \cdot 3n^2}{3n^3} = 1 \neq 0$

$\sum b_n$ conV (p-Series, $p=2$) implies $\sum a_n$ conV

due to L.C.T.

(c)

$$\sum_{n=2}^{\infty} \frac{2}{n^{61}+1}$$

$a_n = \frac{2}{n^{61}+1} < \frac{2}{n^{61}} = b_n$, $p=61>2$.

$\sum b_n$ is conV $\Rightarrow \sum a_n$ is conV. C.T.

Sec11.5. Alternating Series Test and Absolute Convergence. LecNote8.

Q8 Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

(a)

$$(1) \quad \sum_{n=1}^{\infty} \frac{\cos(5n)}{n^5} \qquad \text{and (2)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n}$$

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

C (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

(1). $\sum_{n=1}^{\infty} \left|\frac{\cos(5n)}{n^5}\right| \leq \sum \frac{1}{n^5}$ $p=5$, conV. Comparison Test $\Rightarrow \sum \left|\frac{\cos(5n)}{n^5}\right|$ conV.

$\Rightarrow \sum \frac{\cos(5n)}{n^5}$ ABS conV.

(2) $\sum \left|\frac{(-1)^{n-1}}{3n}\right| = \sum \frac{1}{3n}$ DIV. $\sum \frac{(-1)^{n-1}}{3n}$ conV as an alternating series.

(b) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1) \quad \sum_{n=1}^{\infty} \frac{\sin(n)+1}{2^n} \qquad \text{and (2)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

C (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

(1). $\left|\frac{\sin(n)+1}{2^n}\right| \leq \frac{2}{2^n}$ $\sum \frac{2}{2^n}$ conV $\Rightarrow \sum \left|\frac{\sin(n)+1}{2^n}\right|$ conV.

$\Rightarrow \sum \frac{\sin n+1}{2^n}$ ABS conV

(2). $\sum \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum \frac{1}{\sqrt{n}}$ DIV $p=\frac{1}{2}$. $\sum \frac{(-1)^n}{\sqrt{n}}$ conV (Alternating Series).

Sec11.6. Ratio Test. *LecNote8.*

Q9 Determine whether the following series converge or diverge.

(a)

$$\sum_{n=1}^{\infty} \frac{2^n(n^2+1)}{3^n}$$

$a_n = \frac{2^n \cdot (n^2+1)}{3^n}$, $a_{n+1} = \frac{2^{n+1}((n+1)^2+1)}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}\cdot[(n+1)^2+1]}{3^{n+1}} \cdot \frac{3^n}{2^n \cdot (n^2+1)}$$

$$= \lim_{n\to\infty} \frac{2}{3} \cdot \frac{(n+1)^2+1}{n^2+1} = \frac{2}{3} < 1 \Rightarrow \sum a_n \text{ conV.}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(n+1)!}{e^n}$$

$a_n = \frac{(n+1)!}{e^n}$, $a_{n+1} = \frac{(n+2)!}{e^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)!}{e^{n+1}} \cdot \frac{e^n}{(n+1)!} = \lim_{n\to\infty} \frac{(n+2)}{e} = \infty > 1.$$

$\sum a_n$ is divergent.

(c)

$$\sum_{n=1}^{\infty} \frac{n}{(-2)^n}$$

$a_n = \frac{n}{(-2)^n}$, $a_{n+1} = \frac{n+1}{(-2)^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{(-2)^{n+1}} \cdot \frac{(-2)^n}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{-2} \right| = \left| 1 \cdot \frac{1}{-2} \right| = \frac{1}{2} < 1.$$

$\sum a_n$ is ABS conV (therefore, is convergent).

Sec11.8. Power Series. LecNote9.

Q10 Consider the following power series

$$\sum_{n=0}^{\infty}(n+3)\Big(\frac{2x-3}{3}\Big)^n$$

(a) Find the radius of convergence of the series and the OPEN interval of the convergence.

$a_n = (n+3)\left(\frac{2x-3}{3}\right)^n$, $a_{n+1} = (n+1+3)\cdot\left(\frac{2x-3}{3}\right)^{n+1} = (n+4)\cdot\left(\frac{2x-3}{3}\right)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+4)\left(\frac{2x-3}{3}\right)^{n+1}}{(n+3)\cdot\left(\frac{2x-3}{3}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+4)}{(n+3)}\cdot\left(\frac{2x-3}{3}\right)\right|$$

$$= \left|\frac{2x-3}{3}\right| < 1.$$

$\Rightarrow |2x-3|<3 \Rightarrow \left|x-\frac{3}{2}\right|<\frac{3}{2}.$

$\boxed{\text{radius } R=\frac{3}{2}.}$

$-3<2x-3<3 \Rightarrow 0<2x<6 \Rightarrow 0<x<3.$

$\boxed{\text{open interval } (0,3)}$

(b) Test the Left and Right Endpoints of the open interval in Part (a) for convergence or divergence.

Left endpoint. $x=0.$

$$\sum_{n=0}^{\infty}(n+3)\cdot\left(\frac{0-3}{3}\right)^n = \sum_{n=0}^{\infty}(n+3)\cdot(-1)^n \quad \boxed{\text{DIV}} \quad \text{Test for DIV.}$$

$$\lim_{n\to\infty}(n+3)(-1)^n \neq 0$$

Right endpoint $x=3.$

$$\sum_{n=0}^{\infty}(n+3)\cdot\left(\frac{2\cdot3-3}{3}\right)^n = \sum_{n=0}^{\infty}(n+3). \quad \boxed{\text{DIV}}$$

$$\lim_{n\to\infty}(n+3)=\infty\neq 0 \quad \text{Test for DIV.}$$

Sec11.9. Power Series Representation. LecNote9.

Q11 Consider the function $f(x) = \frac{x}{1+x}$

(a) Find the first FOUR non-zero terms of the power series representation of the function $f(x)$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n = 1 - x + x^2 - x^3 + \cdots\cdots$$

$$\frac{x}{1+x} = x\cdot(1-x+x^2-x^3+\cdots)$$

first four non-zero terms: $x(1-x+x^2+x^3) = \boxed{x - x^2 + x^3 - x^4}$

(b) Use the expression in Part (a) to find the first THREE non-zero terms of the power series representation of the DERIVATIVE function of f,

$$f'(x) = \sum_{n=0}^{\infty} c_n x^n$$

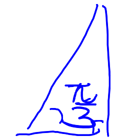
$$(x - x^2 + x^3)' = \boxed{1 - 2x + 3x^2}$$

(c) Use the expression in Part (a) to find the first THREE non-zero terms of the power series representation of the indefinite INTEGRAL of f,

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n x^n + C$$

$$\int x - x^2 + x^3 \, dx = \boxed{\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4}$$

Sec11.10/11. Taylor and Maclaurin Series. LecNote10. **Q12** Find the 3rd degree Taylor polynomial of $f(x) = x\cos(2x)$ centered at $x = \pi/6$



$$f\left(\tfrac{\pi}{6}\right) = \tfrac{\pi}{6}\cdot\cos\tfrac{\pi}{3} = \tfrac{\pi}{12}$$

$$f'(x) = \cos(2x) + x\cdot[-\sin(2x)]\cdot 2, \quad f'\left(\tfrac{\pi}{6}\right) = \cos\tfrac{\pi}{3} - 2\cdot\tfrac{\pi}{6}\cdot\sin\tfrac{\pi}{3} = \tfrac{1}{2} - \tfrac{\sqrt{3}}{6}\pi$$

$$\sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}, \quad \cos\tfrac{\pi}{3} = \tfrac{1}{2}$$

$$f''(x) = -2\sin 2x - 2\sin 2x - 2x\cdot\cos 2x\cdot 2 = -4\sin 2x - 4x\cdot\cos 2x.$$

$$f''\left(\tfrac{\pi}{6}\right) = -4\sin\tfrac{\pi}{3} - 4\cdot\tfrac{\pi}{6}\cdot\cos\tfrac{\pi}{3} = -2\sqrt{3} - \tfrac{\pi}{3}$$

$$f'''(x) = -8\cos 2x - 4\cos 2x - 4x\cdot(-2\sin 2x) = -12\cos 2x + 8x\cdot\sin 2x.$$

$$f'''\left(\tfrac{\pi}{6}\right) = -12\cos\tfrac{\pi}{3} + 8\cdot\tfrac{\pi}{6}\cdot\sin\tfrac{\pi}{3} = -6 + \tfrac{2}{3}\pi\sqrt{3}.$$

$$\boxed{T_3(x) = \tfrac{\pi}{12} + \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{6}\pi\right)\left(x - \tfrac{\pi}{6}\right) + \frac{\left(-2\sqrt{3} - \frac{\pi}{3}\right)}{2}\left(x - \tfrac{\pi}{6}\right)^2 + \frac{\left(-6 + \frac{2\sqrt{3}}{3}\pi\right)}{3!}\left(x - \tfrac{\pi}{6}\right)^3}$$

Q13 Consider the function $f(x) = \ln(1 - 2x)$.

(a) Find the Maclaurin series of $f(x)$.

$$\ln(1-2x) = \boxed{\sum_{n=1}^{\infty} \frac{(-2x)^n}{n}} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n}\cdot x^n$$

$$\left(\frac{-2x}{1} + \frac{4x^2}{2} - \frac{8x^3}{3} + \cdots \right)$$

(b) 'Evaluate'(Guess) the limit $\lim_{x\to 0} \frac{\ln(1-2x)}{x}$ by the power series expression in Part (a).

$$\lim_{x\to 0} \frac{(-2)x + \frac{4x^2}{2} - \frac{8x^3}{3} + \cdots}{x} = \boxed{-2}.$$

(c) Verify the answer in (b) by l'Hopital's Rule.

$$\lim_{x\to 0} \frac{\ln(1-2x)}{x} \qquad \frac{\ln 1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{1-2x}\cdot(-2)}{1} = \frac{1}{1-2\cdot 0}\cdot(-2) = \boxed{-2}.$$

Q14 Consider the function $f(x) = xe^x$.

(a) Find the power series expansion of the function $f(x) = xe^x$ centered at $x = 0$.

$$x \cdot e^x = x \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^{n+1}$$

$$= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

(b) Find the 3rd degree Taylor polynomial $T_3(x)$ of $f(x)$ at $x = 0$. Find $T_3(0.5)$ for estimating $f(0.5)$.

$$\boxed{T_3(x) = x + x^2 + \frac{x^3}{2!}}, \quad \boxed{T_3(0.5) = 0.5 + 0.5^2 + \frac{0.5^3}{2}}$$

(c) Suppose we know that $|f^4(x)| \leq 15$ for all $|x| \leq 1$, what the error in Part (b) when we use $T_3(0.5)$ to approximate $f(0.5)$? And what's the maximal error for estimating $f(x)$ via $T_3(x)$ on $[-1, 1]$.

$n = 3$, $a = 0$, $d = 1$ in Taylor's inequality, $M = 15$.

$$|R_3(0.5)| \leq \frac{M}{(3+1)!} \cdot |0.5 - 0|^{3+1} = \boxed{\frac{15}{4!} \cdot 0.5^4}$$

For any x, $|x| \leq 1$.

$$|R_3(x)| \leq \frac{M}{(3+1)!} \cdot |x - 0|^{3+1} \leq \frac{15}{4!} \cdot 1^4 = \boxed{\frac{15}{4!}} \leftarrow \text{maximal error on } [-1, 1].$$