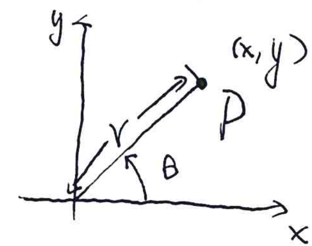
§10.3 Polar Coordinates.

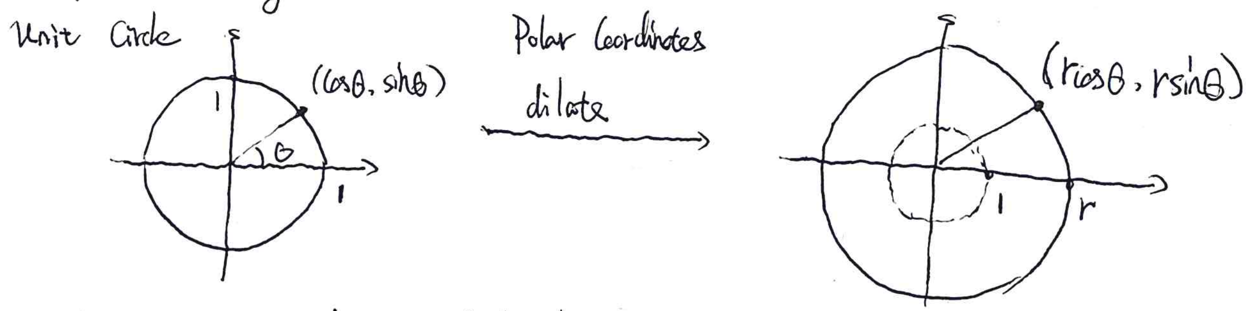
- We introduce two new variables (r, θ) to represent the point (x, y) on XY plane through the equations:

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases} \longleftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$



(r, θ) are the Polar Coordinates for the point P. The radius r is the "distance" from the origin. The angle θ is measured counterclockwise.

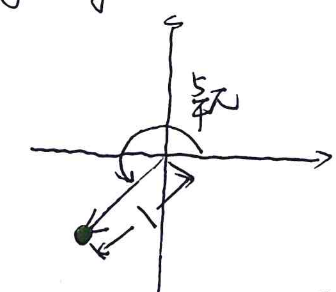
- Motivation from dilating the unit circle.



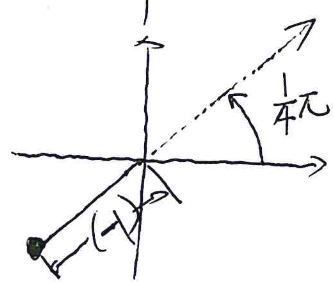
eg1. Give the following Polar coordinates, find the corresponding (x, y) coordinates and plot the points.

(1). $(r, \theta) = (1, \frac{5\pi}{4})$ $\Rightarrow$ $\begin{cases} x = 1\cdot\cos\frac{5}{4}\pi = -\frac{\sqrt{2}}{2} \\ y = 1\cdot\sin\frac{5}{4}\pi = -\frac{\sqrt{2}}{2} \end{cases}$

(or $r = 1$, $\theta = \frac{5}{4}\pi$)



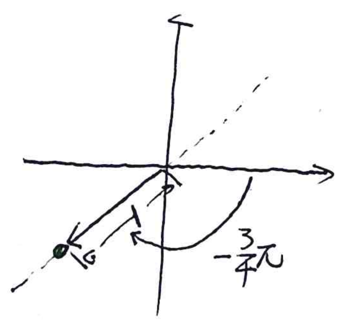
(2). $(r, \theta) = (-1, \frac{1}{4}\pi)$ $\Rightarrow$ $\begin{cases} x = (-1)\cdot\cos\frac{1}{4}\pi = -\frac{\sqrt{2}}{2} \\ y = (-1)\cdot\sin\frac{1}{4}\pi = -\frac{\sqrt{2}}{2} \end{cases}$



positive ray.

negative radius means to move out along the ray in the opposite direction.

(3). $(r, \theta) = (1, -\frac{3}{4}\pi)$ $\Rightarrow$ $\begin{cases} x = 1\cdot\cos(-\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} \\ y = 1\cdot\sin(-\frac{3}{4}\pi) = \frac{\sqrt{2}}{2} \end{cases}$

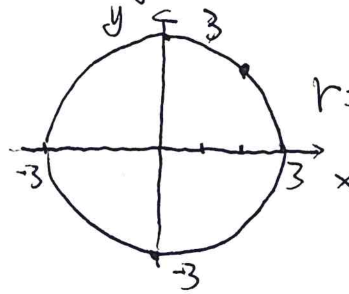


negative angle means to rotate the ray clockwise

- Polar curves given by polar equation $r = f(\theta)$.
- We are most interested in the polar curves given by $r = a + b\sin\theta$ or $r = a + b\cdot\cos\theta$.

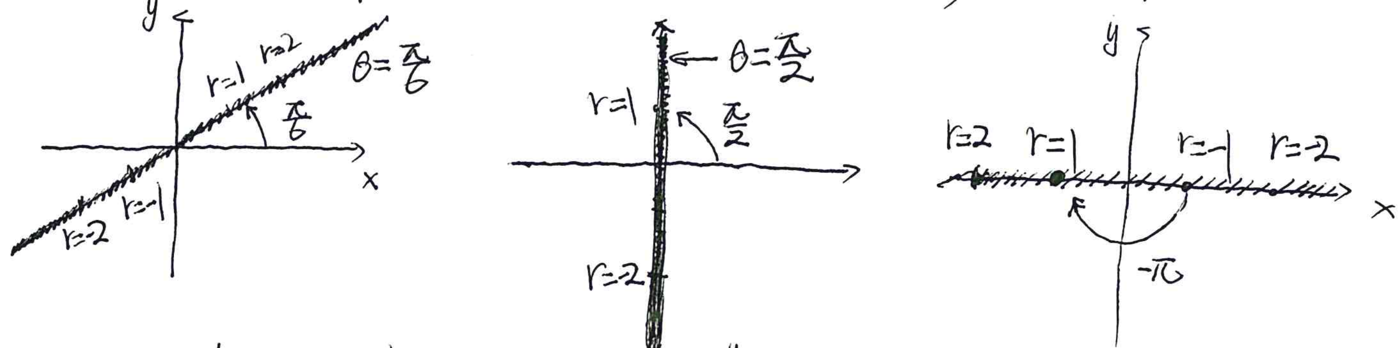
eg.2. What curve is represented by the polar equation $r = 3$?
(Find its Cartesian equation via the relation between (r,θ) and (x,y)).

$r^2 = x^2 + y^2 \Rightarrow x^2 + y^2 = 3^2$: A circle centered at the origin with radius 3.



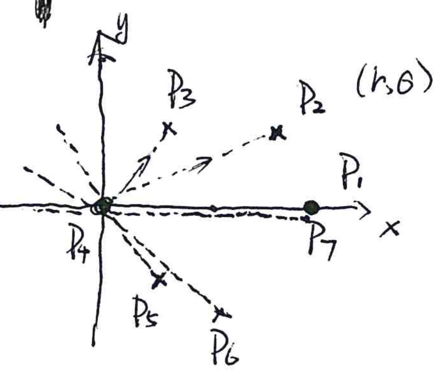
Remark: There is no restriction on θ, which means θ can be any
If there are some restrictions on θ, for example,
$r = 3$, $0 \le \theta \le \pi$, then it would be part of the circle (half).

eg3. ~~What~~ Sketch the polar curve $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$, $\theta = -\pi$



- More nontrivial examples

eg.4. $r = 2\cos\theta$



Circle centered at $(1,0)$ with radius
$(x-1)^2 + y^2 = 1$.

	θ	$r = 2\cos\theta$	(x, y)
P_1	0	$2\cos 0 = 2$	$(2, 0)$
P_2	$\frac{\pi}{6}$	$2\cos\frac{\pi}{6} = \sqrt{3}$	$(\frac{3}{2}, \frac{\sqrt{3}}{2})$
P_3	$\frac{\pi}{3}$	$2\cos\frac{\pi}{3} = 1$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
P_4	$\frac{\pi}{2}$	$2\cos\frac{\pi}{2} = 0$	$(0, 0)$
P_5	$\frac{2\pi}{3}$	$2\cos\frac{2\pi}{3} = -1$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
P_6	$\frac{3\pi}{4}$	$2\cos\frac{3\pi}{4} = -\sqrt{2}$	$(1, -1)$
P_7	π	$2\cos\pi = -2$	$(2, 0)$

Derive the Cartesian equation:

$r = 2\cdot\cos\theta \Rightarrow r\cdot r = r\cdot(2\cos\theta)$ (multiply r both sides)

$\Rightarrow r^2 = 2\cdot r\cos\theta$

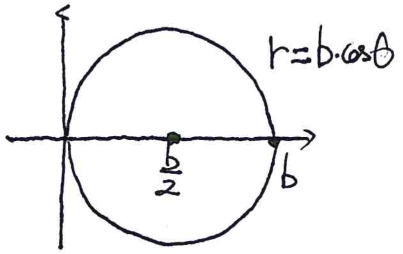
$x^2 + y^2 = 2\cdot x$

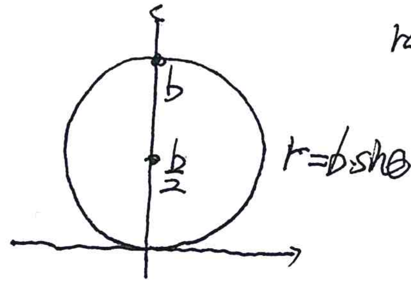
$\Rightarrow x^2 - 2x + 1 + y^2 = 1$

$\Rightarrow (x-1)^2 + y^2 = 1$

☆ Conclusion: In general, $r = b\cdot\cos\theta$ $(b>0)$ is a circle centered at $(\frac{b}{2}, 0)$ with radius $\frac{b}{2}$.

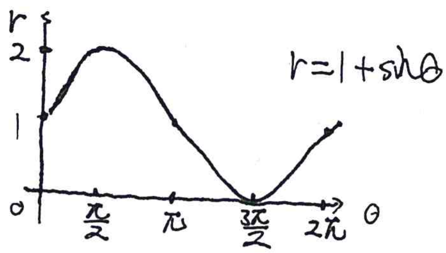
Similarly, $r = b\cdot\sin\theta$ is a circle centered at $(0, \frac{b}{2})$ with radius $\frac{b}{2}$.





• eg5. Cardioid. $r = 1+\sin\theta$ (Heart shaped curve).

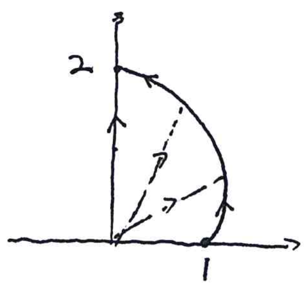
On r-θ plane, the curve $r=1+\sin\theta$ looks like



Remark: This is not the graph on XY plan

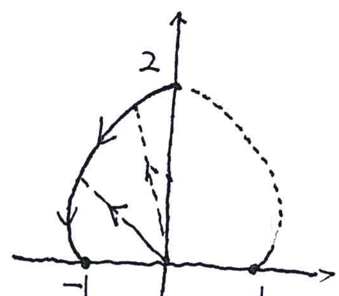
$\theta \in [0, \frac{\pi}{2}]$

r increases from 1 to 2.



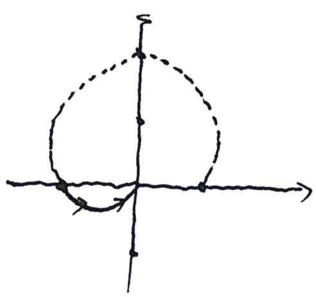
$\theta \in [\frac{\pi}{2}, \pi]$

r decreases from 2 to 1



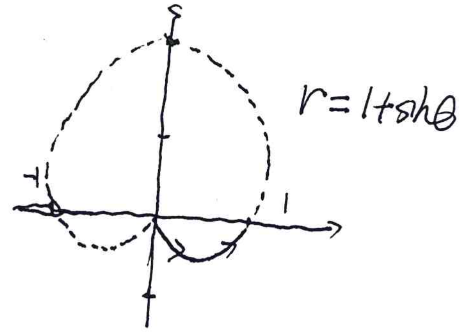
$\theta \in [\pi, \frac{3\pi}{2}]$

r decreases from 1 to 0

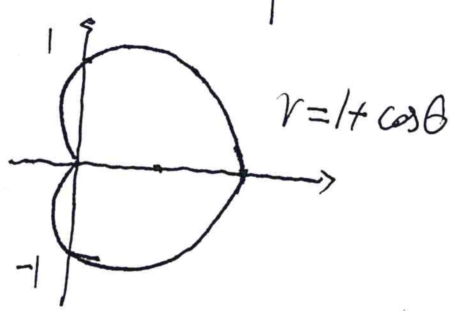


$\theta \in [\frac{3\pi}{2}, 2\pi]$

r increases from 0 to 1



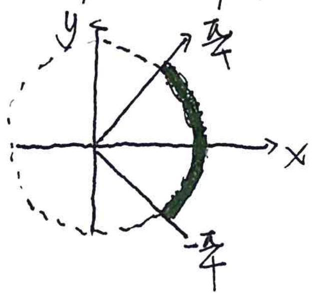
Remark: In the similar way, we can find $r = 1+\cos\theta$ is also a cardioid. looks like:



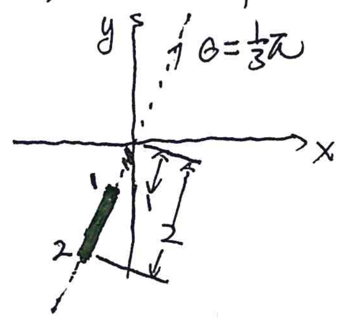
- Plane region (Circular sector) given by polar inequalities.

eg6. Sketch the curves/regions given by the following polar equations/inequalities

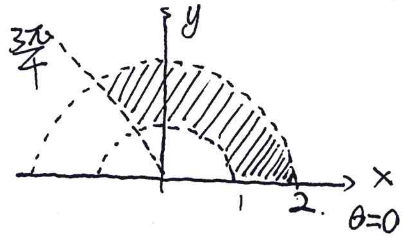
(1). $r=3$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.



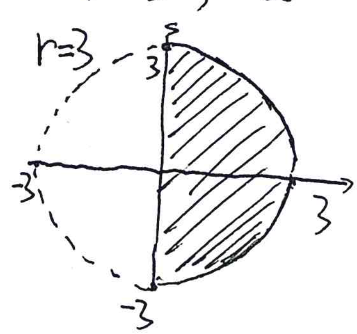
(2) $\theta = \frac{1}{3}\pi$, $-2 < r \le -1$



(3). $1 \le r \le 2$, $0 \le \theta \le \frac{3}{4}\pi$



(4) $0 \le r \le 3$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



- Tangents to Polar Curves

eg7 (ww8). Find the equation in x and y for the tangent line to the polar curve $r = 3\cdot\cos 2\theta$ at $\theta = \frac{\pi}{2}$.

Point-slope formula for the tangent line (in XY coordinates) $y - y_0 = k(x - x_0)$

$\theta = \frac{\pi}{2}$, $r = 3\cdot\cos 2\cdot\frac{\pi}{2} = -3$. Polar coordinates $(-3, \frac{\pi}{2})$ $\xrightarrow[y=r\sin\theta]{x=r\cos\theta}$ $\begin{cases} x_0 = -3\cdot\cos\frac{\pi}{2} = 0 \\ y_0 = -3\sin\frac{\pi}{2} = -3 \end{cases}$
(r,θ)

Point: (in xy) $(0,-3)$.

slope: Parametric equations in θ: $\begin{cases} x = r\cdot\cos 2\theta = 3\cos 2\theta\cdot\cos\theta = \boxed{3\cos 2\theta\cdot\cos\theta} \\ y = r\cdot\sin 2\theta = 3\cos 2\theta\cdot\sin\theta = \boxed{3\cos 2\theta\cdot\sin\theta} \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(3\cos 2\theta\cdot\sin\theta)'}{(3\cos 2\theta\cdot\cos\theta)'} = \frac{-6\sin 2\theta\cdot\sin\theta + 3\cos 2\theta\cdot\cos\theta}{-6\sin 2\theta\cdot\cos\theta - 3\cos 2\theta\cdot\sin\theta} \overset{\theta=\frac{\pi}{2}}{=\!=} 0$$

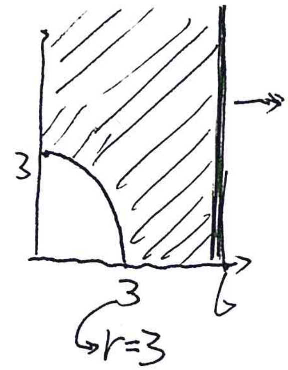
Tangent line: $y-(-3) = 0\cdot(x-0) \Rightarrow \boxed{y=-3}$

Hints for Webwork:

Part I. #5. $r^2 \cdot \sin 2\theta = 11$. Double angle formula: $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow r^2 \cdot 2 \cdot \sin\theta \cos\theta = 11 \Rightarrow 2\cdot(r\sin\theta)\cdot(r\cdot\cos\theta) = 11 \Rightarrow 2x\cdot y - 11 = 0$$

#12.



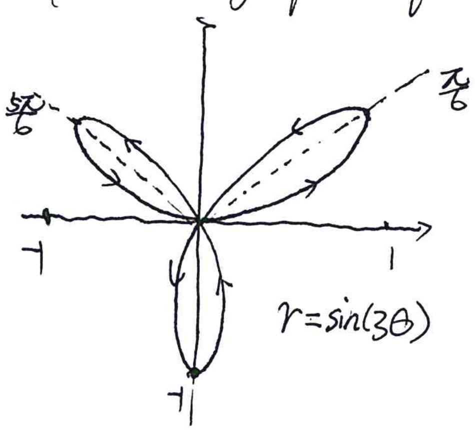
$x = 6 \iff r\cdot\cos\theta = 6 \iff r = \frac{6}{\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$

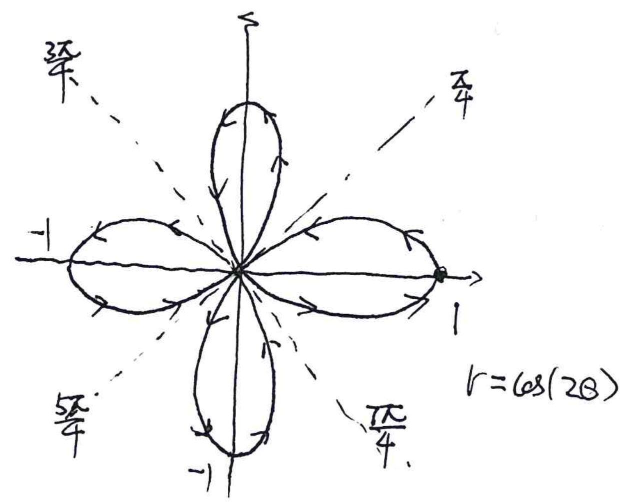
$$\boxed{3 \le r \le \frac{6}{\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}}$$

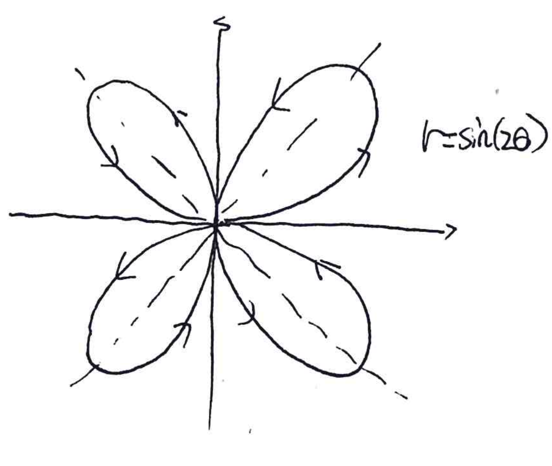
Part 2. #2, 3, 4, 5. Use the online program to plot the graph: desmos.com/calculator.

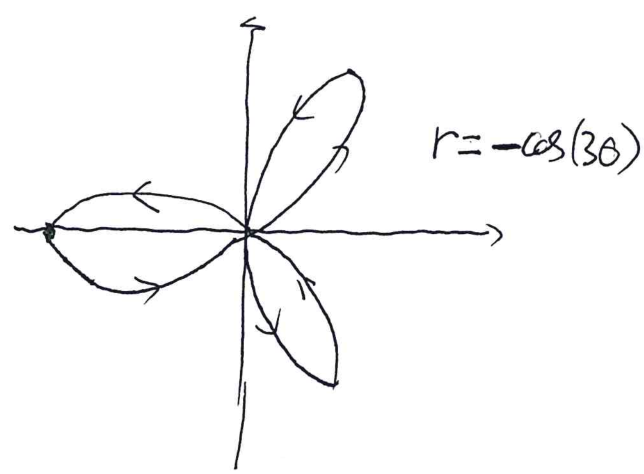
#5. $r^2 = 64\cos(2\theta)$. consider the two branches $r = \pm 8\cdot\sqrt{\cos(2\theta)}$

#6, #7. Polar equations in the form $r = \cos(k\theta)$ or $r = \sin(k\theta)$ (as $r = \sin(3\theta)$ or $r = \cos(2\theta)$) give the graph of roses.







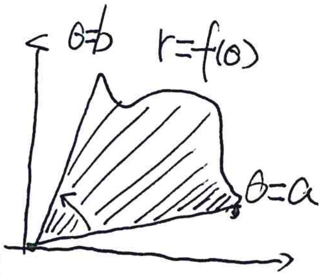


☆ § 10.4. Areas of Circular Sector in Polar Coordinates

Area Formula: Give a polar curve $r=f(\theta)$, $a\le\theta\le b$.

The circular sector region $0\le r\le f(\theta)$, $a\le\theta\le b$ has area

☆ $$A=\int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$$

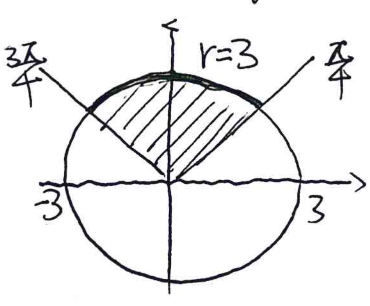


Remark: In the formula sheet, the formula is given as $A=\int_a^b \frac{1}{2} r(\theta)^2\,d\theta$

where $r=r(\theta)$ is the polar equation

eg.1 Find the area of the region bounded by the polar curve $r=3$, $\theta=\frac{\pi}{4}$, $\theta=\frac{3\pi}{4}$.

Remark: It is equivalent to ask "Find the area of the region: $0\le r\le 3$, $\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$"

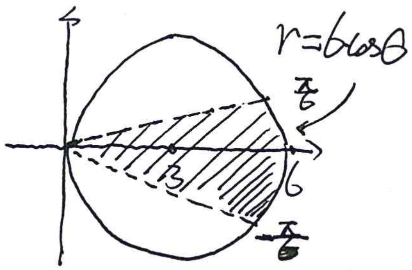


$$\text{Area}=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}\cdot[3]^2\,d\theta=\frac{9}{2}\left[\frac{3\pi}{4}-\frac{\pi}{4}\right]=\frac{9}{4}\pi.$$

Actually, the region is the quarter disk with radius 3

$$\text{Area}=\frac{1}{4}[\text{Area of the Disk}]=\frac{1}{4}\cdot\pi\cdot 3^2=\frac{9}{4}\pi$$

eg.2. Find the area of the region given by: $0\le r\le 6\cdot\cos\theta$, $-\frac{\pi}{6}\le\theta\le\frac{\pi}{6}$.



$$\text{Area}=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}[6\cdot\cos\theta]^2\,d\theta$$. Double angle formula

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cdot 36\cdot\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 9+9\cos 2\theta\,d\theta=9\theta+9\cdot\frac{1}{2}\sin 2\theta\,\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$=9\cdot\left[\frac{\pi}{6}+\frac{\pi}{6}\right]+\frac{9}{2}\left[\sin\frac{\pi}{3}+\sin\frac{\pi}{3}\right]$$

$$=3\pi+\frac{9}{2}\cdot 2\cdot\frac{\sqrt{3}}{2}$$

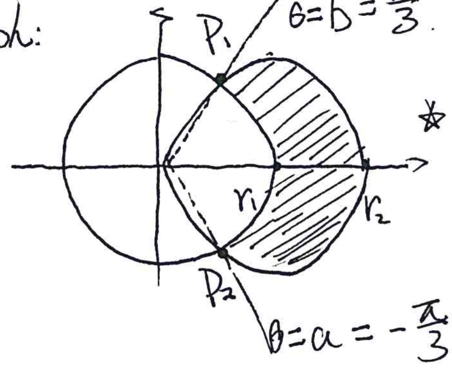
$$=\boxed{3\pi+\frac{9}{2}\sqrt{3}}$$

$\sin(-\frac{\pi}{3})=-\sin\frac{\pi}{3}$

Remark: One trick for this problem is to use symmetry of the graph and evaluate from $\theta=0$ to $\theta=\frac{\pi}{6}$. Then multiply the area by 2.

eg.3 Find the area of the region OUTSIDE $r=3$ and INSIDE $r=6\cdot\cos\theta$.

Graph:



Need to find the angles for the two intersections P_1, P_2

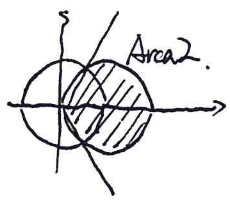
★ Set the two polar equations equal.

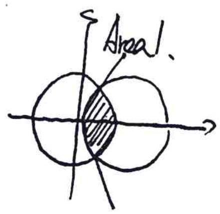
$$r=3=6\cos\theta \Rightarrow \cos\theta=\frac{1}{2}$$

$$\Rightarrow \theta=-\frac{\pi}{3} \text{ and } \theta=\frac{\pi}{3}.$$

$r_1=3 \quad r_2=6\cos\theta$

$$\text{Area} = \text{Area2} - \text{Area1} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}[6\cdot\cos\theta]^2 d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2} 3^2\cdot d\theta$$

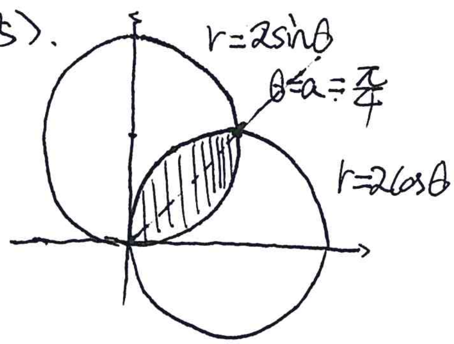




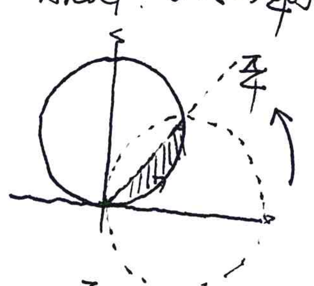
$$= [9\theta+\frac{9}{2}\sin 2\theta]\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \frac{9}{2}\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 9\cdot\frac{2}{3}\pi + \frac{9}{2}\cdot 2\cdot\sin\frac{2}{3}\pi - \frac{9}{2}\cdot\frac{2}{3}\pi = \boxed{3\pi+9\cdot\frac{\sqrt{3}}{2}}$$

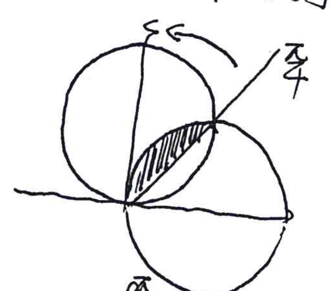
eg.4 Find the area SHARED by the polar curves $r=2\sin\theta$ and $r=2\cos\theta$
(ww5).



Area1. $\theta\in[0,\frac{\pi}{4}]$



Area2. $\theta\in[\frac{\pi}{4},\frac{\pi}{2}]$



Intersection angle:

$r=2\sin\theta=2\cos\theta$

$\sin\theta=\cos\theta \Rightarrow \theta=\frac{\pi}{4}.$

$$\text{Area1} = \int_0^{\frac{\pi}{4}} \frac{1}{2}[2\sin\theta]^2 d\theta \overset{\text{Symmetry}}{=\!=} \text{Area2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}[2\cos\theta]^2 d\theta$$

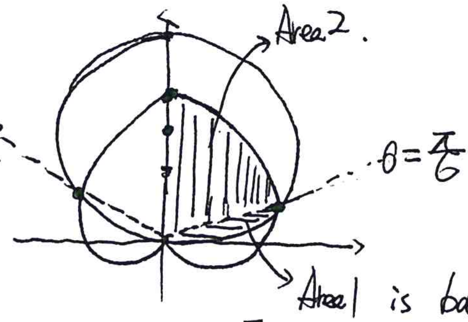
$$\text{Area1} = \int_0^{\frac{\pi}{4}} 2\cdot\sin^2\theta\cdot d\theta = \int_0^{\frac{\pi}{4}} 2\cdot\frac{1-\cos 2\theta}{2}\cdot d\theta = \theta-\frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2} = \boxed{\frac{\pi}{4}-\frac{1}{2}}$$

$$\text{Area2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cdot\frac{1+\cos 2\theta}{2} d\theta = \theta+\frac{1}{2}\sin 2\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2}+\frac{1}{2}\sin\pi-(\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2})$$

$$= \boxed{\frac{\pi}{4}-\frac{1}{2}}$$

$$\text{Area} = 2\cdot\text{Area1} = \boxed{2\cdot(\frac{\pi}{4}-\frac{1}{2})}$$

☆ eg5. Find the area shared by $r_1 = 3\sin\theta$ and $r_2 = 1+\sin\theta$.



$3\sin\theta = 1+\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$, $\theta=\frac{\pi}{6}$, $\theta=\frac{5\pi}{6}$.

Due to symmetry, it is enough to consider the area from $\theta=0$ to $\theta=\frac{\pi}{2}$ (then double the area)

Area1 is bounded by $r=3\sin\theta$, $\theta=0$ and $\theta=\frac{\pi}{6}$

$$\text{Area } 1 = \int_0^{\frac{\pi}{6}} \frac{1}{2}(3\sin\theta)^2 d\theta = \int_0^{\frac{\pi}{6}} \frac{9}{2}\cdot\frac{1-\cos 2\theta}{2} d\theta = \frac{9}{4}\cdot\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{6}}$$

$$= \frac{9}{4}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{9}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$\text{Area } 2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left[1+2\sin\theta+\sin^2\theta\right] d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1+2\sin\theta+\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left[\theta + (-2\cos\theta) + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{3}{2}\cdot\frac{\pi}{2} - \underbrace{2\cos\frac{\pi}{2}}_{=0} - \underbrace{\frac{1}{4}\sin\pi}_{=0}\right] - \frac{1}{2}\left[\frac{3}{2}\cdot\frac{\pi}{6} - 2\cos\frac{\pi}{6} - \frac{1}{4}\sin\frac{\pi}{3}\right]$$

$$= \frac{3}{8}\pi - \frac{1}{8}\pi + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{16} = \frac{1}{4}\pi + \frac{9}{16}\sqrt{3}$$

Shadow Area

$= \text{Area } 1 + \text{Area } 2$

$= \frac{9}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{1}{4}\pi + \frac{9}{16}\sqrt{3} = \frac{5}{8}\pi$.

Total Area

$= 2\times\frac{5}{8}\pi = \boxed{\frac{5}{4}\pi}$

Arc-length in Polar coordinates (WW8) Formula: $L = \int_a^b \sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

eg Find the arc-length of the curve given by $r=e^{6\theta}$ from $\theta=0$ to $\theta=4\pi$

$r = e^{6\theta}$, $\frac{dr}{d\theta} = 6\cdot e^{6\theta}$.

$$L = \int_0^{4\pi} \sqrt{(e^{6\theta})^2 + (6e^{6\theta})^2}\cdot d\theta = \int_0^{4\pi} \sqrt{e^{12\theta} + 36\cdot e^{12\theta}}\, d\theta$$

$$= \int_0^{4\pi} \sqrt{37\cdot e^{12\theta}}\, d\theta$$

$$= \int_0^{4\pi} \sqrt{37}\cdot e^{6\theta} d\theta = \sqrt{37}\cdot\frac{1}{6}e^{6\theta}\Big|_0^{4\pi} = \boxed{\frac{\sqrt{37}}{6}\left[e^{24\pi} - 1\right]}$$