§7.4. Partial Fraction Decomposition (P.F.D.)

Goal: Evaluate the integral of a rational function $\int \frac{\text{Polynomial 1}}{\text{Polynomial 2}}\,dx$.

Rmk: We will restrict our attention to Poly 2 with degree ≤ 3.

Basic Formulas: (omit C)

☆ ① $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$; ☆ ② $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

③ $\int \frac{1}{(ax+b)^2}\,dx = \frac{1}{a}\left(-\frac{1}{ax+b}\right)$; ④ $\int \frac{x}{k\cdot x^2+m}\,dx = \frac{1}{2k}\ln|kx^2+m|$

Rmk: ①, ② are direct linear u-sub. ③ is based on $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$ and $u=ax+b$.

④ is base on $\int \frac{1}{x}\,dx = \ln|x|$ and $u=k\cdot x^2+m$.

P.F.D. cases: Examples.

☆ case 1: $\frac{\blacksquare}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$. $a\neq b$. two distinct roots. $\frac{1}{(x+1)(x-1)}$

case 2: $\frac{\blacksquare}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$ two repeated roots. $\frac{1}{(x-1)^2}$

case 3: $\frac{\blacksquare}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$, a, b, c three distinct roots. $\frac{1}{(x-1)(x+1)(x+2)}$

case 4 $\frac{\blacksquare}{(x-a)\cdot(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$ one distinct, two repeated roots. $\frac{1}{(x-1)(x+1)^2}$

case 5 $\frac{\blacksquare}{(x-a)(x^2+c^2)} = \frac{A}{x-a} + \frac{Bx}{x^2+c^2} + \frac{C}{x^2+c^2}$ one root. $\frac{1}{(x-1)(x^2+1)}$

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eg. 1. (s17). Evaluate $\int \frac{1}{x(x-1)}\,dx$

P.F.D.: $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, A, B unkown constants to be determined

$\Leftrightarrow \quad 1 = A(x-1)+Bx$ times $x(x-1)$

Plug in $\boxed{x=0}$ $1=A(-1) \Rightarrow A=-1$.

$\boxed{x=1}$ $1=B\cdot 1 \Rightarrow B=1$.

$\Rightarrow \frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$.

$= \int \frac{-1}{x} + \frac{1}{x-1}\,dx$

$= \boxed{-\ln|x| + \ln|x-1| + C}$

(or $= \ln\left|\frac{x-1}{x}\right| + C$)

e.g.2 (f16). Evaluate $\int \frac{2}{x^2-2x-8}\,dx$.

Factorize: $x^2-2x-8=(x-4)(x+2)$

P.F.D.

$$\frac{2}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} = \boxed{\frac{\frac{1}{3}}{x-4} + \frac{-\frac{1}{3}}{x+2}}$$

times $(x-4)(x+2)$

$$2 = A(x+2)+B(x-4)$$

Plug in $x=4$. $2=A\cdot 6 \Rightarrow A=\frac{1}{3}$

$x=-2$. $2=B(-6)\Rightarrow B=-\frac{1}{3}$

P.F.D. $\int \frac{\frac{1}{3}}{x-4} + \frac{-\frac{1}{3}}{x+2}\,dx$

$$=\frac{1}{3}\ln|x-4| - \frac{1}{3}\ln|x+2| + C$$

e.g.3 (s16).

$$\int_0^1 \frac{x}{(x+1)^2}\,dx.$$

Two Repeated Roots (case 2)

P.F.D. $\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{1}{x+1} - \frac{1}{(x+1)^2}$

times $(x+1)^2$

$$x = A(x+1) + B$$

$1\cdot x+0 = Ax + A+B$ set the coefficients (of x) equal.

$A=1$, $A+B=0 \Rightarrow A=1$, $B=-1$.

$$=\int_0^1 \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$$

$$=\left[\ln|x+1| + \frac{1}{x+1}\right]\Big|_0^1$$

$$=\ln 2+\frac{1}{2}-(\ln 1+1)$$

$$=\boxed{\ln 2-\frac{1}{2}}$$

Rmk: Actually, the P.F.D. above can be found from the following observation.

$$\frac{x}{(x+1)^2} = \frac{x+1-1}{(x+1)^2} = \frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2} = \boxed{\frac{1}{x+1} - \frac{1}{(x+1)^2}}$$

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e.g.4. (f15, case 3)

Evaluate $\int \frac{x+4}{2x^3-8x}\,dx$.

P.F.D. $2x^3-8x = 2x(x^2-4) = 2x(x+2)(x-2)$.

$$\frac{x+4}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x+2} + \frac{C}{x-2} = \frac{-1}{2x} + \frac{\frac{1}{8}}{x+2} + \frac{\frac{3}{8}}{x-2}.$$

$$x+4 = A(x+2)(x-2) + B\,2x(x-2) + C\,2x(x+2)$$ times $2x(x+2)(x-2)$

$x=0$, $4=A\cdot 2\cdot(-2) \Rightarrow A=-1$.

$x=-2$, $2=B\cdot(-4)(-4) \Rightarrow B=\frac{1}{8}$.

$x=2$, $6=C\cdot 4\cdot 4 \Rightarrow C=\frac{3}{8}$

$$\int \frac{x+4}{2x^3-8x}\,dx = \int \frac{-1}{2x} + \frac{\frac{1}{8}}{x+2} + \frac{\frac{3}{8}}{x-2}\,dx = \boxed{-\frac{1}{2}\ln|x| + \frac{1}{8}\ln|x+2| + \frac{3}{8}\ln|x-2| + C}$$

eg. 5: (Related WW5).
$$\int \frac{4}{(x-1)(x^2-1)}dx.$$
$$(x-1)(x^2-1)=(x-1)(x-1)(x+1)=(x+1)(x-1)^2$$
Case 4.

P.F.D.
$$\frac{4}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$
Cross Multiplication:
$$4=A(x-1)^2+B(x+1)(x-1)+C(x+1)$$
$$4=A(x^2-2x+1)+B(x^2-1)+C(x+1)$$
$$4=(A+B)x^2+(-2A+C)x+A-B+C.$$
$$\begin{cases} A+B=0 \\ -2A+C=0 \\ A-B+C=4 \end{cases} \Rightarrow \begin{matrix} B=-A \\ C=2A \end{matrix} \Longrightarrow A+A+2A=4 \Rightarrow A=1 \Rightarrow \begin{cases} B=-1 \\ C=2. \end{cases}$$

$$=\int \frac{1}{x+1}-\frac{1}{x-1}+\frac{2}{(x-1)^2}dx$$
$$=\boxed{\ln|x+1|-\ln|x-1|-\frac{2}{x-1}+C}$$

eg.6
$$\int \frac{x^2+x-2}{x\cdot(x^2+2)}dx$$
P.F.D.
$$\frac{x^2+x-2}{x\cdot(x^2+2)}=\frac{A}{x}+\frac{Bx}{x^2+2}+\frac{C}{x^2+2}=\frac{-1}{x}+\frac{2x}{x^2+2}+\frac{1}{x^2+2}$$
$$x^2+x-2=A(x^2+2)+Bx\cdot x+C\cdot x.$$
$$1\cdot x^2+1\cdot x-2=(A+B)x^2+C\cdot x+2A$$
$$\begin{cases} 1=A+B \\ 1=C \\ -2=2A \end{cases} \Rightarrow \begin{cases} B=1 \\ C=1 \\ A=-1 \end{cases}$$

$$=\int \frac{-1}{x}+\frac{2x}{x^2+2}+\frac{1}{x^2+2}dx$$
$$=\boxed{-\ln|x|+\ln|x^2+2|+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C}$$

Rmk:
$$\int \frac{2x}{x^2+2}dx \xrightarrow[du=2x\,dx.]{u=x^2+2} \int \frac{du}{u}=\boxed{\ln|x^2+2|}$$
$$\int \frac{1}{x^2+2}dx \xrightarrow[dx=\sqrt{2}\,du]{x=\sqrt{2}u} \int \frac{\sqrt{2}\,du}{2(1+u^2)}=\frac{1}{\sqrt{2}}\tan^{-1}u+C=\boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C}$$

Hints for WW.

WW2: (Degree > 3) P.F.D. for $\frac{6x^3+4x^2+8}{x\cdot(3x-2)^2\cdot(x^2+2x+5)^5}$. Total terms in P.F.D. $\boxed{8}$

Non-repeated (reduced) terms: x, $3x-2$, x^2+2x+5.
with degree $1+2+5=\boxed{8}$

WW7.
$$\frac{2x^3-4x^2+1}{x^2-2x}=f(x)+\frac{p(x)}{x^2-2x}.$$
Use LONG DIVISION to find $f(x)$ and $p(x)$.

Numerator has higher degree than denominator. Then apply P.F.D. to $\frac{p(x)}{x^2-2x}$.

§7.8 Improper Integral.

Geometric meaning: Area of an unbounded region

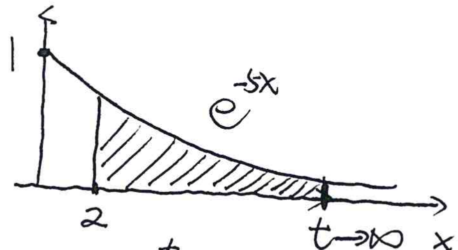
Notation: Definite Integral with "singularities" (bad points)

$\int_a^b f(x)\,dx$. a, b might be $(\pm)\infty$ or there is c in $[a,b]$, $f(c)=\infty$.

"Definition": Improper Integral = Limit of a definite integral.

· Horizontal unbounded region (singularity at $\pm\infty$)

eg.1. Evaluate $\int_2^{+\infty} e^{-5x}\,dx$. Graph:
(f16).



Definition: (Step 0).

$$\int_2^{+\infty} e^{-5x}\,dx = \lim_{t\to\infty} A(t)$$

$$= \lim_{t\to\infty} \int_2^t e^{-5x}\,dx.$$

Fix t, $A(t)=\int_2^t e^{-5x}\,dx$ is a definite integral which represents the area of the shadow region. We want to study what happens as $t\to+\infty$.

We need to evaluate the definite integral first. The result will be a function of t. Then we can take the limit as t tends infinity.

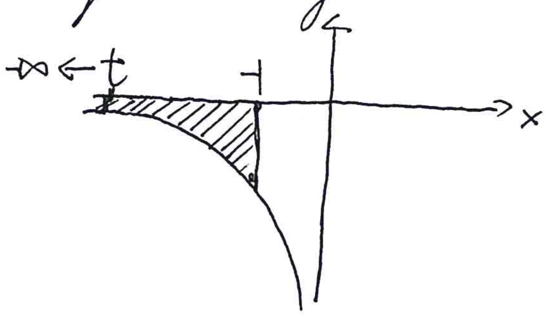
(Step 1): $\int_2^t e^{-5x}\,dx = -\frac{1}{5}e^{-5x}\Big|_2^t = -\frac{1}{5}(e^{-5t}-e^{-10}) \leftarrow$ function of t.

(Step 2): $\lim_{t\to\infty} -\frac{1}{5}(e^{-5t}-e^{-10}) = -\frac{1}{5}(e^{-\infty}-e^{-10}) = \frac{1}{5}e^{-10}$ Hint: $e^{-\infty}=0$.

Conclusion (Step 3): $\int_2^{+\infty} e^{-5x}\,dx = \lim_{t\to\infty}\int_2^t e^{-5x}\,dx = \boxed{\frac{1}{5}e^{-10}}$

Rmk: If the limit in Step 2 is a finite number C, then we say the improper integral is $\boxed{\text{convergent}}$ and equals C. If the limit is ∞ or D.N.E, then the improper integral is $\boxed{\text{DIVERGENT}}$!

e.g.2. Determine whether $\int_{-\infty}^{-1} \frac{1}{x}\,dx$ is convergent or divergent



SLN: $\int_{-\infty}^{-1} \frac{1}{x}\,dx = \lim_{t\to-\infty} \int_{t}^{-1} \frac{1}{x}\,dx$

$= \lim_{t\to-\infty} \ln|x| \Big|_{t}^{-1}$

$= \lim_{t\to-\infty} \ln 1 - \ln|t| = \lim_{t\to-\infty} 0 - \ln|t| = \boxed{-\infty}$

Conclusion: $\int_{-\infty}^{-1} \frac{1}{x}\,dx$ $\boxed{\text{diverges}}$.

e.g.3 Evaluate the improper integral $\int_0^{\infty} \frac{1}{1+4x^2}\,dx$ if it converges or explain why it diverges (S16).

SLN: $\int_0^{\infty} \frac{1}{1+4x^2}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{1}{1+4x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\cdot\tan^{-1}(2t) = \frac{1}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{4}}$

$\int_0^{t} \frac{1}{1+4x^2}\,dx \underset{2dx=du}{\overset{2x=u}{=\!=}} \int_0^{2t} \frac{1}{1+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\tan^{-1}u\Big|_0^{2t} = \frac{1}{2}\tan^{-1}(2t) - \frac{1}{2}\tan^{-1}(0) = \frac{1}{2}\tan^{-1}(2t)$

Hint: $\tan 0 = 0 \Rightarrow \tan^{-1}(0) = 0$.

e.g.4. Evaluate the improper integral $\int_3^{\infty} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx$ and whether it is conv or div? (F16).

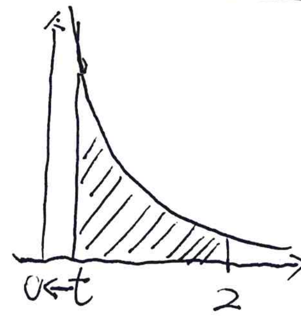
SLN: $\int_3^{\infty} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = \lim_{t\to\infty} \frac{-2}{\sqrt{t-2}} + 2 = \boxed{2}$ conV.

$\int_3^{t} (x-2)^{-\frac{3}{2}}\,dx = \frac{1}{-\frac{3}{2}+1}\,(x-2)^{-\frac{3}{2}+1}\Big|_3^{t} = -2(x-2)^{-\frac{1}{2}}\Big|_3^{t} = -2(t-2)^{-\frac{1}{2}} + 2(3-2)^{-\frac{1}{2}}$

$= -2\frac{1}{\sqrt{t-2}} + 2.$

- Vertical unbounded region (singularity at finite value)

e.g.5. $\int_0^{2} \frac{1}{x^2}\,dx$ 0 is a singularity since $\lim_{x\to 0} \frac{1}{x^2} = \infty$



$= \lim_{t\to 0^+} \int_t^{2} \frac{1}{x^2}\,dx.$ $\boxed{\text{Divergent}}$.

$= \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^{2} = \lim_{t\to 0^+} -\frac{1}{2} + \frac{1}{t} = -\frac{1}{2} + \frac{1}{0^+} = \boxed{+\infty}$

e.g.6. $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$. 1 is a singularity since $\frac{1}{\sqrt{1-x^2}} \to +\infty$ as $x \to 1$.

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1^-} \sin^{-1}x \Big|_0^t = \lim_{t\to 1^-} \sin^{-1}t - \sin^{-1}0$$

$$= \sin^{-1}1 - \sin^{-1}0 = \boxed{\frac{\pi}{2}} \quad \text{conV.}$$

☆ Rmk: If the singularity (the point where the denominator is zero) is inside the interval, we need break the integral up into two via $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c is the singularity. The original integral is conV only if both converge.

e.g.7. whether $\int_{-2}^2 \frac{1}{x^2}dx$ is convergent or divergent?

SLN: 0 is inside $(-2,2)$. Rewrite $\int_{-2}^2 \frac{1}{x^2}dx = \int_{-2}^0 \frac{1}{x^2}dx + \int_0^2 \frac{1}{x^2}dx$.

e.g.5 shows that $\int_0^2 \frac{1}{x^2}dx$ is divergent $\Rightarrow \int_{-2}^2 \frac{1}{x^2}dx$ is divergent.

Caution: The following computation IS WRONG:

$\int_{-2}^2 \frac{1}{x^2}dx = -\frac{1}{x}\Big|_{-2}^2 = -\frac{1}{2} - (-\frac{1}{-2}) = -1$. since $\frac{1}{x^2}$ is not continuous at 0. FTC does not apply.

WW Hints:

ww4: $\int_{-\infty}^{+\infty} \frac{2}{1+64x^2}dx = \int_{-\infty}^0 \frac{2}{1+64x^2}dx + \int_0^{+\infty} \frac{2}{1+64x^2}dx$. The left hand side converges if both integrals on the right converge.

ww5: $\int_{-1}^1 \frac{dx}{|x|^{\frac{4}{5}}} = \int_{-1}^0 \frac{dx}{(-x)^{\frac{4}{5}}} + \int_0^1 \frac{dx}{x^{\frac{4}{5}}}$, $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

ww8,9. If $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$ and $\int_a^\infty g(x)dx$ conV, then $\int_a^\infty f(x)dx$ conV.

ww8: $\frac{6}{5x+e^{3x}} \leq \frac{6}{e^{3x}}$ and prove $\int_0^\infty \frac{6}{e^{3x}}dx$ converges

ww9: $\frac{9}{\sqrt{x^6+3}} \leq \frac{9}{\sqrt{x^6}} = \frac{9}{x^3}$ and prove $\int_1^\infty \frac{9}{x^3}dx$ converges.

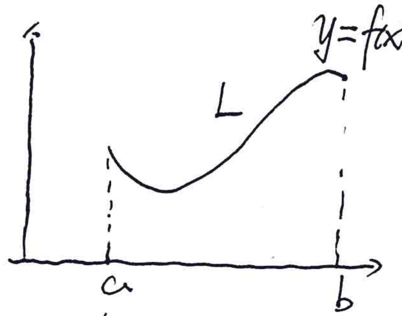
§8.1 Arc length. (NOT covered in Mid 1).

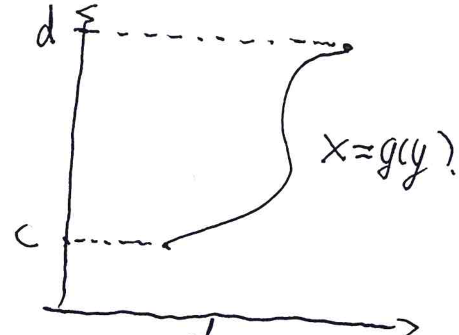
- Arc length Formula (Formula Sheet).

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

The formula computes the arc length of a curve $y=f(x)$ from $x=a$ to $x=b$

Graph:





Horizontal Version: $L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx.$

Vertical Version: $L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy.$

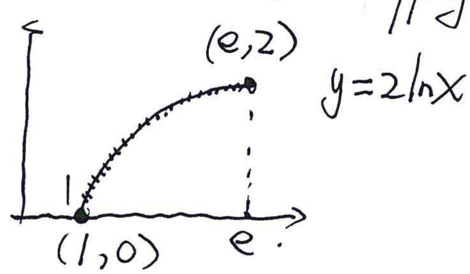
eg1. Set up the integral (Do not evaluate) for the arc length of the curve $y=2\ln x$ from the point $(1,0)$ to $(e,2)$.

SLN: $y=f(x)=2\ln x$, $a=1$, $b=e$ (x-coordinates of the two points)

$f'(x)=\frac{2}{x}$. Arc-length $L=\boxed{\int_1^e \sqrt{1+\left(\frac{2}{x}\right)^2}\,dx}$

Rmk: The graph is shown below, but not necessary for the answer.

All we need is to apply the formula $\circledast$ for correct $f(x)$ and a, b.



e.g.2. Find the exact arc length of $x=\frac{2}{3}\cdot y^{\frac{3}{2}}$, $0\leq y\leq 1$.

SLN: Vertical Version: $g(y)=\frac{2}{3}y^{\frac{3}{2}}$, $c=0$, $d=1$ range of y-variable

$g'(y)=\frac{2}{3}\cdot\frac{3}{2}\cdot y^{\frac{1}{2}}=y^{\frac{1}{2}}$, $L=\int_0^1\sqrt{1+[g'(y)]^2}\,dy=\int_0^1\sqrt{1+y}\cdot dy=\frac{2}{3}(1+y)^{\frac{3}{2}}=\boxed{\frac{2}{3}\cdot 2^{\frac{3}{2}}-\frac{2}{3}}$

e.g.3. Find the arc length of $y=3+\frac{1}{2}\cosh(2x)$ from $x=0$ to $x=5$.

SLN: $y'=\frac{1}{2}\sinh(2x)\cdot 2=\sinh(2x)$, $a=0$, $b=5$.

Hints: $(\sinh x)'=\cosh x$

$(\cosh x)'=\sinh x$.

$1+\sinh^2 x=\cosh^2 x$.

$$L=\int_0^5\sqrt{1+[\sinh(2x)]^2}\,dx$$

$$=\int_0^5\sqrt{\cosh^2(2x)}\,dx=\int_0^5\cosh(2x)\,dx=\frac{1}{2}\sinh(2x)\Big|_0^5$$

$$=\frac{1}{2}\sinh(10)-0=\boxed{\frac{e^{10}-e^{-10}}{2}}$$

☆ e.g.4 Find the arc length of $x=\frac{1}{4}\cdot y^4+\frac{1}{8y^2}-1$ from $y=1$ to $y=2$.

SLN: Vertical Version: $x'=\frac{dx}{dy}=y^3-\frac{1}{4}\cdot y^{-3}$, $c=1$, $d=2$.

$$L=\int_1^2\sqrt{1+[y^3-\tfrac{1}{4}\cdot y^{-3}]^2}\,dy$$

$$=\int_1^2\sqrt{1+[y^3]^2-\tfrac{1}{2}+[\tfrac{1}{4}\cdot y^{-3}]^2}\,dy$$

$$=\int_1^2\sqrt{[y^3]^2+\tfrac{1}{2}+[\tfrac{1}{4}\cdot y^{-3}]^2}\,dy$$

$$=\int_1^2\sqrt{[y^3]^2+2\cdot[y^3][\tfrac{1}{4}y^{-3}]+[\tfrac{1}{4}\cdot y^{-3}]^2}\,dy$$

$$=\int_1^2\sqrt{[y^3+\tfrac{1}{4}\cdot y^{-3}]^2}\,dy$$

$$=\int_1^2 y^3+\tfrac{1}{4}\cdot y^{-3}\,dy$$

$$=\tfrac{1}{4}\cdot y^4-\tfrac{1}{8}\cdot y^{-2}\Big|_1^2$$

$$=\boxed{\tfrac{1}{4}\cdot 2^4-\tfrac{1}{8}\cdot 2^{-2}-\left(\tfrac{1}{4}-\tfrac{1}{8}\right)}$$

Hints: $(a+b)^2=a^2+2ab+b^2$

$$1+(a-\tfrac{1}{4a})^2=a^2+\tfrac{1}{2}+(\tfrac{1}{4a})^2$$

$$=(a+\tfrac{1}{4a})^2$$