Classification of Integrals §7.5

- Trig-Integrals (§7.2)

※13 $\int \sin^5 t \cdot \cos^4 t \cdot dt$ ☆ P5. ; ※38. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\theta \cdot \cot\theta}{\sec\theta} d\theta$ P13. ☆☆

※1. $\int \frac{\cos x}{1-\sin x} \cdot dx$ ☆+ P1 ; ※4. $\int \frac{\sin^3 x}{\cos x} \cdot dx$ ☆☆☆+ P2.

- Trig-Sub (§7.3)

※11 $\int \frac{dx}{x^3 \cdot \sqrt{x^2-1}}$ ☆☆☆ P4 ; ※16 $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$ P6 ; ※60 $\int \frac{dx}{x^2 \cdot \sqrt{4x^2-1}}$ P15.

- Partial Fractions (§7.4)

※25. $\int_0^1 \frac{1+12t}{1+3t} dt$ ☆ P9 ; ※9 $\int_2^4 \frac{x+2}{x^2+3x-4} dx$ ☆☆ P3 ; ※12 $\int \frac{2x-3}{x^3+3x} dx$ ☆☆☆+ P5.

- General u-Sub (in Calculus I)

☆+ ※2 $\int_0^1 (3x+1)^{\sqrt{2}} dx$ P1 ; ※7 $\int_{-1}^1 \frac{e^{\tan^{-1} y}}{1+y^2} dy$ P3 ; ※18 $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$ P7

☆☆ ※19 $\int e^{x+e^x} dx$ P8 ; ※32. $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2} dx$ P12. ☆+

☆☆☆☆ ※27 $\int \frac{dx}{1+e^x}$ P11 ; ※71 $\int \frac{e^{2x}}{1+e^x} dx$ P15

- IBP (§7.1)

☆☆ ※3 $\int_1^4 \sqrt{y} \cdot \ln y \, dy$ P1 ; ※15 $\int x \cdot \sec x \cdot \tan x \cdot dx$ ☆☆ P6

☆☆☆+ ※8 $\int t \sin t \cdot \cos t \cdot dt$ P3 ; ※17 $\int_0^{\pi} t \cdot \cos^2 t \cdot dt$ P7.

☆☆☆+ ※14 $\int \ln(1+x^2) dx$ P6 ; ※21 $\int \tan^{-1}(\sqrt{x}) dx$. P8

- Other

※5. $\int \frac{t}{t^4+2} dt$ ☆☆☆ P2 ; ※6 $\int_0^1 \frac{x}{(2x+1)^3} dx$ ☆☆☆ P2 ; ※23 $\int_0^1 (1+\sqrt{x})^8 dx$, P11

Ex § Section 7.5 (Ed8 Ed7)

☆☆+1. $\int \frac{\cos x}{1-\sin x}dx$, $u=\sin x$, $du=\cos x \cdot dx$

$= \int \frac{1}{1-\sin x}\cdot \cos x \cdot dx = \int \frac{1}{1-u}du = \int \frac{1}{-u+1}du$

$= \frac{1}{-1}\cdot \ln|-u+1|$

$= -\ln|-u+1|$

$\boxed{= -\ln|-\sin x+1| + C}$

Hint: formula $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$ with $a=-1$, $b=1$

Or: $u=1-\sin x$, $du=-\cos x \cdot dx \Rightarrow -du=\cos x\cdot dx$.

$\int \frac{\cos x}{1-\sin x}dx = \int \frac{-du}{u} = -\int \frac{du}{u} = -\ln|u| = \boxed{-\ln|1-\sin x| + C}$

☆+2. $\int_0^1 (3x+1)^{\sqrt{2}}dx$ $u=3x+1$, $du=3\cdot dx$

$= \int_1^4 u^{\sqrt{2}}\cdot \frac{du}{3} = \int_{u=1}^{u=4} \frac{1}{3}u^{\sqrt{2}}du = \frac{1}{3}\cdot\frac{1}{\sqrt{2}+1}\cdot u^{\sqrt{2}+1}\Big|_1^4$

$= \boxed{\frac{1}{3}\cdot\frac{1}{\sqrt{2}+1}\cdot 4^{\sqrt{2}+1} - \frac{1}{3}\cdot\frac{1}{\sqrt{2}+1}}$

☆☆+3. $\int_1^4 \sqrt{y}\cdot \ln y\, dy$ (IBP). $u=\ln y$, $dv=\sqrt{y}\cdot dy$

$du=\frac{1}{y}dy$, $v=\frac{2}{3}\cdot y^{\frac{3}{2}}$

$= \int u\cdot dv = u\cdot v - \int v\cdot du$

$= \ln y\cdot\frac{2}{3}\cdot y^{\frac{3}{2}} - \int \frac{2}{3}\cdot y^{\frac{3}{2}}\cdot\frac{1}{y}dy$

$= \ln y\cdot\frac{2}{3}\cdot y^{\frac{3}{2}} - \int \frac{2}{3}\cdot y^{\frac{1}{2}}dy = \ln y\cdot\frac{2}{3}y^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{3}\cdot y^{\frac{3}{2}}\Big|_1^4$

$= \boxed{\ln 4\cdot\frac{2}{3}\cdot 4^{\frac{3}{2}} - \frac{4}{9}\cdot 4^{\frac{3}{2}} - \left(0 - \frac{4}{9}\right)}$

$= \frac{32}{3}\ln 2 - \frac{28}{9}$

Hint: $4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^3 = 8$

$\ln 4 = \ln 2^2 = 2\cdot\ln 2$

★★★+ 4. $\int \frac{\sin^3 x}{\cos x}\,dx$, $u=\cos x$, $du=-\sin x \cdot dx \Rightarrow \frac{du}{-\sin x}=dx$

$$= \int \frac{\sin^3 x}{u}\cdot\frac{du}{-\sin x} = \int \frac{-\sin^2 x}{u}\,du$$

$\sin^2 x = 1-\cos^2 x$
$= 1-u^2$
$-\sin^2 x = u^2-1$

$$= \int \frac{u^2-1}{u}\,du$$

$$= \int u - \frac{1}{u}\,du = \frac{1}{2}u^2 - \ln|u| + C$$

$$= \boxed{\frac{1}{2}\cdot\cos^2 x - \ln|\cos x| + C}$$

★★★★ 5. $\int \frac{t}{t^4+2}\,dt$, $u=t^2$, $du=2t\,dt$
$t^4=u^2$

$$= \int \frac{\frac{1}{2}\cdot du}{u^2+2}$$

$$= \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right)+C$$

Hint: formula
$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
with $a=\sqrt{2}$.

★★★6. $\int_0^1 \frac{x}{(2x+1)^3}\,dx$. $u=2x+1$, $du=2\cdot dx$, $\int_{x=0}^{x=1} \xrightarrow{u=2x+1} \int_{u=1}^{u=3}$

$\boxed{x=\frac{u-1}{2}}$

$$= \int_{u=1}^{u=3} \frac{x\,\frac{1}{2}du}{u^3}$$

$$= \int_{u=1}^{u=3} \frac{\frac{u-1}{2}\cdot\frac{1}{2}du}{u^3} = \int_{u=1}^{u=3} \frac{u-1}{4\cdot u^3}\,du = \frac{1}{4}\int_1^3 \frac{u}{u^3} - \frac{1}{u^3}\,du$$

$$= \frac{1}{4}\cdot\int_1^3 u^{-2} - u^{-3}\,du$$

$$= \frac{1}{4}\cdot\left(\frac{1}{-1}\cdot u^{-1} - \frac{1}{-2}u^{-2}\right)\Big|_1^3$$

$$\boxed{= \frac{1}{4}\left(\frac{1}{2}\cdot\frac{1}{9} - \frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{2}-1\right)}$$

$$= \frac{1}{18}$$

☆☆ 7. $\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2} \cdot dy$. $u = \arctan y$, $du = \frac{1}{1+y^2} dy$

$= \int_{-1}^{1} e^u \cdot du = e^u = e^{\arctan y} \Big|_{-1}^{1} = e^{\arctan 1} - e^{\arctan(-1)}$

$= e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}}$

(Hint: $\tan\frac{\pi}{4} = 1$, $\tan(-\frac{\pi}{4}) = -1$)

☆☆☆+ 8. $\int t \cdot \sin t \cdot \cos t \, dt$.   Hint: $\sin t \cdot \cos t = \frac{1}{2}\sin 2t$

$= \int t \cdot \frac{1}{2}\sin 2t \cdot dt$ IBP. $u = \frac{t}{2}$, $dv = \sin 2t \cdot dt$

$du = \frac{1}{2} \cdot dt$, $v = \frac{1}{-2} \cdot \cos 2t$

$= u \cdot v - \int v \, du$

$= \frac{t}{2} \cdot (\frac{1}{-2} \cdot \cos 2t) - \int \frac{1}{-2} \cdot \cos 2t \cdot \frac{1}{2} dt$

$= -\frac{1}{4} t \cdot \cos 2t + \frac{1}{4} \cdot \int \cos 2t \, dt = \boxed{-\frac{1}{4} t \cdot \cos 2t + \frac{1}{4} \frac{1}{2} \sin 2t + C}$

☆☆+ 9. $\int_2^4 \frac{x+2}{x^2+3x-4} \cdot dx$ PF. $\frac{x+2}{x^2+3x-4} = \frac{x+2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$x+2 = A(x-1) + B(x+4)$

$x = -4$, $-2 = A \cdot (-5) \Rightarrow A = \frac{2}{5}$

$x = 1$, $3 = B \cdot 5 \Rightarrow B = \frac{3}{5}$

$= \int_2^4 \frac{\frac{2}{5}}{x+4} + \frac{\frac{3}{5}}{x-1} dx$

$= \frac{2}{5} \cdot \ln|x+4| + \frac{3}{5} \ln|x-1| \Big|_2^4$

$= \boxed{\frac{2}{5} \cdot \ln 8 + \frac{3}{5} \cdot \ln 3 - (\frac{2}{5}\ln 6 + \frac{3}{5}\ln 1)}$ Hint: $\ln 8 = \ln 2^3 = 3\ln 2$
$\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$

$= \frac{4}{5}\ln 2 + \frac{1}{5}\ln 3$

☆☆☆+ 10. $\int \frac{\cos(\frac{1}{x})}{x^3} dx$ $\quad u = \frac{1}{x}, \quad du = -\frac{1}{x^2} dx.$

$-x^2 \cdot du = dx.$

$= \int \frac{\cos u}{x^3} \cdot (-x^2) \, du$

$= \int \cos u \cdot (-\frac{1}{x}) \cdot du = \int \cos u \cdot (-u) \cdot du.$ $\quad w = -u, \quad dv = \cos u \cdot du$

$dw = -du, \quad v = \sin u.$

$= \int w \cdot dv$

$= w \cdot v - \int v \cdot dw$

$= -u \cdot \sin u - \int \sin u \cdot (-du)$

$= -u \cdot \sin u + \int \sin u \cdot du = \boxed{-u \cdot \sin u - \cos u + C}$

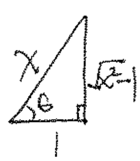
$= \boxed{-\frac{1}{x} \sin(\frac{1}{x}) - \cos(\frac{1}{x}) + C}$

☆☆☆+ 11. $\int \frac{1}{x^3 \sqrt{x^2-1}} dx$ $\quad$ Trig-Sub. $x = \sec\theta, \quad dx = \sec\theta \cdot \tan\theta \cdot d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta.$

$= \int \frac{1}{\sec^3\theta \cdot \tan\theta} \cdot \sec\theta \cdot \tan\theta \, d\theta$

$= \int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta \cdot d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$



$\sec\theta = x = \frac{x}{1}, \quad \theta = \sec^{-1}(x)$

$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$\cos\theta = \frac{1}{x}.$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$= 2 \cdot \frac{\sqrt{x^2-1}}{x} \cdot \frac{1}{x}.$

$= \int \frac{1}{2} + \frac{1}{2} \cdot \cos 2\theta \, d\theta$

$= \frac{1}{2}\theta + \frac{1}{2} \cdot \frac{1}{2} \sin 2\theta + C$

$= \boxed{\frac{1}{2} \sec^{-1}(x) + \frac{1}{4} \cdot 2 \cdot \frac{\sqrt{x^2-1}}{x} \cdot \frac{1}{x} + C}$

☆☆☆☆ 12. $\int \frac{2x-3}{x^3+3x}\,dx$. P.F.: $\frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx}{x^2+3} + \frac{C}{x^2+3}$

P.F. $\int \frac{-1}{x} + \frac{x}{x^2+3} + \frac{2}{x^2+3}\,dx$

$2x-3 = A(x^2+3) + Bx\cdot x + C\cdot x$.

$2x-3 = (A+B)^2 + Cx + 3A$

$A+B=0$, $C=2$, $3A=-3$

$B=1 \longleftarrow A=-1$

$\int \frac{-1}{x}dx = -\ln|x| + C$

$\int \frac{x\,dx}{x^2+3} \quad \frac{u=x^2+3}{du=2x\,dx} \quad \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\cdot\ln|x^2+3| + C$

$\int \frac{2\,dx}{x^2+3} \quad \frac{\text{formula:}}{\int\frac{dx}{x^2+a^2} = \frac{1}{a}\cdot\tan^{-1}(\frac{x}{a})}$ with $a=\sqrt{3}$ $\quad 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}})$.

$= \boxed{-\ln|x| + \frac{1}{2}\cdot\ln|x^2+3| + \frac{2}{\sqrt{3}}\cdot\tan^{-1}(\frac{x}{\sqrt{3}}) + C}$

☆☆+ 13. $\int \sin^5 t\cdot\cos^4 t\cdot dt$ sin odd: $u=\cos t$, $du=-\sin t\cdot dt$

$\sin^2 t = 1-\cos^2 t = 1-u^2$

$\sin^4 t = (1-\cos^2 t)^2 = (1-u^2)^2$

$= \int \sin^5 t\cdot u^4\cdot\frac{du}{-\sin t} = \int -\sin^4 t\cdot u^4\cdot du$

$= \int -(1-u^2)^2\cdot u^4\cdot du$

$= \int -(1-2u^2+u^4)\cdot u^4\cdot du$

$= \int -u^4 + 2\cdot u^6 - u^8\,du$

$= -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$

$= \boxed{-\frac{1}{5}\cos^5 t + \frac{2}{7}\cdot\cos^7 t - \frac{1}{9}\cdot\cos^9 t + C}$

☆☆☆+ 14. $\int \ln(1+x^2)\cdot dx$,

Try: ① u-sub: $u=\ln(1+x^2)$, $du=\frac{2x}{1+x^2}dx$, $\int u\cdot\frac{(1+x^2)}{2x}\cdot du$ DOES NOT WORK!

Try ② IBP: $u=\ln(1+x^2)$, $du=\frac{2x}{1+x^2}dx$

$dv=dx$. $v=x$.

$$\int \ln(1+x^2)\cdot dx = \ln(1+x^2)\cdot x - \int x\cdot\frac{2x}{1+x^2}dx$$

$$= \ln(1+x^2)\cdot x - \int \frac{2x^2}{1+x^2}dx.$$

long division: $\frac{2x^2}{1+x^2} = 2 + \frac{-2}{1+x^2}$

$$= \ln(1+x^2)\cdot x - \int 2 + \frac{-2}{1+x^2}dx$$

$$= \ln(1+x^2)\cdot x - 2x + 2\int \frac{1}{1+x^2}dx.$$

$$= \boxed{\ln(1+x^2)\cdot x - 2x + 2\tan^{-1}x + C}$$

☆☆+ 15. $\int x\cdot\sec x\cdot\tan x\cdot dx$ IBP: $u=x$. $dv=\sec x\cdot\tan x\,dx$

$du=dx$, $v=\sec x$.

$$= u\cdot v - \int v\cdot du$$

$$= x\cdot\sec x - \int \sec x\cdot dx = \boxed{x\sec x - \ln|\tan x+\sec x| + C}$$

☆☆☆+ 16 $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}dx$. $x=\sin\theta$, $dx=\cos\theta\cdot d\theta$, $\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta$.

$$= \int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\cdot d\theta$$

$\int_{x=0}^{x=\sqrt{2}/2} \xrightarrow{x=\sin\theta} \int_{\theta=0}^{\frac{\sqrt{2}}{2}=\sin\theta,\ \theta=\frac{\pi}{4}}$.

$$= \int \sin^2\theta\, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta \Big|_0^{\pi/4} = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\sin(2\cdot\frac{\pi}{4}) - 0 = \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

☆☆☆☆ 17. $\int_0^{\pi} t \cdot \cos^2 t \cdot dt$. Double angle formula first.

$= \int t \cdot \frac{1+\cos 2t}{2} \cdot dt$

$= \int \frac{1}{2}t + \frac{t}{2} \cdot \cos 2t \, dt$

$= \frac{1}{2} \cdot \frac{1}{2}t^2 + \int \frac{t}{2} \cdot \cos 2t \, dt$ IBP: $u = \frac{t}{2}$, $dv = \cos 2t \, dt$

$du = \frac{1}{2}dt$, $v = \frac{1}{2}\sin 2t$

$= \frac{1}{4}t^2 + \left[u \cdot v - \int v \cdot du \right]$

$= \frac{1}{4}t^2 + \left[\frac{t}{2} \cdot \frac{1}{2}\sin 2t - \int \frac{1}{2}\sin 2t \cdot \frac{1}{2} \cdot dt \right]$

$= \frac{1}{4} \cdot t^2 + \left[\frac{t}{4} \cdot \sin 2t - \frac{1}{4} \cdot \left(-\frac{1}{2}\cos 2t\right) \right] \Big|_0^{\pi}$

$= \frac{1}{4} \cdot t^2 + \frac{t}{4} \cdot \sin 2t + \frac{1}{8}\cos 2t \Big|_0^{\pi}$

$= \frac{1}{4} \cdot \pi^2 + \frac{\pi}{4} \cdot \sin 2\pi + \frac{1}{8}\cos 2\pi - \left(0 + 0 + \frac{1}{8} \cdot \cos 0\right)$. $\sin 2\pi = 0$, $\cos 2\pi = \cos 0 = 1$

$= \boxed{\frac{1}{4}\pi^2}$

☆☆+ 18. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} \cdot dt$ u-sub: $u = \sqrt{t}$, $du = \frac{1}{2} \cdot t^{-\frac{1}{2}} dt$

$= \frac{1}{2\sqrt{t}} dt$

$2du = \frac{1}{\sqrt{t}} \cdot dt$

$= \int e^{\sqrt{t}} \cdot \boxed{\frac{1}{\sqrt{t}} \cdot dt}$

$= \int_1^2 e^u \cdot 2du$ $\int_{t=1}^{t=4} \xrightarrow{u=\sqrt{t}} \int_{u=\sqrt{1}=1}^{u=\sqrt{4}=2}$

$= 2 \cdot e^u \Big|_1^2$

$= \boxed{2 \cdot e^2 - 2 \cdot e^1}$

☆☆+ 19. $\int e^{x+e^x}\,dx$. $u=e^x$, $du=e^x\cdot dx$

$=\int e^x\cdot e^{e^x}\,dx$

$=\int e^{e^x}\cdot \underline{e^x dx} = \int e^u\cdot du = e^u = e^{e^x}+C$

20. $\int e^2\,dx$ Hint: e^2 constant.

$= e^2\cdot x + C$

☆☆
☆☆+ 21. $\int \arctan(\sqrt{x})\cdot dx$. (similar to ☆14)

IBP: $u=\arctan\sqrt{x}$, $dv=dx$.

$du=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\,dx$, $v=x$.

$=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,dx$.

$=\tan^{-1}\sqrt{x}\cdot x-\int x\cdot\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,dx$

$=\tan^{-1}\sqrt{x}\cdot x-\int\frac{1}{2}\cdot\frac{\sqrt{x}}{1+x}\,dx$

$=\tan^{-1}\sqrt{x}\cdot x-\left[\sqrt{x}-\tan^{-1}\sqrt{x}\right]+C$

$=\boxed{\tan^{-1}\sqrt{x}\cdot x-\sqrt{x}+\tan^{-1}\sqrt{x}+C}$

u-sub again: $u=\sqrt{x}$, $du=\frac{1}{2\sqrt{x}}\,dx$.

$\int\frac{1}{2}\frac{\sqrt{x}}{1+x}\,dx=\int\frac{1}{2}\cdot\frac{u}{1+u^2}\cdot\underline{2\sqrt{x}\,du}$

$=\int\frac{u^2}{1+u^2}\,du$

$=\int 1-\frac{1}{1+u^2}\,du$

$=u-\tan^{-1}u$

$=\sqrt{x}-\tan^{-1}\sqrt{x}$.

Remark: you can u-sub $\sqrt{x}$ first, then IBP, for this problem.

☆☆ ☆☆+ 22. $\int \frac{\ln x}{x\cdot\sqrt{1+(\ln x)^2}}\cdot dx$. u-sub first: $u=\ln x$, $du=\frac{1}{x}dx$

$= \int \frac{\ln x}{\sqrt{1+(\ln x)^2}}\,\frac{1}{x}dx$.

$= \int \frac{u}{\sqrt{1+u^2}}\cdot du$. u-sub again, $v=1+u^2$
$dv = 2\cdot u\cdot du$

$= \int \frac{\frac{1}{2}\cdot dv}{\sqrt{v}}$

$= \int \frac{1}{2}\cdot v^{-\frac{1}{2}}dv = \frac{1}{2}\cdot 2\cdot v^{\frac{1}{2}} \overset{v=1+u^2}{=} (1+u^2)^{\frac{1}{2}}$

$\overset{u=\ln x}{=} \boxed{(1+(\ln x)^2)^{\frac{1}{2}} + C}$

☆☆☆+ 23. $\int_0^1 (1+\sqrt{x})^8 dx$. $u=1+\sqrt{x}$. $du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\cdot du$
$= 2(u-1)\cdot du$

$= \int_{u=1+\sqrt{0}}^{u=1+\sqrt{1}} u^8\cdot 2\cdot(u-1)du$

$= \int_1^2 2u^9 - 2u^8\, du = 2\cdot\frac{1}{10}\cdot u^{10} - 2\cdot\frac{1}{9}\cdot u^9 \Big|_1^2$

$= \boxed{\frac{1}{5}\cdot 2^{10} - \frac{2}{9}\cdot 2^9 - \left(\frac{1}{5}-\frac{2}{9}\right)}$

$= \frac{4097}{45}$

☆+ 25. $\int_0^1 \frac{1+12t}{1+3t}dt$ long division: $3t+1 \overline{)\,12t+1}$, quotient 4; $12t+4$; remainder -3. $\frac{1+12t}{1+3t} = \frac{4(3t+1)-3}{3t+1}$
$= 4 - \frac{3}{3t+1}$

$= \int_0^1 4 - \frac{3}{3t+1}\,dt$

$= 4t - 3\cdot\frac{1}{3}\ln|3t+1|\Big|_0^1 = 4 - \ln 4 - (0-\ln 1) = \boxed{4-\ln 4}$

(Hint: formula $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|$)

☆☆☆☆☆+ 26. $\int \frac{3x^2+1}{x^3+x^2+x+1}\,dx.$

Factorize : $(x^3+x^2)+(x+1) = x^2(x+1) + x+1$

$= (x^2+1)\cdot(x+1)$

P.F. $\frac{3x^2+1}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx}{x^2+1} + \frac{C}{x^2+1}$

multiply by $(x^2+1)(x+1)$ $\quad 3x^2+1 = A(x^2+1) + Bx\cdot(x+1) + C(x+1)$

$3x^2+1 = (A+B)x^2 + (B+C)\cdot x + A+C$

$3x^2: \; A+B=3$
$0\cdot x: \; B+C=0$
$1: \; A+C=1$

$\Rightarrow A-C=3,\; A+C=1 \Rightarrow A=2,\; C=-1 \Rightarrow B=1$

$\frac{3x^2+1}{(x^2+1)(x+1)} = \frac{2}{x+1} + \frac{x}{x^2+1} + \frac{-1}{x^2+1}$

$\int \frac{2}{x+1}\,dx = 2\ln|x+1|$

$\int \frac{x}{x^2+1}\,dx \xrightarrow[du=2x\,dx]{u=x^2+1} \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u| \xrightarrow{u=x^2+1} \frac{1}{2}\ln|x^2+1|$

$\int \frac{-1}{x^2+1}\,dx = -\tan^{-1}x.$

$\int \frac{3x^2+1}{x^3+x^2+x+1}\,dx \xrightarrow{\text{Step 1: P.F.}} \int \frac{2}{x+1} + \frac{x}{x^2+1} + \frac{-1}{x^2+1}\,dx$

$= \boxed{2\ln|x+1| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + C}$

★★★★ 7. $\int \frac{dx}{1+e^x}$ $u = 1+e^x$, $du = e^x \cdot dx$.

$$\overset{u=1+e^x}{=\!=} \int \frac{1}{u} \cdot dx \overset{dx=\frac{du}{e^x}}{=\!=} \int \frac{1}{u} \cdot \frac{du}{e^x} \overset{e^x=u-1}{=\!=} \int \frac{1}{u} \cdot \frac{1}{u-1} du$$

PF. $\frac{1}{u \cdot (u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{-1}{u} + \frac{1}{u-1}$

$1 = A(u-1) + B \cdot u$, $u=1 \Rightarrow B = 1$

$u = 0 \Rightarrow A = -1$

$$\overset{P.F.}{=\!=} \int \frac{-1}{u} + \frac{1}{u-1} du$$

$$= -\ln|u| + \ln|u-1| + C$$

$$= -\ln|1+e^x| + \ln|e^x+1-1| + C$$

$$= -\ln|1+e^x| + \ln e^x + C$$

$$= -\ln|1+e^x| + x + C.$$

Remark: Substitute $u = e^x$ also works: $du = e^x dx$

$$\int \frac{dx}{1+e^x} \overset{u=e^x}{=\!=} \int \frac{1}{1+u} \cdot dx \overset{dx=\frac{du}{e^x}=\frac{1}{u}du}{=\!=} \int \frac{1}{1+u} \cdot \frac{1}{u} du$$

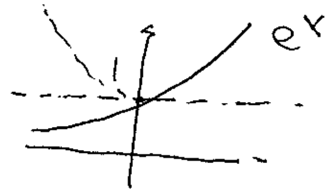
$$= \int \frac{1}{u} - \frac{1}{u+1} du$$

$$= \ln|u| - \ln|u+1| + C$$

$$\overset{u=e^x}{=\!=} \ln e^x - \ln(1+e^x) + C$$

$$= x - \ln(1+e^x) + C$$

☆☆☆+30. $\int_{-1}^{2} |e^x - 1| \, dx$.



$e^x > 1, \ x > 0$
$e^x < 1, \ x < 0$

$\Rightarrow |e^x - 1| = \begin{cases} e^x - 1, & x > 0 \\ 1 - e^x, & x < 0 \end{cases}$

$$= \int_{-1}^{0} |e^x - 1| \, dx + \int_{0}^{2} |e^x - 1| \, dx$$

$$= \int_{-1}^{0} (1 - e^x) \, dx + \int_{0}^{2} (e^x - 1) \, dx$$

$$= x - e^x \Big|_{-1}^{0} + e^x - x \Big|_{0}^{2}$$

$$= [0 - e^0 - (-1 - e^{-1})] + [e^2 - 2 - (e^0 - 0)]$$

$$= \boxed{[-1 + 1 + e^{-1}] + [e^2 - 2 - 1]} = \boxed{e^{-1} + e^2 - 3}$$

☆☆+32. $\int_{1}^{3} \frac{e^{3/x}}{x^2} \, dx$. u-sub, $u = \frac{3}{x}$, $du = \frac{-3}{x^2} dx$,

$\frac{du}{-3} = \frac{1}{x^2} dx$,

$$= \int_{1}^{3} e^{\frac{3}{x}} \cdot \frac{1}{x^2} dx,$$

$$= \int_{u=\frac{3}{1}}^{u=\frac{3}{3}} e^u \cdot \left(\frac{du}{-3}\right) = \frac{1}{-3} \cdot \int_{3}^{1} e^u \, du = \frac{1}{-3} \cdot e^u \Big|_{3}^{1} = \frac{1}{-3} \cdot e^1 - \left(\frac{1}{-3} \cdot e^3\right)$$

$$= \boxed{\frac{1}{3} e^3 - \frac{1}{3} e^1}$$

☆☆+37. $\int_{0}^{\pi/4} \tan^3\theta \cdot \sec^2\theta \cdot d\theta$, $u = \tan\theta$, $du = \sec^2\theta \cdot d\theta$

$$= \int_{0}^{1} u^3 \, du \qquad \int_{u=\tan 0 = 0}^{u = \tan\frac{\pi}{4} = 1}$$

$$= \frac{1}{4} \cdot u^4 \Big|_{0}^{1} = \frac{1}{4}$$

☆☆☆+ 38. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cdot \cot\theta}{\sec\theta}\, d\theta$ Hint: $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

$\frac{1}{\sec\theta} = \cos\theta$

$$= \int_{\pi/6}^{\pi/3} \sin\theta \cdot \frac{\cos\theta}{\sin\theta} \cdot \cos\theta \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \cos^2\theta \cdot d\theta = \int_{\pi/6}^{\pi/3} \frac{1+\cos 2\theta}{2} = \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta \Big|_{\pi/6}^{\pi/3}$$

Hint: $\sin\frac{\pi}{3} = \sin\frac{2}{3}\pi = \frac{\sqrt{3}}{2}$

$$= \frac{1}{2}\cdot\frac{\pi}{3} + \frac{1}{4}\sin\frac{2}{3}\pi - \left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{4}\sin\frac{1}{3}\pi\right)$$

$$= \boxed{\frac{\pi}{12}}$$

☆☆☆☆+ 39. $\displaystyle\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\, d\theta$ $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$$= \int \frac{1}{\sec^2\theta - \sec\theta} \cdot \boxed{\sec\theta\tan\theta\, d\theta}$$

$$= \int \frac{1}{u^2 - u}\, du \overset{P.F.}{=} \int \frac{1}{u(u-1)}\, du$$

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

$u=1$, $B=1$

$u=0$ $A=-1$

$$= \int \frac{A}{u} + \frac{B}{u-1}\, du$$

$$= \int \frac{-1}{u} + \frac{1}{u-1}\, du$$

$$= -\ln|u| + \ln|u-1| + C$$

$$= \boxed{-\ln|\sec\theta| + \ln|\sec\theta - 1| + C}$$

☆☆☆ ☆☆+ 42. $\int \frac{\tan^{-1}x}{x^2}\,dx$. IBP: $u=\tan^{-1}x$, $dv=\frac{1}{x^2}dx$.

$du=\frac{1}{1+x^2}dx$, $v=-\frac{1}{x}$

$= \int u \cdot dv$

$= u\cdot v - \int v\cdot du = \tan^{-1}x\cdot(-\frac{1}{x}) - \int -\frac{1}{x}\cdot\frac{1}{x^2+1}\,dx$.

(P.F.)

P.F. $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx}{x^2+1} + \frac{C}{x^2+1}$

$1 = A(x^2+1) + Bx^2 + Cx$

$= (A+B)x^2 + C\cdot x + A$

$\Rightarrow$ $A+B=0$, $B=-1$; $C=0$; $A=1$

$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$

$= \tan^{-1}x\cdot(-\frac{1}{x}) + \int \frac{1}{x(x^2+1)}\,dx$

$= -\tan^{-1}x\cdot\frac{1}{x} + \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx$

$= -\tan^{-1}x\cdot\frac{1}{x} + \ln|x| - \frac{1}{2}\ln|x^2+1| + C$

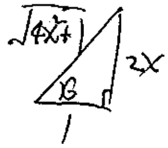
☆☆☆☆☆+ 51. $\int \frac{1}{x\cdot\sqrt{4x^2+1}}\,dx$. Trig-Sub: $\sqrt{4x^2+1} = \sqrt{(2x)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$

$2x=\tan\theta$

$x=\frac{\tan\theta}{2}$, $dx=\frac{1}{2}\sec^2\theta\,d\theta$

$= \int \frac{1}{\frac{1}{2}\tan\theta\cdot\sec\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$

$= \int$ [scribbled out] $\cdot\frac{1}{\tan\theta}\cdot\sec\theta\,d\theta$



$= \int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}\,d\theta$

$= \int \csc\theta\cdot d\theta = -\ln|\cot\theta+\csc\theta| + C = -\ln\left|\frac{1}{2x} + \frac{\sqrt{4x^2+1}}{2x}\right| + C$

☆☆☆☆ 60. $\int \frac{dx}{x^2 \cdot \sqrt{4x^2-1}}$

$\sqrt{4x^2-1} = \sqrt{(2x)^2-1} = \sqrt{\sec^2\theta - 1}$

$2x = \sec\theta$

$= \sqrt{\tan^2\theta}$

$= \tan\theta$

$x = \frac{1}{2} \cdot \sec\theta$

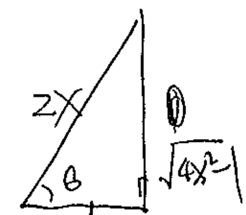
$dx = \frac{1}{2} \cdot \tan\theta \cdot \sec\theta \, d\theta$

$= \int \frac{1}{(\frac{1}{2}\sec\theta)^2 \cdot \tan\theta} \cdot \frac{1}{2} \tan\theta \cdot \sec\theta \cdot d\theta$

$= \int \frac{1}{\frac{1}{4} \cdot \sec^2\theta \cdot \tan\theta} \cdot \frac{1}{2} \cdot \tan\theta \sec\theta \cdot d\theta$

$= \int \frac{2}{\sec\theta} \cdot d\theta = \int 2 \cdot \cos\theta \cdot d\theta = 2 \cdot \sin\theta + C$

$= \boxed{2 \cdot \frac{\sqrt{4x^2-1}}{2x} + C}$



$\sec\theta = 2x = \frac{2x}{1}$

$\Rightarrow \sin\theta = \frac{\sqrt{4x^2-1}}{2x}$

☆☆☆+ 71. $\int \frac{e^{2x}}{1+e^x} dx$.

u-sub

$u = e^x$, $du = e^x dx$.

$u^2 = (e^x)^2 = e^{2x}$

$\overset{u=e^x}{=} \int \frac{u^2}{1+u} \cdot \frac{du}{e^x} = \int \frac{u^2}{1+u} \cdot \frac{du}{u} = \int \frac{u}{1+u} du$

$= \int 1 - \frac{1}{1+u} du$ (long division)

$= u - \ln|1+u| + C$

$= \boxed{e^x - \ln(1+e^x) + C}$