Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V=\int_c^d \pi\cdot[g(y)]^2\,dy \qquad V = \int_a^b A(x)\,dx = \int_a^b \pi\cdot[f(x)]^2\,dx.$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\,dx = \int_a^b 6\cdot s(y)\cdot A(y)\,dy$$

- $\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$ $\quad \int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1$

- $\int \tan x\,dx = \ln|\sec x| + C$, $\quad (\tan x)' = \sec^2 x$.

- $\int \sec x\,dx = \ln|\sec x + \tan x| + C$, $\quad (\sec x)' = \tan x\cdot\sec x$

- $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad$ for $a \neq 1$

- **Integration by Parts**

$$\int u\,dv = uv - \int v\,du$$

- Arc Length Formula:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Derivatives

- $\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$

- Inverse Trigonometric Functions:

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}}$$

- $y = f(x)^{g(x)} \Rightarrow \ln y = g(x)\cdot\ln f(x) \Rightarrow \frac{y'}{y} = [g(x)\cdot\ln f(x)]'$

- l'Hospital's rule

Hyperbolic and Trig Identities

- Hyperbolic Functions

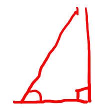
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

right triangle



- $\sin(2x) = 2\sin x\cos x$
- $\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Parametric

- $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0}$ $\quad y = y_0 + k(x - x_0)$

- Arc Length: $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Polar

- $x = r\cos\theta \qquad y = r\sin\theta$
- $r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$
- Area: $A = \int_a^b \frac{1}{2}r(\theta)^2\,d\theta$

Graphs of $\sin x$, $\cos x$, e^x, $\ln x$.

- $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Series

- $\boxed{n\text{th term test for divergence}}$: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- $\boxed{\text{The } p\text{-series}}$: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- $\boxed{\text{Geometric}}$ If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

  (i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

  (ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

n-th degree Taylor Polynomial $T_n(x)$

Remainder $R_n(x) = f(x) - T_n(x)$

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

  (i) $0 < b_{n+1} \leq b_n$ for all n

  (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.

- $\boxed{\text{The Ratio Test}}$

  (i) If $\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1}$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series:** $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

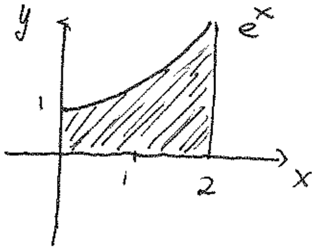
- Some Power Series

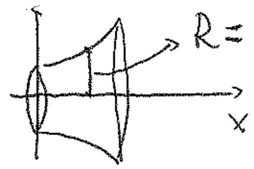
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$ $\qquad R = 1$

Practice Final Version A, Sec61

Q1[*Sec5.2*, rotating solid]

(a)[**horizontal axis**] Sketch the region R bounded by $y = e^x, y = 0,\ x = 0,\ x = 2$. Set up an integral for the volume of the solid rotating R about the $\boxed{x\text{-axis}}$. Do not evaluate the integral.

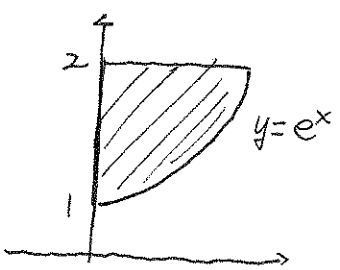


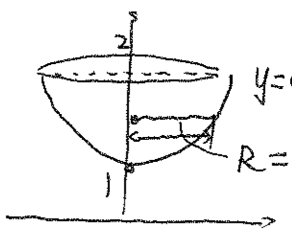


$A(x) = \pi \cdot R^2 = \pi [e^x]^2 = \pi \cdot e^{2x}$

$$\boxed{V = \int_0^2 \pi \cdot e^{2x}\, dx.}$$

(b)[**vertial axis**] Sketch the region R bounded by $y = e^x, x = 0,\ y = 2$. Set up an integral for the volume of the solid rotating R about the $\boxed{y\text{-axis}}$. Do not evaluate the integral.

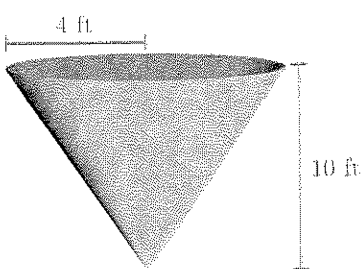


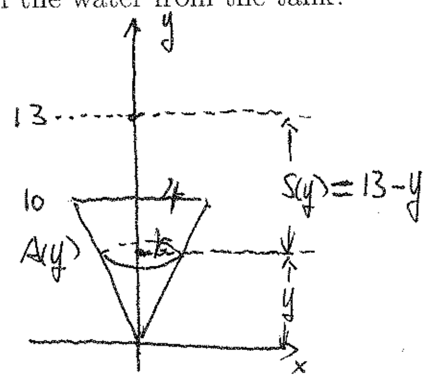


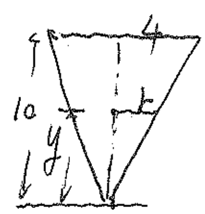
$y = e^x \Leftrightarrow x = \ln y$. $\quad A(y) = \pi R^2 = \pi [\ln y]^2$

$$\boxed{V = \int_1^2 \pi \cdot [\ln y]^2\, dy}$$

Q2[*Sec5.4, Water-Pumping*] A conical water tank with a top diameter of 8 feet and height of 10 feet is standing at ground level as shown in the sketch below. Water weighing 60 pounds per cubic foot is pumped from the tank to an outlet $\boxed{\text{3 feet above}}$ the top of the tank. If the tank is full, how many foot-pounds of work are required to pump all of the water from the tank?







Similar triangle:

$\frac{r}{y} = \frac{4}{10} \Rightarrow r = \frac{2}{5} y$

$A(y) = \pi \cdot r^2 = \pi \left(\frac{2}{5} y\right)^2$

$$\text{Work} = \int_0^{10} 6 \cdot S(y) \cdot A(y)\, dy$$

$$= \int_0^{10} 60 \cdot (13 - y) \cdot \pi \cdot \left(\frac{2}{5} y\right)^2 dy$$

$$= 60 \cdot \pi \cdot \frac{4}{25} \int_0^{10} (13 - y) \cdot y^2\, dy$$

$$= \frac{48}{5} \pi \cdot \int_0^{10} 13y^2 - y^3\, dy$$

$$= \frac{48}{5} \pi \cdot \left[13 \cdot \frac{1}{3} y^3 - \frac{1}{4} \cdot y^4\right] \Big|_0^{10}$$

$$= \boxed{\frac{48}{5} \pi \cdot \left[\frac{13}{3} \cdot 1000 - \frac{1}{4} \cdot 10000\right]}$$

Q3[*Sec6.1, derivative formula for inverse functions*] Let $f(x) = x\ln x + x^2 - 1, x > 0$. Find $(f^{-1})'(0)$. *Hint:* $f(1) = 0$.

$$(f^{-1})'(a) = \frac{1}{f'[f^{-1}(a)]} \Rightarrow (f^{-1})'(0) = \frac{1}{f'[f^{-1}(0)]} = \frac{1}{f'(1)} = \boxed{\frac{1}{3}}$$

$$\left.\begin{array}{l} f(1)=0 \Rightarrow f^{-1}(0)=1 \\ f'(x) = (x\cdot\ln x + x^2 - 1)' = 1\cdot\ln x + x\cdot\frac{1}{x} + 2x \end{array}\right\} f'(1) = \ln 1 + 1 + 2 = 3$$

Q4[*Sec6.2-6.4, exp/log functions*] Find the derivative of the following functions.

(a)[**exp-differential rule**]

$f(x) = (\sec x)^{\ln(x^2+1)}$

$$\ln f(x) = \ln(\sec x)^{\ln(x^2+1)} = [\ln(x^2+1)]\cdot[\ln\sec x]$$

Take derivatives both sides.

$$\frac{f'(x)}{f(x)} = \frac{2x}{x^2+1}\cdot\ln\sec x + \ln(x^2+1)\cdot\frac{1}{\sec x}\cdot\tan x\cdot\sec x.$$

$$= \frac{2x}{x^2+1}\cdot\ln\sec x + \ln(x^2+1)\cdot\tan x.$$

$$\boxed{f'(x) = \sec x^{\ln(x^2+1)}\cdot\left[\frac{2x}{x^2+1}\cdot\ln\sec x + \ln(x^2+1)\cdot\tan x\right]}$$

(b)[**log-property**]

$f(x) = \ln\left(\frac{x}{\tan^{-1}x}\right) = \ln x - \ln\tan^{-1}x$

$$\boxed{f'(x) = \frac{1}{x} - \frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}}$$

Q5[*Sec6.5/9.3, initial value problem*] Consider the following differential equation

$$\sec x\,\frac{dy}{dx} - \sqrt{y} = 0,\ y(0) = 4$$

Find the solution of $y = y(x)$

$$\sec x\cdot\frac{dy}{dx} = \sqrt{y}$$

$$\Leftrightarrow \frac{1}{\sqrt{y}}dy = \frac{1}{\sec x}dx.$$

$$\Leftrightarrow \frac{1}{\sqrt{y}}dy = \cos x\cdot dx.$$

$$\Leftrightarrow \int\frac{1}{\sqrt{y}}dy = \int\cos x\cdot dx$$

$$\Rightarrow 2\sqrt{y} = \sin x + C.$$

$y(0)=4 \Leftrightarrow x=0, y=4.$

$2\sqrt{4} = \sin 0 + C$

$\Rightarrow C = 2\sqrt{4} = 4.$

Therefore, $2\sqrt{y} = \sin x + 4.$

$$\Rightarrow \boxed{y = \left(\frac{1}{2}\sin x + 2\right)^2}$$

Q6[*Sec6.8/10.1, L'Hospital's Rule*] Evaluate the following limits.

(a)[∞^0**-type**]

$$\lim_{n\to\infty} \sqrt[n]{n^2+1} = \lim_{n\to\infty} e^{\ln (n^2+1)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln(n^2+1)}{n}} = e^0 = \boxed{1}.$$

$$\lim_{n\to\infty} \frac{\ln(n^2+1)}{n} \overset{l'Hosp}{=\!=} \lim_{n\to\infty} \frac{\frac{1}{n^2+1}\cdot 2n}{\cancel{n'}1} = \lim_{n\to\infty} \frac{2n}{n^2+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{l'Hosp}{=\!=} \lim_{n\to\infty} \frac{2}{2n} = 0$$

(b)[$\infty \cdot 0$**-type**] $\ln(\cos 0) = \ln 1 = 0$

$$\lim_{n\to\infty} n \ln\left(\cos\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln(\cos\frac{1}{n})}{\frac{1}{n}}$$

$$\overset{l'Hosp}{=\!=} \lim_{n\to\infty} \frac{\frac{1}{\cos(\frac{1}{n})}\cdot(-\sin(\frac{1}{n}))\cdot\cancel{(-\frac{1}{n^2})}}{\cancel{-\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{-\sin(\frac{1}{n})}{\cos(\frac{1}{n})} = \frac{-\sin 0}{\cos 0} = \frac{-0}{1} = \boxed{0}$$

(c)[∞/∞**-type or leading term rule**]

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{2n^4+2}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{\sqrt{2n^4+2}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{2n^4+2}{n^4}}} \quad n^2 = \sqrt{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2+\frac{2}{n^4}}} = \boxed{\frac{1}{\sqrt{2}}}$$

OR $\lim_{n} \frac{n^2}{\sqrt{2n^4+2}} = \lim_{n} \sqrt{\frac{n^4}{2n^4+2}} = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}}$

$$\lim_{n} \frac{n^4}{2n^4+2} = \lim_{n\to\infty} \frac{4\cancel{n^3}}{2\cdot 4\cancel{n^3}} = \frac{1}{2}$$

Q6[*Sec7.1-7.2, integration by parts and trig-integral*] Evaluate the following integrals.

(a) *[Sec7.1,IBP for* $\underbrace{\textit{polynomial}}_{u}\times\underbrace{\sin/\cos/exp}_{dv}$*-type]*

$$\int \underbrace{(x+1)}_{u}\underbrace{\sin x \, dx}_{dv}$$

$u = x+1 \quad du = dx$

$dv = \sin x \cdot dx, \quad V = -\cos x$

$$= u \cdot v - \int v \cdot du$$

$$= (x+1)(-\cos x) - \int (-\cos x)\, dx$$

$$= -(x+1)\cdot\cos x + \int \cos x \cdot dx$$

$$= \boxed{-(x+1)\cdot\cos x + \sin x + C}$$

(b) *[Sec7.1,IBP for* $\underbrace{\ln/\tan^{-1}/\sin^{-1}}_{u}$*-type]*

$u = \ln x, \quad dv = (2x+1)\, dx.$

$du = \frac{1}{x} dx, \quad v = (x^2+x)$

$$\int \underbrace{(2x+1)}_{dv}\underbrace{\ln x}_{u} \, dx$$

$$= \ln x \cdot (x^2+x) - \int (x^2+x)\cdot\frac{1}{x}\, dx$$

$$= \ln x \cdot (x^2+x) - \int (x+1)\, dx$$

$$= \boxed{\ln x \cdot (x^2+x) - \left(\frac{1}{2}x^2+x\right) + C}$$

(c) *[Sec7.2, Odd/Even rule for* $\sin - \cos$*-type]*

$$\int_0^{\pi/6} (2+\cos\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4 + 4\cos\theta + \cos^2\theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4 + 4\cdot\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{9}{2} + 4\cos\theta + \frac{1}{2}\cdot\cos 2\theta \, d\theta$$

$$= \frac{9}{2}\theta + 4\sin\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta \Big|_0^{\frac{\pi}{6}} = \frac{9}{2}\cdot\frac{\pi}{6} + 4\sin\frac{\pi}{6} + \frac{1}{4}\sin\frac{\pi}{3} = \boxed{\frac{3\pi}{4} + 2 + \frac{\sqrt{3}}{8}}$$

Q7[*Sec7.3-7.4, trig-sub and partial fraction decomposition*] Evaluate the following integrals.

(a) ***[U-Sub VS trig-sub]***

$$\int x^3\sqrt{1-x^2}\,\mathrm{d}x$$

u-Sub: $u = 1-x^2 \Rightarrow x^2 = 1-u$

$du = -2x\cdot dx$.

$= \int x^2\cdot\sqrt{1-x^2}\cdot x\cdot dx$.

$= \int (1-u)\cdot\sqrt{u}\cdot\frac{du}{-2}$

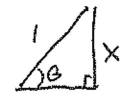
$= \frac{1}{-2}\int u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= \frac{1}{2}\cdot\left(\frac{2}{3}\cdot u^{\frac{3}{2}} - \frac{2}{5}\cdot u^{\frac{5}{2}}\right) + C$

$= \boxed{-\frac{1}{3}\cdot(1-x^2)^{\frac{3}{2}} + \frac{1}{5}(1-x^2)^{\frac{5}{2}} + C}$

Trig-Sub: $x = \sin\theta$, $dx = \cos\theta\cdot d\theta$, $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$= \int \sin^3\theta\cdot\sqrt{1-\sin^2\theta}\cdot\cos\theta\cdot d\theta$



$\cos\theta = \sqrt{1-x^2}$

$= \int \sin^3\theta\cdot\cos\theta\cdot\cos\theta\, d\theta$

$= \int \sin^3\theta\cdot\cos^2\theta\cdot d\theta$. ODD rule: $u = \cos\theta$

$du = -\sin\theta\cdot d\theta$

$= \int \sin^3\theta\cdot u^2\cdot\frac{du}{-\sin\theta}$

$= \int -\sin^2\theta\cdot u^2\cdot du \qquad -\sin^2\theta = \cos^2\theta - 1 = u^2 - 1$

$= \int (u^2-1)\cdot u^2\, du$

$= \int u^4 - u^2\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$

$= \boxed{\frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C}$

(b) ***[Sec7.3,*** $\sin^{-1}$ ***formula]***

$$\int \frac{100}{\sqrt{9-25x^2}}\,\mathrm{d}x$$

$= \int \frac{100}{\sqrt{9\cdot(1-\frac{25x^2}{9})}}dx$

$= \int \frac{100}{\sqrt{9}\cdot\sqrt{1-\frac{25x^2}{9}}}dx$.

$= \frac{100}{3}\cdot\int \frac{1}{\sqrt{1-(\frac{5x}{3})^2}}dx$.

$u = \frac{5x}{3}$, $du = \frac{5}{3}dx \Rightarrow dx = \frac{3}{5}du$

$= \frac{100}{3}\cdot\int \frac{1}{\sqrt{1-u^2}}\cdot\frac{3}{5}du$

$= \frac{100}{\cancel{3}}\cdot\frac{\cancel{3}}{5}\cdot\sin^{-1}u + C$

$= \boxed{20\cdot\sin^{-1}\left(\frac{5x}{3}\right) + C}$

(c) ***[Sec7.4, P.F.D. linear product type]***

$$\int \frac{2}{t^2-1}\,\mathrm{d}t$$

P.F.D. $\frac{2}{t^2-1} = \frac{2}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$. times (t^2-1) both sides.

$2 = A(t+1) + B\cdot(t-1)$

$t=-1 \qquad 2 = A\cdot 0 + B\cdot(-2) \Rightarrow B = -1$

$t=1 \qquad 2 = A\cdot 2 + B\cdot 0 \Rightarrow A = 1$.

$\int \frac{2}{t^2-1}dt = \int \frac{1}{t-1} - \frac{1}{t+1}\, dt = \boxed{\ln|t-1| - \ln|t+1| + C}$

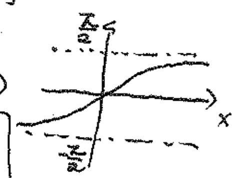
Q8[*Sec7.8, improper integral*] Determine whether each of the improper integral is convergent or divergent. Evaluate the improper integral if it is convergent.

(a)[**critical at** ∞]

$$\int_0^\infty \frac{4x^2}{1+4x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{4x^2}{1+4x^2}dx = \lim_{t\to\infty}\left[x-\tfrac12\tan^{-1}(2x)\right]\Big|_0^t = \lim_{t\to\infty}\left[t-\tfrac12\cdot\tan^{-1}(2t)\right]$$

$$= \infty - \tfrac12\cdot\tan^{-1}(\infty)$$

$$\boxed{= \infty - \tfrac{\pi}{4} = \infty \quad \text{Divergent.}}$$



$$\int \frac{4x^2}{1+4x^2}dx = \int \frac{(4x^2+1)-1}{1+4x^2}dx$$
$$= \int 1 - \frac{1}{1+4x^2}dx.$$
$$= \int 1\cdot dx - \int\frac{1}{1+4x^2}dx = x - \tfrac12\tan^{-1}(2x)+C$$

$$\int 1\cdot dx = x$$

$$\int \frac{1}{1+4x^2}dx \;\underset{du=2dx}{\overset{u=2x}{=}}\; \int \frac{1}{1+u^2}\frac{du}{2} = \tfrac12\tan^{-1}u = \tfrac12\tan^{-1}(2x)$$

(b)[**critical at finite**]

$$\int_0^{1/2}\frac{1}{(1-2x)^{1/3}}\,dx = \lim_{t\to(\frac12)^-}\int_0^t \frac{1}{(1-2x)^{\frac13}}dx = \lim_{t\to\frac12}\left[-3(1-2t)^{\frac23}+3\right] = -3\cdot(1-2\cdot\tfrac12)^{\frac23}+3$$

$$\boxed{= 0+3 = 3 \quad \text{Converges to } 3}$$

$$\int \frac{1}{(1-2x)^{\frac13}}dx \;\underset{du=-2\cdot dx}{\overset{u=1-2x}{=}}\; \int \frac{1}{u^{\frac13}}\cdot\frac{du}{-2}$$
$$= \frac{1}{-2}\int u^{-\frac13}du$$
$$= \frac{1}{-2}\cdot\frac{1}{-\frac13+1}\cdot u^{-\frac13+1}$$
$$= -3\cdot(1-2x)^{\frac23}$$

$$\int_0^t \frac{1}{(1-2x)^{\frac13}}dx = -3\cdot(1-2x)^{\frac23}\Big|_0^t = -3\cdot(1-2t)^{\frac23}+3\cdot(-1)^{\frac23} = -3(1-2t)^{\frac23}+3$$

Q9[*Sec8.1, arc-length formula*] Find the arc-length of the curve $y=\frac23(x+1)^{3/2}$ from $x=1$ to $x=2$.

$$y' = \frac23\cdot\frac32(x+1)^{\frac12} = (x+1)^{\frac12},\quad {y'}^2 = \left[(x+1)^{\frac12}\right]^2 = x+1$$

$$\text{Arc-length} = \int_1^2\sqrt{1+{y'}^2}\,dx = \int_1^2\sqrt{1+(x+1)}\,dx = \int_1^2\sqrt{x+2}\,dx \qquad \begin{matrix}u=x+2\\ du=dx.\end{matrix}$$
$$= \int u^{\frac12}du$$
$$= \frac23 u^{\frac32} = \frac23(x+2)^{\frac32}\Big|_1^2$$
$$\boxed{= \frac23\cdot4^{\frac32} - \frac23\cdot3^{\frac32}}$$
$$= \frac{16}{3} - \frac23\cdot3^{\frac32}$$

$$4^{\frac32} = (\sqrt4)^3 = 2^3 = 8$$

Q10[*Sec11.2*] Determine whether each of the series is convergent or divergent. Please show your work and name any test(s) that are used.

(a) ***[Sec11.2,n-th term test for DIV]***

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{3n^2}\right)$$

$\lim_{n\to\infty} \cos\left(\frac{1}{3n^2}\right) = \cos\left(\frac{1}{\infty}\right) = \cos 0 = 1 \neq 0$

According to $\boxed{\text{n-th term test for divergence}}$,

$\sum \cos\left(\frac{1}{3n^2}\right)$ is $\boxed{\text{divergent.}}$

(b) ***[Sec11.4,(limit) Comparison Test]***

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{3n^2}\right)$$

Let $a_n = \sin\left(\frac{1}{3n^2}\right)$, $b_n = \frac{1}{3n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{3n^2}\right)}{\frac{1}{3n^2}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{3n^2}\right)\cdot\frac{-2}{3n^3}}{\frac{-2}{3n^3}} = \lim_{n\to\infty} \cos\left(\frac{1}{3n^2}\right) = \cos 0 = 1 \neq 0$

$\sum b_n = \sum \frac{1}{3n^2}$ is a p-Series with $p=2>1$, convergent. Therefore, $\boxed{\sum \sin\left(\frac{1}{3n^2}\right) \text{ is also convergent.}}$

(c) ***[Sec11.4,(limit) Comparison Test]***

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^4+2}}$$, $a_n = \frac{n}{\sqrt{2n^4+2}}$, $b_n = \frac{n}{\sqrt{2n^4}} = \frac{n}{\sqrt{2}\cdot n^2} = \frac{1}{\sqrt{2}\cdot n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{2n^4+2}}\cdot\sqrt{2}\cdot n = \lim_{n\to\infty} \frac{\sqrt{2}\cdot n^2}{\sqrt{2n^4+2}} = \lim_{n\to\infty} \frac{\sqrt{2}}{\sqrt{2+\frac{2}{n^4}}} = \frac{\sqrt{2}}{\sqrt{2+0}} = 1 \neq 0$

$\sum b_n = \sum \frac{1}{\sqrt{2}n}$ is divergent. (p-Series with $p=1$) $\Rightarrow \boxed{\sum a_n = \sum \frac{n}{\sqrt{3n^4+2}} \text{ is divergent}}$.

(d) ***[Sec11.6, Ration Test]***

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$ $a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n} = \frac{1}{2}\cdot\frac{n+1}{n}$.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n+1}{n} = \frac{1}{2} < 1$.

$\boxed{\text{According to Ratio Test, } \sum \frac{n}{2^n} \text{ is convergent.}}$

Q11[*Sec11.6,11.8, ratio test for the radius of convergence of power series*] Consider the following power series. Find its center and radius of convergence.

$$\sum_{n=1}^{\infty} \frac{3^n(x+5)^n}{n+1}$$

Center : $x+5=0 \Leftrightarrow x=-5$. Center is $\boxed{-5}$

Apply Ratio Test to $a_n = \frac{3^n(x+5)^n}{n+1}$, $a_{n+1} = \frac{3^{n+1}\cdot(x+5)^{n+1}}{n+2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+5)^{n+1}}{n+2}\cdot\frac{n+1}{3^n(x+5)^n}\right| = \lim_{n\to\infty}\frac{3(n+1)}{n+2}\cdot|x+5|$$

$$= 3\cdot|x+5|$$

According to Ratio Test, the power series is convergent if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\cdot|x+5| < 1$

i.e. $3|x+5| < 1 \Leftrightarrow |x+5| < \boxed{\frac{1}{3}}$

The radius of convergence is $\boxed{R = \frac{1}{3}}$

Q12[*Sec11.9/11.10, power series representation and its integral*]

$$\text{Let } F(x) = \int_0^x \frac{2}{2+t}dt.$$

Find the first four non-zero terms of the Maclaurin series for $F(x)$ and $F(-x)$.

Expand $\frac{2}{2+t}$ first: $\frac{2}{2+t} = \frac{1}{1+\frac{t}{2}} = \frac{1}{1-(-\frac{t}{2})} = \sum_{n=0}^{\infty}\left(-\frac{t}{2}\right)^n$

$$= 1 - \frac{t}{2} + \left(\frac{-t}{2}\right)^2 + \left(\frac{-t}{2}\right)^3 + \cdots$$

$$= 1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots$$

$$F(x) = \int_0^x \frac{2}{2+t}dt = \int_0^x 1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots dt$$

$$= \left(t - \frac{1}{4}t^2 + \frac{1}{4\cdot 3}\cdot t^3 - \frac{1}{8\cdot 4}\cdot t^4 + \cdots\right)\Big|_0^x$$

$$\boxed{F(x) = x - \frac{1}{4}x^2 + \frac{1}{12}\cdot x^3 - \frac{1}{32}\cdot x^4 + \cdots}$$

$$F(-x) = -x - \frac{1}{4}(-x)^2 + \frac{1}{12}(-x)^3 - \frac{1}{32}\cdot(-x)^4 + \cdots$$

$$\boxed{F(-x) = -x - \frac{1}{4}\cdot x^2 - \frac{1}{12}\cdot x^3 - \frac{1}{32}\cdot x^4 + \cdots}$$

Q12, Taylor series Let $f(x) = \cos(x)$. Consider its Taylor series at $x = \pi/4$.

(a)[***Sec11.10, n-th degree Taylor polynomial, derivative table***] Find $T_3(x)$, the 3rd degree Taylor polynomial of $f(x)$ centered at $\pi/4$.

Derivative Table: (at $\frac{\pi}{4}$ up to order 3)

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{4})$
$n=0$	$\cos x$	$\frac{\sqrt{2}}{2}$
$n=1$	$-\sin x$	$-\frac{\sqrt{2}}{2}$
$n=2$	$-\cos x$	$-\frac{\sqrt{2}}{2}$
$n=3$	$\sin x$	$\frac{\sqrt{2}}{2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

$$T_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3$$

$$\boxed{T_3(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{2}\cdot\frac{1}{3!}\cdot\left(x-\frac{\pi}{4}\right)^3}$$

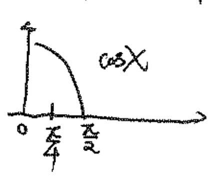
(b)[***Sec11.11, Taylor's Inequality***] Use Taylor's Inequality to estimate the maximum possible error in approximating $f(x)$ by $T_3(x)$ for $x \in [0, \pi/2]$.

$$f(x) - T_3(x) = R_3(x)$$

$$|R_3(x)| \le \frac{M}{(3+1)!}\cdot\left|x-\frac{\pi}{4}\right|^{3+1} \quad \text{for } x \in \left[0, \frac{\pi}{2}\right] \Leftrightarrow \left|x-\frac{\pi}{4}\right| \le \frac{\pi}{4}$$

where M = maximum of $|f^{(3+1)}(x)|$ for $x \in [0, \frac{\pi}{2}]$

$$f^{(4)}(x) = \cos x$$



therefore, $M = 1$

we have

$$\boxed{|R_3(x)| \le \frac{M}{4!}\cdot\left|x-\frac{\pi}{4}\right|^4 \le \frac{1}{4!}\cdot\left|\frac{\pi}{4}\right|^4}$$

Q13[*Sec10.1,10.2, derivative formula for parametric equations*] Find the tangent line to the parametric curve

$$x(t) = \ln\left(\frac{t+1}{t}\right), \quad y(t) = \sqrt{t+3} \qquad \text{at} \qquad t = 1$$

$t=1$, $x(1)=\ln 2$, $y(1)=\sqrt{4}=2$

$x'(t)=\frac{dx(t)}{dt} = \left(\ln(t+1)-\ln t\right)' = \frac{1}{t+1}-\frac{1}{t}$, $x'(1)=\frac{1}{2}-1=-\frac{1}{2}$

$y'(t)=\frac{dy}{dt} = \left(\sqrt{t+3}\right)' = \frac{1}{2}\cdot\frac{1}{\sqrt{t+3}}$, $y'(1)=\frac{1}{2}\cdot\frac{1}{\sqrt{4}}=\frac{1}{4}$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$

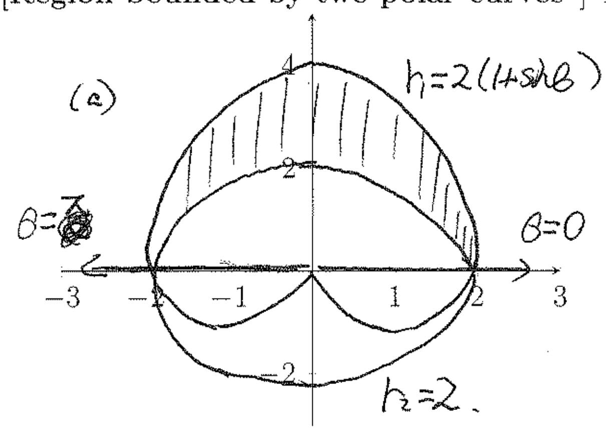
Tangent Line: $\boxed{y = 2-\frac{1}{2}(x-\ln 2) = -\frac{1}{2}x+2+\frac{1}{2}\ln 2}$

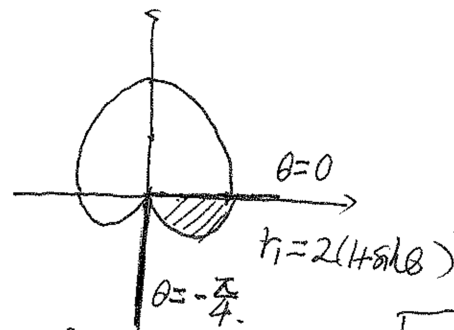
Q14[*Sec10.3,10.4, polar curve*] Consider the following polar curves given by $r_1 = 2(1+\sin\theta)$, and $r_2 = 2$.

(a)[**Circle and Cardioid in polar coordinates**] Sketch both curves r_1 and r_2.

(b)[**Area in polar coordinates**] Set up the integral for the area of the region bounded by r_1, $\theta = 0$ and $\theta = -\frac{\pi}{4}$. (Do not evaluate.)

(c)[**Region bounded by two polar curves**] Find the area of the region inside r_1 and outside r_2.





$$\text{Area} = \int_{-\frac{\pi}{4}}^{0} \frac{1}{2}\left[2(1+\sin\theta)\right]^2 d\theta = \boxed{\int_{-\frac{\pi}{4}}^{0} 2(1+\sin\theta)^2 d\theta}$$

(c)
$$\text{Area} = \int_0^{\pi} \frac{1}{2}\cdot r_1^2 - \frac{1}{2}\cdot r_2^2 \, d\theta$$
$$= \int_0^{\pi} \frac{1}{2}\left[2(1+\sin\theta)\right]^2 - \frac{1}{2}\cdot 2^2 \, d\theta$$
$$= \int_0^{\pi} 2(1+\sin\theta)^2 - 2 \, d\theta$$
$$= \int_0^{\pi} 2+4\sin\theta+2\sin^2\theta-2 \, d\theta$$
$$= \int_0^{\pi} 4\sin\theta + 2\cdot\frac{1-\cos 2\theta}{2} \, d\theta$$
$$= \int_0^{\pi} 4\sin\theta + 1 - \cos 2\theta \, d\theta$$
$$= -4\cos\theta + \theta - \frac{1}{2}\cdot\sin 2\theta \Big|_0^{\pi}$$
$$= -4\cos\pi + \pi - 0 - (-4\cos 0 + 0 - 0)$$
$$= 4 + \pi + 4$$
$$= \boxed{8+\pi}$$