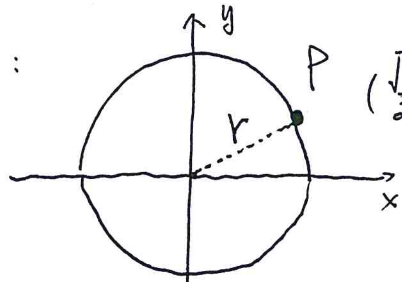
Chapter 10. Parametric Equations and Polar Coordinates

- Motivation: Cartesian Coordinates system VS Other system

Unit circle:



$P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \iff \begin{cases} x = \frac{\sqrt{3}}{2} \\ y = \frac{1}{2} \end{cases}$ VS $\begin{cases} r = 1 \\ \theta = \frac{\pi}{6} \end{cases}$

$$x^2 + y^2 = 1 \iff r = 1 \iff \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$

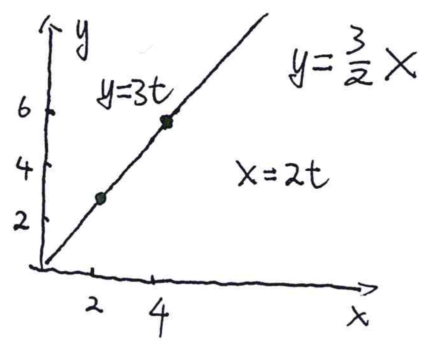
- Equations of plane curve with parameter t

e.g. A car is moving on the XY plane starting from the origin.

It moves at 2 m/s along x-direction and 3 m/s along y-direction.

Describe (Study) the motion of the car after t seconds.

position \ time	0	1	2	3
x	0	2	4	6
y	0	3	6	9



The equations of x, y as functions of t: $\begin{cases} x = 2t \\ y = 3t \end{cases}$ Parametric Equations

The equation relating x and y: $y = \frac{3}{2}x$. Cartesian Equation

Rmk: The car moves along the straight line $y = \frac{3}{2}x$ on XY plane. In this Cartesian equation, we could not see the variable t. The parametric equations describe the same curve (motion) with more information.

§10.1. Curves Defined by Parametric Equations

• Definition: x, y are both given as functions of t (called parameter):

$$x = f(t), \quad y = g(t). \quad \text{(called parametric equations)}$$

On XY plane, the point $(x, y) = (f(t), g(t))$ varies as t varies.

The point traces out a curve C, we call C a parametric curve.

Key points (Goals):

- Give parametric equations, sketch the parametric curve and describe the motion.
- Connections between Cartesian equation and parametric equations
- Calculus (derivative/integral) of parametric equations.

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eg.1. What curves are represented by the following parametric equations

① $x = \cos t, \ y = \sin t, \quad 0 \le t \le 2\pi$

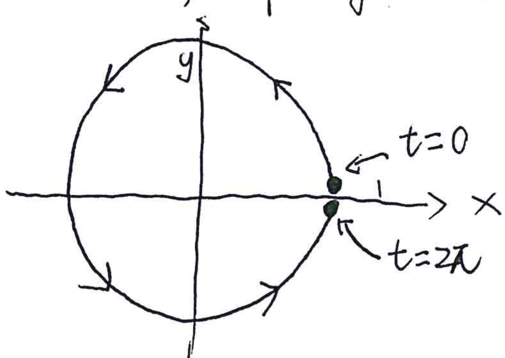
sln: $x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$

on XY plane, it's the unit circle

$t = 0$, $(x, y) = (1, 0)$ (starting point)

$t = 2\pi$, $(x, y) = (1, 0)$ (end point)

t $0 \to 2\pi$, the point goes counterclockwise

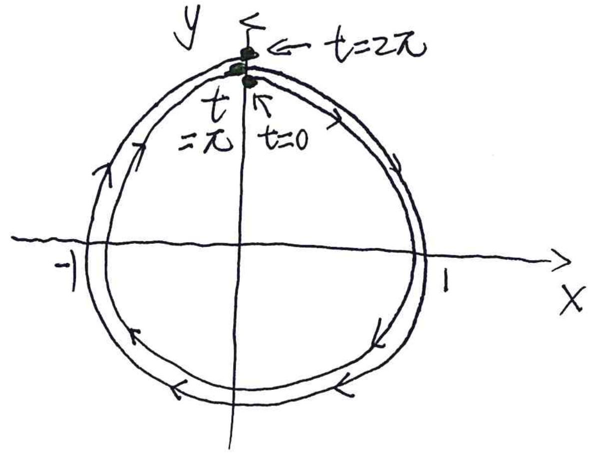


② $x = \sin 2t, \ y = \cos 2t, \quad 0 \le t \le 2\pi$.

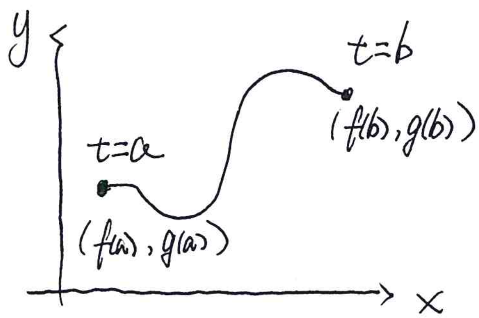
sln: the cartesian equation is the same as ①

$x^2 + y^2 = 1$.

The circle starts at $t = 0$, $(x, y) = (0, 1)$ and goes clockwise twice around the circ



- Rmk: the domain for t is important to parametric curve
  $t \in [a,b]$ $(a \le t \le b)$ tells us the curve starts from $t=a$, $(f(a), g(a))$ and ends at $t=b$, $(f(b), g(b))$.



It also tells the "direction" of the motion t varies from a to b.

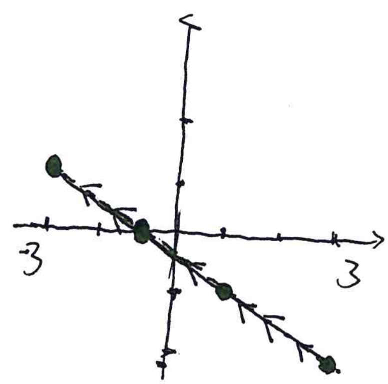
- Parametric equations for line segments

eg. Sketch the curve and find the Cartesian equation.

$$\begin{cases} x = 1-2t \\ y = t-1 \end{cases} \quad -1 \le t \le 2$$

sln:

t	x	y
-1	3	-2
0	1	-1
1	-1	0
2	-3	1



Actually, eliminate the variable t we have

$y = t-1 \Rightarrow t = y+1$

$x = 1-2t = 1-2(y+1)$

$\Leftrightarrow \boxed{x = -2y-1}$, $y = -\frac{1}{2}x - \frac{1}{2}$

Rmk: Without the restriction $-1 \le t \le 2$, the equations would represent the entire line

eg. Find ~~the~~ a parametrization of the line segment starting at $(x,y) = (1,-1)$ and ending at $(x,y) = (3,-2)$ for $t \in [0,1]$.

sln: line segment has the following affine (linear) functions as parametric equation

$$\begin{cases} x = a + b \cdot t \\ y = c + d t \end{cases} \quad \begin{matrix} t=0 \Rightarrow a=1, c=-1 \\ t=1 \Rightarrow b=2, d=-1 \end{matrix} \Rightarrow \begin{cases} x = 1+2t \\ y = -1-t \end{cases} \quad t \in [0,1]$$

More examples

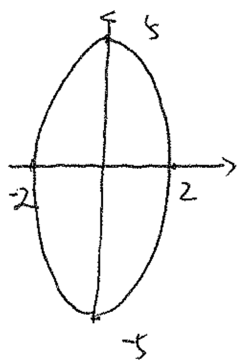
e.g. Ellipse: A curve is prepresented parametrically by

$x = -2\cos 3t$, $y = 5\cdot\sin 3t$, $t \in [0, \frac{\pi}{6}]$

sketch the graph (indicate the direction of the motion) and find its Cartesian equation.

sln: Eliminate t via trig-identity: $\cos^2\Box + \sin^2\Box = 1$

$$\left(\frac{x}{-2}\right)^2 + \left(\frac{y}{5}\right)^2 = (\cos 3t)^2 + (\sin 3t)^2 = 1 \quad \text{ie.} \quad \boxed{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1}$$



Starts ~~from~~ at $t=0$

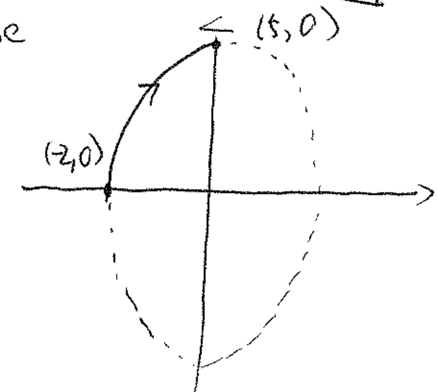
$x = -2$, $y = 0$

Ends at $t = \frac{\pi}{6}$

$x = -2\cdot\cos\frac{\pi}{2} = 0$, $y = 5\sin\frac{\pi}{2} = 5$

$\frac{1}{4}$-ellipse

clockwise.

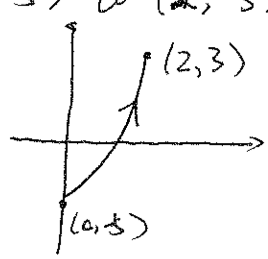


Remark: the range for y-variable is $[0,5]$ ie. $0 \le y \le 5$

e.g. One 'trivial' parametric equations.

curve: $y = 2x^2 - 5$. from $(0,-5)$ to $(2,3)$ can be parameterized via

$$\begin{cases} x = t \\ y = 2t^2 - 5 \end{cases} \quad t \in [0, 2]$$



Remark: In general, any function $y = f(x)$ can be parameterized as $\begin{cases} x = t \\ y = f(t) \end{cases}$.

Hint for ww7: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The natural way to parametrize it is via $x = a\cdot\cos t$, $y = b\sin t$ (or $x = \pm a\cos t$, $y = \pm b\sin t$)

In ww7, since it is the bottom part, you need to consider $y = -b\cdot\sin t$.

§10.2 Calculus with Parametric Curves

(Differentiation)

- Tangents: Give parametric equations $x=f(t)$, $y=g(t)$.

Then $$\boxed{\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ ie. } y'(x)=\frac{g'(t)}{f'(t)}}$$ (in the formula sheet)

Remark: the above formula follows from Chain Rule. $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$

for $y=y(x)=y\circ x(t)$

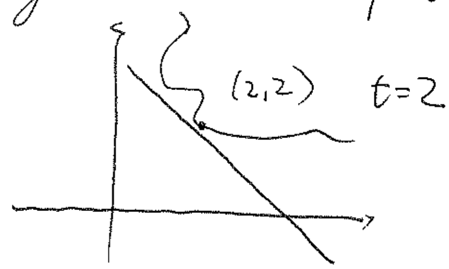
e.g.1. Consider $x=6-t^2$, $y=t^3-3t$. Find the derivative of y with respect to x as a function of t.

$$y'(x)=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-3}{-2t}$$

- The most important application for the above formula is to EVALUATE the derivative at some specific point, i.e. find the SLOPE of the tangent line at this point and find the FORMULA of the tangent line of the curve.

Remark: Tangent line of $y=y(x)$ at (x_0,y_0): $y=y_0+y'(x_0)\cdot(x-x_0)$

e.g.2. Consider the parametric equations in eg.1. What are the coordinates of the curve at $t=2$? What's the slope of the tangent line at that point? Find the tangent line.



$t=2$ $x=6-2^2=2$, $y=2^3-3\cdot2=2$.

Slope: $\frac{dy}{dx}=\frac{3t^2-3}{-2t}=\frac{3\cdot4-3}{-4}=-\frac{9}{4}$

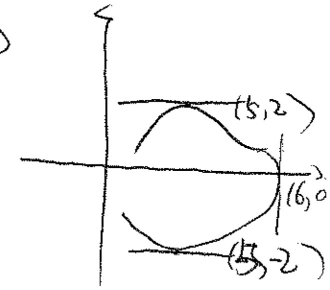
tangent line: $\boxed{y=2-\frac{9}{4}\cdot(x-2)=-\frac{9}{4}x+\frac{13}{2}}$

e.g.3. Find all the points where the curve has horizontal tangent line in eg 1,2.

Hint: horizontal tangent line $\iff \frac{dy}{dx} = 0 \iff \frac{dy}{dt} = 0$

i.e. $y'(x) = \frac{3t^2-3}{-2t} = 0 \Rightarrow 3t^2-3 = 0 \Rightarrow t^2 = 1 \Rightarrow t=1$ or $t=-1$.

$t=1$, $(x,y) = (5,-2)$. $t=-1$, $(x,y) = (5,2)$



e.g.4 Points with vertical tangent line.

Hint: Vertical tangent line: $\frac{dy}{dx} = \infty \iff \frac{dx}{dt} = 0$.

$$y'(x) = \frac{3t^2-3}{-2t} = \infty \Rightarrow -2t = 0 \Rightarrow t=0 \Rightarrow (x,y) = (6,0)$$

Remark: Vertical tangent line can also be viewed as $\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = 0$.

e.g.5. Implicit differential rule (Hint for ww 6).

The curve is defined IMPLICITLY via the following parametric equations.

$x^3 - 2\cdot t^2 = 7$, $2y^3 + t = 18$. Find the slope of the tangent line at $t=2$.

Solution: $t=2$, want to compute $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at $t=2$, $(x,y) = (1,2)$

Take derivate w.r.t. t IMPLICITLY in both equations.

$$\frac{d}{dt}(x^3-2t^2) = \frac{d}{dt}(7) \Rightarrow 3\cdot x^2 \cdot \boxed{\frac{dx}{dt}} - 4\cdot t = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{4t}{3x^2} = \frac{4\cdot 2}{3\cdot 1} = \frac{8}{3}$$

$$\frac{d}{dt}(2y^3+t) = 0 \Rightarrow 2\cdot 3y^2 \cdot \boxed{\frac{dy}{dt}} + 1 = 0 \Rightarrow \frac{dy}{dt} = -\frac{1}{6y^2} = -\frac{1}{24}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{24}}{\frac{8}{3}} = \boxed{-\frac{1}{64}}$$

☆ For $(x, y) = (f(t), g(t))$, the tangent line can also be represented parametrically at $t = a$, the parametric formula for the tangent line is

$$\begin{cases} x = f(a) + f'(a)\, t \\ y = g(a) + g'(a)\, t \end{cases}$$

☆ eg 6 (Final 14). Consider the parametric curve given by

$x = \cos t$, $y = 1 + \sin t$, $t \in [0, 2\pi]$. (a). Give the sketch of the curve

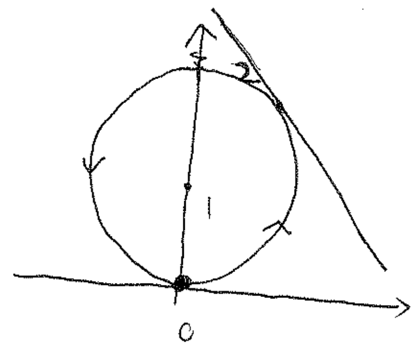
(b) Give the parametric formula for the tangent line at $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

Solution: (a). Cartesian equation: $x^2 + (y-1)^2 = 1$.

A circle of radius 1 centered at $(0, 1)$

counter-clockwise

Starting and ending at the same point $(0, 0)$



☆ (b). Find t value for the given point: $\frac{\sqrt{3}}{2} = \cos t$, $\frac{3}{2} = 1 + \sin t \Leftrightarrow \frac{1}{2} = \sin t$

$$\Rightarrow \boxed{t = \frac{\pi}{6}}$$

Compute $f'(t)$, $g'(t)$ at $t = \frac{\pi}{6}$

$$\frac{dx}{dt} = x'(t) = (\cos t)' = -\sin t = -\sin\frac{\pi}{6} = -\frac{1}{2}.$$

$$\frac{dy}{dt} = y'(t) = (1 + \sin t)' = \cos t = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Therefore, the parametric tangent line at $t = \frac{\pi}{6}$ is

$(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$$\boxed{\begin{cases} x = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot t \\ y = \frac{3}{2} + \frac{\sqrt{3}}{2} \cdot t. \end{cases}} \quad ☆$$

$t \in (-\infty, \infty)$

Remark: the Cartesian tangent line of (b): $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$.

$y = \frac{3}{2} - \sqrt{3} \cdot \left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = -\sqrt{3}x + 3$ can be parameterized as

$$\begin{cases} x = t \\ y = -\sqrt{3}t + 3 \end{cases}$$

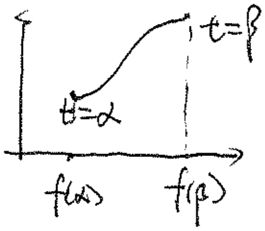
which is also ok. (equivalent to Answer ☆)

• (Integration). Arc-length. (Area will be discussed later in §10.4)

If a curve C is described by the parametric equations $x=f(t)$, $y=g(t)$, $\alpha \le t \le \beta$, then the length of C is

★ $$\boxed{L=\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt} = \int_\alpha^\beta \sqrt{[f'(t)]^2+[g'(t)]^2}\cdot dt.$$

Remark: the above formula can be derived from the previous Arc-L. formula and u-sub.

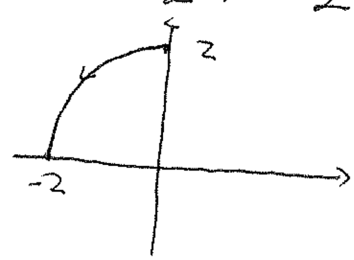


$$L=\int_{f(\alpha)}^{f(\beta)}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx \xrightarrow[\frac{dy}{dx}=\frac{dy/dt}{dx/dt}]{x=f(t)} \int_\alpha^\beta \sqrt{1+\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot\frac{dx}{dt}\cdot dt$$

$$dx=\frac{dx}{dt}\cdot dt \qquad =\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt.$$

e.g. Curve $x=2\cos t$, $y=2\sin t$. Compute the arc-length from $t=\frac{\pi}{2}$ to $t=\frac{3\pi}{2}$.

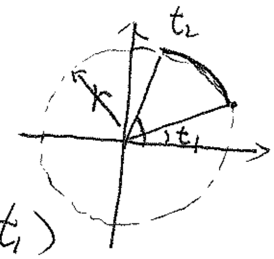
$\frac{dx}{dt}=-2\sin t$, $\frac{dy}{dt}=2\cdot\cos t$.



$$\text{Arc-length}=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sqrt{4}\,dt$$
$$=2\cdot\left(\frac{3\pi}{2}-\frac{\pi}{2}\right)=\pi.$$

Remark: the arclength of a quarter circle is not easy to compute via Cartesian equation $y=\sqrt{2-x^2}$, $-2\le x\le 0$ via $L=\int_{-2}^{0}\sqrt{1+(y')^2}\cdot dx$.

And in general, above computation works for any segment of the circle



$$L=r\cdot(t_2-t_1)$$

Hint for ww7: Use double angle formula $\cos^2\square=\frac{1+\cos2\square}{2}$ to simplify the integrand.

Actually, $\sqrt{15^2\cdot2+15^2\cdot2\cdot\cos5t}=\sqrt{15^2\cdot2\cdot(1+\cos5t)}=\sqrt{15^2\cdot2\cdot2\cdot\cos^2\left(\frac{5t}{2}\right)}=15\cdot2\cdot\cos\left(\frac{5t}{2}\right)$.