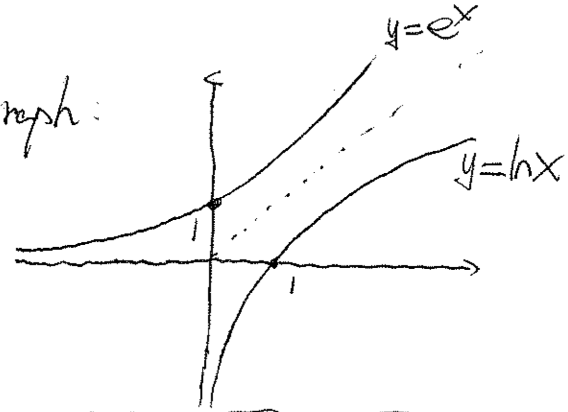
§6.2, 6.3 Natural logarithm / Exponential

	Domain	Range			
$y = e^x$	$(-\infty, +\infty)$	$(0, +\infty)$	$e^0 = 1$	$e^{a+b} = e^a \cdot e^b$	$(e^a)^b = e^{a \cdot b}$
$y = \ln x$	$(0, +\infty)$	$(-\infty, \infty)$	$\ln 1 = 0$	$\ln(x \cdot y) = \ln x + \ln y$ $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$	$\ln x^r = r \cdot \ln x$

Graph:



$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to -\infty} e^x = 0$$

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0^+} \ln x = -\infty$$

$$\ln e^x = x \qquad e^{\ln x} = x \quad \text{(Inverse Relation)}$$

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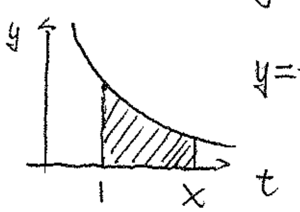
§6.2 Calculus properties of $y = \ln x$.

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad (\ln|x|)' = \frac{1}{x}$$

Rmk: An alternative way to define $\ln x$ is the area below $\frac{1}{t}$ from 1 to x, i.e.,



$$y = \frac{1}{t}. \qquad \ln x = \int_1^x \frac{1}{t}\,dt$$

eg 1 (Algebra review) Simplify the following expression (as a single logarithm)

$$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x+1)^2(x+2)^2\right]$$

$$= \ln \frac{(x+2)^{3\cdot\frac{1}{3}} \cdot x^{\frac{1}{2}}}{\left[(x+1)^2 \cdot (x+2)^2\right]^{\frac{1}{2}}} = \ln \frac{\cancel{(x+2)} \cdot x^{\frac{1}{2}}}{(x+1)\cdot\cancel{(x+2)}} = \boxed{\ln \frac{x^{\frac{1}{2}}}{(x+1)}}$$

$$\left(= \frac{1}{2}\ln x - \ln(x+1) \right)$$

eg 2. Compute $\frac{d}{dx}\ln(3x^2-1)$

(Chain rule): $\frac{d}{dx}\ln(3x^2-1) = \frac{1}{3x^2-1}\cdot(3x^2)' = \boxed{\frac{1}{3x^2-1}\cdot 6x}$

- Log - Implicit defferenciation.

☆ eg. 3 (f16, mid1) Find $f'(x)$ if $f(x)=x^{\sqrt{x}}$

Hint: $\ln \square^{O} = O\cdot\ln\square$, ln turns a power function into a product

Step 1: $f = x^{\sqrt{x}} \Rightarrow \ln f = \ln x^{\sqrt{x}} = \sqrt{x}\cdot\ln x$.

Step 2: $(\ln f)' = (\sqrt{x}\cdot\ln x)'$

L.H.S. $= \frac{1}{f}\cdot f'$ Caution: $f(x)$ is a function of x. chain rule.

R.H.S $= (\sqrt{x})'\cdot\ln x + \sqrt{x}\cdot(\ln x)'$ product rule.

$= \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$

Step 3: $\frac{1}{f}\cdot f' = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \Rightarrow f'(x) = \boxed{x^{\sqrt{x}}\left[\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right]}$

- Integration

eg. 4. (S17). Evaluate $\int \ln e^x\,dx \overset{\text{algebra}}{=} \int x\,dx = \boxed{\frac{1}{2}x^2 + C}$

eg. 5. $\int_e^{e^2} \frac{5}{x\cdot\ln x}\,dx$.   Hint: $(\ln x)' = \frac{1}{x}$. u-sub

$u = \ln x$, $du = \frac{1}{x}\cdot dx$. $\int_{x=e}^{x=e^2} \xrightarrow{u=\ln x}$ $u = \ln e^2 = 2$, $\ln e = 1$

$= \int_{\ln e}^{\ln e^2} \frac{5}{u}\cdot du = 5\ln|u|\Big|_1^2 = 5\ln 2 - 5\ln 1 = \boxed{5\ln 2}$

Hints for Webwork 8-13: u-sub

WW10: $\int \tan 4x\,dx$. $\tan 4x = \frac{\sin 4x}{\cos 4x}$, $u = \cos 4x$. $du = -4\sin 4x\,dx$

WW11: $\int \frac{\tan(\ln(4x^2))}{2x}\,dx$, $u = \ln(4x^2)$, $du = \frac{2}{x}\cdot dx$.

WW12: long-division.

S6.3. Properties of $y=e^x$. $e^a=b \Leftrightarrow a=\ln b$, $e^{\ln \blacksquare}=\blacksquare$, $\ln e^{\blacksquare}=\blacksquare$

$(e^x)'=e^x$, $\int e^x dx = e^x + C$, $(e^{ax+b})' = a\cdot e^{ax+b}$, $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$.

- eg.6 (s17). Solve for k, t if $a_9 = e^{2k}$ and $a_5 = e^{k\cdot t}$

SLN: $a_9 = e^{2k} \Rightarrow 2k = \ln a_9 \Rightarrow \boxed{k=\frac{1}{2}\ln a_9}$

$a_5 = e^{(\frac{1}{2}\ln a_9)\cdot t} \Rightarrow (\frac{1}{2}\ln a_9)\cdot t = \ln a_5 \Rightarrow \boxed{t = \frac{\ln a_5}{\frac{1}{2}\ln a_9}}$

- eg.7 (f16) Find $f'(x)$ if $f(x) = \tan^{-1}(e^x)$. Formula: $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$.

SLN: (Chain rule. Outer function: $\tan^{-1}\square$, Inner: e^x)

$f'(x) = \text{out}'(\text{inner})\cdot(\text{inner})' = \frac{1}{(e^x)^2+1}\cdot(e^x)' = \boxed{\frac{1}{e^{2x}+1}\cdot e^x}$

- eg.8 (s17). Integrate $\int \frac{e^{\sin^{-1}y}}{\sqrt{1-y^2}}\cdot dy$. Formula: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

SLN: u-sub: $u=\sin^{-1}y \Rightarrow du = \frac{1}{\sqrt{1-y^2}}dy$

$\int \frac{e^{\sin^{-1}y}}{\sqrt{1-y^2}}dy = \int e^u\cdot du = e^u + C = \boxed{e^{\sin^{-1}y}+C}$

Hints for Webwork:

☆ WW9: (Implicit diff). Find y' if $\ln y = e^{6y}\cdot\sin(3x)$. $(\ln y)' = (e^{6y}\cdot\sin 3x)'$

$\frac{1}{y}\cdot y' = e^{6y}\cdot 6y'\cdot\sin(3x) + e^{6y}\cdot\cos(3x)\cdot 3$. Solve for y'.

$\left[\frac{1}{y} - 6e^{6y}\cdot\sin(3x)\right]\cdot y' = e^{6y}\cdot\cos(3x)\cdot 3 \Rightarrow \boxed{y' = \frac{e^{6y}\cdot\cos(3x)\cdot 3}{\frac{1}{y} - 6\cdot e^{6y}\cdot\sin(3x)}}$

☆ WW12 (Initial Value). Find $y(x)$ if $\frac{d^2y}{dx^2} = 8\cdot e^{-2x}$, $y'(0)=0$, $y(0)=0$

$y'(x) = \int y''\,dx = \int 8\cdot e^{-2x}dx = 8\cdot\frac{1}{-2}\cdot e^{-2x} + C_1$, $y'(0)=0 \Rightarrow \boxed{C_1=4}$

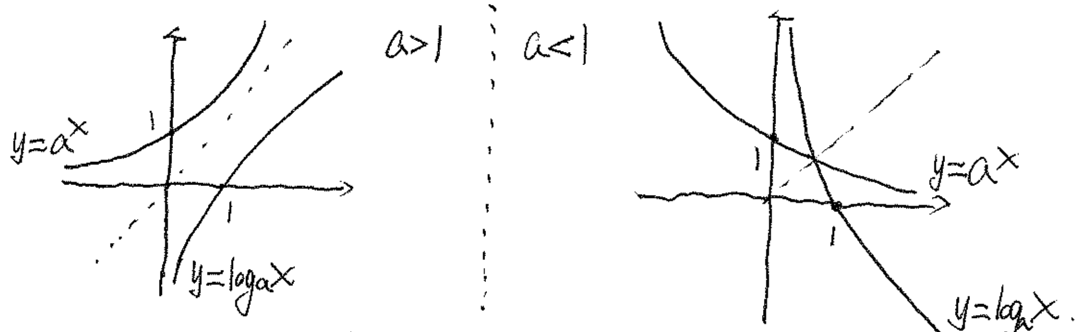
$= -4\cdot e^{-2x} + C_1$

$y(x) = \int y'\,dx = \int(-4\cdot e^{-2x}+4)dx = 2\cdot e^{-2x} + 4x + C_2$, $y(0)=0 \Rightarrow \boxed{C_2=-2}$

$= 2e^{-2x} + 4x - 2$

§6.4 General log/exp functions.

- $e^x \longrightarrow a^x$, $\ln x \longrightarrow \log_a x$, $a>0$, $a\neq 1$. ($\ln x = \log_e x$)
- $y=a^x \xleftrightarrow{\text{Inverse}} y=\log_a x$. $\blacksquare = a^{\blacktriangle} \iff \log_a \blacksquare = \blacktriangle$
- Graph:



e.g. $a=2$, $y=2^x$, $y=\log_2 x$ $\quad a=\frac{1}{2}$, $y=(\frac{1}{2})^x$, $y=\log_{\frac{1}{2}} x$.

☆ • Relation with Natural Log/Exp: $a^x = e^{x\cdot\ln a}$, $\log_a x = \dfrac{\ln x}{\ln a}$

☆ Diff/Integration: $\boxed{(a^x)' = \ln a \cdot a^x,\ (\log_a x)' = \dfrac{1}{\ln a\cdot x},\ \int a^x\,dx = \dfrac{1}{\ln a}\cdot a^x + C}$

eg.9. (S17) Find y' if ① $y=\log_5(3x^2-1)$ ② $y=4^{2x-5}$

SLN: (Chain rule) ① Outer: $\log_5 \blacksquare$. Inner $(3x^2-1)$. $y' = \dfrac{1}{\ln 5\cdot(3x^2-1)}\cdot(3x^2-1)' = \boxed{\dfrac{6x}{\ln 5\cdot(3x^2-1)}}$

② Outer: $4^{\blacksquare}$. Inner: $2x-5$. $y' = \ln 4\cdot 4^{2x-5}\cdot(2x-5)' = \boxed{\ln 4\cdot 4^{2x-5}\cdot 2}$

eg.10 (S16). Find $f'(x)$, ~~$2^{\cos x}$~~ if $f(x)=2^{\cos x}$

SLN: Chain rule. Outer: $2^{\blacksquare}$, Inner: $\cos x$. $f'(x) = \underbrace{\ln 2\cdot 2^{\cos x}}\cdot\underbrace{(\cos x)'} = \boxed{\ln 2\cdot 2^{\cos x}(-\sin x)}$

Hints for WeBwork:

ww7 $\int_0^{\frac{\pi}{4}} (\frac{1}{4})^{\tan x}\sec^2 x\,dx$. u-sub: $u=\tan x$, $du=\sec^2 x\,dx$. Formula: $\int (\frac{1}{4})^u\,du = \dfrac{(\frac{1}{4})^u}{\ln\frac{1}{4}}$.

ww8. $\int_1^e \dfrac{5^{\ln(x^3)}}{7x}\,dx$. u-sub: $u=\ln(x^3)=3\ln x \Rightarrow du=\frac{3}{x}\cdot dx$.

ww11. Find y' if $y=(\ln(4x))^{5x}$. Log-Differentiation (Page 2, eg.3)

$\Rightarrow \ln y = 5x\cdot\ln[\ln(4x)]$ Product and Chain rule

§6.5/9.3. Initial Value Problems.

- Motivation: Exponential growth problem (6.5).

Give a function $y=y(t)$ satisfying:

① The rate of change of y is proportional to y with ratio k (constant).

② The initial value of y is $y(0)=C$. (constant)

Find $y(t)$ as a function of time t.

SLN: $\begin{cases} \frac{dy}{dt} = k\cdot y \\ y(0)=C \end{cases}$ It is easy to verify that $\boxed{y(t)=C\cdot e^{kt}}$ solves the equation

k: growth rate. $C=y(0)$: initial value.

eg. 1. (517). A sample of tritium-3 decayed to 94.5% of its original amount after one year. How many years will it take to decay to half of the original amount.

Solution: Tritium amount: $A(t)$. Initial amount $A(0)$. Decay rate k.

Equation: $A(t)=A(0)\cdot e^{k\cdot t}$. Condition: $A(1)=A(0)\cdot 94.5\%$

$\Rightarrow \cancel{A(0)}\cdot 0.945 = \cancel{A(0)}\cdot e^{k\cdot 1} \Rightarrow k=\ln 0.945$.

Q: $\cancel{A(0)}\cdot 0.5 = A(t) = \cancel{A(0)}\cdot e^{kt} \Rightarrow 0.5 = e^{kt} \Rightarrow 0.5 = e^{(\ln 0.945)t} \Rightarrow \boxed{t=\frac{\ln 0.5}{\ln 0.945}}$

- General separable equations and the Method to solve the eqns.

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \iff \underbrace{\boxed{h(y)\cdot dy = g(x)\cdot dx}}_{\text{separable form}} \xrightarrow[\text{Both Sides}]{\text{Integrate}} \boxed{\int h(y)\,dy = \int g(x)\,dx}$$

eg 2 Rewrite the following equations in their separable forms:

① $3x - 10\cdot y\cdot\sqrt{x^2+1}\cdot\frac{dy}{dx}=0 \iff 3x = 10y\cdot\sqrt{x^2+1}\,\frac{dy}{dx} \iff \boxed{\frac{3x}{\sqrt{x^2+1}}\,dx = 10\cdot y\cdot dy}$

② $6\cdot\sec x\cdot\frac{dy}{dx} = e^{y+\sin x} \iff 6\sec x\cdot\frac{dy}{dx} = e^{y}\cdot e^{\sin x}$

$\iff \cancel{6\sec x}\ \boxed{6\cdot e^{-y}\,dy = \frac{1}{\sec x}e^{\sin x}dx = \cos x\cdot e^{\sin x}\cdot dx}$

• "Initial" Value Problem. $h(y)\,dy = g(x)\,dx \iff \int h(y)\,dy = \int g(x)\,dx$. given $y(x_0)=y_0$.

eg3. ~~Initial~~ (s17). Find the solution to the initial value problem

$$y' = \frac{\ln x}{xy}, \quad y(1)=2.$$

SLN: $y' = \frac{dy}{dx} = \frac{\ln x}{xy}$. Step 1: Rewrite as a separable equation.

$$y \cdot dy = \frac{\ln x}{x} \cdot dx.$$

Step 2: Integrate: $\int y \cdot dy = \int \frac{\ln x}{x}\,dx$.

L.H.S. $= \frac{1}{2}y^2$. R.H.S. $\xrightarrow[du=\frac{1}{x}dx]{u=\ln x} \int u \cdot du = \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C.$$

Step 3: Use initial condition $y(1)=2$ to find constant C.

$y(1)=2 \Rightarrow x=1,\ y=2$ Plug in.

$$\frac{1}{2}\cdot 2^2 = \frac{1}{2}(\ln 1)^2 + C \Rightarrow 2 = 0 + C \Rightarrow C=2.$$

Step 4: Solve for y. $\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2 \Rightarrow y^2 = (\ln x)^2 + 4$.

$\Rightarrow y = \pm\sqrt{(\ln x)^2+4}$, $y = -\sqrt{(\ln x)^2+4}$ does not meet $y(1)=2$

$$\Rightarrow \boxed{y = +\sqrt{(\ln x)^2+4}}$$

eg4: Solve the equation in eg. 2. ② if $y(0)=1$.

(ww3) $\int 6e^{-y}\,dy = -6\cdot e^{-y}$, $\int \cos x \cdot e^{\sin x}\,dx \xrightarrow[du=\cos x\,dx]{u=\sin x} \int e^u\,du = e^u = e^{\sin x}$

$\iff -6\cdot e^{-y} = e^{\sin x} + C$. $y(0)=1 \iff x=0,\ y=1$.

Plug in $x=0$, $y=1$ to solve for C. $-6e^{-1} = e^0 + C \Rightarrow C = -6\cdot e^{-1} - 1$

$\Rightarrow -6\cdot e^{-y} = e^{\sin x} - 6e^{-1} - 1$. solve for y

$$e^{-y} = -\frac{1}{6}e^{\sin x} + e^{-1} + \frac{1}{6}$$

$$-y = \ln\left(-\frac{1}{6}e^{\sin x} + e^{-1} + \frac{1}{6}\right) \Rightarrow \boxed{y = -\ln\left(-\frac{1}{6}e^{\sin x} + e^{-1} + \frac{1}{6}\right)}$$