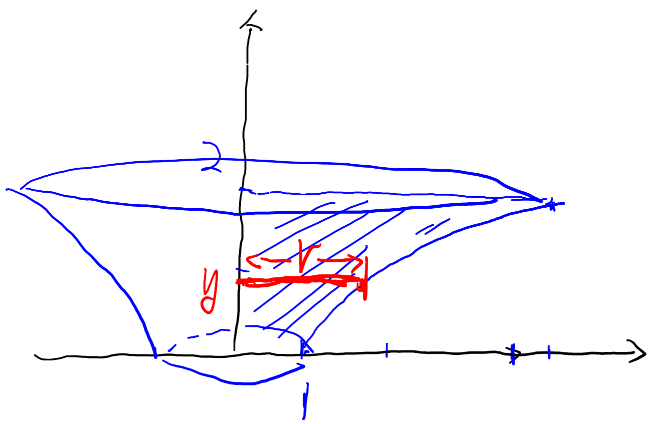
Sec5.3. Volume. LecNote1.

Q1 Find the volume of the following rotating solids.

(a)(Vertical Axe) The region R is bounded by $y = \sqrt{x-1}, y = 2, x = 0, y = 0$. The solid is generated by revolving the region R about the y axis. Sketch the region R and the rotating solid S. Find the volume of the rotating solid.



$y = \sqrt{x-1} \iff y^2 + 1 = x$.

$\Rightarrow r = y^2 + 1$.

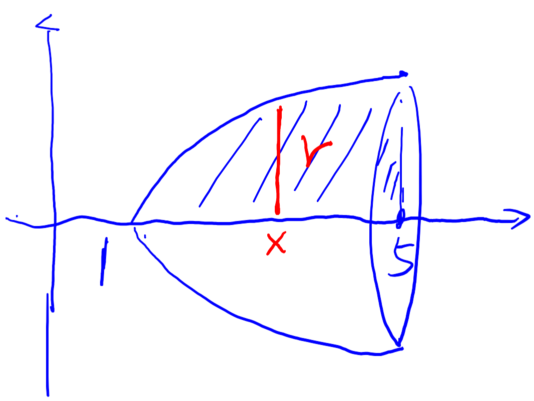
$$V = \int_0^2 \pi \cdot r^2 \cdot dy$$

$$= \int_0^2 \pi \cdot (y^2+1)^2 dy$$

$$= \pi \cdot \int_0^2 y^4 + 2y^2 + 1 \, dy$$

$$= \pi \left(\frac{1}{5} y^5 + \frac{2}{3} y^3 + y\right)\Big|_0^2 = \boxed{\pi\left(\frac{2^5}{5} + \frac{2^4}{3} + 2\right)}$$

(b)(Horizontal Axe) The region R is bounded by $y = \sqrt{x-1}, y = 0, x = 5$. The solid is generated by revolving the region R about the axis $y = -1$. Sketch the region R and the rotating solid S. Find the volume of the rotating solid.



$r = \sqrt{x-1}$

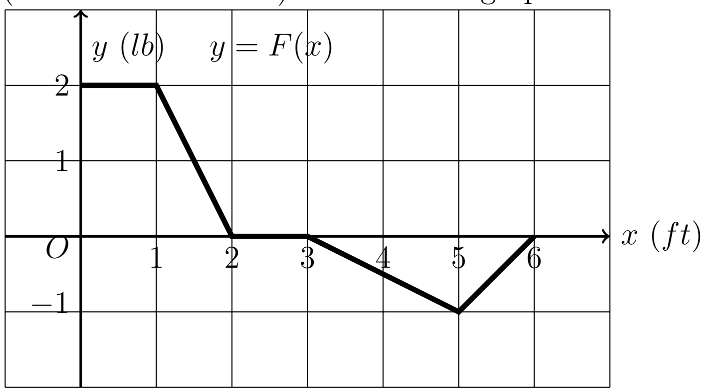
$$V = \int_1^5 \pi \cdot r^2 \cdot dx$$

$$= \int_1^5 \pi \cdot (x-1) dx$$

$$= \pi \left(\frac{1}{2} x^2 - x\right)\Big|_1^5$$

$$= \boxed{\pi\left(\frac{25}{2} - 5\right) - \pi\left(\frac{1}{2} - 1\right)}$$

Q2(Definition of Work.) Below is the graph of a force function $F(x)$ (in lbs).



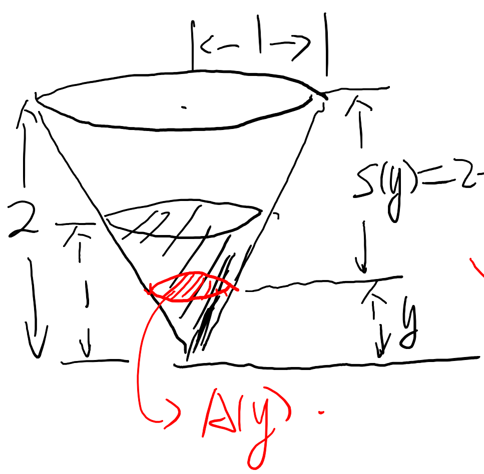
(a) How much work is done by the force in moving an object from $x = 0$ to $x = 3$?

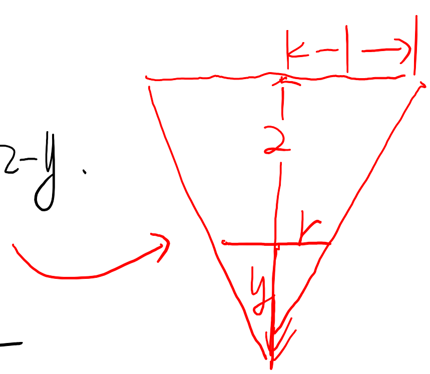
$$W = \frac{1}{2}\cdot 2\cdot(1+2) + 0 = \boxed{3 \text{ ft-lbs.}}$$

(b) How much work is done by the force in moving an object from $x = 0$ to $x = 5$?

$$W = \frac{1}{2}\cdot 2\cdot(1+2) - \frac{1}{2}\cdot 1\cdot 2 = 3 - 1 = \boxed{2 \text{ ft-lbs.}}$$

Q3(Water-Pumping) A tank is in the shape of a downward-pointing cone (vertex at the bottom) has height 2 ft and radius 1 ft. It is filled of soda half the height of the full tank (1 ft above the bottom.) The soda weighs 63 lbs/ft^3. How much work does it take to pump all of the soda from a tank to an outlet which is at the level of the top of the tank.





$$\frac{r}{y} = \frac{1}{2}, \Rightarrow r = \frac{1}{2}y$$

$$A(y) = \pi\cdot r^2 = \pi\cdot\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}\cdot y^2$$

$$\sigma = 63, \quad s(y) = 2-y.$$

$$W = \int_0^1 \sigma\cdot s(y)\cdot A(y)\cdot dy$$

$$= \int_0^1 63\cdot(2-y)\,\frac{\pi}{4}\cdot y^2\, dy$$

$$= \frac{63\pi}{4}\int_0^1 2y^2 - y^3\, dy$$

$$= \frac{63\pi}{4}\left(\frac{2}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$$

$$= \boxed{\frac{63\pi}{4}\cdot\left(\frac{2}{3} - \frac{1}{4}\right) \text{ ft-lbs}}$$

Sec6.1. LecNote2. Sec6.2-6.4. LecNote3. Sec6.6-6.7. LecNote4.

Q4 Derivatives of the inverse functions/inverse trig/log/exp/hyperbolic functions.

(a)(Sec6.1,6.4) $f(x) = x^2 + \log_2(x+1) + 1$, find $(f^{-1})'(1)$ given $f(0) = 1$.

$f'(x) = 2x + \frac{1}{\ln 2}\cdot\frac{1}{x+1}$, $f(0)=1 \Rightarrow f^{-1}(1)=0$

$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{0+\frac{1}{\ln 2}\cdot\frac{1}{0+1}} = \boxed{\ln 2}$

(b)(Sec6.4,6.6) $f(x) = 3^{\sin^{-1}(x)}$, find $f'(x)$ and $f'(\frac{1}{2})$.

$f'(x) = \ln 3 \cdot 3^{\sin^{-1}(x)} \cdot (\sin^{-1}x)' = \ln 3 \cdot 3^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$f'(\frac{1}{2}) = \ln 3 \cdot 3^{\sin^{-1}(\frac{1}{2})} \cdot \frac{1}{\sqrt{1-(\frac{1}{2})^2}} = \boxed{\ln 3 \cdot 3^{\frac{\pi}{6}} \cdot \frac{1}{\sqrt{\frac{3}{4}}}}$

(c)(Sec6.3,6.6) $y = [\tan^{-1} x]^{\ln(\sqrt{x})}$, find y' and $y'(1)$.

$\ln y = \ln(\tan^{-1}x)^{\ln\sqrt{x}} = \ln\sqrt{x} \cdot \ln(\tan^{-1}x) = \frac{1}{2}\cdot\ln x \cdot \ln\tan^{-1}x$

Take derivative: $\frac{1}{y}\cdot y' = \frac{1}{2}\cdot\frac{1}{x}\cdot\ln\tan^{-1}x + \frac{1}{2}\ln x \cdot \frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}$

$\Rightarrow \quad y' = y\cdot\left[\frac{1}{2}x\cdot\ln\tan^{-1}x + \frac{\ln x}{2}\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}\right]$

$\tan\frac{\pi}{4} = 1$

$\Rightarrow \tan^{-1}1 = \frac{\pi}{4}$ $\quad y'(1) = \cancel{(\tan^{-1}1)^{\overset{0}{\overset{\shortparallel}{\ln 1}}}}\left[\frac{1}{2}\cdot 1\cdot\ln\tan^{-1}1 + \cancel{\frac{\overset{0}{\overset{\shortparallel}{\ln 1}}}{2}\cdot\frac{1}{\tan^{-1}1}}\cdot\frac{1}{2}\right] = \boxed{\frac{1}{2}\ln\frac{\pi}{4}}$

(d)(Sec6.2,6.7) $y = \sinh(2x)$, find $y'(0)$ and $y''(0)$.

$y' = \cosh(2x)\cdot 2$, $y'(0) = \cosh(0)\cdot 2 = \frac{e^0+e^{-0}}{2}\cdot 2 = \boxed{2}$.

$y'' = \sinh(2x)\cdot 4$, $y''(0) = \sinh(0)\cdot 4 = \frac{e^0-e^{-0}}{2}\cdot 4 = \boxed{0}$

Sec6.5/9.3. Initial Value Problems. *LecNote3.*

Q5 A population $P(t)$ of insects increases according to the following law $P'(t) = k(P - 100)$. Suppose there are 500 insects at time $t = 0$, and 700 insects 5 days later. Find an expression for the number $P(t)$ of insects at time $t > 0$(in days). How many insects will there be in 5 more days?

$$\frac{dP}{dt} = k(P-100) \iff \frac{dP}{P-100} = k\,dt \iff \int \frac{1}{P-100}\,dP = \int k\,dt$$

$$\iff \ln|P-100| = kt + C_1$$

$$\iff |P-100| = e^{kt+C_1} = e^{kt} \cdot C_2.$$

$$\Rightarrow P(t) = 100 + C_2 \cdot e^{kt}$$

$$P(0) = 100 + C_2 = 500 \Rightarrow C_2 = 400$$

$$P(5) = 100 + 400 \cdot e^{5k} = 700 \Rightarrow e^{5k} = \frac{600}{400} = \frac{3}{2} \Rightarrow k = \frac{1}{5}\ln\frac{3}{2}.$$

$$\Rightarrow \boxed{P(t) = 100 + 400 \cdot e^{\frac{1}{5}\ln\frac{3}{2} \cdot t}}$$

$$P(10) = 100 + 400 \cdot e^{\frac{1}{5}\ln\frac{3}{2} \cdot 10} = 100 + 400 \cdot e^{2\ln\frac{3}{2}} = 100 + 400 \cdot e^{\ln\left(\frac{3}{2}\right)^2} = 100 + 400 \cdot \left(\frac{3}{2}\right)^2$$

$$= 100 + 900 = \boxed{1000}$$

Q6 Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{xe^{x^2}}{y}, \quad y(0) = -3$$

$$y \cdot dy = x \cdot e^{x^2}\,dx.$$

$$\int y \cdot dy = \int x \cdot e^{x^2}\,dx.$$

$$\xrightarrow{u = x^2} \int e^u \tfrac{1}{2}\,du = \tfrac{1}{2}e^u$$

$$\frac{1}{2}y^2 = \frac{1}{2}e^{x^2} + C$$

$$y^2 = e^{x^2} + 2C$$

$$y(0) = -3 \Rightarrow x = 0,\ y = -3$$

$$(-3)^2 = e^0 + 2C \Rightarrow 9 = 1 + 2C \Rightarrow C = 4.$$

$$\Rightarrow y^2 = e^{x^2} + 8$$

$$\Rightarrow y = \pm\sqrt{e^{x^2} + 8}$$

$$\boxed{\begin{gathered} y(0) = -3 \\ \Rightarrow y = -\sqrt{e^{x^2} + 8} \end{gathered}}$$

Sec6.8. l'Hopital Rule. LecNote4.

Q7 Determine whether the following limits exist or not. Find the limit if it exists.

(a)

$$\lim_{x\to 0}\frac{\sec x - 1}{e^x - 1} \quad \left(\frac{0}{0}\right) \leftarrow \text{Caution: You have to state it is a } \frac{0}{0} \text{ case indeterminate form}.$$

$$\overset{l'H}{=} \lim_{x\to 0}\frac{(\sec x - 1)'}{(e^x - 1)'}$$

$$= \lim_{x\to 0}\frac{\tan x \cdot \sec x}{e^x}$$

$$= \frac{\tan 0 \cdot \sec 0}{e^0} = \frac{0\cdot 1}{1} = \boxed{0}$$

(b)

$$\lim_{x\to 0} x\ln(x^2)$$

$$= \lim_{x\to 0}\frac{\ln(x^2)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\right).$$

$$\overset{l'H.}{=} \lim_{x\to 0}\frac{(2\ln|x|)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0}\frac{\frac{2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -2x = \boxed{0}$$

(c)

$$\lim_{x\to +\infty}(2x)^{\frac{1}{x}} = \lim_{x\to +\infty} e^{\ln(2x)^{\frac{1}{x}}} = e^{\lim_{x\to\infty}\frac{\ln(2x)}{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{l'H}{=} e^{\lim_{x\to\infty}\frac{(\ln(2x))'}{(x)'}}$$

$$= e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = \boxed{e^0 = 1}$$

Caution:

$$[\ln(2x)]' = \frac{1}{2x}\cdot(2x)' = \frac{1}{x}$$

Not $\frac{2}{x}$.

Sec7.1-7.4. Methods of Integration. *LecNote5.* *LecNote6.*

Q8 Evaluate the following integrals

(a)(Sec7.1.IBP)

$$\int \underbrace{(\ln x)^2}_{u}\underbrace{\mathrm{d}x}_{dv}$$

$u=(\ln x)^2$, $du = 2\ln x \cdot \frac{1}{x}dx$

$dv = dx$, $v = x$.

$= (\ln x)^2 \cdot x - \int \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} dx$.

$= (\ln x)^2 \cdot x - \int 2\ln x \cdot dx$.

$u = 2\ln x$, $du = \frac{2}{x}dx$

$dv = dx$, $v = x$

$= (\ln x)^2 \cdot x - \left[2\ln x \cdot x - \int \cancel{x} \cdot \frac{2}{\cancel{x}} dx\right]$

$= \boxed{(\ln x)^2 \cdot x - \left[2\ln x \cdot x - 2x\right] + C}$

$= (\ln x)^2 \cdot x - 2\ln x \cdot x + 2x + C$.

(b)(Sec7.2.TrigInt)

$$\int \sin^3 x \cdot \cos^{61} x \, \mathrm{d}x$$

(Both odd, sub the higher one?)

$= \int \sin^2 x \cdot \underline{\sin x} \cdot \cos^{61} x \cdot \underline{dx}$

$u = \cos x$, $du = -\sin x \cdot dx$.

$\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$= \int (1-u^2) \cdot u^{61} (-du)$

$= \int -u^{61} + u^{63} \, du$

$= -\frac{1}{62} \cdot u^{62} + \frac{1}{64} \cdot u^{64} + C$

$= \boxed{-\frac{1}{62} \cdot (\cos x)^{62} + \frac{1}{64} \cdot (\cos x)^{64} + C}$

(c)(Sec7.3.TrigSub)

$$\int \frac{x^3}{\sqrt{x^2+1}}\mathrm{d}x$$

$\sqrt{x^2+1}$ $\quad \dfrac{x=\tan\theta}{dx=\sec^2\theta\, d\theta}$ $\quad \sqrt{\tan^2\theta+1}=\sqrt{\sec^2\theta}=\sec\theta.$

$$=\int \frac{\tan^3\theta}{\sec\theta}\cdot \sec^2\theta\, d\theta$$

$$=\int \tan^3\theta\cdot\sec\theta\, d\theta = \int \tan^2\theta \cdot \underbrace{\tan\theta\cdot\sec\theta\cdot d\theta}$$

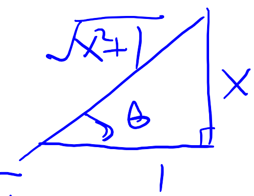
$u=\sec\theta,\ du=\tan\theta\cdot\sec\theta\, d\theta$

$\tan^2\theta=\sec^2\theta-1=u^2-1$

$$=\int (u^2-1)\cdot du.$$

$$=\frac{1}{3}u^3-u+C$$

$x=\tan\theta \Rightarrow$



$$=\frac{1}{3}\sec^3\theta-\sec\theta+C.$$

$\sec\theta=\sqrt{x^2+1}.$

$$=\boxed{\frac{1}{3}\left(\sqrt{x^2+1}\right)^3-\sqrt{x^2+1}+C}$$

(d)(Sec7.4.PartialFraction.)

$$\int_0^2 \frac{10}{x^2-4x-21}\mathrm{d}x$$

$$\frac{10}{(x-7)(x+3)}=\frac{A}{x-7}+\frac{B}{x+3}=\frac{1}{x-7}+\frac{-1}{x+3}$$

$$10 = A(x+3)+B\cdot(x-7)$$

$x=-3 \Rightarrow 10=B\cdot(-10) \Rightarrow B=-1$

$x=7 \Rightarrow 10=A\cdot 10 \Rightarrow A=1.$

$$=\int_0^2 \frac{1}{x-7}-\frac{1}{x+3}\,dx$$

$$=\ln|x-7|-\ln|x+3|\Big|_0^2$$

$$=(\ln 5-\ln 5)-(\ln 7-\ln 3)$$

$$=\boxed{-\ln 7+\ln 3}$$

Sec7.8. Improper Integral. LecNote6.

Q9 Determine whether the improper integral is convergent or divergent. Evaluate the integral if it is convergent.

(a)

$$\int_0^2 \frac{1}{(x-1)^2}\mathrm{d}x = \int_0^1 \frac{1}{(x-1)^2}dx + \int_1^2 \frac{1}{(x-1)^2}dx.$$

l.h.s. converges only if both integrals on the right converge.

It is enough to check one of them.

$$\int_0^1 \frac{1}{(x-1)^2}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^2}dx = \lim_{t\to 1^-} \frac{-1}{x-1}\Big|_0^t$$

$$= \lim_{t\to 1^-} \frac{-1}{t-1} - \frac{-1}{0-1} = \boxed{+\infty}$$

$\boxed{\text{Therefore, } \int_0^2 \frac{1}{(x-1)^2}dx \text{ is divergent.}}$

Rmk: It is also OK to test the second one as.

$$\int_1^2 \frac{1}{(x-1)^2}dx = \lim_{t\to 1^+}\int_t^2 \frac{1}{(x-1)^2}dx = \lim_{t\to 1^+} \frac{-1}{(x-1)}\Big|_t^2 = \lim_{t\to 1^+} \frac{-1}{2-1} - \frac{-1}{t-1} = +\infty$$

(DIV)

(b)

$$\int_4^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$$

$$= \lim_{t\to+\infty}\int_4^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx,$$

$u=-\sqrt{x}$, $du = -\frac{1}{2\sqrt{x}}dx \Rightarrow \frac{dx}{\sqrt{x}} = -2du$

$$= \lim_{t\to+\infty}\int e^u(-2du)$$

$$= \lim_{t\to+\infty} -2e^u$$

$$= \lim_{t\to\infty} -2e^{-\sqrt{x}}\Big|_4^t = \lim_{t\to\infty} -2e^{-\sqrt{t}} + 2e^{-\sqrt{4}} \overset{e^{-\infty}=0}{=\!=} \boxed{2e^{-2}}$$