Integration of RATIONAL Functions by PARTIAL FRACTIONS

Warm up exercises.

1. Fill in the blanks.

   (a) If $\int_1^4 f(x)dx = 2$, find $\int_0^1 f(3x+1)dx = \boxed{}$.

   (b) If $\int f(x)dx = F(x) + C$, find $\int f(3x+1)dx = \boxed{}$

   (c) If $\int f(x)dx = F(x) + C$, find $\int f(ax+b)dx = \boxed{}$

2. Based on the previous part, fill in the following blanks:

   (a)

   $$\int \frac{1}{x-2}dx = \boxed{}, \quad \int \frac{1}{3x-1}dx = \boxed{}, \quad \int \frac{1}{ax+b}dx = \boxed{}$$

   (b)

   $$\int \frac{1}{x^2}dx = \boxed{}, \quad \int \frac{1}{(x+1)^2}dx = \boxed{}, \quad \int \frac{1}{(2x+1)^2}dx = \boxed{}$$

   (c)

   $$\int \frac{1}{1+x^2}dx = \boxed{}, \quad \int \frac{1}{1+(2x)^2}dx = \boxed{}, \quad \int \frac{1}{1+3x^2}dx = \boxed{}$$

   $$\int \frac{1}{1+\frac{1}{4}x^2}dx = \boxed{}, \quad \int \frac{1}{4+x^2}dx = \boxed{}, \quad \int \frac{1}{a^2+x^2}dx = \boxed{}$$

   (d)

   $$\int \frac{x}{1+x^2}dx = \boxed{}, \quad \int \frac{x}{3+2x^2}dx = \boxed{}, \quad \int \frac{x}{a+bx^2}dx = \boxed{}$$

3. Based on the previous part, evaluate the following integrals:

   (a)

   $$\int \frac{1}{x-2} + \frac{1}{3x-1} \quad dx = \underline{\hspace{6cm}}$$

   (b)

   $$\int \frac{1}{x} + \frac{1}{(x+1)^2} \quad dx = \underline{\hspace{6cm}}$$

   (c)

   $$\int \frac{1}{x} + \frac{1}{4+x^2} + \frac{x}{4+x^2} \quad dx = \underline{\hspace{6cm}}$$

Answers to the warm up exercises in the previous page.

1. (a) If $\int_1^4 f(x)dx = 2$, then $\int_0^1 f(3x+1)dx \xlongequal{u=3x+1} \int_1^4 f(u)\frac{du}{3} = \frac{2}{3}$.

   (b) If $\int f(x)dx = F(x)+C$, then

$$\int f(3x+1)dx \xlongequal{u=3x+1} \int f(u)\frac{1}{3}du = \frac{1}{3}F(u)+C \xlongequal{u=3x+1} \frac{1}{3}F(3x+1)+C$$

   (c) If $\int f(x)dx = F(x)+C$, then (repeat the above u-sub)

$$\int f(ax+b)dx \boxed{\xlongequal{u=ax+b} \int f(u)\frac{1}{a}du = \frac{1}{a}F(u)+C \xlongequal{u=ax+b} \frac{1}{a}F(ax+b)+C}$$

2. (a)

$$\int \frac{1}{x}dx = \ln|x|+C \Rightarrow \int \frac{1}{ax+b}dx \boxed{\xlongequal{u=ax+b} \int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\ln|u|+C \xlongequal{u=ax+b} \frac{1}{a}\ln|ax+b|+C}$$

   Then $\int \frac{1}{x-2}dx = \ln|x-2|+C$, and $\int \frac{1}{3x-1}dx = \frac{1}{3}\ln|3x-1|+C$.

   (b)

$$\int \frac{1}{x^2}dx = \frac{-1}{x} \Rightarrow \int \frac{1}{(ax+b)^2}dx \boxed{\xlongequal{u=ax+b} \int \frac{1}{u^2}\frac{du}{a} = \frac{1}{a}\cdot\frac{-1}{u}+C \xlongequal{u=ax+b} \frac{1}{a}\cdot\frac{-1}{ax+b}+C}$$

   Then $\int \frac{1}{(x+1)^2}dx = -\frac{1}{x+1}+C$, $\int \frac{1}{(2x+1)^2}dx = \frac{1}{2}\cdot(-\frac{1}{2x+1})+C$

   (c)

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x \Rightarrow \int \frac{1}{1+(2x)^2}dx \xlongequal{u=2x} \int \frac{1}{u^2}\frac{du}{2} = \frac{1}{2}\tan^{-1}u \xlongequal{u=2x} \frac{1}{2}\tan^{-1}(2x)$$

$$\int \frac{1}{1+3x^2}dx = \int \frac{1}{1+(\sqrt{3}x)^2}dx \xlongequal{u=\sqrt{3}x} \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x)+C$$

$$\int \frac{1}{1+\frac{1}{4}x^2}dx \xlongequal{u=\frac{1}{2}x} \frac{1}{\frac{1}{2}}\tan^{-1}(\frac{1}{2}x)+C = 2\tan^{-1}(\frac{1}{2}x)+C$$

$$\int \frac{1}{4+x^2}dx = \int \frac{1}{4\cdot(1+\frac{1}{4}x^2)}dx = \frac{1}{4}\int \frac{1}{1+\frac{1}{4}x^2}dx = \frac{1}{4}\Big(2\tan^{-1}(\frac{1}{2}x)\Big)+C$$

$$\boxed{\int \frac{1}{a^2+x^2}dx} = \int \frac{1}{a^2\cdot(1+\frac{1}{a^2}x^2)}dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}dx \xlongequal[du=\frac{x}{a}]{u=\frac{x}{a}} \frac{1}{a^2}\Big(\int \frac{1}{1+u^2}\cdot a du\Big)$$

$$= \frac{1}{a^2}\cdot a\int \frac{1}{1+u^2}du = \frac{1}{a}\cdot\Big(\tan^{-1}(u)\Big)+C \xlongequal{u=\frac{1}{a}x} \boxed{\frac{1}{a}\tan^{-1}(\frac{x}{a})+C}$$

$$\text{OR}: \int \frac{1}{a^2+x^2}dx \xlongequal[dx=a\sec^2\theta d\theta]{x=a\tan\theta} \int \frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta} = \int \frac{d\theta}{a} = \frac{1}{a}\theta \xlongequal[\theta=\tan^{-1}(\frac{x}{a})]{\frac{x}{a}=\tan\theta} \frac{1}{a}\tan^{-1}(\frac{x}{a})+C$$

   (d)

$$\int \frac{x}{a+bx^2}dx = \int \frac{1}{a+bx^2}\cdot xdx \xlongequal[du=2b\cdot xdx]{u=a+bx^2} \int \frac{1}{u}\frac{du}{2b} = \frac{1}{2b}\ln|u| \xlongequal{u=a+bx^2} \frac{1}{2b}\ln|a+bx^2|+C$$

   Therefore, $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln|1+x^2|+C$, $\int \frac{x}{3+2x^2}dx \xlongequal[du=4\cdot xdx]{u=3+2x^2} \frac{1}{4}\ln|3+2x^2|+C$

3. Sum up the answers in part 2.