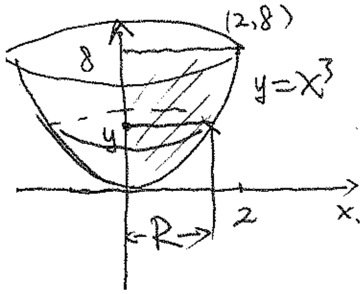
Solution Section 61.

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Consider the region in the xy-plane above the curve $y = x^3$, below the line $y = 8$, and to the right of the y-axis.

   (a) (9 points) Sketch this region, and compute the volume of the solid obtained by rotating this region around the y-axis



y-axis : integral is in terms of y

$R = x = y^{\frac{1}{3}} \quad (\Leftarrow y = x^3)$, $A(y) = \pi \cdot R^2$

$$V = \pi \int_0^8 R^2\, dy = \pi \int_0^8 y^{\frac{2}{3}}\, dy$$

$$= \pi \cdot \frac{1}{\frac{5}{3}} y^{\frac{2}{3}+1} \Big|_0^8$$

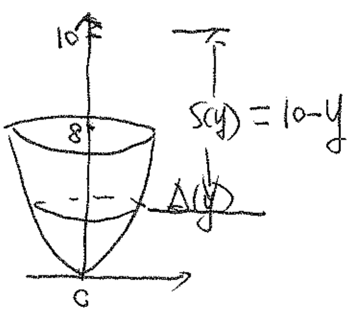
$$= \pi \cdot \frac{3}{5} \cdot y^{\frac{5}{3}} \Big|_0^8$$

$$= \pi \cdot \frac{3}{5} \cdot 8^{\frac{5}{3}} = \boxed{\pi \cdot \frac{96}{5}}$$

Hint. $8 = 2^3$

$(8^{\frac{5}{3}}) = 2^{3 \cdot \frac{5}{3}} = 2^5 = 32$

   (b) (9 points) Assume the solid in part (a) is a storage tank (with all lengths in feet), full of a liquid weighing 4 pounds per cubic foot. Compute the work (in ft-lbs) needed to pump all the liquid to the 2 feet above of the top of the tank.



$\sigma = 4$, $S(y) = 10 - y$, $A(y) = \pi \cdot y^{\frac{2}{3}}$ $(\pi \cdot R^2,\ R = y^{\frac{1}{3}})$

$$W = \int_0^8 \sigma \cdot S(y) \cdot A(y)\, dy.$$

Caution: the range of the integral is from $y=0$ to $y=8$ (NOT to $y=10$)

$$= \int_0^8 4 \cdot (10-y) \cdot \pi \cdot y^{\frac{2}{3}}\, dy$$

$$= 4\pi \cdot \int_0^8 10 \cdot y^{\frac{2}{3}} - y^{\frac{5}{3}}\, dy$$

$$= 4\pi \cdot \left(10 \cdot \frac{3}{5} \cdot y^{\frac{5}{3}} - \frac{3}{8} \cdot y^{\frac{8}{3}}\right)\Big|_0^8$$

$$\boxed{= 4\pi \cdot \left(6 \cdot 8^{\frac{5}{3}} - \frac{3}{8} \cdot 8^{\frac{8}{3}}\right)}$$

$$= 4\pi \cdot \left(6 \cdot 32 - \frac{3}{8} \cdot 16 \cdot 16\right) = \boxed{384\pi \text{ ft-lb}}$$

Hint: $8^{\frac{1}{3}} = 2$

$8^{\frac{5}{3}} = 2^5 = 32$

$8^{\frac{8}{3}} = 2^8 = 16 \cdot 16$

2. Evaluate the following limits.

(a) (6 points) $\lim_{x \to 0} \frac{x^2}{\ln(\sec(3x))}$

Hint: $\sec 0 = 1$, $\ln \sec 0 = \ln 1 = 0$. $\frac{0}{0}$ type.

$$\overset{\text{1st l'Hosp}}{=} \lim_{x \to 0} \frac{2x}{\frac{1}{\cancel{\sec(3x)}} \tan(3x) \cdot \cancel{\sec(3x)} \cdot 3}$$

$$= \lim_{x \to 0} \frac{2x}{3 \cdot \tan(3x)}$$

$$\overset{\text{2nd l'H}}{=} \lim_{x \to 0} \frac{2}{3 \cdot \sec^2(3x) \cdot 3}$$

$\sec 0 = 1$.

$$= \frac{2}{3 \cdot 1 \cdot 3} = \boxed{\frac{2}{9}}$$

(b) (6 points) $\lim_{x \to \infty} \frac{\ln x}{\log_5(2x)}$

Hint: $\log_a x = \frac{\ln x}{\ln a}$

$$= \lim_{x \to \infty} \frac{\ln x}{\frac{\ln(2x)}{\ln 5}}$$

$$= \lim_{x \to \infty} \ln 5 \cdot \frac{\ln x}{\ln 2x}$$

$$\overset{\text{l'H}}{=} \lim_{x \to \infty} \ln 5 \cdot \frac{\frac{1}{x}}{\frac{1}{2x} \cdot 2}$$

$$= \boxed{\ln 5}$$

Remark: You can also use formula $(\log_a x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$, which leads to the same answer.

☆ (c) (6 points) $\lim_{x \to \infty} (2x)^{3/x}$

$$(2x)^{\frac{3}{x}} = e^{\ln(2x)^{\frac{3}{x}}} = e^{\frac{3}{x} \cdot \ln(2x)}$$

$$\lim_{x \to \infty} \frac{3 \ln(2x)}{x} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{3 \cdot \frac{1}{2x} \cdot 2}{1} = \lim_{x \to \infty} \frac{3}{x} = 0$$

$$\Rightarrow \lim_{x \to \infty} (2x)^{\frac{3}{x}} = e^{\lim_{x \to \infty} \frac{3}{x} \ln(2x)} = e^0 = \boxed{1}$$

3. Find the derivative for each of the following functions.

(a) (4 points) $y = 2\sin(6xe^{3x})$

$$y' = 2\cos(6x \cdot e^{3x}) \cdot [6x \cdot e^{3x}]'$$

$$= \boxed{2\cos(6x \cdot e^{3x}) \cdot [6 \cdot e^{3x} + 6x \cdot e^{3x} \cdot 3]}$$

product rule

(or) $= 2\cos(6x \cdot e^{3x})(6 + 18x) \cdot e^{3x}$

(b) (4 points) $g(t) = e^{t^2+2\sqrt{t}}$

$$g'(t) = e^{t^2+2\sqrt{t}} (t^2 + 2\sqrt{t})'$$

$(\sqrt{t})' = (t^{\frac{1}{2}})' = \frac{1}{2} \cdot t^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{t}}$

$$= e^{t^2+2\sqrt{t}} \cdot (2t + 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{t}})$$

$$= \boxed{e^{t^2+2\sqrt{t}} \cdot (2t + \frac{1}{\sqrt{t}})}$$

(c) (5 points) $g(x) = 7^{\ln(x^2-3x+1)}$

Hint: $(a^x)' = \ln a \cdot a^x$

$$g'(x) = \ln 7 \cdot 7^{\ln(x^2-3x+1)} \cdot [\ln(x^2-3x+1)]'$$

$$= \boxed{\ln 7 \cdot 7^{\ln(x^2-3x+1)} \cdot \frac{1}{x^2-3x+1} \cdot (2x-3)}$$

chain rules

☆ (d) (5 points) $y(x) = x^{7\sin(2x)}$

$$\ln y = \ln x^{7\sin(2x)} = 7 \cdot \sin(2x) \cdot \ln x.$$

$$\frac{y'}{y} = 7 \cdot \cos(2x) \cdot 2 \cdot \ln x + 7 \cdot \sin(2x) \cdot \frac{1}{x}.$$

$$\Rightarrow \boxed{y' = x^{7\sin(2x)} \cdot \left[14\cos(2x) \cdot \ln x + \frac{7\sin(2x)}{x}\right]}$$

4. Evaluate the following integrals.

(a) (6 points) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ with $|x| < 1$.

$= \sin^{-1} x$. (anti-Derivative formula directly)

☆ (b) (6 points) $\int \sqrt{1-x^2}\,dx$ with $|x| \le 1$.

Trig Sub: $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$= \int \sqrt{1-\sin^2\theta} \cdot \cos\theta \cdot d\theta$

$= \int \sqrt{\cos^2\theta}\, \cos\theta \cdot d\theta$

$= \int \cos\theta \cdot \cos\theta \cdot d\theta$

$= \int \cos^2\theta \cdot d\theta$

$\overset{\text{D.A.F.}}{=} \int \frac{1+\cos(2\theta)}{2}\, d\theta$

$= \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$

$= \frac{\theta}{2} + \frac{1}{2} \cdot \frac{1}{2}\sin(2\theta) + C$

$= \frac{\theta}{2} + \frac{1}{4} \cdot 2\sin\theta \cdot \cos\theta + C$

$= \boxed{\frac{1}{2}\sin^{-1}(x) + \frac{1}{2} \cdot x \cdot \sqrt{1-x^2} + C}$

Hint: D.A.F.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-x^2}$

☆ (c) (6 points) $\int \frac{x^2+4}{x^3+x^2}\,dx$

P.F.D. $\frac{x^2+4}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$x^2+4 = Ax\cdot(x+1) + B(x+1) + Cx^2$

$x^2+4 = Ax^2 + Ax + Bx + B + Cx^2$

$x^2+4 = (A+C)\cdot x^2 + (A+B)\cdot x + B$

$\begin{cases} A+C=1 \\ A+B=0 \\ B=4 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=4 \\ C=5 \end{cases}$

$\int \frac{x^2+4}{x^3+x^2}\,dx = \int \frac{-4}{x}\,dx + \int \frac{4}{x^2}\,dx + \int \frac{5}{x+1}\,dx$

$= \boxed{-4\ln|x| + 4\cdot\left(-\frac{1}{x}\right) + 5\ln|x+1| + C}$

5. Solve the following initial value problems.

(a) (9 points) $\frac{dy}{dt} = te^t, \quad y(0) = 1$

(Seperate t and y): $dy = t \cdot e^t \cdot dt$

(integrate both sides): $\int dy = \int t e^t dt$ Hint: $\int dy = \int 1 \cdot dy = y$.

$y = \int t \cdot e^t dt$ $\boxed{IBP}$ type: Poly: t, e^t

$y = u \cdot v - \int v \cdot du$ $u = t$, $dv = e^t dt$

$y = te^t - \int e^t dt$ $du = dt$, $v = e^t$

$\boxed{y = t \cdot e^t - e^t + C}$

$y(0) = 1 \Rightarrow \begin{cases} t = 0 \\ y = 1 \end{cases} \Rightarrow 1 = 0 \cdot e^0 - e^0 + C \Rightarrow \boxed{C = 2}$.

$\boxed{y = t \cdot e^t - e^t + 2}$

(b) (9 points) $y''(x) = 12e^{2x} + 2, \quad y(0) = 4, \quad y'(0) = 5$

$y'(x) = \int y''(x) dx = \int 12e^{2x} + 2 \, dx = 12 \cdot \frac{1}{2} e^{2x} + 2x + C_1$

$y'(0) = 5 \Rightarrow \begin{cases} x = 0 \\ y' = 5 \end{cases} \Rightarrow 5 = 6 \cdot e^0 + 0 + C_1 \Rightarrow C_1 = -1$.

$\Rightarrow y'(x) = 6e^{2x} + 2x - 1$.

$y(x) = \int y'(x) dx = \int 6 \cdot e^{2x} + 2x - 1 \, dx$

$= 6 \cdot \frac{1}{2} e^{2x} + x^2 - x + C_2$

$y(0) = 4 \Rightarrow \begin{cases} x = 0 \\ y = 4 \end{cases} \Rightarrow 4 = 3 \cdot e^0 + 0 - 0 + C_2 \Rightarrow C_2 = 1$

$\Rightarrow \boxed{y(x) = 3e^{2x} + x^2 - x + 1}$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (7 points) Mark the correct answer for the integral given by $\int_{-a}^{a} \sin(\sin(x^3))dx$.

   A. 0 ✓
   B. ∞
   C. $2a$
   D. a
   E. None of the above.

$f(x) = \sin(\sin(x^3))$

$f(-x) = \sin(\sin(-x)^3) = -\sin(\sin(x^3)) = -f(x)$

$f(x)$ is ODD. $\Rightarrow \int_{-a}^{a} f(x)dx = 0$

7. (7 points) Compute the improper integral and mark the correct answer $\int_3^6 \frac{1}{\sqrt{x-3}}dx$

   bad point.

   A. 0
   B. ∞
   C. $2\sqrt{3}$ ✓
   D. $\sqrt{3}$
   E. None of the above.

$\int_t^6 \frac{1}{\sqrt{x-3}} dx$

$= \int_t^6 (x-3)^{-\frac{1}{2}} dx$. $u = x-3$

$= \int u^{-\frac{1}{2}} du$

$= 2u^{\frac{1}{2}} = 2(x-3)^{\frac{1}{2}}\Big|_t^6 = 2(6-3)^{\frac{1}{2}} - 2(t-3)^{\frac{1}{2}}$

$= 2\sqrt{3} - 2\sqrt{t-3} \xrightarrow{t\to 3} \boxed{2\sqrt{3} - 0}$

8. (7 points) Compute the $\frac{d}{dx}\ln(\sinh(7x))$ and mark the correct answer

   A. $7\frac{\cosh x}{\sinh x}$ ✓
   B. $\frac{\sinh x}{\cosh x}$
   C. $\frac{\cos x}{\sinh x}$
   D. 0
   E. None of the above.

$= \frac{1}{\sinh(7x)} [\cosh(7x)] \cdot 7$

12. (7 points) It took 1600 J of work to stretch a spring from its natural length of 3 m to a length of 8 m. The spring constant k is

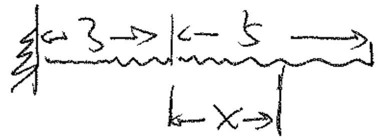
A. 320

B. $\frac{640}{11}$

C. 200

D. 128

E. None of the above.



$F = k \cdot x.$

$W = \int_0^5 k \cdot x\, dx = \frac{1}{2} \cdot k x^2 \Big|_0^5$

$W = \frac{1}{2} k \cdot 25$

$W = 1600 \Rightarrow 1600 = \frac{1}{2} \cdot k \cdot 25 \Rightarrow k = \boxed{128}$

13. (7 points) Let $P(t)$ grams be the population of bacteria in a tank at t hours. Suppose the population doubles every 3 hours, and $P(1) = 2$. Find $P(t)$.

A. $P(t) = 2^t$

B. $P(t) = 2^{(t+2)/3}$

C. $P(t) = 2 \cdot 2^{t/3}$

D. $P(t) = 2 \cdot e^{2t}$

E. None of the above.

$P(t) = C \cdot e^{kt}$

$P(1) = C \cdot e^k = 2$. after 3 hours, P doubles.

$P(4) = C \cdot e^{4k} = 4$ i.e. $P(4) = 4$.

$\Rightarrow \frac{Ce^{4k}}{Ce^k} = \frac{4}{2} \Rightarrow e^{3k} = 2.$ $e^k = 2^{\frac{1}{3}}$

$\Rightarrow C 2^{\frac{1}{3}} = 2 \Rightarrow C = 2^{1-\frac{1}{3}} = 2^{\frac{2}{3}}$

$P = 2^{\frac{2}{3}} \cdot 2^{\frac{t}{3}}$

$= 2^{\frac{t+2}{3}}$

14. (7 points) When evaluating the integral $\int \frac{x^3}{\sqrt{x^2+36}}\, dx$, which of the following would be the best substitution for x?

A. $x = 36 \sin t$

B. $x = 6 \sin t$

C. $x = 6 \tan t$

D. $x = 6 \sec t$

E. None of the above.

$x^2 + 36 = x^2 + 6^2$ $\tan^2\theta + 1 = \sec^2\theta$

$\boxed{x = 6 \cdot \tan\theta}$

9. (7 points) Mark the correct simplification for the expression $\dfrac{e^{3\ln(x-1)^2}}{\ln e^{(x-1)^4}}$

A. $\frac{e^3 \ln(x-1)^2}{4\ln e^{(x-1)}}$

B. $\frac{2e^3\ln(x-1)}{4(x-1)}$

C. $\frac{e^3}{(x-1)^2}$

✓D. $(x-1)^2$

E. None of the above.

$$= \frac{\left[e^{\ln(x-1)^2}\right]^3}{\ln e^{(x-1)^4}}$$

$$= \frac{\left[(x-1)^2\right]^3}{(x-1)^4} = \frac{(x-1)^{2\cdot 3}}{(x-1)^4} = \frac{(x-1)^6}{(x-1)^4}$$

$$= \boxed{(x-1)^2}$$

10. (7 points) If $f(3) = 2$, $f'(3) = 7$, then $(f^{-1})'(2)$ is

A. $\frac{3}{2}$

B. $\frac{7}{2}$

C. $\frac{2}{7}$

✓D. $\frac{1}{7}$

E. None of the above.

$$f^{-1}(a) = \frac{1}{f'[f^{-1}(a)]}$$

$a=2$.

$$f(3)=2 \Rightarrow f^{-1}(2)=3 \Rightarrow f^{-1}(2) = \frac{1}{f'[f^{-1}(2)]} = \frac{1}{f'[3]} = \frac{1}{7}.$$

11. (7 points) The partial fraction decomposition of $\dfrac{3}{(x-1)(x+1)}$ can be written in the form $\dfrac{A}{x-1} + \dfrac{B}{x+1}$ where

A. $A = 2, B = 1$.

✓B. $A = \frac{3}{2}$, $B = -\frac{3}{2}$.

C. $A = -\frac{3}{2}$, $B = \frac{3}{2}$.

D. $A = 3(x-1), B = 2(x+1)$.

E. None of the above.

$$\frac{3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1},$$

$$3 = A(x+1) + B(x-1)$$

$$x=1 \Rightarrow 3 = A\cdot 2 + B\cdot 0 \Rightarrow \boxed{A = \frac{3}{2}}$$

$$x=-1 \Rightarrow 3 = A\cdot 0 + B(-2) \Rightarrow \boxed{B = -\frac{3}{2}}$$