Hw §7.3 Ed 8, Ed 7

☆☆☆☆ 1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$, $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

(Ed8, Ed7)

$4 - x^2 = 4 - (2\sin\theta)^2$

$= 4 - 4\sin^2\theta = 4\cos^2\theta$

$$= \int \frac{2\cos\theta \cdot d\theta}{(2\sin\theta)^2 \cdot \sqrt{4\cos^2\theta}}$$

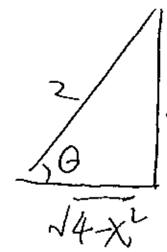
$$= \int \frac{2\cos\theta}{(2\sin\theta)^2 \cdot 2\cos\theta} \cdot d\theta$$

$$= \int \frac{1}{4} \cdot \frac{1}{\sin^2\theta}\, d\theta$$

$\sin\theta = \frac{1}{\csc\theta}$

$$= \int \frac{1}{4} \cdot \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$$

$$= \boxed{-\frac{1}{4} \cdot \frac{\sqrt{4-x^2}}{x} + C}$$



$\sin\theta = \frac{x}{2}$

$\cot\theta = \frac{\sqrt{4-x^2}}{x}$

☆☆☆☆ 2. $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\, dx$,

(Ed8, Ed7)

$$\boxed{\begin{array}{l} x = 2\tan\theta, \quad dx = 2\sec^2\theta \cdot d\theta \\ x^2+4 = (2\tan\theta)^2 + 4 = 4\sec^2\theta \\ \sqrt{x^2+4} = \sqrt{4\sec^2\theta} = 2\sec\theta. \end{array}}$$

$$= \int \frac{(2\tan\theta)^3 \cdot 2\sec^2\theta\, d\theta}{2\sec\theta}$$

$$= \int 8\tan^3\theta \cdot \sec\theta\, d\theta$$

tan θ odd, $u = \sec\theta$

$du = \tan\theta \cdot \sec\theta\, d\theta$

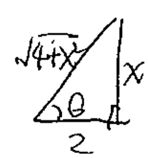
$\tan^2\theta = \sec^2\theta - 1 = u^2 - 1$

$$= \int 8 \cdot \tan^3\theta \cdot \sec\theta \cdot \frac{du}{\tan\theta \cdot \sec\theta}$$

$$= \int 8 \cdot \tan^2\theta \cdot du$$

$u \to \theta \mapsto x$

$\tan\theta = \frac{x}{2}$



$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$$= \int 8 \cdot (u^2 - 1)\, du$$

$$= 8\left(\frac{1}{3}u^3 - u\right) + C = \frac{8}{3}\sec^3\theta - 8 \cdot \sec\theta + C = \boxed{\frac{8}{3}\left[\frac{\sqrt{4+x^2}}{2}\right]^3 - 8 \cdot \frac{\sqrt{4+x^2}}{2} + C}$$

☆☆☆
☆☆ 3. $\int \frac{\sqrt{x^2-4}}{x}\,dx$, $x = 2\sec\theta$, $dx = 2\tan\theta\cdot\sec\theta\,d\theta$

(Ed8, Ed7)

$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4}$

$= \sqrt{4\tan^2\theta}$

$= 2\tan\theta.$

$$= \int \frac{2\tan\theta}{2\sec\theta}\cdot 2\tan\theta\cdot\sec\theta\cdot d\theta$$

$$= \int 2\tan^2\theta\, d\theta$$

$\sec\theta$ power 0, even,
$u = \tan\theta$, $du = \sec^2\theta\cdot d\theta$

$$= \int 2\cdot u^2\cdot\frac{du}{u^2+1}$$

$d\theta = \frac{du}{\sec^2\theta} = \frac{du}{\tan^2\theta+1}$

$= \frac{du}{u^2+1}$

$$= 2\int \frac{u^2}{u^2+1}\,du$$

$$= 2\cdot\int \frac{u^2+1-1}{u^2+1}\,du$$

$$= 2\int \frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}\,du$$

$$= 2\int 1 - \frac{1}{u^2+1}\,du$$

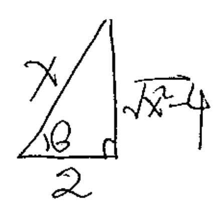
$$= 2(u - \tan^{-1}u) + C$$

$$= 2u - 2\tan^{-1}u + C$$

$u = \tan\theta \iff \tan^{-1}u = \theta$

$$= 2\tan\theta - 2\cdot\theta + C$$

$$= \boxed{2\cdot\frac{\sqrt{x^2-4}}{2} - 2\cdot\sec^{-1}\left(\frac{x}{2}\right) + C}$$



$\sec\theta = \frac{x}{2}$.

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$.

$\theta = \sec^{-1}\left(\frac{x}{2}\right)$

☆☆☆☆ 4. (Ed 8)

$$\int \frac{x^2}{\sqrt{9-x^2}}\cdot dx$$

$9 - x^2 = 3^2 - x^2$

$x = 3\cdot\sin\theta \quad , \quad d\cancel{x} = 3\cdot\cos\theta\cdot d\theta$

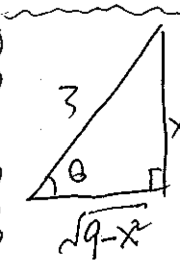
$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cdot\cos\theta$

$$= \int \frac{(3\sin\theta)^2\cdot 3\cos\theta\cdot d\theta}{3\cos\theta}$$

$$= \int 9\cdot\sin^2\theta\ d\theta$$

$$= \int 9\cdot\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \int \frac{9}{2} - \frac{9}{2}\cos 2\theta\ d\theta = \frac{9}{2}\theta - \frac{9}{2}\cdot\frac{1}{2}\sin 2\theta + C$$



$x = 3\sin\theta$

$\Downarrow$

$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\cdot\theta - \frac{9}{2}\cdot\frac{1}{2}\cdot 2\sin\theta\cdot\cos\theta + C$$

$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C}$$

☆☆☆ 5 (Ed 8)

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx$$

$x = \sec\theta \qquad dx = \tan\theta\cdot\sec\theta\cdot d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$= \int \frac{\tan\theta}{\sec^4\theta}\cdot\tan\theta\cdot\sec\theta\cdot d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^3\theta}\cdot d\theta$$

Hint: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

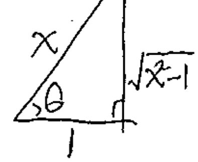
$$= \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta\cdot d\theta$$

$u = \sin\theta$, $du = \cos\theta\cdot d\theta$

$$= \int \sin^2\theta\cdot\cos\theta\cdot d\theta$$

$$= \int u^2\cdot du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\sin^3\theta + C.$$



$\sec\theta = x = \frac{x}{1}$

$\sin\theta = \frac{\sqrt{x^2-1}}{x}$,

$$= \boxed{\frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C}$$

☆☆ 6. $\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$
(Ed8, Ed7)

Note: DO NOT USE TRIG-SUB for this one

u-sub directly, $u = 36 - x^2, \quad du = -2x \cdot dx$

$\frac{du}{-2} = x \cdot dx$

$$= \int_0 \frac{\frac{du}{-2}}{\sqrt{u}} = \int \frac{1}{-2} \cdot u^{-\frac{1}{2}}\,du \qquad \int_{x=0}^{x=3} \xrightarrow{u=36-x^2} \int_{u=36}^{u=27}$$

$$= \frac{1}{-2} \cdot 2\,u^{\frac{1}{2}} \Big|_{36}^{27}$$

$$= -u^{\frac{1}{2}} \Big|_{36}^{27} = \boxed{-\sqrt{27} - (-\sqrt{36}) = -\sqrt{27} + 6}$$

☆☆☆ 8 $\int \frac{dt}{t^2 \cdot \sqrt{t^2-16}}$
(Ed8, Ed7)

$t^2 - 16 = t^2 - 4^2$

$t = 4 \cdot \sec\theta, \quad dt = 4 \tan\theta \cdot \sec\theta \cdot d\theta$

$\sqrt{t^2-16} = \sqrt{16\sec^2\theta - 16}$

$= \sqrt{16 \cdot \tan^2\theta} = 4 \cdot \tan\theta$

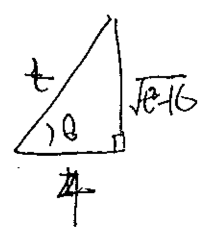
$$= \int \frac{4\tan\theta \cdot \sec\theta \cdot d\theta}{(4\sec\theta)^2 \cdot 4\tan\theta}$$

$$= \int \frac{1}{16} \cdot \frac{1}{\sec\theta} \cdot d\theta$$

$$= \int \frac{1}{16} \cdot \cos\theta \cdot d\theta$$

$$= \frac{1}{16} \cdot \sin\theta + C$$

$$= \boxed{\frac{1}{16} \cdot \frac{\sqrt{t^2-16}}{t} + C}$$



$t = 4\sec\theta$

$\sec\theta = \frac{t}{4}$

$\sin\theta = \frac{\sqrt{t^2-16}}{t}$

☆☆☆ 9、 $\displaystyle\int_2^3 \frac{dx}{(x^2-1)^{3/2}}$, $x = \sec\theta$, $dx = \tan\theta \cdot \sec\theta \cdot d\theta$
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$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

(Ed8)

$$= \int \frac{\tan\theta \cdot \sec\theta \cdot d\theta}{(\tan^2\theta)^{\frac{3}{2}}} = \int \frac{\tan\theta \cdot \sec\theta \cdot d\theta}{(\tan\theta)^{2 \times \frac{3}{2}}}$$

$$= \int \frac{\sec\theta}{\tan^2\theta} \cdot d\theta$$

$\cancel{\sec\theta = 2 \Leftrightarrow \frac{1}{\cos\theta} = 2}$
$\cancel{\Leftrightarrow \cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}}$

$$= \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} d\theta$$

$$= \int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$\cancel{\int_{x=2}^{x=3}}$ $\cancel{x = \sec\theta}$

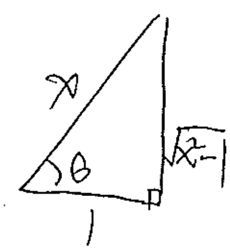
$$= \int \frac{\cos\theta \cdot d\theta}{\sin^2\theta}$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= \int \frac{du}{u^2}$$

$$= -\frac{1}{u}$$

$$= -\frac{1}{\sin\theta} ,$$



$\sec\theta = \frac{x}{1}$

$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$$= -\frac{x}{\sqrt{x^2-1}} \Big|_2^3$$

$$= -\frac{3}{\sqrt{9-1}} - \left(-\frac{2}{\sqrt{4-1}}\right)$$

$$\boxed{\begin{aligned} &= -\frac{3}{\sqrt{8}} + \frac{2}{\sqrt{3}} \\ &= -\frac{3\sqrt{2}}{4} + \frac{2\sqrt{3}}{3} \end{aligned}}$$

☆☆☆ 10. $\int_0^{2/3} \sqrt{4-9x^2}\, dx.$

(Ed 8)

$4-9x^2 = 2^2-(3x)^2$

$3x = 2\sin\theta$

$3\,dx = 2\cos\theta \cdot d\theta, \quad dx = \frac{2\cos\theta}{3}\, d\theta$

$\sqrt{4-9x^2} = \sqrt{4-4\sin^2\theta}$

$= 2\cdot\cos\theta$

$= \int 2\cos\theta \cdot \frac{2\cos\theta}{3}\, d\theta$

$= \int \frac{4}{3}\cdot\frac{1+\cos 2\theta}{2}\cdot d\theta$

$= \int \frac{2}{3} + \frac{2}{3}\cos 2\theta\, d\theta$

$3x = 2\sin\theta$

$x=0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x = \frac{2}{3} \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$= \frac{2}{3}\theta + \frac{2}{3}\cdot\frac{1}{2}\sin 2\theta \Big|_0^{\frac{\pi}{2}}$

$= \frac{2}{3}\cdot\frac{\pi}{2} + \frac{1}{3}\sin 2\pi - 0$

$= \boxed{\frac{\pi}{3}}$

☆☆☆ 11. $\int_0^{\frac{1}{2}} x\cdot\sqrt{1-4x^2}\, dx.$

(Ed 8)

$u = 1-4x^2, \quad du = -4\cdot 2\cdot x\cdot dx.$

$\frac{du}{-8} = x\cdot dx$

$= \int \sqrt{u}\cdot\frac{du}{-8}$

$\int_{x=0}^{x=\frac{1}{2}} \xrightarrow{u=1-4x^2} \int_{u=1-0=1}^{u=1-4\cdot\frac{1}{4}=0}$

$= -\frac{1}{8}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} \Big|_{u=1}^{u=0}$

$= -\frac{1}{12}\cdot 0 - \left(-\frac{1}{12}\cdot 1\right)$

$= \boxed{\frac{1}{12}}$

☆☆☆ ☆☆ 12. (Ed 8)

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}}$$

$t = 2\tan\theta$. $dt = 2\cdot\sec^2\theta\cdot d\theta$

$\sqrt{4+t^2} = \sqrt{4+4\tan^2\theta} = \sqrt{4\cdot\sec^2\theta}$
$= 2\cdot\sec\theta$.

$$= \int \frac{2\sec^2\theta\cdot d\theta}{2\sec\theta}$$

$$= \int \sec\theta \cdot d\theta$$

$$= \ln|\tan\theta + \sec\theta| \Big|_{t=0}^{t=2}$$

$\boxed{t=0 \Rightarrow \tan\theta = 0, \quad \sec\theta = 1}$

$\boxed{t=2 \Rightarrow \tan\theta = 1, \quad \sec\theta = 2}$

$$= \ln|1+2| - \ln|0+1|$$

$$= \boxed{\ln 3}$$

☆☆ ☆☆ 14. (Ed8, Ed7)

$$\int_0^1 \frac{dx}{(x^2+1)^2}$$

$x = \tan\theta$, $dx = \sec^2\theta\cdot d\theta$

$x^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$$= \int \frac{\sec^2\theta\cdot d\theta}{\sec^4\theta}$$

$$= \int \frac{1}{\sec^2\theta}\cdot d\theta = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta \Big|_{x=0}^{x=1}$$

$\tan\theta = 1$, $\theta = \frac{\pi}{4}$

$\tan\theta = 0$, $\theta = 0$

$$= \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{4}\sin\frac{\pi}{2} - \left(\frac{1}{2}\cdot 0 + 0\right)$$

$$= \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

☆☆ 17. $\int \frac{x\,dx}{\sqrt{x^2-7}}$ u-Sub

(Ed8, Ed7)

$u=x^2-7$ $du=2\cdot x\cdot dx$.

$\frac{du}{2}=x\cdot dx$

$= \int \frac{\frac{du}{2}}{\sqrt{u}}$

$= \int \frac{1}{2}\cdot u^{-\frac{1}{2}}du = \frac{1}{2}\cdot 2\cdot u^{\frac{1}{2}}+C$

$= u^{\frac{1}{2}}+C$

$=\sqrt{x^2-7}+C$

☆☆ 20 $\int \frac{x\,dx}{\sqrt{1+x^2}}$, $u=1+x^2$, $du=2\cdot x\cdot dx$.

(Ed8, Ed7)

$= \int \frac{\frac{1}{2}\cdot du}{\sqrt{u}} = \int \frac{1}{2}\cdot u^{-\frac{1}{2}}du = \frac{1}{2}\cdot 2\cdot u^{\frac{1}{2}}+C$

$= u^{\frac{1}{2}}+C$

$= \boxed{(1+x^2)^{\frac{1}{2}}+C}$

Ed7 ☆☆4 $\int_0^1 x^3\sqrt{1-x^2}\cdot dx$ $x=\sin\theta$. $dx=\cos\theta\,d\theta$, $\sqrt{1-\sin^2\theta}=\cos\theta$

$= \int \sin^3\theta\cdot\cos\theta\cdot\cos\theta\,d\theta$, $u=\cos\theta$. $du=-\sin\theta\,d\theta$

$= \int \sin^3\theta\cdot u^2\cdot\frac{du}{-\sin\theta}$ $-\sin^2\theta=-(1-\cos^2\theta)=\cos^2\theta-1=u^2-1$

$= \int -\sin^2\theta\cdot u^2\cdot du$

$x=0$ $\sin\theta=0$ $\cos\theta=1$

$= \int (u^2-1)\,u^2\,du$

$x=1$ $\sin\theta=1$ $\cos\theta=0$

$= \int u^4-u^2\,du$

$= \frac{1}{5}u^5-\frac{1}{3}u^3 = \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta\Big|_{x=0}^{x=1} = \frac{1}{5}\cdot 0-\frac{1}{3}\cdot 0-\left(\frac{1}{5}-\frac{1}{3}\right)=\boxed{\frac{2}{15}}$