§11.4. Comparison Tests.

• **The Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i). If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

☆ Rmk: (i) ConV of LARGER series implies ConV of SMALLER series. $\sum a_n \le \sum b_n < \infty$
(C.T.)

(ii) DIV of SMALLER series implies DIV of LARGER series. $\sum a_n \ge \sum b_n = \infty$.

• **The Limit Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c>0$, then either both series converge or both diverge.

☆ Rmk: In particularly, suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. If $\sum b_n$ conV, then $\sum a_n$ conV
(L.C.T.)
If $\sum b_n$ ~~conV~~ DIV, then $\sum a_n$ DIV.

Motivation and Goal: Compare a given series $\sum a_n$ with a G.S. or p-Series $\sum b_n$

☆ **Key**: Choose b_n accordingly and draw the conclusion based on G.S. and p-Series.

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e.g.1 Test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for ConV/DIV. Hint: $\frac{1}{n^2+1}$ is similar to $\frac{1}{n^2}$. Compare them.
(sim).

(Smaller ~~Large~~) $a_n = \frac{1}{n^2+1} < b_n = \frac{1}{n^2}$ (larger)

According to p-Series, $\sum \frac{1}{n^2}$ is ConV ($p=2>1$). According to C.T. (i), $\sum a_n = \sum \frac{1}{n^2+1}$ is also ConV.

eg2. Test $\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}(\sqrt{n}-1)}$, Hint: $a_n = \frac{3}{\sqrt{n}(\sqrt{n}-1)} > \frac{3}{\sqrt{n}\sqrt{n}} = \frac{3}{n} = b_n$

According to p-Series, $\sum b_n = \sum \frac{3}{n}$ is DIV, and $a_n > b_n$

According to C.T., $\sum a_n = \sum \frac{3}{\sqrt{n}(\sqrt{n}-1)}$ is DIV.

☆ eg3. $\sum_{n=1}^{\infty} \frac{1}{n+1}$. Remark: It is natural to relate this series to $\sum \frac{1}{n}$ and guess that $\sum \frac{1}{n+1}$ is divergent as $\sum \frac{1}{n}$.

$\boxed{\text{WRONG choice of } b_n: \ b_n = \frac{1}{n}}$

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{n}$, $n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$, i.e., $a_n < b_n$

We know that $\sum b_n = \sum \frac{1}{n}$ is divergent (p-Series, $p=1$). However, the comparison test is inconclusive for THIS CHOICE of b_n. (Neither (i) nor (ii) can be applied.)

$\boxed{\text{Correct Choice}: \ b_n = \frac{1}{2n}}$

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{2n}$, $n+1 \leq 2n \Rightarrow \frac{1}{n+1} \geq \frac{1}{2n}$, i.e., $a_n \geq b_n$.

$\sum a_n$ is larger, $\sum b_n$ is smaller. $\sum b_n = \sum \frac{1}{2n} = \frac{1}{2} \cdot \sum \frac{1}{n}$ is divergent. ($p=1$)

Therefore, according to Comparision Test (ii), $\sum b_n$ DIV implies $\sum \frac{1}{n+1}$ also DIV.

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eg 4. (Application of ~~Limiting~~ Limit comparison Test)

(S16) Determine whether $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ is CONV or DIV. State the test you are using.

Solution: $a_n = \frac{3n^2+n}{n^4+\sqrt{n}}$, $b_n = \frac{3n^2}{n^4}$ (leading terms of a_n)

(Draw the conclusion for $\sum b_n$:) $b_n = \frac{3}{n^2}$ p-Series, $p=2>1$, $\sum b_n$ is convergent.

(Compute $\lim_{n\to\infty} \frac{a_n}{b_n}$)

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n}} = \lim_{n\to\infty} \frac{3n^2+n}{n^4+\sqrt{n}} \cdot \frac{n^2}{3} = \lim_{n\to\infty} \frac{3n^4+\cancel{n^3}}{3n^4+\cancel{3\sqrt{n}}} = \lim_{n\to\infty} \frac{3n^4}{3n^4} \boxed{= 1}$$

Therefore, by the limit comparison test, since $\sum b_n$ is convergent, $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ is also CONV.

Remark: If the formula of a_n is such a ratio of two polynomials, then CHOOSE b_n via the $\boxed{\text{"leading term" rule}}$. Under this choice of b_n, $\lim_{n\to\infty} \frac{a_n}{b_n}$ will be 1, and $\sum a_n$ & $\sum b_n$ will be both CONV or DIV. Moreover, b_n (after simplification) is a p-Series (up to some constant.)

☆ eg5. (S17). Test $\sum_{n=1}^{\infty} \frac{1}{2n+\sqrt{n}}$ for CONV/DIV.

Solution 1: (C.T.): $\sqrt{n} \le n \Rightarrow \frac{1}{2n+\sqrt{n}} \ge \frac{1}{2n+n} = \frac{1}{3n}$, $\sum \frac{1}{3n}$ is DIV $\Rightarrow \sum \frac{1}{2n+\sqrt{n}}$ is DIV.

Solution 2: (L.C.T.): $a_n = \frac{1}{2n+\sqrt{n}}$, leading term: $2n$, $b_n = \frac{1}{2n}$, $\sum \frac{1}{2n}$ is DIV.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n}{2n+\sqrt{n}} \overset{\text{leading term}}{\underset{\text{rule}}{=\!=}} 1 \ (\neq 0) \Rightarrow \sum \frac{1}{2n+\sqrt{n}} \text{ is DIV.}$$

WW Hints:

#5. $\sum -\frac{\sqrt{n+7}}{5n}$, It is enough to test $\sum \frac{\sqrt{n+7}}{5n}$. L.C.T. by choosing $b_n = \frac{\sqrt{n}}{5n} = \frac{1}{5\sqrt{n}}$

#7. (Comparing with G.S.) Test $\sum \frac{3^n}{8^n - 2n}$ Hint: 8^n is the leading term compared with $2n$.

$a_n = \frac{3^n}{8^n-2n}$, $b_n = \frac{3^n}{8^n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{8^n}{8^n - 2n} \overset{l'H}{=\!=} \lim_{n\to\infty} \frac{\ln 8 \cdot 8^n}{\ln 8 \cdot 8^n - 2}$

$\sum \frac{3^n}{8^n}$ CONV (G.S. $r = \frac{3}{8} < 1$) $\Rightarrow \sum \frac{3^n}{8^n - 2n}$ CONV. $\overset{l'H}{=\!=} \lim_{n\to\infty} \frac{\ln 8 \cdot \ln 8 \cdot 8^n}{\ln 8 \cdot \ln 8 \cdot 8^n} = 1$

#8. Test $\sum_{n=1}^{\infty} \frac{4}{n^{1+\frac{1}{n}}}$ Hint: $n^{1+\frac{1}{n}} \sim n^{1+0}$. Choose $b_n = \frac{1}{n}$ and apply L.C.T.

#9. Test $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$. Hint: $\sin x \sim x$ when x is small ($\lim_{x\to 0} \frac{\sin x}{x} = 1$)

Choose $b_n = \frac{1}{n^2}$ and compute $\lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{\frac{1}{n^2}}$ $(\frac{0}{0})$ by l'H. Then apply L.C.T.

$\boxed{\text{Question}}$ Test $\sum \cos(\frac{1}{n^2})$. Hint: $\cos 0 = 1$.

$\lim_{n\to\infty} \cos(\frac{1}{n^2}) = \cos 0 = 1 \neq 0$. According to Test for DIV, $\sum \cos(\frac{1}{n^2})$ is DIV

#10. Test $\sum_{n=1}^{\infty} \frac{\ln(4n)}{7n}$. Hint: $\ln(4n) \ge \ln 4 > 1 \Rightarrow \frac{\ln(4n)}{7n} > \frac{1}{7n} = b_n$.

(Actually, Integral Test also works)

#12. $\sum \frac{\cos^2(7n)}{6^n}$. Hint: $-1 \le \cos\theta \le 1$ for any $\theta \Rightarrow 0 \le \cos^2\theta \le 1$ for any θ.

(Part 1). §11.6 Ratio Test for positive series

- Ratio Test: Give $\sum_{n=1}^{\infty} a_n$, where a_n are all positive. Consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, we have the following three cases:
  (i) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, series $\sum a_n$ is convergent.
  (ii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, series $\sum a_n$ is divergent.
  (iii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L = 1$, (ratio test) is $\boxed{\text{inconclusive.}}$

Motivation from Geometric Series:

Suppose $\frac{a_{n+1}}{a_n} = L$ (without limit). Then $a_n = a \cdot L^{n-1}$ is a Geometric Sequence ($a_1 = a$) and L is the common ratio (r). We have $L \geq 1$, DIV and $|L| < 1$ CONV.

Remark: (i) $L < 1$ includes the case $L = 0$, i.e., $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

(ii) $L > 1$ includes the case $L = \infty$, i.e., $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$

(iii) Do not apply Ratio Test to p-Series or (p-Series like). You will get limit 1.

(vi) Ratio Test should be applied for a_n with n-factorial

$$\boxed{n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n}.$$

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e.g. 1. Test $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ for convergence via ratio test.

Step 0: (compute $\frac{a_{n+1}}{a_n}$) $a_n = \frac{n^2}{3^n}$, $a_{n+1} = \frac{(n+1)^2}{3^{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} = \frac{(n+1)^2 \cdot 3^n}{3^{n+1} \cdot n^2}$

(combine 'Similar parts') $= \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}}$

Step 1: (compute $\lim \frac{a_{n+1}}{a_n}$) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}} = \left[\lim_{n\to\infty} \frac{(n+1)^2}{n^2}\right] \cdot \frac{1}{3}$.

$= 1 \cdot \frac{1}{3}$ (The limit is 1 according to leading term rule).

i.e. $L = \frac{1}{3} < 1$.

Step 2: $L < 1$. (Draw conclusion)

$\sum a_n = \sum \frac{n^2}{3^n}$ is convergent because of Ratio Test.

eg 2. $\sum_{n=0}^{\infty} \frac{9^n}{(2n)!}$

(§17)

Solution: Step 0: $a_n = \frac{9^n}{(2n)!}$, $a_{n+1} = \frac{9^{n+1}}{(2(n+1))!}$.

Rmk: $2(n+1) = 2n+2$.

$(2n+2)! = \underbrace{1 \times 2 \times 3 \times \cdots \times (2n-1) \times (2n)}_{(2n)!} \times (2n+1) \times (2n+2)$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{9^{n+1}}{(2n+2)!}}{\frac{9^n}{(2n)!}} = \frac{9^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{9^n} = \boxed{\frac{9^{n+1}}{9^n}} \cdot \boxed{\frac{(2n)!}{(2n+2)!}}$$

$$= 9 \cdot \frac{(2n)!}{(2n)! \cdot (2n+1)(2n+2)} = \frac{9}{(2n+1)(2n+2)}.$$

Step 1: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{9}{(2n+1)(2n+2)} = 0 < 1$

Step 2: ($L=0$ in Ratio Test) $\Rightarrow \sum \frac{9^n}{(2n)!}$ is Conv.

eg 3. $\sum_{n=1}^{\infty} n! \cdot e^{-n}$, $a_n = n! \cdot e^{-n}$, $a_{n+1} = (n+1)! \cdot e^{-(n+1)}$

(§15).

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)! \cdot e^{-(n+1)}}{n! \cdot e^{-n}} = \lim_{n\to\infty} (n+1) \cdot e^{-1} = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} n! e^{-n} \text{ is DIV}$$

($L = \infty$)

eg 4. $\sum_{n=1}^{\infty} \frac{n^8}{8^n}$, $a_n = \frac{n^8}{8^n}$ $a_{n+1} = \frac{(n+1)^8}{8^{n+1}}$

(§16).

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^8}{8^{n+1}}}{\frac{n^8}{8^n}} = \lim_{n\to\infty} \frac{(n+1)^8}{n^8} \cdot \frac{8^n}{8^{n+1}} = 1 \cdot \frac{1}{8} < 1$$

Hint: $\lim_{n\to\infty} \frac{(n+1)^8}{n^8} = 1$ leading term rule.

$\sum_{n=1}^{\infty} \frac{n^8}{8^n}$ is Conv according to ratio test.

☆ eg 5. Can we use Ratio Test to determine $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$?

(§17)

$a_n = \frac{1}{n^2+1}$, $a_{n+1} = \frac{1}{(n+1)^2+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2+1}{n^2+1} \xlongequal[\text{rule}]{\text{leading term}} \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

Ratio Test is $\boxed{\text{inconclusive}}$ for $\sum \frac{1}{n^2+1}$.

§11.5 Alternating Series. (A.S.)

- Notation: $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot b_n = b_1 - b_2 + b_3 - b_4 + \cdots$, where all $b_n > 0$ (positive), is called an alternating series.
- Remark 1: One key feature of A.S. is that POSITIVE and NEGATIVE terms appear alternatively. Therefore, $(-1)^{n-1}$ can be replaced by $(-1)^n$, $(-1)^{n+1}$ etc, i.e., $\sum(-1)^n\cdot b_n$, $\sum(-1)^{n+1} b_n$ are all A.S..

- Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot b_n$ satisfies:
  (A.S.T.) (i) $0 < b_{n+1} \le b_n$ for all n. (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

- eg.0. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. Hint: $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n} = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\cdots$, alternating with $b_n = \frac{1}{n}$.

  Consider A.S. Test for $b_n = \frac{1}{n}$. (i) $0 < \frac{1}{n+1} \le \frac{1}{n}$ for all n (ii) $\lim_{n\to\infty}\frac{1}{n} = 0$

  Therefore, $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ is convergent according to A.S. Test.

- Remark 2: $\underbrace{\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}}_{(A.S.)}$ and $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{(p\text{-series}, p=1)}$ are two DIFFERENT series. The first CONV and the second DIV.

- eg.1. $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2n+3}{3n+1}$. Alternating S. $b_n = \frac{2n+3}{3n+1}$, $\lim_{n\to\infty}\frac{2n+3}{3n+1} = \frac{2}{3} \neq 0$

  According to $\boxed{\text{nth term test for DIV}}$, $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2n+3}{3n+1}$ is divergent. (A.S.T. is inconclusive and not necessary)

- Remark 3: Flow chart to test Alternating Series $\sum(-1)^n\cdot b_n$.

  $\boxed{\text{① Find } b_n} \Longrightarrow \boxed{\text{② Compute } \lim_{n\to\infty} b_n}$
  $\Longrightarrow \boxed{\text{3.1 limit} \neq 0,\ \sum(-1)^n b_n \text{ DIV (DIV Test). Done} \times\times}$
  $\Longrightarrow \boxed{\text{3.2 limit} = 0, \text{(ii) is satisfied}} \Longrightarrow \boxed{\text{④ 'check' } b_n \text{ is decreasing}} \Downarrow \boxed{\text{⑤ conclusion. A.S. is CONV.}}$

- eg.2. $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3}\cdot(-1)^n$

  Solution: $b_n = \frac{(n+1)^2}{n^3}$, $\lim_{n\to\infty}\frac{(n+1)^2}{n^3} = \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^3}} = \frac{1}{\infty} = 0$. and $\boxed{b_n \text{ is decreasing}}$ (do not need to check).

  According to A.S.T., $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3}\cdot(-1)^n$ is convergent.

- eg.3. Find the sum of $\sum_{n=0}^{\infty}\frac{4\cdot(-1)^n(3)^n}{5^n}$. Remark: This is AN A.S., but also a GEOMETRIC Series.

(S15, 14pts)

$$a_n = \frac{4\cdot(-1)^n\cdot(3)^n}{5^n} = 4\cdot\left(\frac{-3}{5}\right)^n,\ n=0,1,2,\cdots,\quad \text{G.S: } a=4,\ \boxed{r=\frac{-3}{5}}$$

G.S. formula: $$\sum_{n=0}^{\infty}a_n = \frac{a}{1-r} = 4\cdot\frac{1}{1-\left[-\frac{3}{5}\right]} = 4\cdot\frac{1}{1+\frac{3}{5}} = 4\cdot\frac{1}{\frac{8}{5}} = 4\cdot\frac{5}{8} = \boxed{\frac{5}{2}}$$

(Part II) §11.6 Absolute Convergence and the Ratio Test (Part II)

- The Ratio Test (Full version)

(i). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is $\boxed{\text{absolute convergent.}}$

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive

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☆ $\boxed{\text{Absolute Convergence}}$: We say $\sum a_n$ is ABSOLUTE convergent if $\sum |a_n|$ is convergent.

Rmk 1: If a_n are all positive, then ABS ConV = ConV. We are interested in whether an alternating series $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$ is ABS or NOT. According to the definition $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ is convergent absolutely if $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n| = \sum b_n$ is convergent.

Rmk 2: $\boxed{\text{ABS conV} \Rightarrow \text{ConV}}$; If $\sum a_n$ is ABS conV, then $\sum a_n$ is conV.

Caution: If $\sum a_n$ is NOT conV, then $\sum a_n$ may be convergent or divergent.

☆ e.g. 1 Study whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is ① ConV ② ABS ConV.

①: According to A.S.T., both $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ are convergent

② $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is DIV $(p=1)$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is NOT ABS conV.

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is ConV $(p=2)$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is ABS ConV.

Conclusion, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but NOT ABS ConV.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent and also ABS conV.

Rmk: If $\sum a_n$ is ConV but NOT ABS conV, then $\sum a_n$ is called conditionally convergent.

☆ eg 2. Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$ is absolutely convergent, conditionally convergent, or divergent.
(f16)

Rmk: Check ABS conV first. ie, check $\sum \left|\frac{\cos n}{n^2+1}\right|$ first.

Hint: $\left|\frac{\cos n}{n^2+1}\right| = \frac{|\cos n|}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$

By (Direct) Comparison Test, $\sum \frac{1}{n^2}$ is convergent implies that $\sum \left|\frac{\cos n}{n^2+1}\right|$ is ConV.

ie. $\sum \frac{\cos n}{n^2+1}$ is ABS convergent.

(We do not need to check $\sum \frac{\cos n}{n^2+1}$ separately.)

eg 3 Test $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k-3}$ for ABS conV and ConV. Hint: Use Ratio Test ~~first~~.
(S16)

$a_k = \frac{(-1)^k}{2^k-3}$, $a_{k+1} = \frac{(-1)^{k+1}}{2^{k+1}-3}$, $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(-1)^{k+1}}{2^{k+1}-3}}{\frac{(-1)^k}{2^k-3}}\right| = \left|\frac{2^k-3}{2^{k+1}-3}\right|$

$\lim_{k\to\infty} \frac{2^k-3}{2^{k+1}-3} \overset{l'H}{=} \lim_{k\to\infty} \frac{\ln 2 \cdot 2^k}{\ln 2 \cdot 2^{k+1}} = \frac{1}{2}$. since $|(-1)^k| = |(-1)^{k+1}| = 1$ for any k

i.e. $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$. According to Ratio Test, $\sum \frac{(-1)^k}{2^k-3}$ is ABS conV and therefore is ConV.

Hints for WW:

\#3. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{7n}$, $\cos\pi = -1$, $\cos 2\pi = +1$, $\cos 3\pi = -1$, $\cos 4\pi = +1$, $\cdots \Rightarrow \boxed{\cos n\pi = (-1)^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{7n}$. Ratio Test is inclusive since $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$. We have to check $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{7n}\right| = \sum_{n=1}^{\infty} \frac{1}{7n}$ directly (DIV and NOT ABS conV)

\#4: $(6n)! = 1\times 2\times \cdots \times (6n)$, $(6(n+1))! = (6n+6)! = \underbrace{1\times 2\times \cdots \times (6n)}_{(6n)!} \times \underbrace{(6n+1)\times(6n+2)\times(6n+3)\times\cdots\times(6n+6)}$

\#5. Check ①, ②, ④, ⑤ directly. Do NOT use Ratio Test for these

③. $\sum \frac{(n+1)\cdot(6^2-1)^n}{6^{2n}}$ Ratio Test or use Test for DIV directly

Show $\lim_{n\to\infty} \frac{(6^2-1)^n}{6^{2n}} = 1$.