Landscape theory for finite $\mathbb{Z}^1$ lattice,

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Theorem 0.1 (Landscape theory for finite matrix. M. L. Lyra, S. Mayboroda and M. Filoche)**.** *Let*

$$H = \begin{pmatrix} v_1 & -1 & 0 & \cdots & 0 \\ -1 & v_2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & v_{n-1} & -1 \\ 0 & \cdots & 0 & -1 & v_n \end{pmatrix}. \tag{0.1}$$

If $v_j \geq 2, j = 1, \cdots, n$, and $H\vec{x} = \lambda\vec{x}$, then for all $j = 1, \cdots, n$,

$$\frac{|x_j|}{\max\limits_{1 \leq k \leq n} |x_k|} \leq \lambda u_j, \tag{0.2}$$

where $\vec{u} \in \mathbb{R}^n$ is the landscape function satisfying

$$H\vec{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} =: \vec{1}. \tag{0.3}$$

1 Useful lemmas

Lemma 1.1. *Let H be given in* (0.1)*. If If $v_j \geq 2, j = 1, \cdots, n$, then H is invertible. Let $G_{ij} = H^{-1}(i,j)$ be the (i,j) entry of the inverse of H. As a consequence, there is always a $\vec{u} \in \mathbb{R}^n$ satisfying eq.* (0.3)*, with explicit expression as*

$$u_j = \sum_{k=1}^{n} G_{jk}. \tag{1.1}$$

Moreover, all the eigenvalues of H are strictly positive.

Lemma 1.2. *Let H, $G_{ij} = H^{-1}(i,j)$ and u_j be as above. $G_{ij} > 0$ for all i, j. As a consequence,*

$$u_j > 0. \tag{1.2}$$

Exercise 1.3. Use Lemma 1.2 to prove Theorem 0.2.

2 Proof of Lemma 1.1: Existence of Green's function

Let $H = V - H_0$, where

$$V = \begin{pmatrix} v_1 & 0 & 0 & \cdots & 0 \\ 0 & v_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & v_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & v_n \end{pmatrix}, \quad \text{and} \quad H_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{2.1}$$

Exercise 2.1. Consider the usual matrix 2-norm $\|\cdot\|$ (defined in HW1). Prove that $\|H_0\| \le 2$. As a consequence, all eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_n\}$ of H_0 are contained in $[-2, 2]$, i.e. $|\mu_j| \le 2$ for all j.

Proof. Let

$$R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

Clearly, $H_0 = L + R$. For any $\vec{x} = (x_1, x_2, \cdots, x_n)^T$, direct computation shows that

$$R\vec{x} = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} \tag{2.3}$$

Therefore,

$$\|R\vec{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \le x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = \|\vec{x}\|^2 \tag{2.4}$$

which implies $\|R\vec{x}\| \le \|\vec{x}\|$. According to the definition of the matrix norm and (2.4):

$$\|R\| = \max_{\vec{x} \ne \vec{0}} \frac{\|R\vec{x}\|}{\|\vec{x}\|} \le 1. \tag{2.5}$$

Exact the same argument shows that $\|L\| \le 1$. Therefore, by the property (triangle inequality) of the matrix norm, we have that

$$\|H_0\| \le \|L\| + \|R\| \le 2, \tag{2.6}$$

which completes the proof. □

Exercise 2.2. Assume first that $v_j > 2, j = 1, \cdots, n$, (all v_j are strictly greater than 2). Prove that $H = V - H_0$ is invertible.

Exercise 2.3. Prove that $|\mu_j| < 2$, i.e., -2 and 2 are not eigenvalues of H_0.

Hint: consider the **difference** *equation $H_0\vec{x} = 2\vec{x}$, where $\vec{x} = (x_1, \cdots, x_n)$, as part of the infinite system (see Ex.* (2.6) *below), with* **zero boundary condition** $x_0 = x_{n+1} = 0$.

Exercise 2.4. Prove that $\|H_0\| < 2$.

Hint: prove that for any symmetric $n \times n$ matrix A

$$\|A\| := \sup_{\|\vec{x}\|=1} \|A\vec{x}\| = \max_{1\le j\le n} |\mu_j|, \tag{2.7}$$

where $\{\mu_1, \mu_2, \cdots, \mu_n\}$ are the eigenvalues of A.

Exercise 2.5. Complete the proof of Lemma 1.1 under the assumption $v_j \ge 2, j = 1, \cdots, n,$.

Exercise 2.6. (Supplementary problem) Consider the difference equation (on a infinite lattice)

$$x_{n+1} + x_{n-1} = \lambda\, x_n,\ \ n \in \mathbb{Z},\ \ \lambda \in \mathbb{C} \tag{2.8}$$

1. Prove that the following expression solves eq. 2.8 for all $n \in \mathbb{Z}$

   $$x_n = c_1\ \mu^n + c_2\ \mu^{-n},\quad c_1, c_2 \in \mathbb{C},\quad \mu = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \tag{2.9}$$

   And find c_1 c_2 (in terms of μ) if $x_0 = 0, x_1 = 1$.

2. For any $|\lambda| \ne 2$, consider the two infinite sequences $\vec{\alpha}, \vec{\beta}$ given by μ^n and μ^{-n}, i.e.,

   $$\vec{\alpha} = (\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \cdots) = (\cdots, \mu^{-1}, 1, \mu^1, \mu^2, \cdots) \tag{2.10}$$
   $$\vec{\beta} = (\cdots, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots) = (\cdots, \mu^{1}, 1, \mu^{-1}, \mu^{-2}, \cdots) \tag{2.11}$$

   Prove that

   $$W(\vec{\alpha}, \vec{\beta}) := \det \begin{pmatrix} \alpha_{n+1} & \beta_{n+1} \\ \alpha_n & \beta_n \end{pmatrix} \tag{2.12}$$

   is a constant (independent of n). [1]

3. Putting 1 and 2 together, we actually can say that if $|\lambda| \ne 2$, then all solutions $\vec{x} = \{x_j\}$ to eq. (2.8) is a linear combination of $\vec{\alpha}, \vec{\beta}$,

   $$\vec{x} = c_0\vec{\alpha} + c_1\vec{\beta} \tag{2.13}$$

   This is not the case if $|\lambda| = 2$. Prove that if $|\lambda| = 2$, then $\vec{\alpha} = \vec{\beta} =$ a constant vector. Then find another solution (a non-constant vector) $\vec{\gamma} = \{\gamma_n\}$, which solves eq. (2.8) (for $\lambda = 2$) and satisfies that $W(\vec{\alpha}, \vec{\gamma})$ is a constant.

3 Proof of Lemma 1.2: Positivity of the Green's function

3.1 Maximum principle and the first (original) proof

Lemma 3.1 (Maximum principle)**.** *Let H be given as in* (0.1)*. For any $\vec{x} \in \mathbb{R}^n$, let $\vec{y} = H\vec{x}$. If $y_i \ge 0$ for all $i = 1, 2, \cdots, n$, then*

$$x_i \ge 0,\quad i = 1, 2, \cdots, n. \tag{3.1}$$

[1] W is usually referred to be the Wronski of the system (or simply of eq. (2.8)), which plays important role in the general study of second order difference/differential equations.

Exercise 3.2. Prove Lemma 3.1 by contradiction.

Hint: consider the equation $H\vec{x} = \vec{y}$ as a boundary problem on the extended lattice: $[0, 1, \cdots, n, n+1]$, with zero boundary condition: $x_0 = x_{n+1} = 0$. Fix $\vec{y}$, assume that there is a minimum inside the lattice, that is, there exists $j \in [1, \cdots, n]$ such that $x_j \leq x_{j+1}$ and $x_j \leq x_{j-1}$. Prove that this will contradict the condition that $y_i \geq 0$ for all $i = 1, 2, \cdots, n$.

Lemma 3.3 (Strong Maximum principle)**.** *Following the notation in Lemma 3.1, if we assume additionally that there exist an i_0 such that $y_{i_0} > 0$ (strictly positive), then*

$$x_i > 0, \quad i = 1, 2, \cdots, n. \tag{3.2}$$

Exercise 3.4. Prove Lemma 3.3 by contradiction.

Hint: continue with Lemma 3.1, assume that there exists $j \in [1, \cdots, n]$ such that $x_j = 0$. Prove that this will lead to $y_i \geq 0$ for all $i = 1, 2, \cdots, n$, which contradict the condition that $y_{i_0} > 0$ for some i_0.

Lemma 3.5. *Let $\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, j = 1, 2 \cdots, n$ be the standard basis of $\mathbb{R}^n$, with 0 entries except for 1 in the j-th place. Let*

$$\vec{g} = \begin{pmatrix} G_{1j} \\ G_{2j} \\ \vdots \\ G_{nj} \end{pmatrix} \tag{3.3}$$

be the j-th column vector of H^{-1}, where $H^{-1} = \{G_{ij}\}$ is given as in Lemma 1.1. Prove that for all $j = 1, 2, \cdots, n$,

$$H\vec{g} = \vec{e}_j \tag{3.4}$$

Exercise 3.6. Use Lemma 3.3 and Lemma 3.5 to prove Lemma 1.2.

Hints: apply Lemma 3.3 to each pair of $\vec{g}$ and $\vec{e}_j$.

3.2 Power series expansion and an alternative proof of Lemma 1.2

References

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