Solution

Name: _____

Section: _____   **Recitation Instructor:** _____

INSTRUCTIONS

- Fill in your name, etc. on this first page.

- Without fully opening the exam, check that you have pages 1 through 12.

- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.

- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.

- If you have any questions please raise your hand and a proctor will come to you.

- You will be given exactly 90 minutes for this exam.

- Remove and utilize the formula sheet provided to you at the end of this exam.

- This is a practice exam. The actual exam may differ significantly from this practice exam because there are many varieties of problems that can test each concept.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.

- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).

- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.

- No scratch paper is permitted. If you need more room use the back of a page.

- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the
above instructions and statements
regarding academic honesty: _____
                                                    SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Determine whether the series is convergent, or divergent. State which test you used and all necessary conditions to use the test.

    (a) (3 points) $\sum_{n=1}^{\infty} \dfrac{\cos(n\pi)}{n}$

Notice that $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, $\cdots$

$\Rightarrow \cos(n\pi) = (-1)^n$, $n = 1, 2, \cdots$

Alternating Series : $\sum \dfrac{(-1)^n}{n}$.

$\lim \dfrac{1}{n} = 0$ and $\dfrac{1}{n}$ is decreasing $\Rightarrow \sum \dfrac{\cos(n\pi)}{n}$ is convergent.

    (b) (3 points) $\sum_{n=1}^{\infty} \dfrac{n^2 + n + 5}{\left(\frac{n^2}{2}\right)}$

nth term test for div.

~~limit comparison test a_n~~

$\lim_{n \to \infty} \dfrac{\frac{n^2 + n + 5}{n^2}}{2} = \lim_{n \to \infty} \dfrac{n^2}{\frac{n^2}{2}} = 2 \neq 0$

divergent.

    (c) (4 points) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n + 5^n}$

Alternating : $b_n = \dfrac{1}{n + 5^n}$ is decreasing and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \dfrac{1}{n + 5^n} = 0$

$\sum \dfrac{(-1)^n}{n + 5^n}$ is convergent due to Alt. Series Test.

    (d) (4 points) $\sum_{n=1}^{\infty} \dfrac{9^n}{1 + 15^n}$

Comparison Test.

$a_n = \dfrac{9^n}{1 + 15^n} < \dfrac{9^n}{15^n} = \left(\dfrac{9}{15}\right)^n$ $r = \dfrac{9}{15} = \dfrac{3}{5} < 1$. $\sum \left(\dfrac{9}{15}\right)^n$ is conv

$\sum a_n$ is convergent due to comparison test. (Ratio Test also works)

    (e) (4 points) $\sum_{n=1}^{\infty} \dfrac{\ln n}{n}$

Comparison Test: $\ln n > 1$ for $n \geq 3$, $\dfrac{\ln n}{n} > \dfrac{1}{n}$.

$\sum_{n=3}^{\infty} \dfrac{\ln n}{n} \geq \sum_{n=3}^{\infty} \dfrac{1}{n}$. is divergent ~~due~~ (p-Series $p=1$)

$\Rightarrow \sum_{n=3}^{\infty} \dfrac{\ln n}{n}$ is divergent. $\Rightarrow \sum_{n=1}^{\infty} \dfrac{\ln n}{n}$ is divergent

(integral test also works)

2. Find the series' radius and interval of convergence. *(do not need to test endpoints)*

(a) (6 points) $\sum_{n=1}^{\infty} \frac{nx^n}{n+5}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{n+1}}{n+6}}{\frac{(n)\cdot x^n}{n+5}} \right| = \lim_{n\to\infty} \frac{(n+1)}{(n+6)} \cdot \frac{(n+5)}{n} \cdot |x|$$

$$= |x| < 1.$$

$$\Leftrightarrow \quad -1 < x < 1$$

$$R = 1$$

(open) interval : $x \in (-1, 1)$

(b) (6 points) $\sum_{n=1}^{\infty} (2x)^n$

Geometric Series: $\quad |2x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{2}$

$$\Leftrightarrow \quad -\frac{1}{2} < x < \frac{1}{2}$$

$$R = \frac{1}{2}$$

(open) interval : $\left(-\frac{1}{2}, \frac{1}{2} \right)$

(c) (6 points) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |3x-2|$$

$$= |3x-2| < 1$$

$$R = \frac{1}{3}$$

$$\Leftrightarrow \quad |x - \frac{2}{3}| < \frac{1}{3}.$$

Interval $= \left(\frac{1}{3}, 1 \right)$

$$\Leftrightarrow \quad -\frac{1}{3} < x - \frac{2}{3} < \frac{1}{3}$$

$$\Leftrightarrow \quad -\frac{1}{3} + \frac{2}{3} < x < \frac{2}{3} + \frac{1}{3} = 1$$

$$\frac{1}{3}.$$

3. (18 points) Find the Taylor polynomial of degree 3 generated by $f(x) = \sqrt{x}$ centered at $a = 4$.

$$n = 3$$

Derivative Table of $\sqrt{x}$ at $a=4$ up to $n=3$

$n \downarrow$	$f^{(n)}(x)$	$f^{(n)}(4)$
$n=0$	$\sqrt{x} = x^{\frac{1}{2}}$	$4^{\frac{1}{2}} = 2$
$n=1$	$\frac{1}{2} \cdot x^{-\frac{1}{2}}$	$\frac{1}{2} \cdot 4^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{4^{\frac{1}{2}}} = \frac{1}{4}$
$n=2$	$\frac{1}{2} \cdot (-\frac{1}{2}) x^{-\frac{3}{2}}$	$-\frac{1}{4} \cdot 4^{-\frac{3}{2}} = -\frac{1}{4} \cdot \frac{1}{4^{\frac{3}{2}}} = -\frac{1}{4} \cdot \frac{1}{2^3} = -\frac{1}{4} \cdot \frac{1}{8}$
$n=3$	$\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) x^{-\frac{5}{2}}$	$-\frac{3}{8} 4^{-\frac{5}{2}} = -\frac{3}{8}(4^{\frac{1}{2}})^{-5} = -\frac{3}{8} \cdot 2^{-5}$

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!} \cdot (x-a)^3$$

$$= 2 + \frac{1}{4} \cdot (x-4) - \frac{\frac{1}{32}}{2} \cdot (x-4)^2 - \frac{\frac{3}{8} \cdot 2^{-5}}{3!} \cdot (x-4)^3$$

4. (18 points) Find the first four nonzero terms of the power series representation for the function

$$f(x) = \int_0^x t \sin(t) \, dt$$

$$t \sin t = t \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \right)$$

$$= t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots$$

$$f(x) = \int_0^x t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots \, dt$$

$$= \frac{1}{3} t^3 - \frac{1}{3!} \cdot \frac{1}{5} t^5 + \frac{1}{5!} \cdot \frac{1}{7} t^7 - \frac{1}{7!} \cdot \frac{1}{9} t^9 + \cdots \Big|_0^x$$

$$\boxed{\frac{1}{3} \cdot x^3 - \frac{1}{3!} \cdot \frac{1}{5} \cdot x^5 + \frac{1}{5!} \cdot \frac{1}{7} x^7 - \frac{1}{7!} \cdot \frac{1}{9} \cdot x^9 + \cdots}$$

5. (18 points) Consider the curve given by $\quad y = \dfrac{x^3}{6} + \dfrac{1}{2x}$

    (a) Write an integral that expresses the length of the arc of the curve on $[1,3]$.

$$y' = \frac{1}{6} \cdot 3x^2 + \frac{1}{2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{2} x^2 - \frac{1}{2x^2}$$

$$\text{Arc-L} = \int_1^3 \sqrt{1 + y'^2}\, dx = \int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\, dx .$$

$$= \int_1^3 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}\, dx \qquad = \int_1^3 \sqrt{1 + \frac{1}{4}x^4 - 2 \cdot \frac{1}{2}x^2 \cdot \frac{1}{2x^2} + \frac{1}{4x^4}}\, dx .$$

$$= \int_1^3 \frac{1}{2}x^2 + \frac{1}{2x^2}\, dx \qquad = \int_1^3 \sqrt{1 + \left(\frac{1}{2}x^2\right)^2 - \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{3}x^3 + \frac{1}{2} \cdot \frac{1}{-1} \cdot x^{-1} \Big|_1^3 \qquad = \int_1^3 \sqrt{\left(\frac{1}{2}x^2\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx$$

$$= \frac{1}{6} \cdot 3^3 - \frac{1}{2} \cdot 3^{-1} - \left(\frac{1}{6} \cdot 1^3 - \frac{1}{2} \cdot 1\right) = \frac{9}{2} - \frac{1}{6} - \frac{1}{6} + \frac{1}{2} = \boxed{\frac{14}{3}}$$

    (b) Evaluate the integral from (a) to find the length of the arc.

> **Multiple Choice**. Circle the best answer. No work needed.
> No partial credit available. No credit will be given for choices not clearly marked.

6. (7 points) Evaluate the series: $\sum_{n=1}^{\infty} (-1)^{n+1}(0.1)^n$

Geometric

A. 0

B. ∞

C. 1/11

D. $-1/11$

E. None of the above

$$a = (-1)^2 \cdot a| = a|$$

$$r = \textcircled{.} - a|$$

$$\frac{a}{1-r} = \frac{a|}{1-(-a|)} = \frac{a|}{1.1} = \frac{1}{11}$$

$$= (-1)\cdot(-1)^n \cdot (a|)^n$$

$$= (-1)\cdot(-a|)^n$$

$$= (-1)\cdot(-a|)\cdot(\textcircled{.}-a|)^{n-1}$$

$$= a| \cdot(-a|)^{n-1}$$

7. (7 points) What can be said about the series $\sum_{n=1}^{\infty} \sin(1/n)$?

$b_n = \frac{1}{n}$ $\sum \frac{1}{n}$ div.

A. The Comparison Test does not apply here because the hypothesis is not met.

B. The Comparison Test shows that it converges.

C. The Limit Comparison Test shows that it diverges.

D. The nth Term Test shows that it diverges .

E. None of the above

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$$

$$\overset{l'Hop}{=} \lim_{n} \frac{\cos(\frac{1}{n})\cdot(-\frac{1}{n^2})}{-\frac{1}{n^2}}$$

$$= \lim \cos(\frac{1}{n}) = 1.$$
$$\neq 0$$

8. (7 points) Which of the following is true about the series $\sum_{n=1}^{\infty} \left(\frac{-3n}{n+1}\right)^{5n}$?

A. The series doesn't converge.

B. The series converges absolutely, but doesn't converge.

C. The series converges, but doesn't converge absolutely.

D. The series converges and converges absolutely.

E. None of the above

$$a_n = \left(\frac{-3n}{n+1}\right)^{5n} = (-3)^{5n}\cdot\left[\left(\frac{n}{n+1}\right)^n\right]^5$$

$$\lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \neq 0$$

$$\lim (-3)^{5n} \text{ D.N.E.} \Rightarrow \lim a_n \text{ D.N.E.} \Rightarrow DIV$$

9. (7 points) Find a power series representation for the function $f(x) = \dfrac{5}{1-4x^2}$

    A. $5 + 4x^2 + 16x^4 + 64x^6 + \cdots$

    B. $5 - 4x^2 + 16x^4 - 64x^6 + \cdots$

    C. $5 + 20x^2 + 80x^4 + 320x^5 + \cdots$

    D. $5 + 20x^2 + 80x^4 + 320x^6 + \cdots$

    E. None of the above

$$= 5 \sum_{n=0}^{\infty} (4x^2)^n$$

$$= 5 \left(1 + 4x^2 + 16x^4 + 4^3 \cdot x^6 + \cdots \right)$$

$$= 5 + 20 \cdot x^2 + 80x^4 + 5 \cdot 4^3 \cdot x^6 + \cdots$$

10. (7 points) Determine the interval of convergence of the power series from the previous question

$$f(x) = \frac{5}{1-4x^2}$$

    A. $\left(-\frac{1}{4}, \frac{1}{4}\right)$

    B. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

    C. $\left(-\frac{1}{16}, \frac{1}{16}\right)$

    D. $(-\infty, \infty)$

    E. None of the above

$$|4x^2| < 1$$

$$\Rightarrow \quad x^2 < \frac{1}{4}$$

$$\Rightarrow \quad |x| < \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2} < x < \frac{1}{2}$$

Ratio Test:

$$|a_n| = \frac{(n+6)!}{7! \cdot n! \cdot 7^n}$$

11. (7 points) The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (n+6)!}{7! \cdot n! \cdot 7^n}$ is

    A. divergent by the Divergence Test

    B. divergent by the Comparison Test

    C. convergent by the Limit Comparison Test

    D. convergent by the Ratio Test

    E. None of the above

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+7)!}{7! (n+1)! \cdot 7^{n+1}} \cdot \frac{7! \cdot n! \cdot 7^n}{(n+6)!}$$

$$= \frac{(n+7)!}{(n+6)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{7^n}{7^{n+1}}$$

$$= (n+7) \cdot \frac{1}{n+1} \cdot \frac{1}{7}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{7} < 1$$

12. (7 points) Find the Maclaurin series for $f(x) = \dfrac{x^2}{2+x}$

    A. $x^2/2 + x^3/2 + x^4/2 + x^5/2 + \cdots$

    B. $x^2/2 - x^3/2 + x^4/2 - x^5/2 + \cdots$

    C. $x^2/2 + x^3/4 + x^4/8 + x^5/16 + \cdots$

    D. $x^2/2 - x^3/4 + x^4/8 - x^5/16 + \cdots$

    E. None of the above

$= x^2 \cdot \dfrac{1}{2 \cdot \left[1 - \left(-\frac{x}{2}\right)\right]}$

$= \dfrac{x^2}{2} \cdot \sum_{n=0}^{\infty} \left(-\dfrac{x}{2}\right)^n$

$= \dfrac{x^2}{2}\left(1 - \dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8} + \cdots\right)$

$= \dfrac{x^2}{2} - \dfrac{x^3}{4} + \dfrac{x^4}{8} - \dfrac{x^5}{16} + \cdots$

13. (7 points) Find the Maclaurin series for $f(x) = x\cos\sqrt{x}$

    A. $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^{2n+1}}{(2n)!}$

    B. $\sum_{n=0}^{\infty} (-1)^{n} \dfrac{x^{n}}{(2n)!}$

    C. $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^{n+1}}{(2n)!}$

    D. $\sum_{n=0}^{\infty} (-1)^{n} \dfrac{x^{n+1}}{(2n)!}$

    E. None of the above

$= x \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{(\sqrt{x})^{2n}}{(2n)!}$

$(\sqrt{x})^{2n} = x^{\frac{1}{2}\cdot 2n} = x^n$

$= x \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{x^n}{(2n)!}$

$= \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{x^{n+1}}{(2n)!}$

14. (7 points) What can be said about the limit

$$\lim_{x\to\infty} \dfrac{n^2}{e^{2n}} + \dfrac{n^2+2}{3-2n^2}$$

    A. The limit does not exist because $\frac{n^2}{e^{2n}}$ doesn't have a limit.

    B. The limit does not exist because $\frac{n^2+2}{3-2n^2}$ doesn't have a limit.

    C. The limit L exists and $L = \frac{1}{3}$.

    D. The limit L exists and $L = -\frac{1}{2}$.

    E. None of the above.

$\lim_n \dfrac{n^2}{e^{2n}} \overset{l'H\hat{o}p}{=} \lim_{n\to\infty} \dfrac{2n}{2\cdot e^{2n}}$

$\overset{l'H\hat{o}p}{=} \lim_{n\to\infty} \dfrac{1}{2\cdot e^{2n}}$

$= 0$

$\lim \dfrac{n^2+2}{3-2n^2} = \lim_{n\to\infty} \dfrac{n^2}{-2n^2}$

$= -\dfrac{1}{2}$

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	18	
3	18	
4	18	
5	18	
6	18	
7	21	
8	21	
9	21	
Total:	153	

No more than 150 points may be earned on the exam.