§7.1 Integration by Parts (IBP)

IBP: (uv version in Formula Sheet) $\int u\,dV = u\cdot V - \int V\,du$

(f,g version): $\boxed{\begin{array}{l} u=f(x),\ du=f'(x)dx \\ V=g(x),\ dV=g'(x)dx \end{array}}$ $\int f(x)\cdot g'(x)\cdot dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx.$

(Definite integral version:) $\int_a^b u\cdot dV = u\cdot V\Big|_a^b - \int_a^b V\,du$

Motivation from Product Rule + FTC.

$$\int (f\cdot g)'\,dx = f(x)\cdot g(x) + C$$

Alternating Form

Product Rule: $\Longleftrightarrow$ $\int f'\cdot g + f\cdot g'\,dx = f\cdot g + C \Longleftrightarrow \boxed{\int f'\cdot g\,dx} + \boxed{\int f\cdot g'\,dx} = f\cdot g + C.$

Key step: Carefully choose $u=f(x)$, $V=g(x)$ s.t. $du=f'(x)dx$ has simpler derivative and $V=\int g'(x)dx$ has simpler anti-derivative.

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e.g. 1. (S17) Evaluate $\int \underbrace{x}_{u}\cdot\underbrace{e^x\,dx}_{dV}$. IBP: $u=x$, $du=dx$

$dV=e^x dx$, $V=\int e^x dx = e^x$

$\stackrel{IBP}{=} u\cdot V - \int V\cdot du$

$= x\cdot e^x - \int e^x\cdot dx = \boxed{x\cdot e^x - e^x + C}$

e.g. 2 (F16). Evaluate $\int_0^1 x\cdot e^{2x}dx$ IBP: $u=x$, $du=dx$

$dV=e^{2x}dx$, $V=\int e^{2x}dx = \frac{1}{2}e^{2x}.$

$\stackrel{IBP}{=} u\cdot V\Big|_0^1 - \int_0^1 V\cdot du$

$= x\cdot\frac{1}{2}e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx.$

$= (\frac{1}{2}e^2 - 0) - \frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{2}e^2 - (\frac{1}{4}e^2 - \frac{1}{4}e^0) = \boxed{\frac{1}{4}e^2 + \frac{1}{4}}$

- Typical IBP pair. (Typical choice of u and dv)

$$\int \underbrace{\text{Polynomial}}_{u} \cdot \underbrace{\sin/\cos/\exp}_{dv}\, dx \qquad \int \underbrace{\ln/\tan^{-1}/\sin^{-1}}_{u} \cdot \underbrace{\square\, dx}_{dv}$$

eg3 $\int \underbrace{(2t+1)}_{u} \cdot \underbrace{\sin 3t\, dt}_{dv}$

IBP: $u = 2t+1$, $du = 2dt$

$dv = \sin 3t\, dt$, $v = \int \sin 3t\, dt = -\frac{1}{3}\cos 3t$

$$\overset{IBP}{=} (2t+1)\left(-\frac{1}{3}\cos 3t\right) - \int -\frac{1}{3}\cos 3t \cdot 2dt$$

$$= \boxed{(2t+1)\left(-\frac{1}{3}\cos 3t\right) + \frac{2}{3}\cdot\frac{1}{3}\sin 3t + C}$$

Remark: Three u-sub formulas

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

eg4 $\int \underbrace{4x}_{dv} \cdot \underbrace{\ln x}_{u} \cdot dx$

$u = \ln x$, $du = \frac{1}{x}dx$

$dv = 4x\,dx$, $v = 2x^2$

$$\overset{IBP}{=} \ln x \cdot 2x^2 - \int 2x^2 \cdot \frac{1}{x}\,dx.$$

$$= \ln x \cdot 2x^2 - \int 2x\,dx = \boxed{\ln x \cdot 2x^2 - x^2 + C}$$

eg5. $\int \underbrace{\tan^{-1}x}_{u} \cdot \underbrace{dx}_{dv}$.

IBP: $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}dx$, $dv = dx$, $v = x$.

$$\overset{IBP}{=} \tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2}\,dx.$$

Hint: $\int \frac{x}{1+x^2}dx \xlongequal[du=2x\,dx]{u=1+x^2} \int \frac{\frac{1}{2}du}{u}$

$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|$.

$$\overset{u\text{-sub}}{=} \tan^{-1}x \cdot x - \frac{1}{2}\ln|1+x^2|.$$

Webwork Hints:

ww3: $\int x^2 \cdot \cos 6x\,dx$. IBP twice since $x^2 \xrightarrow{\frac{d}{dx}} 2x \xrightarrow{\frac{d}{dx}} 2$.

ww4: $\int 3x \cdot \sec^2(5x)\,dx$. Hint: $(\tan x)' = \sec^2 x \Rightarrow \int \sec^2(5x)\,dx = \frac{1}{5}\tan(5x)$

☆ ww7. $\int e^{-4x} \cdot \cos 4x\,dx$. Loop Trick.

Apply IBP twice, set up an equation for $\int e^{-4x} \cdot \cos 4x\,dx$

Then solve for $\int e^{-4x} \cdot \cos 4x\,dx$ (as an unknown variable).

☆ ww8: $\int \sin(\ln x)\,dx$. $u = \sin(\ln x)$ and $dv = dx$. Similar loop trick as ww7.

§7.2. Trigonometric Integrals

☆. ① sin-cos product with ODD power term(s): Substitute THE OTHER TERM (sin/cos)

② sin-cos product without ODD power: Double angle formula (D.A.F.)

③ tan, sec and other mixed type.

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① eg1. (5/6) $\int \sin^4 x \cdot \cos^3 x \, dx$. $\cos^3 x$ has ODD power 3, substitute $\sin x$ (NOT $\sin^4 x$). CAUTION!

$u = \sin x$. $du = \cos x \, dx$.

$= \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$.

Trig-Id: $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x = 1 - u^2$

$= \int u^4 \cdot (1-u^2) \cdot du$

$= \int u^4 - u^6 \, du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$

eg2. (4/6) $\int \sin^3 x \cdot \cos^{\frac{3}{2}} x \cdot dx$. $\sin^3 x$ ODD: substitute $\cos x$. (NOT $\cos^{\frac{3}{2}} x$)

$u = \cos x$, $du = -\sin x \, dx$.

$= \int \sin^3 x \cdot u^{\frac{3}{2}} \cdot \frac{du}{-\sin x}$.

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - u^2$

$= \int -\sin^2 x \cdot u^{\frac{3}{2}} \, du$

$\Rightarrow -\sin^2 x = u^2 - 1$

$= \int (u^2 - 1) \cdot u^{\frac{3}{2}} \, du = \int u^{2+\frac{3}{2}} - u^{\frac{3}{2}} \, du$

$= \int u^{\frac{7}{2}} - u^{\frac{3}{2}} \, du = \frac{1}{\frac{7}{2}+1} u^{\frac{7}{2}+1} - \frac{1}{\frac{3}{2}+1} u^{\frac{3}{2}+1} + C$

$= \boxed{\frac{2}{9} \cdot (\cos x)^{\frac{9}{2}} - \frac{2}{5} \cdot (\cos x)^{\frac{5}{2}} + C}$

Rmk: If both sin and cos x have odd powers, then sub the one with HIGHER order.

②. eg3 (4/6) Evaluate $\int_0^{\pi} (2\sin x + 2)^2 - 4 \cdot dx$. For $\sin^2 x$ term, we love the D.A.F.

$= \int_0^{\pi} 4 \cdot \sin^2 x + 8\sin x + 4 - 4 \, dx$. $= \frac{1 - \cos 2x}{2}$. (formula sheet)

$= \int_0^{\pi} 4 \cdot \frac{1 - \cos 2x}{2} + 8 \cdot \sin x \cdot dx$. $\cos^2 x = \frac{1 + \cos 2x}{2}$. (formula sheet)

$= \int_0^{\pi} 2 - 2\cos 2x + 8 \cdot \sin x \, dx = 2x - \sin 2x - 8\cos x \Big|_0^{\pi} = \boxed{2\pi + 16}$

③. tan-Sec and Mixed Type.

Formulas to be memorized: $\sec^2 x = \tan^2 x + 1$, $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \cdot \sec x$.

Formula-Sheet: $\int \tan x \, dx = \ln|\sec x| + C$, $\int \sec x \, dx = \ln|\sec x + \tan x| + C$.

eg4 (s17). $\int \tan x \cdot \sec^3 x \, dx$. u-sub: $u = \sec x$, $du = \tan x \cdot \sec x \, dx$

$= \int \underbrace{\sec^2 x}_{u^2} \cdot \underbrace{\tan x \sec x \, dx}_{du}$.

$= \int u^2 \cdot du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\sec^3 x + C}$

eg5. $\int \tan^2(5x) \, dx$. Hint: $\tan^2 ▨ = \sec^2 ▨ - 1$ and $\int \sec^2 x \, dx = \tan x + C$

$= \int \sec^2(5x) - 1 \, dx = \boxed{\frac{1}{5}\tan(5x) - x + C}$

eg6. (f16) $\int \frac{\sec\theta}{\tan^2\theta} d\theta$ Hint: $\sec\theta = \frac{1}{\cos\theta}$; $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Apply these Trig-ID first.

$= \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} d\theta = \int \frac{\cos\theta}{\sin^2\theta} d\theta \xrightarrow[du = \cos\theta d\theta]{u = \sin\theta} \int \frac{du}{u^2} = -\frac{1}{u} + C = \boxed{-\frac{1}{\sin\theta} + C}$

WW Hints:

ww2. $\int \cos^4(6x) dx$. D.A.F twice: $\cos^4(6x) = \left[\frac{1+\cos(12x)}{2}\right]^2 = \frac{1 + 2\cos(12x) + \cos^2(12x)}{4} = \frac{1 + 2\cos(12x) + \frac{1+\cos(24x)}{2}}{4}$.

ww3. $\int \sin^2(4x) \cos^2(4x) dx$. sin DAF. $\sin ▨ \cdot \cos ▨ = \frac{\sin 2▨}{2} \Rightarrow [\sin 4x \cdot \cos 4x]^2 = \left[\frac{\sin 8x}{2}\right]^2 = \frac{1}{4} \cdot \frac{1 - \cos 16x}{2}$.

ww7. $\int \tan^3(5x) dx = \int \tan(5x) \cdot \tan^2(5x) dx$

$= \int \tan(5x)[\sec^2(5x) - 1] dx = \underbrace{\int \tan(5x) \cdot \sec^2(5x) dx}_{\text{u-sub: } u = \tan(5x)} - \underbrace{\int \tan(5x) dx}_{\text{Formula sheet}}$.

ww8 $\int \tan^4(3x) dx = \int \tan^2(3x)[\sec^2(3x) - 1] dx = \underbrace{\int \tan^2(3x) \cdot \sec^2(3x) dx}_{\text{eg.4 } u = \tan(3x)} - \underbrace{\int \tan^2(3x) dx}_{\text{eg.5; } \tan^2 ▨ = \sec^2 ▨ - 1}$.

☆ $\int \sin 3x \cdot \sin 8x \cdot dx$ Product-Sum formula (formula-sheet): $\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$= \int \frac{1}{2}[\cos(3x - 8x) - \cos(3x + 8x)] dx = \int \frac{1}{2}[\cos(-5x) - \cos(11x)] dx$

§7.3 Trigonometric Substitution

① $\sqrt{a^2-b^2x^2} \xlongequal{bx=a\sin\theta} \sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=\sqrt{a^2\cos^2\theta}=a\cos\theta$, $dx=\frac{a\cos\theta}{b}d\theta$

② $\sqrt{b^2x^2-a^2} \xlongequal{bx=a\sec\theta} \sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2(\sec^2\theta-1)}=\sqrt{a^2\tan^2\theta}=a\tan\theta$, $dx=\frac{a\tan\theta\sec\theta}{b}d\theta$

③ $\sqrt{b^2x^2+a^2} \xlongequal{bx=a\tan\theta} \sqrt{a^2\tan^2\theta+a^2}=\sqrt{a^2(\tan^2\theta+1)}=\sqrt{a^2\sec^2\theta}=a\sec\theta$, $dx=\frac{a\sec^2\theta}{b}d\theta$

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eg.1. (s17, multi-choice). What is the appropriate substitution for integral $\int\frac{\sqrt{25x^2-4}}{x}dx$?

$\sqrt{25\cdot x^2-4}$

$b^2 \quad a^2$

$b=5, \quad a=2.$

Idea: we want. $25x^2=4\sec^2\theta \iff 5x=2\sec\theta \iff \boxed{x=\frac{2}{5}\sec\theta}$

which gives $\sqrt{25x^2-4}=\sqrt{4\sec^2\theta-4}$

$=\sqrt{4(\sec^2\theta-1)}=\sqrt{4\cdot\tan^2\theta}=2\tan\theta$

eg.2 (f16) Evaluate $\int\frac{1}{x^2\sqrt{x^2+4}}dx$.

SLN: $\sqrt{x^2+4} \xlongequal{x=2\tan\theta} \sqrt{4\tan^2\theta+4}=\sqrt{4\sec^2\theta}=2\sec\theta$, $dx=2\sec^2\theta\cdot d\theta$

$$\int\frac{1}{x^2\sqrt{x^2+4}}dx=\int\frac{1}{(2\tan\theta)^2\cdot 2\sec\theta}\cdot 2\sec^2\theta\, d\theta=\int\frac{\sec\theta}{4\tan^2\theta}d\theta \quad \text{(eg.6 in §7.2)}$$

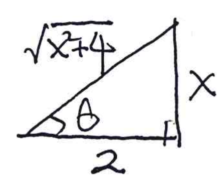
$$=\int\frac{\cos\theta}{4\sin^2\theta}d\theta \qquad u=\sin\theta$$

$$=-\frac{1}{4}\cdot\frac{1}{\sin\theta}+C.$$

Caution: the final answer has to be expressed in terms of x, NOT in θ.

LAST STEP: solving right triangle: $x=2\tan\theta \iff \tan\theta=\frac{x}{2}$.

$\Rightarrow \sin\theta=\frac{x}{\sqrt{x^2+4}}$



Remark: $\theta=\tan^{-1}(\frac{x}{2}) \Rightarrow \sin\theta=\sin(\tan^{-1}\frac{x}{2})$ NOT FULL CREDITS.

$$\int\frac{1}{x^2\sqrt{x^2+4}}dx=-\frac{1}{4}\cdot\frac{1}{\sin\theta}=\boxed{-\frac{1}{4}\cdot\frac{\sqrt{x^2+4}}{x}+C}$$

eg3. $\int\frac{x}{\sqrt{x^2+4}}dx \xlongequal[\substack{u=x^2+4 \\ du=2x\cdot dx}]{\text{Direct u-sub}} \int\frac{\frac{1}{2}du}{\sqrt{u}}=\sqrt{u}=\sqrt{x^2+4}+C$. Trig-sub also works, direct u-sub is much easier.

eg4: (s16) $\int \frac{8\,dx}{x^2\sqrt{16-x^2}}$ $\sqrt{16-x^2} = \sqrt{4^2-x^2} \xrightarrow[x=4\sin\theta]{a=4,\,b=1} \sqrt{4^2(1-\sin^2\theta)} = 4\cos\theta$, $dx = 4\cos\theta\, d\theta$

$$= \int \frac{8}{(4\sin\theta)^2 \cdot 4\cos\theta} \cdot 4\cos\theta \cdot d\theta$$

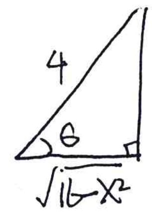
$$= \int \frac{1}{2\sin^2\theta}\, d\theta.$$ Hint: Trig-Id: $\frac{1}{\sin\theta} = \csc\theta$. $(\cot\theta)' = -\csc^2\theta$.

$$= \int \frac{1}{2} \csc^2\theta\, d\theta = -\frac{1}{2}\cot\theta + C.$$

Solving-Triangle: $x = 4\sin\theta \Leftrightarrow \sin\theta = \frac{x}{4}$.

$$= -\frac{1}{2}\cot\theta + C$$

$$= \boxed{-\frac{1}{2} \cdot \frac{\sqrt{16-x^2}}{x} + C}$$



$\cot\theta = \frac{\sqrt{16-x^2}}{x}$.

eg5. (conti-eg1). Evaluate $\int \frac{\sqrt{25x^2-4}}{x}\, dx$, $5x = 2\sec\theta$, $\sqrt{25x^2-4} = 2\tan\theta$.

$\Leftrightarrow x = \frac{2}{5}\sec\theta$, $dx = \frac{2}{5}\tan\theta\sec\theta\, d\theta$

$$= \int \frac{\sqrt{4\sec^2\theta - 4}}{\frac{2}{5}\sec\theta} \cdot \frac{2}{5}\tan\theta \cdot \sec\theta \cdot d\theta$$

$$= \int \frac{2\tan\theta}{\cancel{\frac{2}{5}\sec\theta}} \cdot \cancel{\frac{2}{5}}\tan\theta \cdot \cancel{\sec\theta} \cdot d\theta = \int 2\tan^2\theta \cdot d\theta$$ (eg5 in §7.2)

$$= \int 2(\sec^2\theta - 1) \cdot d\theta$$

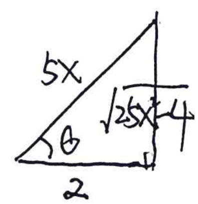
$$= 2(\tan\theta - \theta) + C.$$

$$= \boxed{2\left[\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\left(\frac{5x}{2}\right)\right] + C}$$

Solving-Triangle:

$\sec\theta = \frac{5x}{2}$

$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$.



Rmk: For θ part, it's OK to use $\sec^{-1}$, but not for tan part. (NOT FULL CREDITS)

$\tan\theta = \tan\left(\sec^{-1}\left(\frac{5x}{2}\right)\right)$.

WW Hints:

ALL webwork questions use variable t instead of θ, which will be the same.

WW7: $\int \frac{dx}{\sqrt{x^2-6x-72}}$ Complete the square: $x^2-6x-72 = (x-3)^2 - 81$.

Set $x-3 = 9\sec\theta$, then $\sqrt{x^2-6x-72} = \sqrt{81\sec^2\theta - 81} = 9\tan\theta$.

$dx = 9\tan\theta \cdot \sec\theta \cdot d\theta$

$$= \int \frac{9\tan\theta\sec\theta \cdot d\theta}{9 \cdot \tan\theta}$$

$$= \int \sec\theta \cdot d\theta \xlongequal{\text{Formula Sheet}} \ln|\sec\theta + \tan\theta| = \ln\left|\frac{x-3}{9} + \frac{\sqrt{(x-3)^2-81}}{9}\right| + C.$$