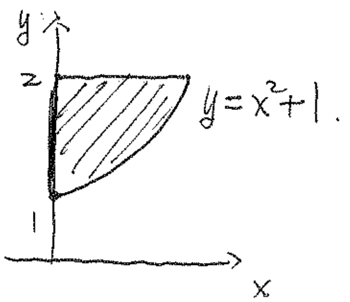
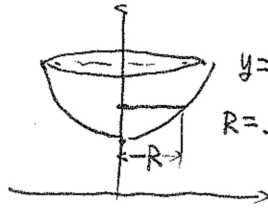
Practice Final Version B, Sec61

Q1 Sketch the region R bounded by $y = x^2 + 1, x = 0,\ y = 2$. Find the volume of the solid rotating R about the $y-$axis.





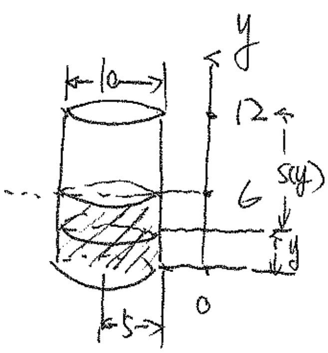
$y=x^2+1 \iff x=\sqrt{y-1}$

$R=\sqrt{y-1}$. $A(y)=\pi R^2=\pi(\sqrt{y-1})^2=\pi(y-1)$

$$V=\int_1^2 \pi(y-1)\,dy = \pi\left(\tfrac12 y^2-y\right)\Big|_1^2=\pi\left(\tfrac12 2^2-2\right)-\pi\left[\tfrac12 1-1\right]$$

$$=\boxed{\tfrac12\pi}$$

Q2 A vertical right-circular cylindrical tank measures 12 ft high and 10 ft in diameter. It is half full of kerosene weighing 20 lb/ft^3. Find the work it would take to pump the kerosene to the top of the tank.



$s(y)=12-y \quad A(y)=\pi r^2=\pi\cdot 5^2$

$$W=\int_0^6 \delta\cdot s(y)\cdot A(y)\,dy$$

$$=\int_0^6 20\cdot(12-y)\cdot\pi 5^2\,dy$$

$$=20\cdot\pi\cdot 25\cdot\int_0^6 12-y\,dy$$

$$=500\cdot\pi\left(12y-\tfrac12 y^2\right)\Big|_0^6=\boxed{500\pi\left(12\cdot6-\tfrac12 6^2\right)=500\pi\cdot54}\ \text{lb-ft.}$$

Q3 Find the arc-length of the curve $x = y^{3/2}$ from $y = 0$ to $y = 2$.

$\frac{dx}{dy}=\frac32\cdot y^{\frac12}$

$$\text{Arc-length}=\int_0^2\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^2\sqrt{1+\left(\tfrac32 y^{\frac12}\right)^2}\cdot dy=\int_0^2\sqrt{1+\tfrac94 y}\,dy$$

$$\xrightarrow[du=\frac94 dy]{u=1+\frac94 y}\int\sqrt{u}\cdot\tfrac49\,du$$

$$=\tfrac49\cdot\tfrac23\cdot u^{\frac32}$$

$$=\tfrac{8}{27}\left(1+\tfrac94 y\right)^{\frac32}\Big|_0^2$$

$$=\boxed{\tfrac{8}{27}\left(1+\tfrac92\right)^{\frac32}-\tfrac{8}{27}}$$

Q4 Evaluate the following integrals.

(a)

$$\int 2\sin^{-1}(x)\ \mathrm{d}x$$

$\boxed{\begin{array}{ll} \text{IBP} & \\ u=2\sin^{-1}x & du=\frac{2}{\sqrt{1-x^2}}dx \\ dv=dx & v=x \end{array}}$

$= 2\sin^{-1}x\cdot x - \int x\cdot\frac{2}{\sqrt{1-x^2}}dx$ $\quad u=1-x^2,\ du=-2x\,dx$

$= 2\sin^{-1}x\cdot x - \int\frac{-du}{\sqrt{u}}$

$= 2\sin^{-1}x\cdot x + 2\sqrt{u} + C$

$= \boxed{2\sin^{-1}x\cdot x + 2\sqrt{1-x^2} + C}$

(b)

$$\int_0^1 xe^{2x}\ \mathrm{d}x$$

IBP: $u=x \quad du=dx$
$dv=e^{2x}dx, \quad v=\frac{1}{2}e^{2x}$

$= x\cdot\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}dx$

$= \boxed{x\cdot\frac{1}{2}e^{2x} - \frac{1}{2}\cdot\frac{1}{2}\cdot e^{2x} + C}$

(c)

$$\int_0^{\pi/4}(\sin\theta+\cos\theta)\cos\theta\ \mathrm{d}\theta$$

$= \int\sin\theta\cdot\cos\theta\cdot d\theta + \int\cos^2\theta\, d\theta$

$u=\sin\theta,\ du=\cos\theta d\theta$

$= \int u\cdot du + \int\frac{1+\cos2\theta}{2}d\theta$

$= \frac{1}{2}u^2 + \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin2\theta$

$= \frac{1}{2}\cdot\sin^2\theta + \frac{1}{2}\cdot\theta + \frac{1}{4}\sin2\theta\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\sin^2\frac{\pi}{4}+\frac{1}{2}\frac{1}{4}\pi+\frac{1}{4}\sin\frac{\pi}{2}-0 = \boxed{\frac{1}{2}\cdot\frac{1}{2}+\frac{\pi}{8}+\frac{1}{4}=\frac{1}{2}+\frac{\pi}{8}}$

Q5 Test the following improper integral. Evaluate it if it is convergent.

$\int_0^1\frac{1}{(x)^{4/3}}\ \mathrm{d}x = \lim_{t\to0^+}\int_t^1\frac{1}{x^{4/3}}dx = \lim_{t\to0^+}(-3+t^{-\frac{1}{3}}) = \lim_{t\to0^+} -3+\frac{1}{\sqrt[3]{t}}$

$\int_t^1 x^{-\frac{4}{3}}dx = \frac{1}{-\frac{1}{3}}\cdot x^{-\frac{1}{3}}\Big|_t^1 = -3\cdot1+3\cdot t^{-\frac{1}{3}}$ $= \boxed{\infty \quad \text{Divergent}}$

Q6 Determine whether each of the series is convergent or divergent.

(a)

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$$

n-th term test for divergence

$$\lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{l'Hosp}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \cos\left(\frac{1}{\infty}\right) = \cos 0 = \boxed{1 \neq 0}$$

$\boxed{\sum n \sin\left(\frac{1}{n}\right) \text{ is divergent}}$ due to n-th term test.

(b)

$$\sum_{n=1}^{\infty} \frac{5^n}{n!} \qquad a_n = \frac{5^n}{n!}$$

ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \lim_{n\to\infty} \frac{5}{(n+1)} = \boxed{0 < 1.}$

$\sum \frac{5^n}{n!}$ is convergent.

Q7 Evaluate the following limits.

(a)

$$\lim_{n\to\infty} (\ln n)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln (\ln n)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln \ln n} = e^0 = \boxed{1}$$

$$\lim_{n\to\infty} \frac{\ln \ln n}{n} \overset{l'Hosp}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{\ln n \cdot n} = \frac{1}{\infty} = 0$$

(b)

$$\lim_{n\to\infty} \ln\left(\frac{3n}{\sqrt{n^2+1}}\right) \boxed{= \ln 3}$$

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{3\cdot\sqrt{n^2}}{\sqrt{n^2+1}} = \lim_{n\to\infty} 3\sqrt{\frac{n^2}{n^2+1}} = 3\sqrt{1} = 3$$

Q8 Find the derivative of $f(x) = (\sin^{-1}(x))^{\sqrt{x}}$

$$\ln f(x) = \ln (\sin^{-1} x)^{\sqrt{x}} = \sqrt{x} \cdot \ln(\sin^{-1} x)$$

Take derivative. $$\frac{f'(x)}{f(x)} = (\sqrt{x} \ln \sin^{-1} x)' = \frac{1}{2\sqrt{x}} \cdot \ln \sin^{-1} x + \sqrt{x} \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{f'(x) = (\sin^{-1} x)^{\sqrt{x}} \cdot \left[\frac{1}{2\sqrt{x}} \ln \sin^{-1} x + \frac{\sqrt{x}}{\sin^{-1} x \cdot \sqrt{1-x^2}} \right]}$$

Q9 Consider the following power series. Find its center and radius of convergence.

$$\sum_{n=1}^{\infty} n(2x+1)^n$$

Center : $2x + 1 = 0 \Rightarrow \boxed{x = -\frac{1}{2}}$

Ratio Test for $a_n = n \cdot (2x+1)^n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1) \cdot (2x+1)^{n+1}}{n \cdot (2x+1)^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot |2x+1| = \boxed{|2x+1| < 1}$$

$$\Rightarrow |x + \tfrac{1}{2}| < \boxed{\tfrac{1}{2}}$$

Radius of Convergence: $\boxed{R = \frac{1}{2}}$

Q10 Consider $g(t) = \frac{1}{1-t^2}$. Find the first three non-zero terms of the Maclaurin series for $g(t)$. And then find the first three non-zero terms of the Maclaurin series for $\int_0^{3x} g(t)dt$.

$$g(t) = \sum_{n=0}^{\infty} (t^2)^n = \sum_{n=0}^{\infty} t^{2n} = \boxed{1 + t^2 + t^4} + \cdots$$

$$\int_0^{3x} g(t)\,dt = \int_0^{3x} 1 + t^2 + t^4 + \cdots \, dt$$

$$= t + \frac{1}{3} t^3 + \frac{1}{5} t^5 + \cdots \Big|_0^{3x}$$

$$= \boxed{3x + \frac{1}{3} \cdot (3x)^3 + \frac{1}{5} (3x)^5 +} \cdots$$

Q11 Let $f(x) = 1/x$. Consider its Taylor series at $x = 3$.

(a) Find $T_2(x)$, the second degree Taylor polynomial of $f(x)$ centered at 3.

Derivative Table:

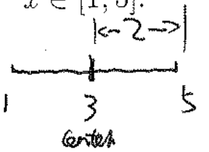
n	$f^{(n)}(x)$	$f^{(n)}(3)$
$n=0$	$\frac{1}{x}$	$\frac{1}{3}$
$n=1$	$-\frac{1}{x^2}$	$-\frac{1}{9}$
$n=2$	$2\cdot\frac{1}{x^3}$	$\frac{2}{27}$.

$$T_2(x) = f(3) + f'(3)\cdot(x-3) + f''(3)\cdot\frac{1}{2}\cdot(x-3)^2$$

$$= \boxed{\frac{1}{3} - \frac{1}{9}\cdot(x-3) + \frac{2}{27}\cdot\frac{1}{2}\cdot(x-3)^2}$$

$$f^{(2)}(x) = \frac{2}{x^3}, \quad f^{(3)}(x) = \frac{2\cdot(-3)}{x^4} = \frac{-6}{x^4}$$

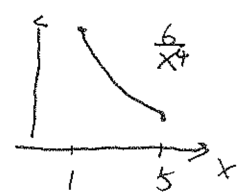
(b) Use Taylor's Inequality to estimate the maximum possible error in approximating $f(x)$ by $T_2(x)$ for $x \in [1, 5]$. $\Leftrightarrow$ $|x-3| < 2$. $\quad n=2$



$$|f(x) - T_2(x)| = |R_2(x)| \le \frac{M}{(2+1)!}\cdot|x-3|^{2+1} \quad \text{for } |x-3| \le 2$$

$$M = \text{maximum of } |f^{(2+1)}(x)| = |f^{(3)}(x)| = \left|\frac{-6}{x^4}\right| \text{ on } x \in [1, 5]$$

$$= \frac{6}{x^4}$$



$M = 6$.

$$\Rightarrow \boxed{|f(x) - T_2(x)| = |R_2(x)| \le \frac{6}{3!}|x-3|^3 \le \frac{6}{3!}\cdot 2^3 = 8}$$

Q12 Solve the following differential equation

$$y'(x) = e^{-2y}x, \; y(0) = 0$$

$$y' = \frac{dy}{dx} = e^{-2y}\cdot x$$

$$\Rightarrow e^{2y}\,dy = x\cdot dx.$$

$$\Rightarrow \int e^{2y}\,dy = \int x\,dx$$

$$\frac{1}{2}e^{2y} = \frac{1}{2}x^2 + C$$

$$y(0) = 0 \Rightarrow x=0, \; y=0$$

$$\frac{1}{2}\cdot e^0 = 0 + C \Rightarrow C = \frac{1}{2}.$$

$$\frac{1}{2}\cdot e^{2y} = \frac{1}{2}x^2 + \frac{1}{2}$$

$$e^{2y} = x^2 + 1$$

$$2y = \ln(x^2+1)$$

$$\boxed{y = \frac{1}{2}\cdot\ln(x^2+1)}$$

Q13 Find the tangent line to the parametric curve

$$x(t) = \ln\left(\sec t\right), \quad y(t) = (t - \pi/4)^2 + 2(t - \pi/4) \qquad \text{at} \qquad t = \pi/4$$

$t = \frac{\pi}{4}$ $\quad x(\frac{\pi}{4}) = \ln \sec(\frac{\pi}{4}) = \ln\sqrt{2}$, $y(\frac{\pi}{4}) = 0$

$\frac{dx}{dt} = \frac{1}{\cancel{\sec t}} \cdot \tan t \cdot \cancel{\sec t} = \tan t = \tan\frac{\pi}{4} = 1$

$\frac{dy}{dt} = 2(t - \frac{\pi}{4}) + 2 = 2(\frac{\pi}{4} - \frac{\pi}{4}) + 2 = 2.$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{1} = 2.$

slope of the tangent line

$\boxed{y = 0 + 2(x - \ln\sqrt{2}) = 2(x - \ln\sqrt{2})}$

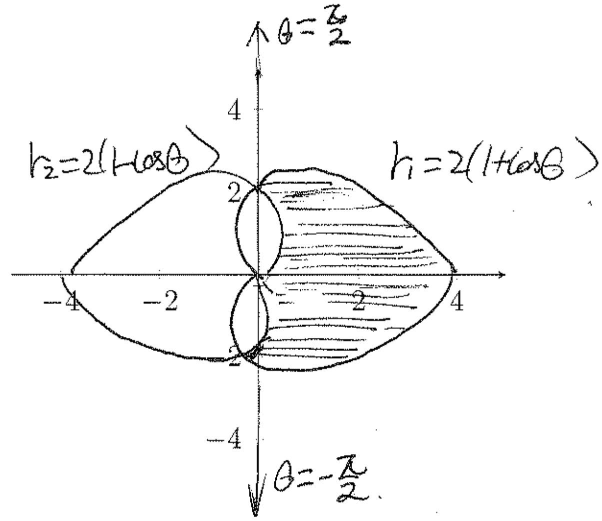
Q14 Consider the following polar curves given by $r_1 = 2(1 + \cos\theta)$, and $r_2 = 2(1 - \cos\theta)$.

(a) Sketch both curves r_1 and r_2.

$r_1 = 2(1+\cos\theta) = 2(1-\cos\theta) = r_2$

$2 + 2\cos\theta = 2 - 2\cos\theta$

$4\cos\theta = 0, \quad \cos\theta = 0, \quad \theta = \frac{\pi}{2}, -\frac{\pi}{2}.$



(b) Find the area of the region inside r_1 and outside r_2.

$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\frac{1}{2} r_1^2}_{\text{inside } r_1} - \underbrace{\frac{1}{2} r_2^2}_{\text{outside } r_2} \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[2(1+\cos\theta)]^2 - \frac{1}{2}[2(1-\cos\theta)]^2 \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(1 + 2\cos\theta + \cos^2\theta) - 2\cdot(1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cdot\cos\theta \, d\theta = 8\cdot\sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 8\cdot\sin\frac{\pi}{2} - 8\cdot\sin(-\frac{\pi}{2})$$

$$= 8 - 8\cdot(-1) = 16$$