Practice Midterm2 Version A, Sec61

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

Q1 Which statement is true about the series

$$\sum_{n=1}^{\infty} e^{\frac{2}{n}}$$

$\lim_{n\to\infty} e^{\frac{2}{n}} = e^{\frac{2}{\infty}} = e^0 = 1 \boxed{\neq 0}$

nth term test $\Rightarrow$ DIV.

A The **nth term test** concludes that the series converges.

✓ B The **nth term test** concludes that the series diverges.

C The **nth term test** hypotheses are not met by this series, so it cannot be applied.

D The **nth term test** hypotheses are met by this series however the test is inconclusive.

E None of the above are true. The nth term test concludes that the series converges.

Q2 Which statement is true about the series

hypotheses: $f(x)$ is positive (✓), continuous (✓)

$\boxed{\text{decreasing}}$: $f'(x) = \frac{2\cdot\frac{1}{x}\cdot x - 2\ln x \cdot 1}{x^2} = \frac{2(1-\ln x)}{x^2} < 0$ $(x>3)$

$$\sum_{n=2}^{\infty} \frac{2\ln n}{n} \sim \int_2^{\infty} \frac{2\ln x}{x}\,dx = \lim_{t\to\infty}\int_2^t \frac{2\ln x}{x}\,dx$$

$u = \ln x$, $du = \frac{1}{x}dx$

$= \lim_{t\to\infty} \int 2u \cdot du$

$= \lim_{t\to\infty} 2\ln\ln x \Big|_2^t = \lim_{t\to\infty} 2\ln\ln t - 2\ln\ln 2$

$= \infty$.

A The **integral test** concludes that the series converges.

✓ B The **integral test** concludes that the series diverges.

C The **integral test** hypotheses are not met by this series, so it cannot be applied.

D The **integral test** hypotheses are met by this series however the test is inconclusive.

E None of the above are true.

Q3 Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1)\ \sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2} \quad \text{and (2)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n}$$

(1). $a_n = \frac{\sin(2n)}{n^2}$, $|a_n| = \frac{|\sin(2n)|}{n^2}$

$|a_n| = \frac{|\sin(2n)|}{n^2} \leq \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ conv. $(p=2)$

(Comparison Test) $\Rightarrow \sum |a_n|$ conv

$\Rightarrow \sum a_n$ ABS conv.

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

✓ C (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

(2). $a_n = \frac{(-1)^{n-1}}{3n}$. $|a_n| = \frac{1}{3n}$.

$\sum |a_n| = \sum \frac{1}{3n}$ is divergent. $\Rightarrow \sum a_n$ NOT ABS conv.

$\sum a_n = \sum (-1)^{n-1} \cdot \frac{1}{3n}$, b_n decreasing and $\lim \frac{1}{3n} = 0$, Alternating Series Test, $\sum (-1)^{n-1} b_n$ conv

$b_n = \frac{1}{3n}$

Therefore, $\sum a_n$ is conditionally conv.

Q4 Determine whether the following series converge or diverge.

(a)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{e^n} \qquad a_n = \frac{\sqrt{n}+1}{e^n}$$

Ratio Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}+1}{e^{n+1}}}{\frac{\sqrt{n}+1}{e^n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}+1}{e^{n+1}} \cdot \frac{e^n}{\sqrt{n}+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}+1}{\sqrt{n}+1} \cdot \frac{1}{e}$$

$$= \frac{1}{e} < 1.$$

According to Ratio Test,

$\sum \frac{\sqrt{n}+1}{e^n}$ is convergent.

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n^3}}{3n^2+7n} \qquad a_n = \frac{\sqrt{n^2+n^3}}{3n^2+7n}, \quad b_n = \frac{\sqrt{n^3}}{3n^2} = \frac{n^{\frac{3}{2}}}{3n^2} = \frac{1}{3\sqrt{n}}$$

Limit Comparison Test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+n^3}}{3n^2+7n} \cdot 3\sqrt{n} = \lim_{n\to\infty} \frac{3\sqrt{(n^2+n^3)\cdot n}}{3n^2+7n} = \lim_{n\to\infty} \frac{3\cdot\sqrt{n^4}}{3\cdot n^2} = 1 \neq 0$$

Since $\sum b_n = \sum \frac{1}{3\sqrt{n}}$ is a p-Series with $p=\frac{1}{2}<1$, divergent,

$\sum a_n$ is also divergent.

(c)

$$\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{4n^5-1}} \qquad a_n = \frac{n+1}{\sqrt{4n^5-1}}, \quad b_n = \frac{n}{\sqrt{4n^5}} = \frac{n}{2n^{\frac{5}{2}}} = \frac{1}{2n^{\frac{3}{2}}}$$

Limit Comparison Test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{\sqrt{4n^5-1}} \cdot \frac{\sqrt{4n^5}}{n} = \lim_{n\to\infty} \frac{n\cdot\sqrt{4n^5}}{\sqrt{4n^5}\cdot n} = 1.$$

$\sum b_n = \sum \frac{1}{2n^{\frac{3}{2}}}$ p-Series $p=\frac{3}{2}>1$, convergent

$\sum a_n$ is also convergent.

Q5 Check the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$$

using integral test. (Note: you need to check the series satisfies ALL the THREE hypotheses of integral test.)

hypotheses: $a_n = f(n) = \frac{2n}{n^2+1}$ is continuous, positive
(✓) (✓)

and decreasing: $f'(n) = \frac{2\cdot(n^2+1) - 2n\cdot 2n}{(n^2+1)^2} = \frac{2n^2+2-4n^2}{(n^2+1)^2} = \frac{-2n^2+2}{(n^2+1)^2} < 0$

(on $[2,+\infty)$)

Consider improper integral:

$$\int_1^{\infty} \frac{2x}{x^2+1}\cdot dx = \lim_{t\to\infty} \int_1^{t} \frac{2x}{x^2+1}\, dx, \quad u = x^2+1, \; du = 2x\cdot dx.$$

$$= \lim_{t\to\infty} \int \frac{du}{u}.$$

$$= \lim_{t\to\infty} \ln|u| = \lim_{t\to\infty} \ln(x^2+1)\Big|_1^t = \lim_{t\to\infty} \ln(t^2+1) - \ln(2) = \infty$$

The improper integral is divergent. According to integral test,

$\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ is also divergent.

Q6 Find the exact arc-length of $f(x) = \frac{2}{3}(x^2+1)^{3/2}$ from $x = 0$ to $x = 2$.

$$f'(x) = \frac{2}{3}\cdot\frac{3}{2}\cdot(x^2+1)^{\frac{1}{2}}\cdot 2x = (x^2+1)^{\frac{1}{2}}\cdot 2x$$

$$\text{Arc-length} = \int_0^2 \sqrt{1+[f'(x)]^2}\, dx = \int_0^2 \sqrt{1+(x^2+1)\cdot 4x^2}\, dx$$

$$= \int_0^2 \sqrt{1+4x^4+4x^2}\, dx$$

Hint: Complete the square $(1+2a)^2 = 1+4a+4a^2$

$$= \int_0^2 \sqrt{(1+2x^2)^2}\, dx$$

$$= \int_0^2 1+2x^2\, dx$$

$$= x + 2\cdot\frac{1}{3}x^3 \Big|_0^2$$

$$\boxed{= 2 + \frac{2}{3}\cdot 8 = \frac{22}{3}}$$

Q7 What does the series $-2+\frac{6}{5}-\frac{18}{25}+\frac{54}{125}+\cdots$ converge to? Find the sum.

$$a=-2, \quad r=\frac{\frac{6}{5}}{-2}=-\frac{3}{5}, \quad a_n=(-2)\cdot\left(-\frac{3}{5}\right)^{n-1}, \; n=1,2,\cdots$$

$$\sum_{n=1}^{\infty}(-2)\cdot\left(-\frac{3}{5}\right)^{n-1}=\frac{a}{1-r}=\frac{-2}{1-\left(-\frac{3}{5}\right)}=\frac{-2}{\frac{8}{5}}=\boxed{-\frac{5}{4}}$$

(or $a_n=(-2)\cdot\left(-\frac{3}{5}\right)^{n}, \; n=0,1,2,\cdots$)

($\sum_{n=0}^{\infty}(-2)\cdot\left(-\frac{3}{5}\right)^{n}=\frac{a}{1-r}=-\frac{5}{4}$)

Q8 Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{9^{n/2}}{3(2^{2n+1})}=\frac{9^{\frac{1}{2}}}{3\cdot 2^3}+\frac{9^{\frac{2}{2}}}{3\cdot 2^5}+\frac{9^{\frac{3}{2}}}{3\cdot 2^7}+\cdots$$

$$a=\frac{9^{\frac{1}{2}}}{3\cdot 2^3}=\frac{3}{3\cdot 8}=\frac{1}{8}, \quad r=\frac{9}{3\cdot 2^5}\Big/\frac{1}{8}=\frac{3}{2^5}\cdot 8=\frac{3}{4}$$

Therefore, $\sum_{n=1}^{\infty}\frac{9^{n/2}}{3\cdot(2^{2n+1})}=\frac{a}{1-r}=\frac{\frac{1}{8}}{1-\frac{3}{4}}=\frac{\frac{1}{8}}{\frac{1}{4}}=\boxed{\frac{1}{2}}$

Alternative way to find a, r.

$$a_n=\frac{9^{\frac{n}{2}}}{3\cdot 2^{2n+1}}=\frac{(9^{\frac{1}{2}})^n}{3\cdot 2^{2n}\cdot 2}=\frac{3^n}{3\cdot 4^n\cdot 2}=\frac{1}{6}\cdot\left(\frac{3}{4}\right)^n=\frac{1}{6}\cdot\frac{3}{4}\cdot\left(\frac{3}{4}\right)^{n-1}=\underset{\underset{a}{\uparrow}}{\frac{1}{8}}\cdot\Big(\underset{\underset{r}{\uparrow}}{\frac{3}{4}}\Big)^{n-1}$$

Q9 Find the radius of convergence of

$$\sum_{n=0}^{\infty}\frac{x^n(n^2+3)}{(-5)^n}$$

Apply Ratio Test to $a_n=\frac{x^n\cdot(n^2+3)}{(-5)^n}$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}\cdot((n+1)^2+3)}{(-5)^{n+1}}\cdot\frac{(-5)^n}{x^n\cdot(n^2+3)}\right|=\frac{(n+1)^2+3}{n^2+3}\cdot\frac{1}{5}\cdot|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)^2+3}{n^2+3}\cdot\frac{1}{5}\cdot|x|=\frac{1}{5}|x|<1$$

$\therefore |x|<5$. Therefore, the radius of convergence is $R=5$.

Q10 Find the first three non-zero terms of the power series representation of the function

$$f(x) = 1 - \frac{x}{1+2x^2}$$

$$\frac{x}{1+2x^2} = x \cdot \frac{1}{1-(-2x^2)} = x \cdot \sum_{n=0}^{\infty} (-2x^2)^n = x \cdot (1 - 2x^2 + 4x^4 - \cdots)$$

$$= x - 2x^3 + 4x^5 - \cdots$$

$$\Rightarrow f(x) = 1 - \frac{x}{1+2x^2} = 1 - (x - 2x^3 + 4x^5 - \cdots)$$

$$= 1 - x + 2x^3 - 4x^5 + \cdots$$

Therefore, the first three non-zero terms of $f(x)$ are

$$\boxed{1 - x + 2x^3}$$

Q11 Find the power series representation and the radius of convergence of the function

$$f(x) = \frac{x^2}{3x+2}$$

$$f(x) = \frac{x^2}{3x+2} = x^2 \cdot \frac{1}{2 \cdot \left[1 - \left(-\frac{3x}{2}\right)\right]} = \frac{x^2}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{3x}{2}\right)^n$$

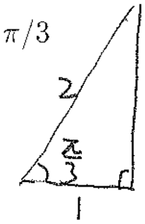
$$= \sum_{n=0}^{\infty} \frac{x^2}{2} \cdot \left(-\frac{3}{2}\right)^n \cdot x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(-\frac{3}{2}\right)^n \cdot x^{n+2}}$$

Radius of Convergence: $\left|-\frac{3x}{2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$

$$\boxed{R = \frac{2}{3}}$$

Q12 Find the 3rd degree Taylor polynomial of $f(x) = 2 + \cos(x)$ centered at $a = \pi/3$



$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Derivative Table

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$
$n=0$	$2+\cos x$	$2+\cos\frac{\pi}{3} = 2+\frac{1}{2} = \frac{5}{2}$
$n=1$	$-\sin x$	$-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$
$n=2$	$-\cos x$	$-\cos\frac{\pi}{3} = -\frac{1}{2}$
$n=3$	$\sin x$.	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

3rd degree Taylor polynomial:

$$f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^2$$

$$= \frac{5}{2} - \frac{\sqrt{3}}{2}\cdot(x-\frac{\pi}{3}) + \frac{-\frac{1}{2}}{2}\cdot(x-\frac{\pi}{3})^2 + \frac{\frac{\sqrt{3}}{2}}{3!}\cdot(x-\frac{\pi}{3})^2$$

Q13 Find the $\boxed{\text{first three non-zero}}$ terms of the Taylor series at $x = 0$ for $f(x) = 3\sin(2x) + x^2$.

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots$$

$$f(x) = 3\cdot\sin(2x) + x^2$$

$$= 3\cdot\left[2x - \frac{2^3\cdot x^3}{3!} + \frac{2^5\cdot x^5}{5!} - \dots\right] + x^2$$

$$= 6x - 4x^3 + \frac{3\cdot 2^5}{5!}\cdot x^5 + \dots + x^2$$

$$= \boxed{6x + x^2 - 4x^3} + \frac{3\cdot 2^5}{5!}\cdot x^5 + \dots$$

The first three non-zero terms are $\boxed{6x + x^2 - 4x^3}$