Solution: HW §7.4. Ed8, Ed7.

☆☆☆ 7. $\int \frac{x^4}{x-1}\,dx$.

Long division

$$\begin{array}{r} x^3+x^2+x+1 \\ x-1 \,\big)\, \overline{x^4+0+0+0+0} \\ \underline{x^4-x^3} \\ x^3+0+0+0 \\ \underline{x^3-x^2} \\ x^2+0+0 \\ \underline{x^2-x} \\ x+0 \\ \underline{x-1} \\ 1 \end{array}$$

$x^4=(x-1)\cdot(x^3+x^2+x+1)+1$

$$\boxed{\frac{x^4}{x-1}=(x^3+x^2+x+1)+\frac{1}{x-1}}$$

$$\int \frac{x^4}{x-1}dx=\int (x^3+x^2+x+1)+\frac{1}{x-1}\,dx=\boxed{\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C}$$

☆☆ 8 $\int \frac{3t-2}{t+1}\,dt$

$$\begin{array}{r} 3 \\ t+1 \,\big)\, \overline{3t-2} \\ \underline{3t+3} \\ -5 \end{array}$$

$(3t-2)=(t+1)\cdot 3 -5$

$$\boxed{\frac{3t-2}{t+1}=3+\frac{-5}{t+1}}$$

$$\int \frac{3t-2}{t+1}dt=\int 3+\frac{-5}{t+1}\,dt=\boxed{3t-5\ln|t+1|+C}$$

☆☆ half 9. $\int \frac{5x+1}{(2x+1)(x-1)}\,dx$.

$=\int \frac{A}{(2x+1)}+\frac{B}{(x-1)}\,dx$, $\begin{cases}A=1\\B=2\end{cases}$

$=\int \frac{1}{2x+1}+\frac{2}{x-1}\,dx$

$=\boxed{\frac{1}{2}\ln|2x+1|+2\cdot\ln|x-1|+C}$

Partial Fraction: $\frac{5x+1}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}$

Cross-multiply: $5x+1=A(x-1)+B(2x+1)$

$x=-\frac{1}{2}$, $-\frac{5}{2}+1=A(-\frac{1}{2}-1)+0$, $\Rightarrow A=1$

$x=1$, $5+1=0+B(2+1)$, $\Rightarrow B=2$

☆☆ half. 10. $\displaystyle\int \frac{y}{(y+4)(2y-1)} \cdot dy$

P.F.

$$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$$

Cross-multiply:

$$y = A(2y-1) + B\cdot(y+4)$$

$$y=-4 \Rightarrow -4 = A\cdot(-8-1)+0, \Rightarrow A=\frac{4}{9}$$

$$y=\frac{1}{2} \Rightarrow \frac{1}{2} = 0 + B\cdot\left(\frac{1}{2}+4\right) \Rightarrow B=\frac{1}{9}$$

$$= \int \frac{A}{y+4} + \frac{B}{2y-1}\, dy$$

$$= \int \frac{\frac{4}{9}}{y+4} + \frac{\frac{1}{9}}{2y-1}\, dy$$

$$= \boxed{\frac{4}{9}\cdot \ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C}$$

☆☆☆ 11. $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\, dx$

Factorize the denominator

$$(2x^2+3x+1) = (2x+1)(x+1)$$

P.F.

$$\frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

CrossM.

$$2 = A(x+1) + B(2x+1)$$

$$x=-\frac{1}{2},\quad 2 = A\cdot\left(-\frac{1}{2}+1\right)+0, \Rightarrow A=4$$

$$x=-1,\quad 2 = 0 + B\cdot(-2+1) \Rightarrow B=-2.$$

$$= \int \frac{A}{2x+1} + \frac{B}{x+1}\, dx$$

$$= \int \frac{4}{2x+1} + \frac{-2}{x+1}\, dx$$

$$= \boxed{4\cdot\frac{1}{2}\ln|2x+1| - 2\cdot\ln|x+1| \Big|_0^1 \cancel{+C}}$$

$$= 2\ln|2x+1| - 2\ln|x+1| \Big|_0^1$$

$$= 2\cdot\ln 3 - 2\cdot\ln 2 - (2\ln 1 - 2\ln 1)$$

$$= \boxed{2\ln 3 - 2\ln 2}$$

☆☆☆ 12. $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

$$= \int_0^1 \frac{x-4}{(x-2)(x-3)}\,dx$$

$$= \int_0^1 \frac{A}{x-2} + \frac{B}{x-3}\,dx.$$

$$= \int_0^1 \frac{2}{x-2} + \frac{-1}{x-3}\,dx$$

$$= 2\int_0^1 \frac{1}{x-2}\,dx - \int_0^1 \frac{1}{x-3}\,dx$$

Factorize: $x^2-5x+6 = (x-2)(x-3)$.

P.F.:

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Cross-multiply:

$$x-4 = A(x-3) + B\cdot(x-2)$$

$x=2$, $2-4 = A\cdot(2-3)+0$, $A=2$

$x=3$. $3-4 = 0 + B\cdot 1$, $B=-1$

$$= 2\ln|x-2| - \ln|x-3|\Big|_0^1 = 2\ln 1 - \ln|2| - [2\ln 2 - \ln 3]$$

$$= 0 - \ln 2 - 2\ln 2 + \ln 3 = \boxed{-3\ln 2 + \ln 3}$$

☆☆☆half. 15. (Ed8) $\displaystyle\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$

Step 1: Long division

$$\begin{array}{r} x+3 \\ x^2-3x+2 \,\big)\, \overline{x^3+0-4x+1} \\ \underline{x^3-3x^2+2x} \\ 3x^2-6x+1 \\ \underline{3x^2-9x+6} \\ 3x-5 \end{array}$$

$x^3-4x+1 = (x^2-3x+2)(x+3) + (3x-5)$

$$\frac{x^3-4x+1}{x^2-3x+2} = x+3 + \frac{3x-5}{x^2-3x+2}.$$

Step 2: Factorize + P.F. $x^2-3x+2 = (x-1)(x-2)$

$$\frac{3x-5}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$3x-5 = A(x-2) + B(x-1)$$

$x=1$ $-2 = A(-1)+0$, $A=2$

$x=2$, $1 = 0 + B\cdot 1$, $B=1$.

$$\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$$

$$= \int_{-1}^{0} x+3 + \frac{2}{x-1} + \frac{1}{x-2}\,dx$$

$$= \frac{1}{2}x^2 + 3x + 2\ln|x-1| + \ln|x-2|\Big|_{-1}^{0}$$

$$= 0 + 2\ln 1 + \ln 2 - \left(\frac{1}{2} - 3 + 2\ln 2 + \ln 3\right)$$

$$\cancel{= \frac{5}{2} - 2\ln 3}$$

$$= \ln 2 - \left(-\frac{5}{2} + 2\ln 2 + \ln 3\right)$$

$$= \boxed{-\ln 2 + \frac{5}{2} - \ln 3}$$

★★★★ 17. $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

Partial Fractions: $\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3}$

Cross-multiply by $y(y+2)(y-3)$

$$y\cdot(y+2)(y-3)\cdot\frac{4y^2-7y-12}{y\cdot(y+2)(y-3)} = \frac{A}{y}\cdot y\cdot(y+2)(y-3) + \frac{B}{y+2}\cdot y\cdot(y+2)(y-3) + \frac{C}{y-3}\cdot y\cdot(y+2)(y-3)$$

$$4y^2-7y-12 = A\cdot(y+2)(y-3) + B\cdot y\cdot(y-3) + C\cdot y\cdot(y+2).$$

$y=0$, $\quad -12 = A\cdot(2)\cdot(-3) \Rightarrow A=2$

$y=-2$, $\quad 16+14-12 = 0 + B(-2)\cdot(-5)+0$

$\quad 18 = 10\cdot B \Rightarrow B=\frac{9}{5}$

$y=3$, $\quad 4\cdot9-7\cdot3-12 = A\cdot0 + B\cdot0 + C\cdot3\cdot5$

$\quad 3 = C\cdot3\cdot5 \Rightarrow C=\frac{1}{5}$

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \frac{2}{y} + \frac{\frac{9}{5}}{y+2} + \frac{\frac{1}{5}}{y-3}\,dy$$

$$= 2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3|\,\Big|_1^2$$

$$= 2\ln2 + \frac{9}{5}\ln4 + \frac{1}{5}\ln1 - \left(2\ln1 + \frac{9}{5}\ln3 + \frac{1}{5}\ln2\right)$$

(Full credits) → $\boxed{= 2\ln2 + \frac{9}{5}\cdot\ln4 + 0 - 0 - \frac{9}{5}\ln3 - \frac{1}{5}\ln2}$

$$= 2\ln2 + \frac{9}{5}\cdot2\ln2 - \frac{9}{5}\ln3 - \frac{1}{5}\ln2$$

$$\boxed{= \frac{27}{5}\cdot\ln2 - \frac{9}{5}\ln3.}$$

☆☆☆☆☆ 23. $\int \frac{10}{(x-1)(x^2+9)} dx$.

Step 1: P.F. $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx}{x^2+9} + \frac{C}{x^2+9}$.

times $(x-1)(x^2+9)$ both sides.

$$10 = A(x^2+9) + Bx\cdot(x-1) + C(x-1).$$

$$10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

$$0\cdot x^2 + 0\cdot x + 10 = \underbrace{(A+B)}\cdot x^2 + (C-B)x + 9A - C$$

$A+B=0$, $C-B=0$, $9A-C=10$

$A=-B$, $C=B$. $-9B-B=10 \Rightarrow B=-1$, $A=1$, $C=-1$

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x}{x^2+9} + \frac{-1}{x^2+9}.$$

Step 2: $\int \frac{1}{x-1} dx = \ln|x-1| + C$.

$$\int \frac{-x\cdot dx}{x^2+9} \xrightarrow[du=2\cdot x dx]{u=x^2+9} \int \frac{-\frac{1}{2}du}{u} = -\frac{1}{2}\ln|u| \xrightarrow{u=x^2+9} -\frac{1}{2}\ln|x^2+9| + C$$

$$\int \frac{-dx}{x^2+9} = \int \frac{-dx}{x^2+3^2} \xrightarrow[a=3]{\text{Formula: } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)} -\frac{1}{3}\cdot\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Step 1 + Step 2:

$$\int \frac{10}{(x-1)(x^2+9)} dx = \int \frac{1}{x-1} + \frac{-x}{x^2+9} + \frac{-1}{x^2+9} dx = \boxed{\ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

☆☆☆☆☆ 24. $\int \frac{x^2-x+6}{x^3+3x}\,dx$.

Step 1: $x^3+3x = x(x^2+3)$

P.F. $\frac{x^2-x+6}{x^3+3x} = \frac{A}{x} + \frac{Bx}{x^2+3} + \frac{C}{x^2+3}$.

Cross-multiply: times $x\cdot(x^2+3)$ both sides.

$$x^2-x+6 = A(x^2+3) + Bx\cdot x + C\cdot x.$$

$$x^2-x+6 = Ax^2+3A+Bx^2+Cx.$$

$$x^2-x+6 = (A+B)\cdot x^2 + C\cdot x + 3A.$$

$A+B=1$; $C=-1$, $3A=6$

$\Downarrow$

$A=2 \Rightarrow 2+B=1 \Rightarrow B=-1$

$$\frac{x^2-x+6}{x^3+3x} = \frac{2}{x} + \frac{-x}{x^2+3} + \frac{-1}{x^2+3}.$$

Step 2: $\int \frac{2}{x}\,dx = 2\ln|x| + C$

$$\int \frac{-x\,dx}{x^2+3} \xrightarrow[du=2\cdot x\,dx]{u=x^2+3} \int \frac{-\frac{1}{2}du}{u} = -\frac{1}{2}\ln|u| \xrightarrow{u=x^2+3} -\frac{1}{2}\ln|x^2+3| + C.$$

$$\int \frac{-1}{x^2+3}\,dx \xrightarrow[x^2+3=3\tan^2\theta+3=3\sec^2\theta]{x=\sqrt{3}\tan\theta,\ dx=\sqrt{3}\sec^2\theta\,d\theta} \int \frac{-\sqrt{3}\sec^2\theta\cdot d\theta}{3\cdot\sec^2\theta} = -\frac{1}{\sqrt{3}}\int d\theta = -\frac{1}{\sqrt{3}}\theta + C$$

OR directly use the formula: with $a=\sqrt{3}$.

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right)}$$

$$\xrightarrow[\theta=\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)]{\tan\theta=\frac{x}{\sqrt{3}}} -\frac{1}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Step 1 + Step 2: $\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{2}{x} + \frac{-x}{x^2+3} + \frac{-1}{x^2+3}\,dx = \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$