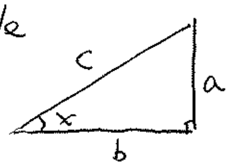
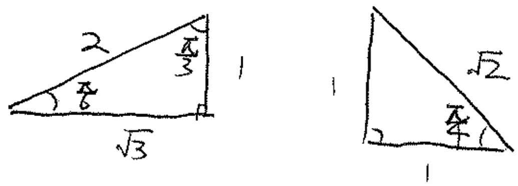
§6.6 Inverse Trigonometric Functions

- Right Triangle



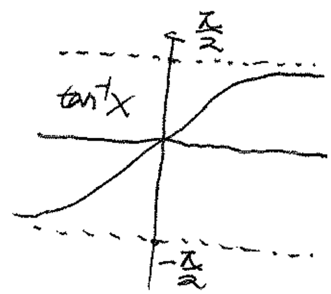
$$\boxed{\sin x = \frac{a}{c} \quad \cos x = \frac{b}{c} \quad \tan x = \frac{a}{b} \quad \sec x = \frac{c}{b}}$$

$$\csc x = \frac{c}{a} \quad \cot x = \frac{b}{a}.$$



- Inverse Trig.

	Inverse	Domain	Range	Derivative
☆ $y=\sin x$, $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$	$y=\sin^{-1}x$ $=\arcsin x$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$(\sin^{-1}x)'=\frac{1}{\sqrt{1-x^2}}$
$y=\cos x$. $x\in[0,\pi]$	$y=\cos^{-1}x$ $=\arccos x$	$[-1,1]$	$[0,\pi]$	$(\cos^{-1}x)'=\frac{-1}{\sqrt{1-x^2}}$
☆ $y=\tan x$, $x\in(-\frac{\pi}{2},\frac{\pi}{2})$	$y=\tan^{-1}x$ $=\arctan x$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$(\tan^{-1}x)'=\frac{1}{x^2+1}$



Rmk: $\sec^{-1}$, $\csc^{-1}$, $\cot^{-1}$ are not frequently used. For detailed definition, please refer to the testbook and the MSU Cal II notes.

Rmk: The derivatives of all the six Inverse Trig are given in the exam formula sheet.

eg.1 Compute $(\tan^{-1}(e^x))'$ and $\int \frac{\sin^{-1}y}{\sqrt{1-y^2}}\,dy$. See Lec Note 2. Page 3 eg2,3.

eg.2 (f16). Evaluate $\int \frac{5}{\sqrt{1-4x^2}}\,dx$.

Hint: Rewrite $\frac{1}{\sqrt{1-4x^2}} = \frac{1}{\sqrt{1-(2x)^2}}$, which allows us to apply $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$ via u-sub.

$u=2x$, $du=2dx$.

$$\int \frac{5}{\sqrt{1-4x^2}}\,dx = \int \frac{5}{\sqrt{1-u^2}}\,\frac{du}{2} = \frac{5}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{5}{2}\sin^{-1}u = \boxed{\frac{5}{2}\sin^{-1}(2x)+C}$$

eg.3 (s16). Find $f'(x)$ if $f(x)=\cos^{-1}(3x)$.

SLN: Outer: $\cos^{-1}\blacksquare \longrightarrow (\cos^{-1}\blacksquare)' = \frac{-1}{\sqrt{1-\blacksquare^2}}$. Inner: $3x \rightarrow (3x)'=3$.

$$f'(x) = \text{Outer}'(\text{inner})\cdot \text{Inner}' = \frac{-1}{\sqrt{1-(3x)^2}}\cdot(3x)' = \boxed{-\frac{3}{\sqrt{1-9x^2}}}$$

eg.4 (s16) Evaluate $\int \frac{1}{1+4x^2}\,dx$.

Hint: Similar to eg 2, $\frac{1}{1+4x^2}$ can be ~~wri~~ rewritten as $\frac{1}{1+(2x)^2}$

$u=2x$, $du=2\cdot dx$.

$$\int \frac{1}{1+4x^2}\,dx = \int \frac{1}{1+u^2}\frac{du}{2} = \frac{1}{2}\tan^{-1}u = \boxed{\frac{1}{2}\tan^{-1}(2x)+C}$$

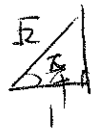
- Formulas Related to $\sin^{-1}$ and $\tan^{-1}$ (via u-sub)

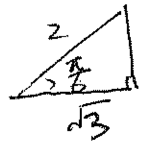
$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+C, \quad \int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)+C$$

Hints for Webwork:

- ww9: $\int_{\frac{2\sqrt{3}}{9}}^{\frac{\sqrt{2}}{3}} \frac{11}{t\cdot\sqrt{9t^2-1}}\,dt$. Formula: $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$.

$u=3t$, $du=3dt$ $\quad = \int_{\frac{2\sqrt{3}}{3}}^{\sqrt{2}} \frac{11}{\frac{u}{3}\sqrt{u^2-1}}\cdot\frac{du}{3} = 11\int_{\frac{2\sqrt{3}}{3}}^{\sqrt{2}} \frac{1}{u\sqrt{u^2-1}}\,du = 11\sec^{-1}u\Big|_{\frac{2\sqrt{3}}{3}}^{\sqrt{2}}$

Hint: 

$\sec(\frac{\pi}{4})=\sqrt{2}$, 

$\sec(\frac{\pi}{6})=\frac{2}{\sqrt{3}}$

$$=11\sec^{-1}(\sqrt{2})-11\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \boxed{11\cdot\frac{\pi}{4}-11\cdot\frac{\pi}{6}}$$

- ww10: $\int \frac{y}{\sqrt{1-4y^4}}\,dy \quad \frac{u=2y^2}{du=4y\cdot dy} \quad \int \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{4}$ (to be completed)

- ww11: $\int \frac{8}{x^2-2x+17}\,dx$. Hint: complete the square $x^2-2x+17=(x-1)^2+4^2$

$$= \int \frac{8}{(x-1)^2+4^2}\,dx \quad \frac{x-1=4u}{dx=4du} \quad \int \frac{8}{4^2u^2+4^2}\,4\,du = 2\int \frac{1}{1+u^2}\,du$$ (to be completed)

§6.7. Hyperbolic Functions.

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad \operatorname{csch}(x) = \frac{1}{\sinh x}$ $\quad \operatorname{sech}(x) = \frac{1}{\cosh x}$.

  $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad \tanh(x) = \frac{\sinh x}{\cosh x}$ $\quad \coth(x) = \frac{\cosh x}{\sinh x}$.

- $\cosh^2 x - \sinh^2 x = 1$, $\frac{d}{dx}(\sinh x) = \cosh x$, $\frac{d}{dx}(\cosh x) = \sinh x$.

eg1: Find (Simplify) $3\sinh(2\ln 4) = 3 \cdot \frac{e^{2\ln 4} - e^{-2\ln 4}}{2}$ $\quad$ Hint: $a \ln b = \ln b^a$
(ww).

$$= 3 \cdot \frac{e^{\ln 4^2} - e^{\ln 4^{-2}}}{2} = 3 \cdot \frac{4^2 - 4^{-2}}{2} = \frac{3}{2}\left(16 - \frac{1}{16}\right) = \boxed{\frac{765}{32}}$$

eg2. Evaluate $\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$. Hint: $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.
(#16).

(u-Sub) $\frac{u = \sqrt{x}}{du = \frac{1}{2\sqrt{x}}dx}$ $\int \cosh u \cdot 2\,du = 2\sinh u = \boxed{2\sinh\sqrt{x} + C}$

☆ eg3 Compute y' if $y = \operatorname{sech}(3x)$.
(\#17). Hint: need to compute $(\operatorname{sech} x)'$ via $\operatorname{sech} x = \frac{1}{\cosh x}$ first.

$$(\operatorname{sech} x)' = \left[(\cosh x)^{-1}\right]' = -1 \cdot (\cosh x)^{-2} \cdot (\cosh x)'$$

$$= -1 \cdot \frac{1}{(\cosh x)^2} \cdot \sinh x = -1 \cdot \frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \cdot \tanh x$$

$$\Rightarrow (\operatorname{sech}(3x))' = \boxed{-\operatorname{sech}(3x) \cdot \tanh(3x) \cdot 3}$$

Webwork Hints:

ww5: $(\tanh x)' = \operatorname{sech}^2 x$. $\Rightarrow \left[e^{\tanh(9x)}\right]' = e^{\tanh(9x)} \cdot [\operatorname{sech}(9x)]^2 \cdot 9$.

ww6: $\int \tanh 6x\,dx$. Hint: $\tanh(6x) = \frac{\sinh(6x)}{\cosh(6x)}$. u-sub: $u = \cosh(6x)$

ww7: $\int \operatorname{sech}^2(x)\,dx = \tanh(x) + C$.

§6.8. L'Hopital Rule

Key points:

① (LH). If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$, then $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$

② (Basic Limits) $\frac{1}{\pm\infty} = 0$, $\frac{1}{0^\pm} = \pm\infty$, $e^{+\infty} = +\infty$, $e^{-\infty} = 0$, $\ln +\infty = +\infty$, $\ln 0^+ = -\infty$

$\tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$

③ (Indeterminate Forms) $\boxed{\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty}$, $\infty - \infty$, 0^0, ∞^0, 1^∞

- Motivation: limits for indeterminate forms (I.F.)

eg.0. Evaluate the following limits ① $\lim_{x\to 0^+} \frac{\cos x}{2x}$ ② $\lim_{x\to 0^+} \frac{\sin x}{2x}$.

① $\overset{\text{Plug in } 0^+}{=\!=} \frac{\cos 0^+}{2\cdot 0^+} = \frac{1}{0^+} = +\infty$; ② $\overset{\text{Plug in } 0^+}{=\!=} \frac{\sin 0^+}{2\cdot 0^+} = \boxed{\frac{0^+}{0^+} = ?}$

The type of the limit $\lim_{x\to 0} \frac{\sin x}{2x}$ is called INDETERMINATE FORM

eg.1. $\lim_{x\to\infty} \frac{x^2+1}{x} = \infty$, $\lim_{x\to\infty} \frac{2x^3-1}{5x^4+6x} = 0$, $\lim_{x\to\infty} \frac{2x+1}{3x-2} = \frac{2}{3}$.

(leading-terms rule for rational functions).

eg.1 shows the answer for indeterminate forms can be anything (0, infinity, finite number)

The general method to deal with I.F. is the l'Hopital Rule (L.H.) which may convert an I.F. limit into a NON-I.F. limit.

②: $\lim_{x\to 0^+} \frac{\sin x}{2x} \overset{\text{LH}}{=\!=} \lim_{x\to 0^+} \frac{(\sin x)'}{(2x)'} = \lim_{x\to 0^+} \frac{\cos x}{2} \overset{\text{Plug in } 0^+}{=\!=} \frac{\cos 0^+}{2} = \boxed{\frac{1}{2}}$

Rmk: $\lim_{x\to 0^+} \frac{\cos x}{2x}$ is NOT an I.F., we cannot apply LH to this limit.

Actually, $\lim_{x\to 0^+} \frac{(\cos x)'}{(2x)'} = \lim_{x\to 0^+} \frac{-\sin x}{2} = \frac{-\sin 0}{2} = \boxed{0 \neq \lim_{x\to 0^+} \frac{\cos x}{2x} = +\infty}$

eg.2. (f16) ($\frac{0}{0}$-case). Evaluate $\lim_{t\to 0} \frac{\sin 4t}{e^{2t}-1}$

$$\overset{\text{L.H.}}{=} \lim_{t\to 0} \frac{(\sin 4t)'}{(e^{2t}-1)'} = \lim_{t\to 0} \frac{4\cdot\cos 4t}{2\cdot e^{2t}} \overset{\text{Plug in } 0}{=} \frac{4\cdot\cos 0}{2\cdot e^{0}} = \frac{4\cdot 1}{2\cdot 1} = \boxed{2}$$

eg.3. (s16) ($\frac{0}{0}$-case) Evaluate $\lim_{x\to 0} \frac{e^x-1}{x^2+3x}$.

$$\overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{(e^x-1)'}{(x^2+3x)'} = \lim_{x\to 0} \frac{e^x}{2x+3} \overset{\text{Plug in } 0}{=} \frac{e^0}{2\cdot 0+3} = \boxed{\frac{1}{3}}$$

☆ eg.4. Evaluate $\lim_{x\to\infty} \frac{e^x-1}{x^2+3x}$. ($\frac{\infty}{\infty}$ case) since $e^{\infty}=\infty$

$$\overset{\text{1st L.H.}}{=} \lim_{x\to\infty} \frac{(e^x-1)'}{(x^2+3x)'} = \lim_{x\to\infty} \frac{e^x}{2x+3}$$

Caution: Plug in ∞, it is still $\frac{\infty}{\infty}$ I.F.
We need L.H. one more time.

$$\overset{\text{2nd L.H.}}{=} \lim_{x\to\infty} \frac{(e^x)'}{(2x+3)'} = \lim_{x\to\infty} \frac{e^x}{2} \overset{\text{Plug in}}{=} \frac{e^{\infty}}{2} = \boxed{+\infty}$$

eg.5 (f16). ($\infty\cdot 0$ case) Evaluate $\lim_{x\to -\infty} x\cdot e^x$.

Hint: $\lim_{x\to-\infty} e^x = 0$. We need to convert the product into a ratio.

In general, $0\cdot\infty = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$ or $0\cdot\infty = \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$.

SLN: Rewrite $\lim_{x\to-\infty} x\cdot e^x = \lim_{x\to-\infty} \frac{x}{e^{-x}}$. $\frac{\infty}{\infty}$ case

$$\overset{\text{L.H.}}{=} \lim_{x\to-\infty} \frac{(x)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} \overset{\text{Plug in}}{=} \frac{1}{-1\cdot e^{+\infty}} = \frac{1}{-\infty} = \boxed{0}$$

eg.6 (s16). ($0\cdot\infty$ case) Evaluate $\lim_{x\to 0^+} x\cdot\ln(2x)$. Hint: $\ln 0^+ = -\infty$

SLN: $\lim_{x\to 0^+} x\cdot\ln(2x) \overset{\text{Rewrite}}{=} \lim_{x\to 0^+} \frac{\ln(2x)}{\frac{1}{x}}$ $\quad \frac{\infty}{\infty} = \frac{\ln 0}{\frac{1}{0}}$

$$\overset{\text{L.H.}}{=} \lim_{x\to 0^+} \frac{(\ln(2x))'}{(\frac{1}{x})'} = \lim_{x\to 0^+} \frac{\frac{1}{2x}\cdot 2}{\frac{-1}{x^2}} \overset{\text{Simplify}}{=} \lim_{x\to 0^+} -x = \boxed{0}$$

• More difficult type

☆ e.g. 7 (s17, $\infty-\infty$ case) $\lim_{x\to1} \frac{x}{x-1} - \frac{1}{\ln x} \overset{\text{Algebra}}{=\!=} \lim_{x\to1} \frac{x\cdot\ln x-(x-1)}{(x-1)\cdot\ln x}$ $\quad \frac{0}{0}$ case

$$\overset{\text{1st L.H.}}{=\!=} \lim_{x\to1} \frac{(x\cdot\ln x - x+1)'}{((x-1)\cdot\ln x)'} = \lim_{x\to1} \frac{1\cdot\ln x + x\cdot\frac{1}{x} - 1}{1\cdot\ln x + (x-1)\cdot\frac{1}{x}}$$

$$(*) \overset{\text{Simplify}}{=\!=} \lim_{x\to1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}}$$ Plug in $x=1$, $\frac{\ln 1}{\ln 1+1-1} = \frac{0}{0}$ case.

$$\overset{\text{2nd L.H.}}{=\!=} \lim_{x\to1} \frac{\frac{1}{x}}{\frac{1}{x}+0+\frac{1}{x^2}} \overset{\text{Plug in}}{=\!=} \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

Rmk: in step (*), if you get $\lim_{x\to1} \frac{x\cdot\ln x}{x\ln x + x - 1}$, L.H. still works ($\frac{0}{0}$ case)

$$\overset{\text{L.H.}}{=\!=} \lim_{x\to1} \frac{1\cdot\ln x + x\cdot\frac{1}{x}}{1\cdot\ln x + x\cdot\frac{1}{x} + 1} = \lim_{x\to1} \frac{\ln x+1}{\ln x+2} = \frac{\ln 1+1}{\ln 1+2} = \boxed{\frac{1}{2}}$$

e.g. 8 (f15, $\infty-\infty$ case) $\lim_{x\to0^+} \frac{3x+1}{x} - \frac{1}{\sin x} =\!= \lim_{x\to0^+} \frac{(3x+1)\sin x - x}{x\cdot\sin x}$ $\quad \frac{0}{0}$ case

$$\overset{\text{1st L.H.}}{=\!=} \lim_{x\to0^+} \frac{[(3x+1)\cdot\sin x - x]'}{[x\cdot\sin x]'} = \lim_{x\to0^+} \frac{3\sin x + (3x+1)\cos x - 1}{\sin x + x\cdot\cos x}$$ $\quad \frac{0+1-1}{0+0\cdot1} = \frac{0}{0}$ case

$$\overset{\text{2nd L.H.}}{=\!=} \lim_{x\to0^+} \frac{3\cdot\cos x + 3\cdot\cos x - (3x+1)\sin x}{\cos x + \cos x - x\cdot\sin x} \overset{\text{Plug in}}{=\!=} \frac{3\cos 0 + 3\cos 0 - 0}{\cos 0 + \cos 0 - 0} = \boxed{\frac{6}{2} = 3}$$

☆ e.g. 9 (s17, 1^∞ case).

$$\lim_{x\to0^+} (\cos x)^{\frac{2}{x}} \underset{e^{\ln \blacksquare} = \blacksquare}{\overset{\text{Algebra}}{=\!=}} \lim_{x\to0^+} e^{\ln(\cos x)^{\frac{2}{x}}} =\!= e^{\lim_{x\to0^+} \frac{2}{x}\ln(\cos x)} \overset{\text{Plug in}}{=\!=} \boxed{e^0 = 1}$$

It is enough to evaluate $\lim_{x\to0^+} \frac{2\ln(\cos x)}{x} \underset{\text{L.H.}}{\overset{\frac{0}{0}\text{ case}}{=\!=}} \lim_{x\to0^+} \frac{\frac{2}{\cos x}\cdot(-\sin x)}{1} = \lim_{x\to0^+} \frac{-2\sin x}{\cos x} = 0$

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ww4. $\lim_{x\to0^+} 8x\cdot(\ln(3x))^2$ $\quad 0\cdot\infty$ case

ww5, 6. $\lim_{x\to\infty} (2x)^{\frac{1}{4x}} \overset{\infty^0\text{ case}}{=\!=} \lim_{x\to\infty} e^{\boxed{\ln(2x)^{\frac{1}{4x}}}}$, $\quad \lim_{x\to\infty} (\ln x)^{5x} \overset{\text{NOT I.F.}}{=\!=} \infty^\infty = \infty$

ww7. $\infty-\infty$ case