§5.4. Work

Key formulas:

① Work = Force × Distance, $W = F \cdot d$, work done by constant force F.

② $W = \int_a^b f(x)\,dx$. The work in moving an object from a to b by force $f(x)$.

☆ ③ Water-Pumping formula: $\text{Work} = \int_a^b \underbrace{\delta}_{\text{Density}} \cdot \underbrace{s(y)}_{\text{Distance}} \cdot \underbrace{A(y)}_{\text{Area of Cross-Section}}\,dy$

- Work done by constant force

eg.0. How much work is done in lifting a 20-lb weight 6ft off the ground?

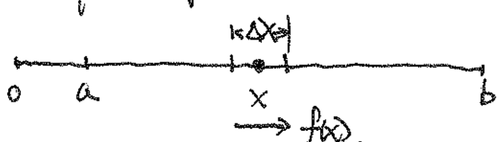
$$W = F \cdot d = 20 \cdot 6 = \boxed{120 \text{ ft-lbs}}$$

Remark: Work has two unit, ft-lb and J = Newton × Meter (joule).

- The force is a function of the position x, $f(x)$. Work done by moving a particle from a to b.

$$W = \int_a^b f(x)\,dx.$$

Idea of the formula:



work done by the

The total work equals the sum of the "constant" force along the path.

$$W \approx \sum f(x) \cdot \Delta x \approx \int_a^b f(x)\,dx.$$

eg.1. (s16, mid1). A variable force of $x^2 - 2x$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work W done in moving the object from $x=2$ to $x=3$ feet.

SLN: $$W = \int_2^3 x^2 - 2x\,dx = \left(\frac{1}{3}x^3 - x^2\right)\Big|_2^3 = \frac{1}{3}\cdot 3^3 - 3^2 - \left(\frac{1}{3}\cdot 2^3 - 2^2\right) = \boxed{\frac{4}{3} \text{ ft-lbs}}$$

Rmk: $\int x^n\,dx = \frac{1}{n+1} \cdot x^{n+1}$, $n \neq -1$.

- Spring (Rubber) and the Hooke's Law. (Examples from physics).

eg.2 (f14, mid1). To hold a spring stretched 2 m beyond its natural length requires a force of 12 Newtons. Compute the work needed to stretch the spring from 2 m beyond its natural length to 3 m beyond its natural length.

SLN: step1: Find the spring constant k via Hooke's Law $f(x) = k \cdot x$.

Plug in the given data: $12 = k \cdot 2 \Rightarrow k = 6 \Rightarrow f(x) = 6x$.

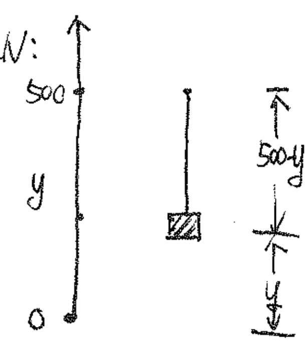
Step2: Compute the work by formula ②

$$W = \int_2^3 f(x)\,dx = \int_2^3 6x \cdot dx = 6 \cdot \tfrac{1}{2}x^2 \Big|_2^3 = 3x^2\Big|_2^3 = 3\cdot 3^3 - 3\cdot 2^2 = \boxed{15\ J}$$

- Work against gravity $\begin{cases} \text{Cable-lifting} & (\text{ww}3) \\ \text{Water-Pumping} & (\text{ww}4,5) \end{cases}$

eg3 (s17, mid1) A cable that weighs 2 lb/ft is used to haul 800 lbs of coal up a shaft that is 500 ft deep. Find the work that is done.

SLN:



Weight (Gravity) at height y:

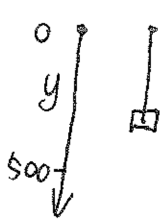
$$800 + 2\cdot(500-y).$$

$$W = \int_0^{500} 800 + 2(500-y)\cdot dy$$

$$= \int_0^{500} 1800 - 2y \cdot dy = 1800y - y^2 \Big|_0^{500}$$

$$= 1800\cdot 500 - (500)^2 = 650{,}000 \text{ ft-lb}$$

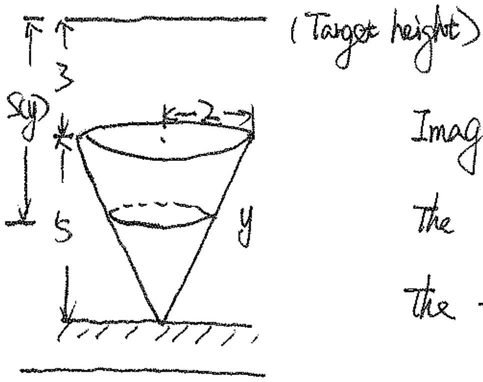
Rmk: The answer will be the same if you set up the axis pointing down.



Actually, one can prove via u-sub that

$$\int_0^{500} 800 + 2\cdot y\, dy = \int_0^{500} 800 + 2(500-y)\, dy$$

☆ e.g. 4 (s15) (Water-Pumping). A tank is in the shape of a downward-pointing cone (inverted circular cone) which has height 5 feet and radius 2 feet. The tank is full of oil weighing 7 lb/ft^3. Find the work it would take to pump the oil from the tank to an outlet 3 feet above the top of the tank.

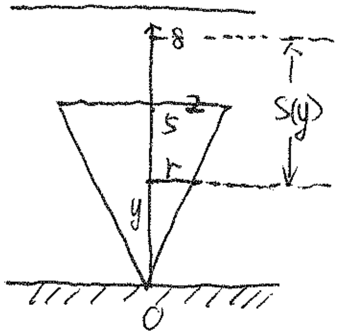


Idea of the Formula: $W = \int_a^b \sigma \cdot s(y) \cdot A(y) \cdot dy$

Imagine the oil can be cut into "horizontal slices"

The work against each slice's gravity is $\sigma \cdot A(y) \cdot s(y)$

The total work will be the sum (integral) of these work.



σ: Density

$s(y)$: distance to pump (distance from the slice at y to desitination)

$A(y)$: The area of the 'slice' (Area of the cross-section)

SLN: $\sigma = 7$, $s(y) = 8 - y$. $(0 \le y \le 5)$

The Cross-Section is a circle. We need to find its radius r via SIMILAR TRIANGLE

$$\frac{r}{2} = \frac{y}{5} \Rightarrow r = \frac{2}{5}y \Rightarrow A(y) = \pi \cdot r^2 = \pi \cdot \left(\frac{2}{5}y\right)^2$$

Formula ③:
$$W = \int_0^5 \sigma \cdot s(y) \cdot A(y) dy = \int_0^5 7 \cdot (8-y) \cdot \pi \left(\frac{2}{5}y\right)^2 dy$$

$$= \int_0^5 \frac{28}{25}\pi \cdot (8-y) \cdot y^2 dy$$

$$= \int_0^5 \frac{28}{25}\pi \cdot 8y^2 - \frac{28}{25}\pi \cdot y^3 dy$$

$$= \frac{28}{25}\pi \cdot 8 \cdot \frac{1}{3}y^3 - \frac{28}{25}\pi \cdot \frac{1}{4}y^4 \Big|_0^5$$

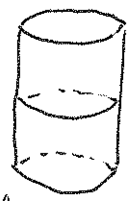
$$= \boxed{\frac{28}{25}\pi \cdot \frac{8}{3} \cdot 5^3 - \frac{28}{25}\pi \cdot \frac{1}{4} \cdot 5^4 \text{ ft} \cdot \text{lb}}$$

Rmk: Several commonly used tanks

- Inverted circular cone.
- Vertical Cylinder
- Rectangular cuboid

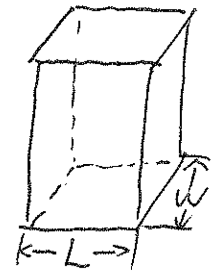
$A(y)=\pi\cdot r^2$

r depends on y.



$A(y)=\pi\cdot r^2$

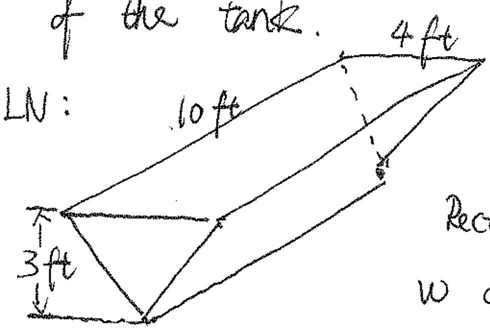
r is a constant.



$A(y)=L\cdot W$ (constant)

☆ eg5. (~~[illegible]~~ f16, mid1) A tank (shown below) with ends that are isosceles triangles is filled with oil weighing 90 lbs/ft³. Find the work required to pump all of the oil out to a height of 2 feet above the top of the tank.

SLN:

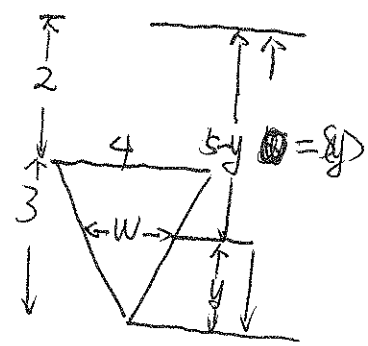


$\sigma=90$, $S(y)=5-y$

Cross-Section:

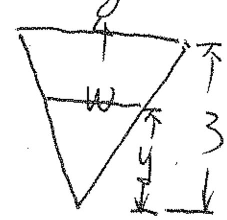
Rectangle with length 10, width w

w changes as y changes:



Similar triangle: $\frac{w}{4}=\frac{y}{3}$

$\Rightarrow w=\frac{4}{3}y$



$A(y)=10\cdot\frac{4}{3}y$

Formula ③: $W=\int_0^3 90\cdot(5-y)\cdot 10\cdot\frac{4}{3}y\cdot dy$

$=\int_0^3 6000\,y-1200\cdot y^2\,dy$

$=6000\cdot\frac{1}{2}\cdot y^2-1200\cdot\frac{1}{3}\cdot y^3\Big|_0^3$

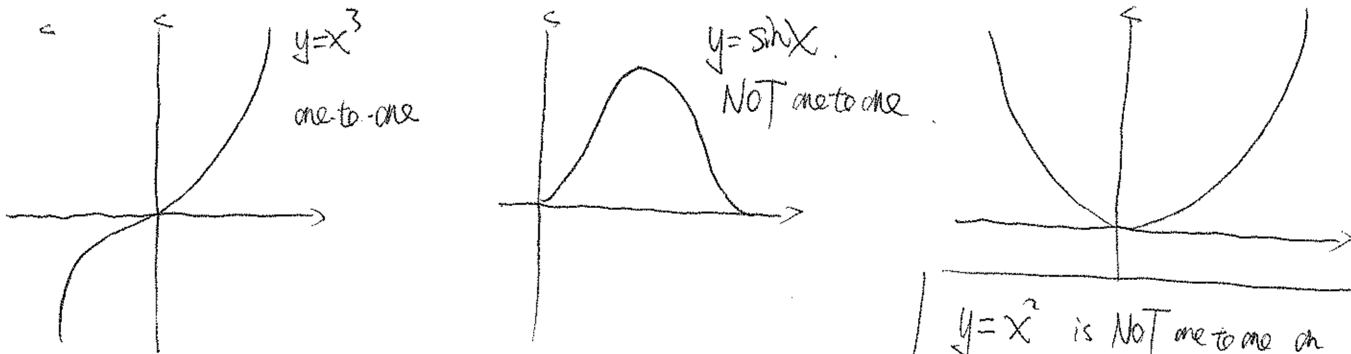
$=\boxed{6000\cdot\frac{1}{2}\cdot 3^2-1200\cdot\frac{1}{3}\cdot 3^3\quad \text{ft-lbs)}}$

§6.1. Inverse function

- $f(x)$ is ONE-TO-ONE if $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

  Equivallent to: No horizontal line intersects the graph of $f(x)$.

eg.1.



$\boxed{y = x^2 \text{ is NOT one to one on } \mathbb{R} \text{; is one to one on } [0, +\infty)}$

- Give a one-to-one function $y = f(x)$ (with domain A and range B)

  Solve the equation $y = f(x)$ for x (in terms of y).

  The solution is called THE INVERSE FUNCTION of $f(x)$.

  Switch x and y in the new function and denote it by $\boxed{y = f^{-1}(x)}$

- Properties of f^{-1}: Domain f^{-1} = Range f. Range f = Domain f^{-1}.

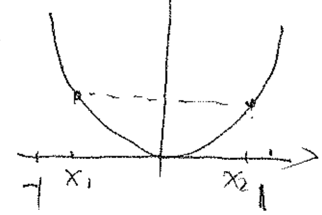
  $$f(f^{-1}(\square)) = \square \;; \quad f^{-1}(f(\triangle)) = \triangle$$

  The graph of f and f^{-1} are $\boxed{\text{SYMMETRIC with respect to } y = x}$

- KEY FORMULA for derivative of f^{-1} at the point $x = a$

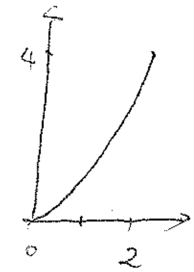
$$☆ \quad (f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

e.g. 2. • $y = x^2$ is NOT ONE-TO-ONE on $[-1, 1]$ since



so has no inverse function on $[-1, 1]$.

• $y = x^2$ is ONE-TO-ONE on $[0, 2]$



Domain : $[0, 2]$ (where x lives)

Range : $[0, 4]$ (where y lives)

(Three Steps to find the inverse of $y = x^2$)

Step 1: Write $y = x^2$

Step 2: Solve ~~for~~ x as a function of y: $x = \sqrt{y}$

Step 3: Interchange x and y in step 2

$x = \sqrt{y} \longrightarrow \boxed{y = \sqrt{x}.}$

(Caution: Actually, there are two solutions: $x = \pm\sqrt{y}$. We drop $-\sqrt{y}$ since $x \in [0, 2]$ is positive)

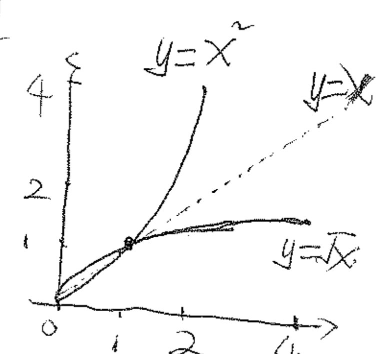
Conclusion: The inverse function of $y = x^2$ on $[0, 2]$ is $y = \sqrt{x}$, whose domain is $[0, 4]$ (the range of $y = x^2$) and whose range is $[0, 2]$ (the domain of $y = x^2$)

e.g. 3: check eg 2 satisfies all properties listed on Page 1

$(f(f^{-1}(\square)) = \square \checkmark :)$ $\quad (\sqrt{\square})^2 = \square$

$(f^{-1}(f(\triangle)) = \triangle \checkmark :)$ $\quad \sqrt{x^2} = x.$

Graph:



e.g. 4: Compute the derivatives of $y = x^2$ and $y = \sqrt{x}$ at $x = 4$. Compare with ✪ formula at $x = 2$

$$(x^2)'\Big|_{x=2} = 2x\Big|_{x=2} = 4 \; ; \quad (\sqrt{x})'\Big|_{x=4} = \frac{1}{2}\frac{1}{\sqrt{x}}\Big|_{x=4} = \frac{1}{4}.$$

Remark 1: The most important types of inverse functions will be discussed in §6.2–6.7, which are log-exp, inverse-trig, inverse-hyp.

Remark 2: Most functions do not have an explicit inverse function as in eg.2. But we can still study the derivative via $\boxed{\text{formula } ⊛}$.

eg.5: Let $f(x)=2x^4+3x-5$ for $x>0$. Find $(f^{-1})'(x)$ at the point $x=0=f(1)$

(s16, mid)

s/n: (Step 1:) Compute the derivative of $f(x)$.

$$f'(x)=2\cdot 4x^3+3$$

(Step 2:) Evaluate $f'(x)$ (at the CORRECT POINT $f^{-1}(a)$.)

In this example, $a=x=0$. $\boxed{f(1)=0 \Rightarrow 1=f^{-1}(0)}$

Therefore, $f^{-1}(0)=1$

and $f'(f^{-1}(0))=f'(1)=2\cdot 4\cdot 1^3+3=11$

(Step 3:) Flip (via ⊛)

$$(f^{-1})'(0)=\frac{1}{f'(f^{-1}(0))}=\boxed{\frac{1}{11}}$$

※

☆☆. eg.6. $y=f(x)=\sin x$. Compute the derivative of $\sin x$ (which is denoted by $\sin^{-1}$), at the point $x=a=\sin b$. In TERMS OF a.

s/n: S1: $f'(x)=(\sin x)'=\cos x$; S2: $a=\sin b$, i.e., $f(b)=a$ i.e. $b=f^{-1}(a)$

S2: $f'(f^{-1}(a))=f'(b)=\cos b$.

S3: $(f^{-1})'(a)=\dfrac{1}{\cos b}$

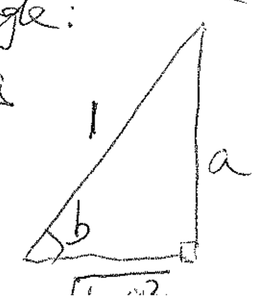
Extra Step: express $\cos b$ in terms of a

i.e. $\boxed{(f^{-1})'(a)=\dfrac{1}{\cos b}=\dfrac{1}{\sqrt{1-a^2}}}$

Solving Triangle:

$\sin b=a=\dfrac{a}{1}$

$\cos b=\sqrt{1-a^2}$



§6.2-6.4. (Natural) Logarithm/Exponential Functions and their Applications.

- log $\boxed{y=\log_a x}$ $\boxed{y=a^x}$ exponential

  nature log $\boxed{y=\ln x}$ $\boxed{y=e^x}$ natural exponential.

- Motivation: Generalization of integer power and its reverse

$$2^n: \quad \begin{array}{l} 2^3 = 2\times2\times2 = 8 \\ 2^4 = 2\times2\times2\times2 = 16 \end{array} \Rightarrow 2^{3.5}\ ? \xRightarrow{\text{exp}} 2^x \text{ for any } x.$$

Reverse (Inverse)

$$\begin{array}{l} 3 \longmapsto 8 \\ 4 \xmapsto{\text{exp}} 16 \end{array} \quad \text{reverse:} \quad \begin{array}{l} 8 \longmapsto 3 \\ 16 \xmapsto{\log} 4 \end{array} \quad \log_2 x.$$

$2 \mapsto a$ $(a>0)$. $\boxed{y=a^x}$ Special case: $a=e$, $\boxed{y=e^x}$.

exp-function with base a. natural exp.

Inverse

$\boxed{y=\log_a x}$ Special case: $a=e$, $\boxed{y=\ln x}$

log-function with base a natural log.

- Graph of a^x and $\log_a x$.

$a>1$ | $0<a<1$