Practice Midterm2 Version B, Sec61

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

Q1 Which statement is true about the series

$$\sum_{n=1}^{\infty}(1+\frac{1}{n})^n$$

A The **nth term test** concludes that the series converges.

B The **nth term test** concludes that the series diverges. ✓

C The **nth term test** hypotheses are not met by this series, so it cannot be applied.

D The **nth term test** hypotheses are met by this series however the test is inconclusive.

E None of the above are true. The nth term test concludes that the series converges.

$a_n = (1+\frac{1}{n})^n = e^{\boxed{\ln(1+\frac{1}{n})^n}}$

$\lim_{n\to\infty} \ln(1+\frac{1}{n})^n = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} \overset{l'Hosp}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \cancel{(-\frac{1}{n^2})}}{\cancel{(-\frac{1}{n^2})}} = \frac{1}{1+\frac{1}{\infty}} = 1$

$\lim a_n = e^1 \neq 0$

Σa_n is divergent.

Q2 Which statement is true about the series

$$\sum_{n=2}^{\infty}\frac{10n}{\sqrt{n^2+2}} \sim \int_2^{\infty} \frac{10x}{\sqrt{x^2+2}}\,dx = \infty \quad \text{DIV.}$$

$f(x) = \frac{10x}{\sqrt{x^2+2}}$ continuous, positive and decreasing.

A The **integral test** concludes that the series converges.

B The **integral test** concludes that the series diverges. ✓

C The **integral test** hypotheses are not met by this series, so it cannot be applied.

D The **integral test** hypotheses are met by this series however the test is inconclusive.

E None of the above are true.

$u = x^2+2$

$du = 2x \cdot dx$

$\int_2^{\infty} \frac{10x}{\sqrt{x^2+2}}\,dx = \lim_{t\to\infty} \int_2^t \frac{10x}{\sqrt{x^2+2}}\,dx$

$= \int \frac{5 \cdot 2x\,dx}{\sqrt{x^2+2}} = \int \frac{5\,du}{\sqrt{u}} = \int 5 \cdot u^{-\frac{1}{2}}\,du$

$= 5 \cdot 2 \cdot \sqrt{u}$

$= \lim_{t\to\infty} 10\sqrt{x^2+2} \Big|_2^t$

$= \lim_{t\to\infty} 10\sqrt{t^2+2} - 10\sqrt{6} = \infty$

Q3 Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1)\ \sum_{n=1}^{\infty} \frac{\sin(n)+1}{2^n} \qquad \text{and (2)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$$

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

✓**C** (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

(1). $a_n = \frac{\sin(n)+1}{2^n}$

$|a_n| = \frac{|\sin(n)+1|}{2^n} \leq \frac{2}{2^n}$

since $-1 \leq \sin(n) \leq 1 \Rightarrow 0 \leq \sin(n)+1 \leq 2$

$\sum \frac{2}{2^n}$ is conv $\Rightarrow \sum |a_n|$ is conv $\Rightarrow \sum a_n$ is ABS conv.

(2). $a_n = \frac{(-1)^n}{\sqrt{n+2}}$.

$|a_n| = \frac{1}{\sqrt{n+2}}$, $\sum |a_n|$ is DIV

compared with $\frac{1}{\sqrt{n}}$. $p = \frac{1}{2}$

NOT ABS conv.

$a_n = \frac{(-1)^n}{\sqrt{n+2}}$ is alternating

$\frac{1}{\sqrt{n+2}} \xrightarrow{n\to\infty} 0$ and decreasing

$\Rightarrow \sum a_n$ conv therefore

is conditionally conv.

Q4 Determine whether the following series converge or diverge.

(a)

$$\sum_{n=1}^{\infty} \frac{2^n(n^2+1)}{3^n}$$

Ratio Test for $a_n = \frac{2^n \cdot (n^2+1)}{3^n}$, $a_{n+1} = \frac{2^{n+1} \cdot [(n+1)^2+1]}{3^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}[(n+1)^2+1]}{3^{n+1}} \cdot \frac{3^n}{2^n \cdot (n^2+1)} = \frac{2}{3} \cdot \frac{(n+1)^2+1}{n^2+1}.$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3} < 1.$$

According to ratio test, $\sum \frac{2^n(n^2+1)}{3^n}$ is convergent.

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+n^3+2n}{\sqrt{9n^8+7n}}$$

Limit Comparison Test for $a_n = \frac{\sqrt{n}+n^3+2n}{\sqrt{9n^8+7n}}$,

choose $b_n = \frac{n^3}{\sqrt{9n^8}} = \frac{n^3}{3 \cdot n^4} = \frac{1}{3n}$, $\sum b_n = \sum \frac{1}{3n}$ is divergent

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}+n^3+2n}{\sqrt{9n^8+7n}} \cdot 3n = \lim_{n\to\infty} \frac{n^3 \cdot 3n}{\sqrt{9n^8}} = 1 \neq 0$$

According to LCT., $\sum a_n$ is divergent since $\sum b_n$ is divergent.

Q5 Check the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+n}$$

using integral test. (Note: you need to check the series satisfies ALL the THREE hypotheses of integral test.)

$a_n = f(n) = \dfrac{2n+1}{n^2+n}$ is positive and continuous

Decreasing: $f'(n) = \dfrac{2\cdot(n^2+n) - (2n+1)\cdot(2n+1)}{(n^2+n)^2} = \dfrac{2n^2+2n-(4n^2+4n+1)}{(n^2+n)^2}$

(quotient rule)

$= \dfrac{-2n^2-2n-1}{(n^2+n)^2} < 0$

$f'(n)<0$ implies $f(n)$ is decreasing.

Now consider the improper integral $\int_1^{+\infty} \dfrac{2x+1}{x^2+x}\,dx$.

$= \lim_{t\to\infty} \int_1^t \dfrac{2x+1}{x^2+x}\,dx$. $u = x^2+x$, $du = (2x+1)dx$.

$= \int \dfrac{du}{u} = \ln|u| = \lim_{t\to\infty} \ln|x^2+x| \,\Big|_1^t = \lim_{t\to\infty} \ln|t^2+t| - \ln 2 = \infty$.

Therefore, $\sum \dfrac{2n+1}{n^2+n}$ is divergent since $\int_1^{+\infty} \dfrac{2x+1}{x^2+x}\,dx = \infty$ is divergent.

Q6 Find the exact arc-length of $f(x) = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $x=1$ to $x=2$.

Formula: Arc-L $= \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

$f'(x) = \frac{1}{2}\cdot 2x - \frac{1}{4}\cdot\frac{1}{x} = x - \frac{1}{4x}$

Arc-L $= \int_1^2 \sqrt{1+\left[x-\frac{1}{4x}\right]^2}\,dx = \int_1^2 \sqrt{1+x^2-2x\cdot\frac{1}{4x}+\frac{1}{16x^2}}\,dx$.

$= \int_1^2 \sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}\,dx$

$= \int_1^2 \sqrt{\frac{1}{2}+x^2+\frac{1}{16x^2}}\,dx$.

$= \int_1^2 \sqrt{\dfrac{8x^2+16x^4+1}{16x^2}}\,dx$.

$= \int_1^2 \sqrt{\dfrac{(4x^2+1)^2}{(4x)^2}}\,dx$.

$= \int_1^2 \dfrac{4x^2+1}{4x}\,dx$

$= \int_1^2 x + \frac{1}{4x}\,dx$

$= \frac{1}{2}x^2 + \frac{1}{4}\ln|x| \,\Big|_1^2$

$= \frac{1}{2}\cdot 2^2 + \frac{1}{4}\ln 2 - (\frac{1}{2}\cdot 1^2 + \frac{1}{4}\ln 1)$

$= \boxed{\frac{3}{2} + \frac{1}{4}\ln 2}$.

Q7 Consider the series $2 - \frac{4}{3e} + \frac{8}{9e^2} - \frac{16}{27e^3} + \cdots$. Give the value of the nth term a_n which would allow us to rewrite this series as $\sum_{n=1}^{\infty} a_n$ and find the sum.

$$a_n = (-1)^{n-1} \cdot \frac{2^n}{(3e)^{n-1}}, \quad \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2^n}{(3e)^{n-1}} = \frac{a}{1-r}$$

$$= \frac{(-1)^{n-1} \cdot 2^{n-1} \cdot 2}{(3e)^{n-1}} \qquad\qquad = \frac{2}{1-\left(\frac{-2}{3e}\right)} = \boxed{\frac{2}{1+\frac{2}{3e}} = \frac{6e}{3e+2}}$$

$$= 2 \cdot \left(\frac{-2}{3e}\right)^{n-1}$$

$$a = 2, \quad r = \frac{-2}{3e}$$

alternative way to find a, r.

a = first term $= 2$

r = common ratio = $\frac{\text{2nd term}}{\text{1st term}} = \frac{\frac{-4}{3e}}{2} = \frac{-2}{3e}$.

Therefore, $a_n = a \cdot r^{n-1} = 2 \cdot \left(\frac{-2}{3e}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$

Q8 Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^{2n+1} - (-1)^{n-1}}{9^n} = \sum_{n=1}^{\infty} \frac{2^{2n+1}}{9^n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{9^n}$$

First Geometric Series: $\frac{2^{2n+1}}{9^n} = \frac{2^{2n} \cdot 2}{9^n} = \frac{4^n \cdot 2}{9^n} = 2 \cdot \frac{4}{9} \cdot \left(\frac{4}{9}\right)^{n-1}$

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{9^n} = \frac{\frac{8}{9}}{1-\frac{4}{9}} = \frac{8}{9-4} = \frac{8}{5} \qquad a = \frac{8}{9}, \quad r = \frac{4}{9}.$$

Second G.S. $\frac{(-1)^{n-1}}{9^n} = \frac{1}{9} \cdot \frac{(-1)^{n-1}}{9^{n-1}} = \frac{1}{9} \cdot \left(-\frac{1}{9}\right)^{n-1}, \quad a = \frac{1}{9}, \quad r = -\frac{1}{9}.$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{9^n} = \frac{\frac{1}{9}}{1-\left(-\frac{1}{9}\right)} = \frac{\frac{1}{9}}{1+\frac{1}{9}} = \frac{1}{10}$$

Therefore, $\sum_{n=1}^{\infty} \frac{2^{2n+1} - (-1)^{n-1}}{9^n} = \frac{8}{5} - \frac{1}{10} = \frac{16}{10} - \frac{1}{10} = \frac{15}{10} = \frac{3}{2}$

Q9 Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(n+3)(2x-3)^n}{3^n}$$

Apply Ratio Test to

$a_n = \frac{(n+3)\cdot(2x-3)^n}{3^n}$, $|a_n| = \frac{(n+3)\,|(2x-3)^n|}{3^n}$, $|a_{n+1}| = \frac{(n+4)\cdot|(2x-3)^{n+1}|}{3^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+4)\cdot|(2x-3)^{n+1}|}{3^{n+1}} \cdot \frac{3^n}{(n+3)\,|(2x-3)^n|} = \frac{n+4}{n+3}\cdot\frac{1}{3}\cdot|(2x-3)|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\cdot|2x-3| \boxed{<1} \Rightarrow |2x-3|<3 \Rightarrow \left|x-\frac{3}{2}\right| < \boxed{\frac{3}{2}}$$

The power series is conv for $\left|x-\frac{3}{2}\right| < \boxed{\frac{3}{2}}$.

Therefore the radius of convergence $R = \frac{3}{2}$

Q11 Find the first three non-zero terms of the Maclaurin series of the function

$$f(x) = xe^x + \cos x$$

The first three non-zero terms are $1 + x + \frac{x^2}{2}$.

$$f(x) = x\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}$$

$$= x\cdot\left(1+x+\frac{1}{2}x^2+\cdots\right) + \left(1-\frac{x^2}{2}+\frac{x^4}{4}+\cdots\right)$$

$$= x+x^2+\frac{1}{2}\cdot x^3+\cdots+1-\frac{x^2}{2}+\frac{x^4}{4}+\cdots = \boxed{1+x+\frac{x^2}{2}}+\frac{1}{2}x^3+\cdots$$

Q12 Consider the function $f(x) = \frac{3x}{2+3x^2}$. Find the power series representation of f and the radius of convergence.

$$f(x) = 3x\cdot\frac{1}{2\left[1+\frac{3x^2}{2}\right]} = \frac{3x}{2}\cdot\frac{1}{1-\left(-\frac{3x^2}{2}\right)} = \frac{3x}{2}\sum_{n=0}^{\infty}\left(-\frac{3x^2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{3x}{2}\cdot\left(-\frac{3}{2}\right)^n\cdot x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{3}{2}\cdot\left(-\frac{3}{2}\right)^n\cdot x^{2n+1}$$

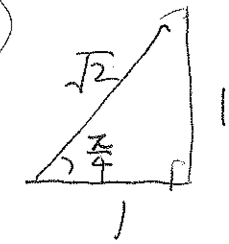
Radius of Conv:

$$\left|-\frac{3x^2}{2}\right| < 1 \Rightarrow x^2 < \frac{2}{3}$$

$$\Rightarrow |x| < \boxed{\sqrt{\frac{2}{3}}}$$

$$\boxed{R = \sqrt{\frac{2}{3}}}$$

Q13 Find the 4th degree Taylor polynomial of $f(x) = 3\sin(2x)$ centered at $a = \pi/8$



$\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

Derivative Table.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{8})$
$n=0$	$3\sin(2x)$	$3\cdot\sin(\frac{\pi}{4}) = 3\cdot\frac{1}{\sqrt{2}}$
$n=1$	$3\cdot2\cdot\cos(2x)$	$3\cdot2\cdot\cos(\frac{\pi}{4}) = 3\cdot2\cdot\frac{1}{\sqrt{2}}$
$n=2$	$3\cdot(-2^2)\sin(2x)$	$-3\cdot2^2\cdot\sin(\frac{\pi}{4}) = -3\cdot2^2\cdot\frac{1}{\sqrt{2}}$
$n=3$	$3\cdot(-2^3)\cos(2x)$	$-3\cdot2^3\cdot\cos(\frac{\pi}{4}) = -3\cdot2^3\cdot\frac{1}{\sqrt{2}}$
$n=4$	$3\cdot2^4\cdot\sin(2x)$	$3\cdot2^4\cdot\sin(\frac{\pi}{4}) = 3\cdot2^4\cdot\frac{1}{\sqrt{2}}$

4th degree Taylor Polynomial $T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$

$$T_4(x) = 3\cdot\frac{1}{\sqrt{2}} + \frac{6}{\sqrt{2}}(x-\frac{\pi}{8}) + \frac{-3\cdot2^2}{\sqrt{2}}\cdot\frac{(x-\frac{\pi}{8})^2}{2!} + \frac{-3\cdot2^3}{\sqrt{2}}\cdot\frac{1}{3!}(x-\frac{\pi}{8})^3 + \frac{3\cdot2^4}{\sqrt{2}\cdot4!}\cdot(x-\frac{\pi}{8})^4$$

Q14 Find the Taylor series at $x = 0$ for $f(x) = x^2e^{-2x}$ (find the general nth term and write it in Sigma notation).

Hint: $e^{\square} = \sum_{n=0}^{\infty}\frac{\square^n}{n!}$

$\square = -2x$.

$$f(x) = x^2\cdot\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$$

$$= x^2\cdot\sum_{n=0}^{\infty}\frac{(-2)^n\cdot x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-2)^n\cdot x^2\cdot x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-2)^n}{n!}\cdot x^{n+2}$$