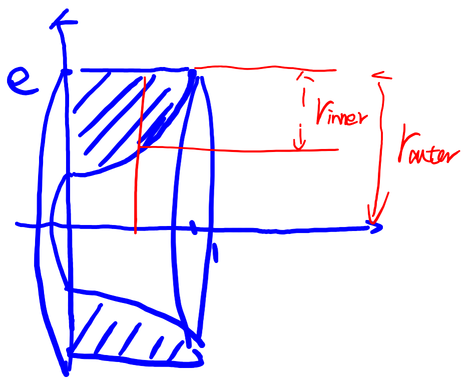
Practice Final Version A, Sec61

Q1[*Sec5.2*, rotating solid, Washer] Sketch the region R bounded by $y = e^x, y = e,\ x = 0$.

(a) Set up an integral for the volume of the solid rotating R about the $x-$axis. Do not evaluate the integral.

$r_{outer} = e\ ,\quad r_{inner} = e^x$

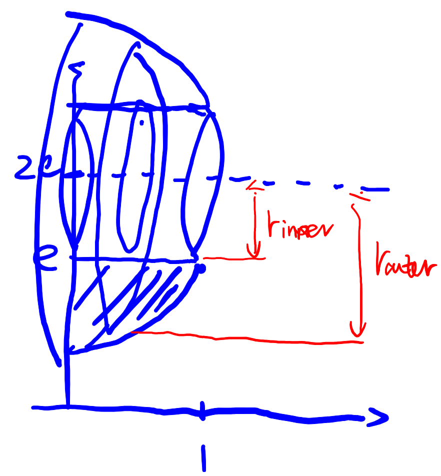
$$\boxed{V = \int_0^1 \pi \cdot \left[e^2 - (e^x)^2 \right] \cdot dx .}$$



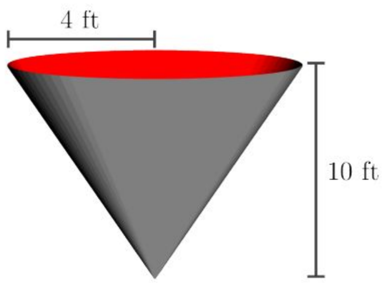
(b) Set up an integral for the volume of the solid rotating R about the axis $y = 2e$. Do not evaluate the integral.

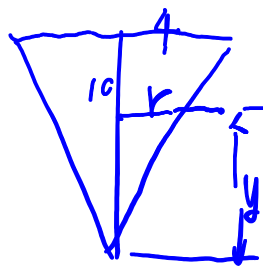
$r_{outer} = 2e - e^x\ ,\quad r_{inner} = 2e - e$

$$\boxed{V = \int_0^1 \pi \left[(2e - e^x)^2 - (2e - e)^2 \right] dx .}$$



Q2[*Sec5.4, Water-Pumping*] A conical water tank with a top diameter of 8 feet and height of 10 feet is standing at ground level as shown in the sketch below. Water weighing 60 pounds per cubic foot is pumped from the tank to an outlet 3 feet above the top of the tank. If the tank is full, how many foot-pounds of work are required to pump all of the water from the tank?





$s(y) = 10 - y$

$\frac{r}{y} = \frac{4}{10}$

$r = \frac{2}{5} y.$

$A(y) = \pi \cdot r^2 = \pi \cdot (\frac{2}{5} y)^2$

$$W = \int_0^{10} \sigma \cdot s(y) \cdot A(y)\, dy = \int_0^{10} 60 \cdot (10-y) \cdot \pi \cdot \frac{4}{25} y^2 \cdot dy$$

$$= \frac{240\pi}{25} \cdot \int_0^{10} 10 \cdot y^2 - y^3\, dy$$

$$= \frac{240\pi}{25} \cdot \left(10 \cdot \frac{1}{3} y^3 - \frac{1}{4} \cdot y^4\right) \Big|_0^{10}$$

$$= \boxed{\frac{240\pi}{25} \cdot \left[10 \cdot \frac{1}{3} \cdot 10^3 - \frac{1}{4} \cdot 10^4\right]}$$

Q3[*Sec6.1, derivative formula for inverse functions*] Let $f(x) = 3^x - 3x, x > 0$. Find $(f^{-1})'(0)$. Notice that $f(1) = 0$. $f^{-1}(0) = 1$

$f'(x) = \ln 3 \cdot 3^x - 3$

$$(f^{-1})'(0) = \frac{1}{f'[f^{-1}(0)]} = \frac{1}{f'(1)} = \boxed{\frac{1}{\ln 3 \cdot 3^1 - 3}}$$

Q4[*Sec6.2-6.4, exp/log functions*] Find the derivative of the following functions.

(a)

$$f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh x} \cdot (\sinh x)' = \frac{1}{\sinh x} \cdot \cosh x$$

(b)

$$f(x) = \sec(\arctan x)$$

$$f'(x) = \tan(\arctan x) \sec(\arctan x) \cdot (\arctan x)'$$

$$= \tan(\tan^{-1} x) \cdot \sec(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

$$= x \cdot \sec(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

Remark: $\tan(\tan^{-1} x) = x$ actually.

Q5[*Sec6.5/9.3, initial value problem*] Find $y = y(x)$ if

$$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} - \sqrt{y} = 0, \; y(0) = 4$$

$$\sec x \cdot \frac{dy}{dx} = \sqrt{y}$$

$$\frac{1}{\sqrt{y}} \cdot dy = \frac{1}{\sec x} dx = \cos x \cdot dx.$$

$$\int \frac{1}{\sqrt{y}} \cdot dy = \int \cos x \cdot dx.$$

$$2\sqrt{y} = \sin x + C. \quad y(0)=4, \Rightarrow x=0, y=4.$$

$$2\sqrt{4} = \sin 0 + C \Rightarrow C = 2\sqrt{4} = 4.$$

$$2\sqrt{y} = \sin x + 4.$$

$$\sqrt{y} = \frac{1}{2}\sin x + 2 \Rightarrow \boxed{y = \left(\frac{1}{2}\sin x + 2\right)^2}$$

Q6[*Sec6.8/10.1, L'Hospital's Rule*] Evaluate the following limits.

(a)[∞^0**-type**]

$$\lim_{n\to\infty} \sqrt[n]{n^2+1} = \lim_{n\to\infty} (n^2+1)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n^2+1)} = e^0 = \boxed{1}$$

$$\lim_{n\to\infty} \frac{\ln(n^2+1)}{n} \overset{L'H}{=\!=} \lim_{n\to\infty} \frac{\frac{1}{n^2+1}\cdot 2n}{1} = \lim_{n\to\infty} \frac{2n}{n^2+1} = 0$$

(b)[$\infty \cdot 0$**-type**]

$$\lim_{n\to\infty} n^2\left(\cos\frac{1}{n} - 1\right) = \lim_{n\to\infty} \frac{\cos(\frac{1}{n}) - 1}{\frac{1}{n^2}} \qquad \frac{\cos 0 - 1}{\frac{1}{\infty}} = \frac{0}{0}$$

$$\overset{L'H}{=\!=} \lim_{n\to\infty} \frac{-\sin(\frac{1}{n})\cdot(-\frac{1}{n^2})}{\frac{-2}{n^3}} \qquad \text{simplify}$$

$$= \lim_{n\to\infty} -\frac{n\cdot\sin(\frac{1}{n})}{2}.$$

$$= \lim_{n\to\infty} -\frac{\sin(\frac{1}{n})}{\frac{2}{n}}. \qquad \frac{\sin 0}{0}$$

$$\overset{L'H}{=\!=} \lim_{n\to\infty} -\frac{\cos(\frac{1}{n})\cdot(\frac{-1}{n^2})}{\frac{-2}{n^2}} = \lim_{n\to\infty} -\frac{1}{2}\cos(\frac{1}{n}) = \boxed{-\frac{1}{2}\cdot\cos 0 = -\frac{1}{2}}$$

Q7[*Sec7.1-7.2, integration by parts and trig-integral*] Evaluate the following integrals.

(a) ***[Sec7.1,IBP for polynomial*** $\times \sin/\cos/exp$***-type]***

$$\int (x+1)\sin x \, \mathrm{d}x$$

$u = x+1, \quad dv = \sin x \cdot dx.$

$du = dx, \quad v = -\cos x.$

$$= (x+1)(-\cos x) - \int (-\cos x) \cdot dx.$$

$$= (x+1)\cdot(-\cos x) + \int \cos x \cdot dx$$

$$= \boxed{-(x+1)\cos x + \sin x + C}$$

(b) ***[Sec7.2, Odd/Even rule for*** sin-cos-***type]***

$$\int_0^{\pi/6} (2+\cos\theta)^2 \, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} 4 + 4\cdot\cos\theta + \cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4 + 4\cos\theta + \frac{1+\cos 2\theta}{2} d\theta.$$

$$= \int_0^{\frac{\pi}{6}} \frac{9}{2} + 4\cos\theta + \frac{1}{2}\cos(2\theta) \, d\theta$$

$$= \frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin(2\theta) \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{9}{2}\cdot\frac{\pi}{6} + 4\sin\frac{\pi}{6} + \frac{1}{4}\cdot\sin\frac{\pi}{3} - 0$$

$$= \boxed{\frac{9\pi}{12} + 4\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2}}$$

(c) ***[Sec7.2,*** tan-sec ***type]***

$$\int \sec^2(2x)\tan(2x) \, \mathrm{d}x$$

$u = \tan(2x). \quad du = 2\cdot\sec^2(2x)\cdot dx$

$$= \int \tan(2x)\cdot \sec^2(2x) dx$$

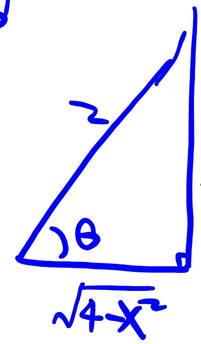
$$= \int u \cdot \frac{1}{2} du = \frac{1}{2}\cdot\frac{1}{2} u^2 + C$$

$$= \boxed{\frac{1}{4}(\tan(2x))^2 + C}$$

Q8[*Sec7.3-7.4, trig-sub and partial fraction decomposition*] Evaluate the following integrals.

(a) ***[Trig-sub]***

$$\int \sqrt{4-x^2}\,\mathrm{d}x \qquad x = 2\cdot\sin\theta. \qquad dx = 2\cos\theta\, d\theta$$



$x = 2\cdot\sin\theta$

$\frac{x}{2} = \sin\theta.$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}.$

$$= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int \sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\, d\theta$$

$$= \int 2\cdot\cos\theta\cdot 2\cdot\cos\theta\, d\theta$$

DAF.

$$= \int 4\cdot\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int 2 + 2\cos 2\theta\, d\theta$$

$$= 2\theta + \sin(2\theta) + C$$

$$= 2\theta + 2\sin\theta\cdot\cos\theta + C.$$

$$= \boxed{2\cdot\sin^{-1}\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C}$$

(c) ***[Sec7.4, P.F.D. linear product type]***

$$\int \frac{2}{t^2-1}\,\mathrm{d}t \qquad \frac{2}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{-1}{t+1} + \frac{1}{t-1}.$$

$$2 = A\cdot(t-1) + B(t+1) \qquad t=1 \Rightarrow B=1$$

$$t=-1 \Rightarrow A=-1.$$

$$= \int \frac{-1}{t+1} + \frac{1}{t-1}\, dt$$

$$= \boxed{-\ln|t+1| + \ln|t-1| + C}$$

Q9[*Sec7.8, improper integral*] Determine whether each of the improper integral is convergent or divergent. Evaluate the improper integral if it is convergent.

(a)

$$\int_0^\infty \frac{2}{1+9x^2}\,\mathrm{d}x$$

$9x^2 = u^2 \quad (3x)^2 = u^2 , \quad 3x = u . \quad x = \frac{u}{3}$

$dx = \frac{1}{3} du .$

$= \lim_{t\to\infty} \int_0^t \frac{2}{1+u^2} \cdot \frac{1}{3} du .$

$= \lim_{t\to\infty} \frac{2}{3} . \arctan(u) .$

$= \lim_{t\to\infty} \frac{2}{3} . \arctan(3x) . \Big|_0^t$

$= \lim_{t\to\infty} \frac{2}{3} \arctan(3t) - \frac{2}{3} \arctan(0)$

$= \frac{2}{3} \arctan(\infty) - 0 = \boxed{\frac{2}{3} \cdot \frac{\pi}{2}}$

$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

$\tan 0 = 0$

$\arctan 0 = 0 .$

(b)

$$\int_0^{1/2} \frac{1}{\sqrt{1-2x}}\,\mathrm{d}x$$

$\int \frac{1}{\sqrt{1-2x}} dx \xrightarrow[du = -2dx .]{u = 1-2x} \int \frac{1}{\sqrt{u}} \cdot (-\frac{1}{2}) du .$

$-\frac{1}{2} du = dx$

$= -\frac{1}{2} \cdot 2\sqrt{u} .$

$= -\sqrt{u} = -\sqrt{1-2x} \Big|_0^t$

$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2x}} dx = \lim_{t\to\frac{1}{2}^-} \int_0^t \frac{1}{\sqrt{1-2x}} dx$

$= -\sqrt{1-2t} + \sqrt{1-0}$

$= \lim_{t\to\frac{1}{2}^-} -\sqrt{1-2t} + \sqrt{1} = -\sqrt{0} + 1 = \boxed{1}$

Q10[*Sec11.2-11.6*] Determine whether each of the series is convergent or divergent. Please show your work and name any test(s) that are used.

(a) ***[Sec11.2,n-th term test for DIV]***

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{3n^2}\right)$$

$$\lim_{n\to\infty} \cos\left(\frac{1}{3n^2}\right) = \cos 0 = \boxed{1 \neq 0}$$

Divergent Test $\Rightarrow$ $\boxed{\text{DIV}}$

(b) ***[Sec11.4,(limit) Comparison Test]***

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+1} = a_n, \quad b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}, \quad p = \frac{3}{2} > 1, \quad \sum b_n \text{ conV.}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \cancel{\frac{\sqrt{n}+1}{n^2+1}} \cdot \cancel{\frac{n^2}{\sqrt{n}}} = 1 \neq 0$$

$\sum b_n$ conV $\Rightarrow$ $\sum a_n$ conV. due to limit Comparison Test.

(c) ***[Sec11.6, Ration Test]***

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} < 1.$$

Ratio Test $\Rightarrow$ conV.

Q11[*Sec11.6,11.8, ratio test for the radius of convergence of power series*] Consider the following power series. Find its center and radius of convergence.

$$\sum_{n=1}^{\infty} \frac{3^n (x+5)^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}\cdot(x+5)^{n+1}}{(n+1)!}}{\frac{3^n\cdot(x+5)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{(x+5)^{n+1}}{(x+5)^n} \right|$$

$$= \lim_{n\to\infty} 3\cdot\frac{1}{n+1}\cdot|x+5| = 0 < 1.$$

Center $x+5=0 \Rightarrow x=-5$ (or $a=-5$).

Radius of conv: ∞

(Interval: $(-\infty, +\infty)$).

Q12[*Sec11.9/11.10, power series representation*] Let

$$f(x) = \frac{2+x}{1+x^2}.$$

Find the **third degree** Maclaurin polynomial of $f(x)$.

$$f(x) = (2+x)\cdot\frac{1}{1-(-x^2)} = (2+x)\cdot\sum_{n=0}^{\infty}(-x^2)^n$$

$$= (2+x)\cdot[1-x^2+x^4-x^6+\cdots]$$

$$= \underbrace{2-2x^2}+2x^4+\cdots+\underbrace{x-x^3}+x^5+\cdots$$

$$= \boxed{2+x-2x^2-x^3}+\cdots$$

Q12, Taylor series Let $f(x) = (1+x^2)\sin x$.

(a) Find the 2nd degree Taylor polynomial of $f(x)$ at $x = \frac{\pi}{2}$.

$\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$

$f(\frac{\pi}{2}) = [1 + (\frac{\pi}{2})^2] \cdot 1 = 1 + \frac{\pi^2}{4}$.

$f'(x) = 2x \cdot \sin x + (1+x^2) \cdot \cos x$, $f'(\frac{\pi}{2}) = 2 \cdot \frac{\pi}{2} \cdot 1 + 0 = \pi$.

$f''(x) = 2 \cdot \sin x + 2x \cdot \cos x + 2x \cdot \cos x + (1+x^2)(-\sin x)$

$f''(\frac{\pi}{2}) = 2 \cdot 1 + (1 + \frac{\pi^2}{4}) \cdot (-1) = 1 - \frac{\pi^2}{4}$.

$T_2(x) = \boxed{1 + \frac{\pi^2}{4} + \pi \cdot (x - \frac{\pi}{2}) + \frac{1 - \frac{\pi^2}{4}}{2} \cdot (x - \frac{\pi}{2})^2}$

(b) Find the first four nonzero terms in the Maclaurin series of $f(x)$.

$f(x) = (1+x^2) \cdot [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots]$

$= \underline{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}} + \cdots + \underline{x^3 - \frac{x^5}{3!} + \frac{x^7}{5!}} - \frac{x^9}{7!} + \cdots$

$= \boxed{x + (-\frac{1}{3!} + 1)x^3 + (\frac{1}{5!} - \frac{1}{3!})x^5 + (-\frac{1}{7!} + \frac{1}{5!})x^7} + \cdots$

Q13[***Sec10.1,10.2, Parametric curves/equations***] Consider the parametric equations given by

$$x(t) = \ln(t+1), \quad y(t) = \sqrt{t+3} \qquad \text{at}$$

(a) Find the equation of the tangent line to the parametric curve at $(x, y) = (0, \sqrt{3})$

$$(x, y) = (\underset{0}{\underset{\shortparallel}{\ln(t+1)}}, \underset{\sqrt{3}}{\underset{\shortparallel}{\sqrt{t+3}}}), \quad t=0$$

$$\frac{dx}{dt} = \frac{1}{t+1}, \quad \frac{dy}{dt} = \frac{1}{2}\frac{1}{\sqrt{t+3}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{t+3}}}{\frac{1}{t+1}} \overset{t=0}{=\!=} \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{3}}}{\frac{1}{0+1}} = \frac{1}{2\sqrt{3}}$$

Point-Slope

$$y - \sqrt{3} = \frac{1}{2\sqrt{3}}\cdot(x-0)$$

$$\boxed{y = \frac{1}{2\sqrt{3}}\cdot x + \sqrt{3}}$$

(b) Set up an integral to find the arc length of the curve from $t = 0$ to $t = 1$. DO NOT EVALUATE THE INTEGRAL.

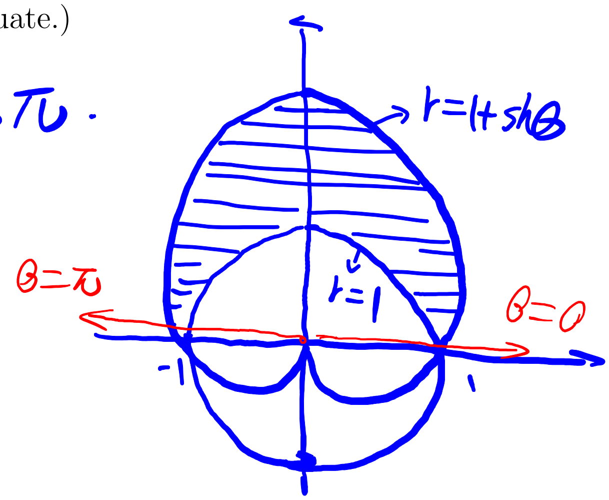
$$\text{Arc-length} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \boxed{\int_0^1 \sqrt{\left(\frac{1}{t+1}\right)^2 + \left(\frac{1}{2}\cdot\frac{1}{\sqrt{t+3}}\right)^2}\, dt}$$

Q14[***Sec10.3,10.4, polar curve***]

(a) Sketch the curves $r = 1 + \sin\theta$, and $r = 1$. Set up the integral for the area of the region inside $r = 1 + \sin\theta$ and outside $r = 1$.(Do not evaluate.)

$r = 1+\sin\theta = 1$, $\sin\theta = 0$, $\theta = 0, \pi$.



$$\text{Area1} = \int_0^{\pi} \frac{1}{2}(1+\sin\theta)^2 \cdot d\theta$$

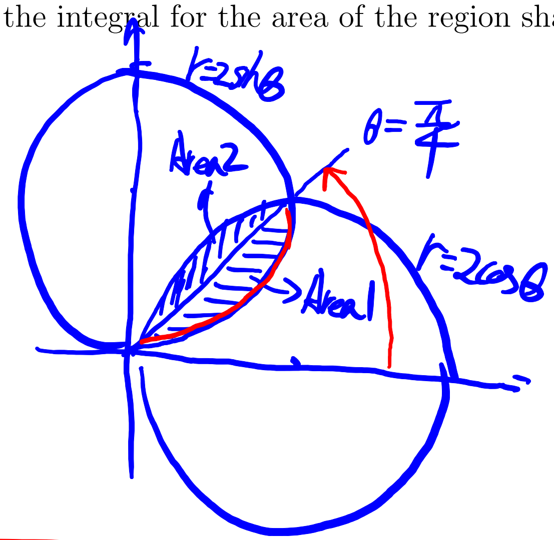
$$\text{Area2} = \int_0^{\pi} \frac{1}{2}\cdot 1^2\, d\theta$$

$$\text{Area} = \text{Area1} - \text{Area2} .$$

$$= \boxed{\int_0^{\pi} \frac{1}{2}\cdot(1+\sin\theta)^2 \cdot d\theta - \int_0^{\pi} \frac{1}{2}\cdot 1^2\, d\theta}$$

(b) Sketch the curves $r = 2\sin\theta$, and $r = 2\cos\theta$. Set up the integral for the area of the region shared by $r = 2\sin\theta$, and $r = 2\cos\theta$. (Do not evaluate.)

$2\sin\theta = 2\cos\theta$, $\theta = \frac{\pi}{4}$, $r = \sqrt{2}$.



$$\text{Area1} = \int_0^{\frac{\pi}{4}} \frac{1}{2}\cdot(2\sin\theta)^2\, d\theta$$

$$\text{Area2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\cdot(2\cos\theta)^2\, d\theta$$

$$\text{Area} = \boxed{\int_0^{\frac{\pi}{4}} \frac{1}{2}\cdot(2\sin\theta)^2\, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 \cdot d\theta}$$

$$\left(\text{or } \text{Area} = 2\cdot\text{Area1} = 2\,\text{Area2} = 2\cdot\int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2 d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 d\theta\right)$$