Classification for Improper Integrals in §7.8

★★+. (Simple linear u-Sub)

5. $\int_3^{+\infty} \frac{1}{(x-2)^{3/2}} dx$. P1 ; 6. $\int_0^{+\infty} \frac{1}{\sqrt[4]{1+x}} dx$ P1 ; 7. $\int_{-\infty}^0 \frac{1}{3-4x} dx$. P1

8. $\int_1^{+\infty} \frac{1}{(2x+1)^3} dx$ P2 ; 9. $\int_2^{+\infty} e^{-5p} dp$. P2 ; 28. $\int_0^5 \frac{1}{\sqrt[3]{5-x}} dx$, P10

29 $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$ P11 . ★★★+ 33. $\int_0^9 \frac{1}{\sqrt[3]{x-1}} dx$ P13

★★+ Direct anti-Derivative formula:

10. $\int_{-\infty}^0 2^r \cdot dr$ P2 ; 31. $\int_{-2}^3 \frac{1}{x^4} dx$ P12 ; 32. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ P12

★★★ Typical u-Sub method.

11. $\int_0^{+\infty} \frac{x^2}{\sqrt{1+x^3}} dx$. P3 ; 14. $\int_1^{+\infty} \frac{e^{-\frac{1}{x}}}{x^2} dx$ P4 ; 16. $\int \sin\theta \cdot e^{\cos\theta} \cdot d\theta$ P4.

21. $\int_1^{+\infty} \frac{\ln x}{x} dx$ P7 24. $\int_e^{+\infty} \frac{1}{x(\ln x)^2} dx$. P8 ; 38 $\int \frac{\cos\theta}{\sqrt{\sin\theta}} d\theta$. P15

★★★+ ~ 4 Stars. IBP method.

19 $\int_{-\infty}^0 z \cdot e^{2z} dz$. P6 ; 20 $\int_2^{+\infty} y \cdot e^{-3y} dy$ P6 ; 22 $\int_1^{+\infty} \frac{\ln x}{x^2} dx$ P7

37. $\int_0^1 r \cdot \ln r \cdot dr$ P15

3 half ~ 4 Stars Partial Fraction

17. $\int_1^{+\infty} \frac{1}{x(x+1)} dx$ P5 ; 18 $\int_2^{+\infty} \frac{dv}{v^2+2v-3}$ P5 ; 34. $\int_0^5 \frac{w}{w-2} dw$ P13 ; 36. $\int \frac{dx}{x^2-x-2}$ P14

Trig-Integral.

3 stars 15. $\int \sin^2\alpha \cdot d\alpha$ P4 4 stars +. 35. $\int \tan^2\theta \cdot d\theta$ P14.

Over 5 stars, super-hard. 23, 25, 26, 30, 39, 40

4 half 6 stars

Solution §7.8

Determine whether each integral is convergent or divergent.

5. $\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to+\infty} \int_3^{t} \frac{1}{(x-2)^{3/2}}\,dx.$

$$\int_3^{t} \frac{1}{(x-2)^{3/2}}\,dx \overset{u=x-2}{=\!=} \int_{3-2}^{t-2} \frac{1}{u^{3/2}}\,du$$

$$= \int_1^{t-2} u^{-\frac{3}{2}}\,du = \frac{1}{-\frac{1}{2}}\cdot u^{-\frac{1}{2}}\Big|_1^{t-2} = -2\cdot u^{-\frac{1}{2}}\Big|_1^{t-2}$$

$$= -2(t-2)^{-\frac{1}{2}} + 2\cdot 1$$

$$= \frac{-2}{\sqrt{t-2}} + 2$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty} \frac{-2}{\sqrt{t-2}} + 2 = \frac{-2}{\infty} + 2$$

$$= 2 \qquad \text{Conv.}$$

6. $\int_0^{+\infty} \frac{1}{\sqrt[4]{1+x}}\,dx \overset{u=1+x}{=\!=} \int \frac{1}{\sqrt[4]{u}}\cdot du = \int u^{-\frac{1}{4}}\,du$

$$= \frac{1}{-\frac{1}{4}+1}\cdot u^{\frac{3}{4}}$$

$$\overset{u=1+x}{=\!=} \frac{4}{3}\cdot(1+x)^{\frac{3}{4}}\Big|_0^{+\infty}$$

DIV.

$$= \lim_{t\to\infty} \frac{4}{3}(1+t)^{\frac{3}{4}} - \frac{4}{3}\cdot(1+0)^{\frac{3}{4}} = \infty$$

7. $\int_{-\infty}^{0} \frac{1}{3-4x}\,dx = \lim_{t\to-\infty} \int_t^{0} \frac{1}{3-4x}\,dx$ $\qquad u = 3-4x,$ $du = (-4)\cdot dx$

$$= \lim_{t\to-\infty} \int_t^{0} \frac{1}{u}\,\frac{du}{-4}$$

$$= \lim_{t\to-\infty} \ln|u|\cdot\frac{1}{-4}$$

DIV

$$= \lim_{t\to-\infty} \ln|3-4x|\cdot\frac{1}{-4}\Big|_t^{0} = \lim_{t\to-\infty} \frac{1}{4}\ln|3-4t| - \frac{1}{4}\ln 3 = +\infty$$

☆☆+ 8 $\int_1^{+\infty} \frac{1}{(2x+1)^3} dx.$ $u=2x+1,\ du=2\cdot dx$

$$= \lim_{t\to+\infty} \int_1^t \frac{1}{(2x+1)^3} dx = \lim_{t\to+\infty} \int_1^t \frac{1}{u^3}\cdot\frac{du}{2}$$

$$= \lim_{t\to+\infty} \frac{1}{2}\cdot\frac{1}{-2}\cdot u^{-2}$$

$$= \lim_{t\to+\infty} \frac{1}{-4}\cdot\frac{1}{(2x+1)^2}\Big|_1^t = \lim_{t\to+\infty} \frac{1}{4}\cdot\frac{1}{(2+1)^2} - \frac{1}{4}\cdot\frac{1}{(2t+1)^2}$$

ConV.

$$= \frac{1}{36} - \frac{1}{4}\cdot\frac{1}{\infty} = \boxed{\frac{1}{36}}$$

☆☆+ 9. $\int_2^{\infty} e^{-5p} dp$ $u=-5p,\quad du=-5\cdot dp$

$$= \lim_{t\to+\infty} \int_2^t e^{-5p} dp = \lim_{t\to+\infty} \int_{\cancel{2}}^{\cancel{t}} e^u\cdot\frac{du}{-5}$$

$$= \lim_{t\to+\infty} \frac{1}{-5}\cdot e^u = \lim_{t\to+\infty} \frac{1}{-5} e^{-5p}\Big|_2^t$$

$$= \lim_{t\to+\infty} \frac{1}{-5}\cdot e^{-5t} + \frac{1}{5}\cdot e^{-10}$$

ConV.

$$= \underbrace{-\frac{1}{5}\cdot e^{-\infty}}_{=0} + \frac{1}{5}e^{-10} = \boxed{\frac{1}{5}e^{-10}}$$

☆☆★ 10 $\int_{-\infty}^{0} 2^r dr.$

$$= \lim_{t\to-\infty} \int_t^0 2^r dr = \lim_{t\to-\infty} \frac{2^r}{\ln 2}\Big|_t^0$$

Hint: formula $\int a^x dx = \frac{a^x}{\ln a}$

$$= \lim_{t\to-\infty} \frac{2^0}{\ln 2} - \frac{2^t}{\ln 2} = \frac{1}{\ln 2} - \frac{2^{-\infty}}{\ln 2} = \frac{1}{\ln 2} - \frac{0}{\ln 2}$$

ConV

$$= \boxed{\frac{1}{\ln 2}}$$

☆☆☆ 11. $\int_0^{+\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx$ $\qquad u=1+x^3,\ du=3\cdot x^2\cdot dx.$

$$= \lim_{t\to+\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx.$$

$$= \lim_{t\to+\infty}\int_{\cancel{0}}^{\cancel{t}} \frac{1}{\sqrt{u}}\cdot\frac{du}{3} = \lim_{t\to+\infty}\int \frac{1}{3}\cdot u^{-\frac{1}{2}}\,du$$

$$= \lim_{t\to+\infty} \frac{1}{3}\cdot 2\cdot u^{\frac{1}{2}}$$

$$= \lim_{t\to+\infty} \frac{2}{3}(1+x^3)^{\frac{1}{2}}\Big|_0^t$$

DIV

$$= \lim_{t\to+\infty} \frac{2}{3}\cdot(1+t^3)^{\frac{1}{2}} - \frac{2}{3}\cdot 1 = \infty$$

☆☆☆ 12. $\int_{-\infty}^{+\infty}(y^3-3y^2)\,dy = \int_{-\infty}^{0} y^3-3y\,dy + \int_0^{+\infty} y^3-3y\,dy$

(Remark, if one of above is DIV, then left hand side is DIV)

$$\int_{-\infty}^0 (y^3-3y)\,dy = \lim_{t\to-\infty}\int_t^0 y^3-3y\,dy$$

$$\boxed{\int_t^0 y^3-3y\,dy = \frac{1}{4}y^4-\frac{3}{2}y^2\Big|_t^0 = -\frac{1}{4}t^4+\frac{3}{2}\cdot t^2}$$

DIV $\qquad = \lim_{t\to-\infty}\left(-\frac{1}{4}t^4+\frac{3}{2}t^2\right) = -\infty$

☆☆☆☆ 13. $\int_{-\infty}^{+\infty} x\cdot e^{-x^2}\,dx = \int_{-\infty}^{0} x\cdot e^{-x^2}\,dx + \int_0^{+\infty} x\cdot e^{-x^2}\,dx = \boxed{-\frac{1}{2}+\frac{1}{2}=0.\ \text{ConV}}$

$$\int x\cdot e^{-x^2}\,dx \underset{du=-2\cdot x\cdot dx}{\overset{u=-x^2}{=\!=}} \int e^u\cdot\frac{du}{-2} = -\frac{1}{2}\cdot e^u = -\frac{1}{2}\cdot e^{-x^2}+C$$

$$\int_{-\infty}^0 x\cdot e^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 x\cdot e^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_t^0 = \lim_{t\to-\infty} -\frac{1}{2}+\frac{1}{2}\cdot e^{-t^2} = -\frac{1}{2}$$

$$\int_0^{+\infty} x\cdot e^{-x^2}\,dx = \lim_{t\to+\infty}\int_0^t x\cdot e^{-x^2}\,dx = \lim_{t\to+\infty}\left(-\frac{1}{2}\cdot e^{-x^2}\right)\Big|_0^t = \lim_{t\to+\infty} -\frac{1}{2}\cdot e^{-t^2}+\frac{1}{2} = \frac{1}{2}$$

☆☆☆ 14. $\int_1^{+\infty} \frac{e^{-\frac{1}{x}}}{x^2} \cdot dx$

$$\int \frac{e^{-\frac{1}{x}}}{x^2} dx \xlongequal[du=\frac{1}{x^2}dx]{u=-\frac{1}{x}} \int e^u \cdot du = e^u + C \xlongequal{u=-\frac{1}{x}} e^{-\frac{1}{x}} + C$$

$$\int_1^{+\infty} \frac{e^{-\frac{1}{x}}}{x^2} dx = \lim_{t\to+\infty} \int_1^t \frac{e^{-\frac{1}{x}}}{x^2} dx = \lim_{t\to+\infty} e^{-\frac{1}{x}} \Big|_1^t$$

$$= \lim_{t\to+\infty} e^{-\frac{1}{t}} - e^{-1}$$

Conv.

$$= e^{-\frac{1}{\infty}} - e^{-1} = e^0 - e^{-1} = \boxed{1 - e^{-1}}$$

☆☆☆ 15. $\int \sin^2\alpha \cdot d\alpha = \int \frac{1-\cos 2\alpha}{2} \cdot d\alpha$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2\alpha\right) d\alpha = \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha + C$$

$$\int_0^{+\infty} \sin^2\alpha \cdot d\alpha = \lim_{t\to+\infty} \int_0^t \sin^2\alpha \cdot d\alpha = \lim_{t\to+\infty} \left[\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right]\Big|_0^t$$

DIV.

Hint: $\lim_{t\to\infty} \sin t$ DNE

$$\boxed{= \lim_{t\to+\infty} \frac{t}{2} - \frac{1}{4}\sin 2t - 0 \quad \text{DNE.}}$$

☆☆☆ 16. $\int \sin\theta \cdot e^{\cos\theta} d\theta \xlongequal[du=-\sin\theta \cdot d\theta]{u=\cos\theta} \int e^u \cdot (-du) = -e^u = -e^{\cos\theta} + C$

$$\int_0^{+\infty} \sin\theta \, e^{\cos\theta} \cdot d\theta = \lim_{t\to+\infty} \int_0^t \sin\theta \cdot e^{\cos\theta} \cdot d\theta = \lim_{t\to+\infty} -e^{\cos\theta} \Big|_0^t$$

$$= \lim_{t\to+\infty} -e^{\cos t} + e^{\cos 0}$$

$\boxed{\text{DIV.}}$

DNE. (Hint: $\cos\infty$ DNE)

☆☆☆☆ 7. $\int \frac{1}{x(x+1)}dx \overset{P.F.}{=} \int \frac{1}{x} - \frac{1}{x+1}\,dx$

$$= \ln|x| - \ln|x+1| + C$$

$$\int_1^{+\infty} \frac{1}{x^2+x}\,dx = \lim_{t\to+\infty}\int_1^t \frac{1}{x(x+1)}\,dx = \lim_{t\to+\infty}\left[\ln|x| - \ln|x+1|\right]\Big|_1^t$$

$$= \lim_{t\to+\infty} \ln|t| - \ln|t+1| - (\ln 1 - \ln 2)$$

$$! \quad = \lim_{t\to+\infty} \ln\frac{|t|}{|t+1|} + \ln 2$$

$$= \ln \lim_{t\to+\infty}\left|\frac{t}{t+1}\right| + \ln 2$$

Conv.

$$= \ln 1 + \ln 2 = \boxed{\ln 2}$$

☆☆☆☆ 18. $\int_2^{+\infty} \frac{dv}{v^2+2v-3}$.

$\int \frac{dv}{v^2+2v-3} = \int \frac{1}{(v+3)(v-1)}\,dv$ P.F. $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$

$= \int \frac{-\frac{1}{4}}{v+3} + \frac{\frac{1}{4}}{v-1}\,dv$ $\quad 1 = A(v-1) + B(v+3)$

$v=1 \Rightarrow B=\frac{1}{4}$

$v=-3 \Rightarrow A=-\frac{1}{4}$

$$= -\frac{1}{4}\cdot\ln|v+3| + \frac{1}{4}\ln|v-1| + C = \boxed{\frac{1}{4}\cdot\ln\left|\frac{v-1}{v+3}\right| + C}$$

$$\int_2^{+\infty} \frac{dv}{v^2+2v-3} = \lim_{t\to+\infty}\int_2^t \frac{dv}{v^2+2v-3} = \lim_{t\to+\infty} \frac{1}{4}\cdot\ln\left|\frac{v-1}{v-3}\right|\Big|_2^t$$

$$= \lim_{t\to+\infty} \frac{1}{4}\cdot\ln\left|\frac{t-1}{t+3}\right| - \frac{1}{4}\cdot\ln\left|\frac{1}{-1}\right|$$

Hint: $\lim_{t\to+\infty}\left|\frac{t-1}{t+3}\right| = 1$ $\quad = \frac{1}{4}\cdot\ln 1 - \frac{1}{4}\cdot\ln 1$

Conv. $\quad = 0$

★★★★ 19. $\int_{-\infty}^{0} z\cdot e^{2z}\,dz$.

$\int z\, e^{2z}\,dz$ IBP: $u=z$. $dv=e^{2z}dz$

$du=dz$, $v=\int e^{2z}=\frac{1}{2}\cdot e^{2z}$

$= u\cdot v-\int v\cdot du$

$= z\cdot\frac{1}{2}\cdot e^{2z}-\int\frac{1}{2}\cdot e^{2z}\cdot dz = \frac{1}{2}\cdot z\cdot e^{2z}-\frac{1}{2}\cdot\frac{1}{2}\cdot e^{2z}+C$

$\int_{-\infty}^{0} z\,e^{2z}\,dz = \lim_{t\to-\infty}\int_{t}^{0} z\cdot e^{2z}dz = \lim_{t\to-\infty}\left[\frac{z}{2}\cdot e^{2z}-\frac{1}{4}\cdot e^{2z}\right]\Big|_{t}^{0}$

$= \lim_{t\to-\infty}\left(-\frac{1}{4}e^{0}\right)-\left(\frac{t}{2}\cdot e^{2t}-\frac{1}{4}\cdot e^{2t}\right)$

Hint: $\lim_{t\to-\infty}\frac{t}{2}\cdot e^{2t}$ ($\infty\cdot 0$ type)

$= \lim_{t\to-\infty}\frac{t}{2e^{-2t}} \overset{LH}{=} \lim_{t\to-\infty}\frac{1}{-4e^{-2t}}$

$= \frac{1}{-4\cdot e^{+\infty}} = 0$

$\lim_{t\to-\infty} e^{2t} = e^{-\infty} = 0$

$= \boxed{-\frac{1}{4}+0+0}$

Conv.

★★★★ 20. $\int y\cdot e^{-3y}\,dy \xrightarrow[dv=e^{-3y}dy]{u=y,\ du=dy} u\cdot v-\int v\cdot du = y\cdot\frac{1}{-3}e^{-3y}-\int\frac{1}{-3}e^{-3y}\,dy$

(IBP:) $v=\frac{1}{-3}\cdot e^{-3y}$

$= -\frac{y}{3}e^{-3y}+\frac{1}{3}\cdot\frac{1}{-3}\cdot e^{-3y}+C$

$\int_{2}^{+\infty} y\cdot e^{-3y}dy = \lim_{t\to+\infty}\int_{2}^{t} y\cdot e^{-3y}dy = \lim_{t\to+\infty}\left[-\frac{y}{3}e^{-3y}-\frac{1}{9}e^{-3y}\right]\Big|_{2}^{t}$

Hint: $\lim_{t\to+\infty} t\cdot e^{-3t}$

$= \lim_{t\to+\infty}\frac{t}{e^{3t}} \overset{LH}{=} \lim_{t\to+\infty}\frac{1}{3e^{3t}} = \frac{1}{\infty} = 0$

$\lim_{t\to+\infty} e^{-3t} = e^{-\infty} = 0$

$= \lim_{t\to+\infty}\left(-\frac{t}{3}\cdot e^{-3t}-\frac{1}{9}e^{-3t}\right)-\left(-\frac{2}{3}e^{-6}-\frac{1}{9}e^{-6}\right)$

$= 0+0+\boxed{\frac{7}{9}\cdot e^{-6}}$

☆☆☆ 21 $\int_1^{+\infty} \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \frac{1}{2}(\ln t)^2 = +\infty$

$$\int_1^t \frac{\ln x}{x}\,dx \ \overset{u=\ln x}{\underset{du=\frac{1}{x}dx}{=\!=}} \ \int_{\ln 1}^{\ln t} u\cdot du = \frac{1}{2}u^2\Big|_0^{\ln t}$$

$$= \boxed{\frac{1}{2}\cdot(\ln t)^2 - 0}$$

DIV.

☆☆☆☆ 22. $\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

$\int_1^t \frac{\ln x}{x^2}\,dx$ IBP: $u=\ln x$, $dv=\frac{1}{x^2}dx$

$du=\frac{1}{x}dx$, $v=-\frac{1}{x}$.

$$= uv - \int v\cdot du$$

$$= \ln x\cdot\left(-\frac{1}{x}\right) - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^t$$

$$= -\frac{\ln t}{t} - \frac{1}{t} + \frac{\ln 1}{1} + 1$$

$$\lim_{t\to+\infty}\left(\frac{\ln t}{t}\right) \overset{L'H}{=} \lim_{t\to+\infty} \frac{\frac{1}{t}}{1}$$

$$= \frac{1}{\infty} = 0$$

$$\lim_{t\to+\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 1\right) = 0 + 0 + 1$$

$$\boxed{\int_1^{+\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to+\infty}\int_1^t \frac{\ln x}{x^2}\,dx = 1}$$

ConV

☆☆☆+ 24. $\int \frac{1}{x(\ln x)^2} dx \underset{du=\frac{1}{x}dx}{\overset{u=\ln x}{=\!=}} \int \frac{1}{u^2} du = -\frac{1}{u}$

$= -\frac{1}{\ln x}$.

$\int_e^{+\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \to +\infty} \int_e^t \frac{1}{x(\ln x)^2} dx$

$= \lim_{t \to +\infty} \frac{1}{-\ln x} \Big|_e^t = \lim_{t \to +\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln e}\right)$

$\boxed{\text{Conv}}$

$= -\frac{1}{\ln \infty} + 1 = \boxed{1}$

☆☆☆ ☆☆ 23. $\int \frac{z}{z^4+4} dz$. $u = z^2$, $du = 2 \cdot z\, dz$

$= \int \frac{\frac{1}{2} du}{u^2+4} = \int \frac{1}{2} \cdot \frac{1}{u^2+2^2} du = \frac{1}{2} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{u}{2}\right) + C$

$= \frac{1}{4} \cdot \tan^{-1}\left(\frac{z^2}{2}\right) + C$

$\int_{-\infty}^0 \frac{z}{z^4+4} dz = \lim_{t \to -\infty} \int_t^0 \frac{z}{z^4+4} dz = \lim_{t \to -\infty} \frac{1}{4} \tan^{-1}\left(\frac{z^2}{2}\right) \Big|_t^0$

$= \lim_{t \to -\infty} \frac{1}{4} \cdot \tan^{-1} 0 - \frac{1}{4} \cdot \tan^{-1}\left(\frac{t^2}{2}\right)$

$= 0 - \frac{1}{4} \cdot \tan^{-1}\left(\frac{(-\infty)^2}{2}\right)$

$= -\frac{1}{4} \cdot \tan^{-1}\left(\frac{+\infty}{2}\right)$

Conv. $\boxed{= -\frac{1}{4} \cdot \frac{\pi}{2}}$

☆☆☆☆☆ 25. $\int_0^{+\infty} e^{-\sqrt{y}}\,dy$

$\int e^{-\sqrt{y}}\,dy$ u-sub first $u=-\sqrt{y}$

$du = -\frac{1}{2\sqrt{y}}\cdot dy$

$= \int e^u \cdot 2u \cdot du$

$dy = -2\sqrt{y}\,du = 2u\,du$

$= w\cdot v - \int v\cdot dw$ IBP: $w=2u$, $dv=e^u\,du$

$dw = 2\,du$, $v=e^u$

$= 2u\cdot e^u - \int e^u \cdot 2\cdot du$

$= 2u\cdot e^u - 2\cdot e^u \overset{u=-\sqrt{y}}{=\!=} -2\sqrt{y}\cdot e^{-\sqrt{y}} - 2\cdot e^{-\sqrt{y}}$

$\int_0^{+\infty} e^{-\sqrt{y}}\,dy = \lim_{t\to+\infty}\int_0^t e^{-\sqrt{y}}\,dy = \lim_{t\to+\infty}\left(-2\sqrt{y}\cdot e^{-\sqrt{y}} - 2\cdot e^{-\sqrt{y}}\right)\Big|_0^t$

$= \lim_{t\to+\infty}\left(-2\sqrt{t}\cdot e^{-\sqrt{t}} - 2e^{-\sqrt{t}}\right) + (0 + 2\cdot e^0)$

$= 0 + 2\cdot 1$

$= \boxed{2}$. Conv ✓

Hint:

$\lim_{t\to\infty}\sqrt{t}\cdot e^{-\sqrt{t}} = \lim_{t\to\infty}\frac{\sqrt{t}}{e^{\sqrt{t}}}$

$= \lim_{t\to\infty}\frac{\frac{1}{2\sqrt{t}}}{e^{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}}}$

$= \lim_{t\to\infty}\frac{1}{e^{\sqrt{t}}} = \frac{1}{e^{\infty}} = 0$

$\lim_{t\to+\infty} e^{-\sqrt{t}} = e^{-\infty} = 0$

☆ 27. $\int_0^1 \frac{1}{x}\,dx = \lim_{t\to0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to0^+} \ln|x|\Big|_t^1$

$= \lim_{t\to0^+} \ln 1 - \ln t$ DZ ✓

$= 0 - \ln 0^+ = -(-\infty) = \boxed{+\infty}$

☆☆ ☆☆+ 26 $\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$ u-sub $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} \cdot dx.$

$$= \int \frac{2\sqrt{x} \cdot du}{\sqrt{x} + x\sqrt{x}} \quad \Rightarrow dx = 2\sqrt{x} \cdot du$$

$$= \int \frac{2du}{1+x} = \int \frac{2du}{1+(\sqrt{x})^2} = \int \frac{2du}{1+u^2}$$

$$= 2 \cdot \tan^{-1} u$$

$$= 2 \cdot \tan^{-1}(\sqrt{x}),$$

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}+x\sqrt{x}} = \lim_{t\to+\infty} \int_1^t \frac{dx}{\sqrt{x}+x\sqrt{x}} = \lim_{t\to+\infty} 2\tan^{-1}(\sqrt{x})\Big|_1^t$$

$$= \lim_{t\to+\infty} 2\tan^{-1}(\sqrt{t}) - 2\tan^{-1} 1$$

$$= 2 \cdot \tan^{-1}(\infty) - 2\tan^{-1} 1$$

Conv.

$$= 2 \cdot \frac{\pi}{2} - 2 \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

☆☆+ 28 $\int_0^5 \frac{1}{\sqrt[3]{5-x}} dx = \lim_{t\to 5^-} \int_0^t \frac{1}{\sqrt[3]{5-x}} dx$ u-sub $u = 5-x.$

$du = -dx$

$$= \lim_{t\to5^-} \int_{5-0}^{5-t} \frac{1}{\sqrt[3]{u}} (-du)$$

$$= \lim_{t\to5^-} \int_5^{5-t} u^{-\frac{1}{3}} (-du)$$

$$= \lim_{t\to5^-} -\frac{3}{2} u^{\frac{2}{3}} \Big|_5^{5-t}$$

$$= \lim_{t\to5^-} -\frac{3}{2} \cdot (5-t)^{\frac{2}{3}} + \frac{3}{2} \cdot 5^{\frac{2}{3}}$$

$$= 0 + \boxed{\frac{3}{2} \cdot 5^{\frac{2}{3}}}$$

Conv.

☆☆+ 29. $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

$$\int \frac{dx}{\sqrt[4]{x+2}} \xrightarrow[du=dx]{u=x+2} \int \frac{du}{\sqrt[4]{u}} = \int u^{-\frac{1}{4}}\,du$$

$$= \frac{4}{3} \cdot u^{\frac{3}{4}}$$

$$= \frac{4}{3} \cdot (x+2)^{\frac{3}{4}}$$

$$\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t \to 2^+} \int_t^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t \to -2^+} \frac{4}{3} \cdot (x+2)^{\frac{3}{4}} \Big|_t^{14}$$

$$= \lim_{t \to -2^+} \frac{4}{3} \cdot 16^{\frac{3}{4}} - \frac{4}{3} \cdot (t+2)^{\frac{3}{4}}$$

$$= \frac{4}{3} \cdot 16^{\frac{3}{4}} - \underbrace{\frac{4}{3} \cdot (-2+2)^{\frac{3}{4}}}_{=0}$$

Hint: $16 = 2^4$

$16^{\frac{3}{4}} = 2^{4 \cdot \frac{3}{4}} = 2^3 = 8$

$$= \boxed{\frac{4}{3} \cdot 16^{\frac{3}{4}}} = \frac{32}{3}$$

☆☆☆☆☆ 30. $\int \frac{x}{(x+1)^2}\,dx = \int \frac{x+1-1}{(x+1)^2}\,dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$

$$\overset{u=x+1}{=} \int \frac{1}{u} - \frac{1}{u^2}\,du$$

$$= \ln|u| - 1\left(-\frac{1}{u}\right) = \ln|x+1| + \frac{1}{x+1}$$

$$\int_{-1}^{2} \frac{x}{(x+1)^2}\,dx = \lim_{t \to -1^+} \int_t^2 \frac{x}{(x+1)^2}\,dx = \lim_{t \to -1^+} \ln|x+1| + \frac{1}{x+1} \Big|_t^2$$

$$= \lim_{t \to (-1)^+} \ln 3 + \frac{1}{3} - \left(\ln|t+1| + \frac{1}{t+1}\right)$$

$$= \infty$$

$\boxed{\text{DIV}}$

☆☆+ 31. $\int \frac{1}{x^4}dx = \int x^{-4}dx = \frac{1}{-3}\cdot x^{-3} = \frac{1}{-3x^3}$

$\int_{-2}^{3} \frac{1}{x^4}dx = \int_{-2}^{0}\frac{1}{x^4}dx + \int_{0}^{3}\frac{1}{x^4}dx$

$\int_{-2}^{0}\frac{1}{x^4}dx = \lim_{t\to 0^-}\int_{-2}^{t}\frac{1}{x^4}dx = \lim_{t\to 0^-}\frac{1}{-3x^3}\Big|_{-2}^{t}$

$= \lim_{t\to 0^-}\frac{1}{-3\cdot t^3} + \frac{1}{3(-2)^3}$

$= \infty$

$\boxed{\text{DIV}}$

☆☆+ 32. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}}$

$\int_0^t \frac{dx}{\sqrt{1-x^2}} \xrightarrow[dx=\cos\theta\, d\theta]{\text{Trig-Sub: } x=\sin\theta} \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta$

$\sqrt{1-x^2} = \cos\theta$

$= \sin^{-1}x\Big|_0^t$

(Hint: or directly use formula: $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$)

$= \sin^{-1}t - \sin^{-1}0$

$= \sin^{-1}t$

$\lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\sin^{-1}t = \sin^{-1}1 = \boxed{\frac{\pi}{2}}$ ConV.

☆☆☆+ 33. $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^9 \frac{dx}{\sqrt[3]{x-1}}$

$\int \frac{1}{\sqrt[3]{x-1}}\,dx \overset{u=x-1}{=\!=} \int \frac{1}{\sqrt[3]{u}}\,du = \int u^{-\frac{1}{3}}\,du$

$= \frac{3}{2} u^{\frac{2}{3}} = \frac{3}{2}\cdot(x-1)^{\frac{2}{3}} + C$

$\int_0^1 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^-} \frac{3}{2}\cdot(x-1)^{\frac{2}{3}}\Big|_0^t$

$= \lim_{t\to 1^-} \frac{3}{2}\cdot(t-1)^{\frac{2}{3}} - \frac{3}{2}\cdot 1$

$\boxed{= 0 - \frac{3}{2}}$

$\int_1^9 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^+} \int_t^9 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^+} \frac{3}{2}\cdot(x-1)^{\frac{2}{3}}\Big|_t^9$

$= \lim_{t\to 1^+} \frac{3}{2}\cdot 8^{\frac{2}{3}} - \frac{3}{2}\cdot(t-1)^{\frac{2}{3}}$

Hint: $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$: $= \frac{3}{2}\cdot 4 - 0 = \boxed{6}$

$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2} + 6 = \boxed{\frac{9}{2}}$ ConV.

☆☆☆+34. $\int \frac{w}{w-2}\,dw = \int \frac{w-2+2}{w-2}\,dw = \int 1 + \frac{2}{w-2}\,dw = w + 2\ln|w-2|$

$\int_0^5 \frac{w}{w-2}\,dw = \int_0^2 \frac{w}{w-2}\,dw + \int_2^5 \frac{w}{w-2}\cdot dw$

$\int_0^2 \frac{w}{w-2}\,dw = \lim_{t\to 2^-} \int_0^t \frac{w}{w-2}\,dw = \lim_{t\to 2^-} \left[w + \ln|w-2|\right]\Big|_0^t$

$= \lim_{t\to 2^-} (t + \ln|t-2|) - \ln 2$

Since $\lim_{t\to 2} \ln|t-2| = \ln 0 = -\infty$

$\boxed{\text{DIV}}$ $= -\infty$

☆☆☆☆ 35. $\int \tan^2\theta \cdot d\theta$

sec even power:
$u = \tan\theta$, $du = \sec^2\theta \cdot d\theta$

$= \int u^2 \cdot \frac{du}{\sec^2\theta}$

$\sec^2\theta = \tan^2\theta + 1 = u^2+1$

$= \int \frac{u^2}{u^2+1} du = \int \frac{u^2+1-1}{u^2+1} du$

$= \int 1 - \frac{1}{1+u^2} du$

$= u - \tan^{-1} u = \tan\theta - \theta$

Hint: $u = \tan\theta$
$\tan^{-1} u = \theta$

$\int_0^{\pi/2} \tan^2\theta \cdot d\theta = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \tan^2\theta \, d\theta$

$= \lim_{t \to (\frac{\pi}{2})^-} \tan\theta - \theta \Big|_0^t = \lim_{t \to (\frac{\pi}{2})^-} \tan t - t - 0$

$= \boxed{+\infty}$

DIV.

☆☆☆ + 36. $\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x-2)(x+1)} \overset{\text{P.F.}}{=} \int \frac{A}{x-2} + \frac{B}{x+1} dx$

$1 = A(x+1) + B(x-2)$

$x=-1 \Rightarrow B = -\frac{1}{3}$

$x = 2 \Rightarrow A = \frac{1}{3}$

$= \int \frac{\frac{1}{3}}{x-2} + \frac{-\frac{1}{3}}{x+1} dx$

$= \frac{1}{3} \ln|x-2| - \frac{1}{3}\ln|x+1| = \cancel{\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|}$

$\int_0^4 \frac{dx}{(x-2)(x+1)} = \int_0^2 \frac{dx}{(x-2)(x+1)} + \int_2^4 \frac{dx}{(x-2)(x+1)}$

$\int_0^2 \frac{dx}{(x-2)(x+1)} = \lim_{t \to 2^-} \int_0^t \frac{dx}{(x-2)(x+1)} = \lim_{t\to 2^-} \left[\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1|\right]\Big|_0^t$

$= \lim_{t \to 2^-} \left[\frac{1}{3}\ln|t-2| - \frac{1}{3}\ln|t+1|\right] - \left(\frac{1}{3}\ln 2 - 0\right)$

Hint: $\lim_{t\to 2^-} \ln|t-2| = -\infty$

$= \boxed{-\infty}$ DIV

☆☆☆☆ 37. $\int r\cdot \ln r \, dr$ IBP: $u=\ln r$ $dv = r\cdot dr$

$du = \frac{1}{r}dr,$ $v=\frac{1}{2}r^2$

$= u\cdot v - \int v\cdot du$

$= \ln r \cdot \frac{1}{2}r^2 - \int \frac{1}{2}\cdot r^2 \cdot \frac{1}{r}\, dr$

$= \ln r\cdot \frac{1}{2}r^2 - \int \frac{1}{2} r\cdot dr = \frac{1}{2}\ln r\cdot r^2 - \frac{1}{2}\cdot\frac{1}{2}\cdot r^2 + C$

$\int_0^1 r\cdot \ln r \cdot dr = \lim_{t\to 0^+} \int_t^1 r\cdot \ln r\, dr$

$= \lim_{t\to 0^+} \left(\frac{1}{2}\ln r\cdot r^2 - \frac{1}{4}\cdot r^2\right)\Big|_t^1$

$= \lim_{t\to 0^+} \left(\cancel{\frac{1}{2}\ln t\cdot t^2 - \frac{1}{4}}\right.(0-\frac{1}{4}) - \left(\frac{1}{2}\ln t\cdot t^2 - \frac{1}{4}t^2\right)$

$= -\frac{1}{4} + 0 + 0$

$= \boxed{-\frac{1}{4}}$ ConV

Hint:

$\lim_{t\to 0^+} \ln t\cdot t^2$

$= \lim_{t\to 0} \frac{\ln t}{t^{-2}}$

$\overset{L'H}{=} \lim_{t\to 0} \frac{\frac{1}{t}}{-2\cdot t^{-3}}$

$= \lim_{t\to 0^+} \frac{t^2}{-2} = 0$

☆☆☆+ 38. $\int \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta$ $u=\sin\theta.$ $du = \cos\theta\cdot d\theta$

$= \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\, du = 2\cdot u^{\frac{1}{2}} = 2\sqrt{\sin\theta} + C$

$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta = \lim_{t\to 0^+}\int_t^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta = \lim_{t\to 0^+} 2\sqrt{\sin\theta}\Big|_t^{\frac{\pi}{2}}$

$= \lim_{t\to 0^+} 2\sqrt{\sin\frac{\pi}{2}} - 2\sqrt{\sin t}$

ConV. $= 2\cdot\sqrt{1} - 2\cdot\sqrt{0} = \boxed{2}$

6 Stars 39, 40

$\int \frac{e^{\frac{1}{x}}}{x^3} dx \qquad u=\frac{1}{x}, \quad du=-\frac{1}{x^2}dx,$

$= \int \frac{e^u}{x^3}\cdot(-x^2\cdot du) = \int e^u\cdot(-\frac{1}{x}) du$

$= \int (-u)\cdot e^u\cdot du$

IBP:

$w=-u, \quad dv=e^u du$

$dw=-du, \quad v=e^u$

$= w\cdot v - \int v\cdot dw$

$= -u\cdot e^u - \int e^u\cdot(-du)$

$= -u\cdot e^u + e^u \underset{u=\frac{1}{x}}{=} -\frac{1}{x}\cdot e^{\frac{1}{x}} + e^{\frac{1}{x}}$

39. $\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^-} [-\frac{1}{x}\cdot e^{\frac{1}{x}} + e^{\frac{1}{x}}]\Big|_{-1}^{t}$

$= \lim_{t\to 0^-} (-\frac{1}{t}e^{\frac{1}{t}} + e^{\frac{1}{t}}) - (e^{-1} + e^{-1})$

Hint: $\lim_{t\to 0^-} \frac{e^{\frac{1}{t}}}{t} = \lim_{t\to 0^-} \frac{\frac{1}{t}}{e^{-\frac{1}{t}}}$

$= \lim_{t\to 0^-} \frac{-\frac{1}{t^2}}{e^{-\frac{1}{t}}\cdot(+\frac{1}{t^2})} = \lim_{t\to 0^-} \frac{-1}{e^{-\frac{1}{t}}} = \frac{-1}{e^{+\infty}} = 0$

$= 0 + 0 \boxed{-2\cdot e^{-1}}$ Conv

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40. $\int_0^1 \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^+} (-\frac{1}{x}\cdot e^{\frac{1}{x}} + e^{\frac{1}{x}})\Big|_t^1$

$= \lim_{t\to 0^+} (-e^1 + e^1) - (-\frac{1}{t}e^{\frac{1}{t}} + e^{\frac{1}{t}})$

Hint: $\lim_{t\to 0^+} \frac{1}{t}\cdot e^{\frac{1}{t}} = +\infty\cdot e^{+\infty}$

$= +\infty$

DIV

DIV.