MTH 320 Section 004 Midterm 1 Solutions

1. (10 points) Let $x_1, \dots, x_n \in \mathbb{R}$. Prove that

$$|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|.$$

Note: This problem may sound trivial, and it has already been used in other problems. Please provide a detailed proof.

Proof. We use the triangle inequality:

$$|x+y| \le |x| + |y|, \quad \forall x, y \in \mathbb{R}.$$

We prove the statement by induction. The induction bases is $|x_1| \le |x_1|$, which is trivial. Suppose the inequality holds for n, i.e.,

$$|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|.$$

Now we prove it for $n+1$. Since $x_1 + \cdots + x_n + x_1 = (x_1 + \cdots + x_n) + x_{n+1}$, by triangle inequality and induction hypothesis,

$$|x_1 + \cdots + x_n + x_1| \le |x_1 + \cdots + x_n| + |x_{n+1}| \le |x_1| + \cdots + |x_n| + |x_{n+1}|.$$

So the inequality also holds for $n+1$. By math induction, the inequality should hold for all $n \in \mathbb{N}$. □

2. (a) (4 points) Define a convergent sequence of real numbers.
   (b) (6 points) Let (s_n) be a convergent sequence of real numbers, and $s = \lim s_n$. Suppose $s < 0$. Prove that there is $N \in \mathbb{N}$ such that $s_n < 0$ for all $n > N$.

Proof. (a) A sequence (s_n) of real numbers is convergent if there is $s \in \mathbb{R}$ (called $\lim s_n$) such that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for any $n > N$, $|s_n - s| < \varepsilon$.

(b) Let $\varepsilon = -s$. Since $s < 0$, $\varepsilon > 0$. By definition, there is $N \in \mathbb{N}$ such that for any $n > N$, $|s_n - s| < \varepsilon$, which implies that $s_n < s + \varepsilon = 0$. □

3. (10 points) Compute the limit

$$\lim_{n\to\infty} \frac{3n^2 - 5n + 2}{2n^2 - \cos(n^3)}.$$

Justify all steps.

Solution. Letting the numerator and the denominator both be divided by n^2, we get

$$\frac{3n^2 - 5n + 2}{2n^2 - \cos(n^3)} = \frac{3 - 5/n + 2/n^2}{2 - \cos(n^3)/n^2}.$$

We learned in class that $\lim_{n\to\infty} \frac{1}{n} = 0$. By limit theorems, we get

$$\lim_{n\to\infty} -\frac{5}{n} = -5 * 0 = 0, \quad \lim_{n\to\infty} \frac{1}{n^2} = 0 * 0 = 0, \quad \lim_{n\to\infty} \frac{2}{n^2} = 2 * 0 = 0.$$

So the numerator $3 - 5/n + 2/n^2$ converges to $3 + 0 + 0 = 3$. Since $|\cos(n^3)| \le 1$, we have

$$-\frac{1}{n^2} \le \frac{\cos(n^3)}{n^2} \le \frac{1}{n^2}.$$

Since $\frac{1}{n^2} \to 0$, we also have $-\frac{1}{n^2} \to 0$. By squeeze lemma, we get $\frac{\cos(n^3)}{n^2} \to 0$. Thus the denominator $2 - \cos(n^3)/n^2$ converges to $2 - 0 = 2$. So the fractal $\frac{3-5/n+2/n^2}{2-\cos(n^3)/n^2}$ converges to $\frac{3}{2}$. □

4. Let (s_n) and (t_n) be two sequences such that $s_n \le t_n$ for each $n \in \mathbb{N}$.

   (a) (6 points) Prove that for any $N \in \mathbb{N}$, $\inf\{s_n : n > N\} \le \inf\{t_n : n > N\}$.

   (b) (4 points) Prove that $\liminf s_n \le \liminf t_n$.

Proof. (a) Let $N \in \mathbb{N}$. For every $n \in \mathbb{N}$ such that $n > N$, we have $t_n \ge s_n \ge \inf\{s_n : n > N\}$. Since $t_n \ge \inf\{s_n : n > N\}$ for every $n > N$, we get $\inf\{t_n : n > N\} \ge \inf\{s_n : n > N\}$.

(b) Let $u_N = \inf\{s_n : n > N\}$ and $v_N = \inf\{t_n : n > N\}$, $N \in \mathbb{N}$. Recall the definition of $\liminf s_n$. If (s_n) is bounded below, then $(u_N)_{N\in\mathbb{N}}$ is an increasing sequence of real numbers, and $\liminf s_n$ is defined as $\lim_{N\to\infty} u_N$, which could be a real number or $+\infty$; and if (s_n) is not bounded below, then $u_N = -\infty$ for every N, and $\liminf s_n$ is defined as $-\infty$. Similarly, if (t_n) is bounded below, then $\liminf t_n = \lim_{N\to\infty} v_N$.

We have proved in (a) that $u_N \le v_N$ for every $N \in \mathbb{N}$. We have learned a theorem in class that for two sequences real numbers (a_n) and (b_n), if $a_n \le b_n$ for every n, and if $\lim a_n$ and $\lim b_n$ both exist (could be a real number or $\pm\infty$), then $\lim a_n \le \lim b_n$. If (s_n) and (t_n) are both bounded below, then (u_N) and (v_N) are two sequences of real numbers, and $u_N \le v_N$ for every $N \in \mathbb{N}$. Applying the above theorem to (u_N) and (v_N), we get $\liminf s_n = \lim u_N \le \lim v_N = \liminf t_n$.

We have assumed that (s_n) and (t_n) are both bounded below. We now deal the other cases. In fact, if (s_n) is not bounded below, then $\liminf s_n = -\infty$. So $\liminf s_n \le \liminf t_n$ always holds regardless of whether (t_n) is bounded below. If (s_n) is bounded below, then there is a lower bound $L \in \mathbb{R}$ of (s_n). Since $t_n \ge s_n$ for every n, L is also a lower bound of (t_n), and so (t_n) is also bounded. Now we have studied all cases, and the proof is done. □

5. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let (s_n) be a sequence of numbers chosen from S such that every element of S appears infinitely many times in (s_n). Specifically,

$$\begin{array}{llllll}
s_1 = 1, & & & & & \\
s_2 = 1, & s_3 = \frac{1}{2}, & & & & \\
s_4 = 1, & s_5 = \frac{1}{2}, & s_6 = \frac{1}{3}, & & & \\
s_7 = 1, & s_8 = \frac{1}{2}, & s_9 = \frac{1}{3}, & s_{10} = \frac{1}{4}, & & \\
s_{11} = 1, & s_{12} = \frac{1}{2}, & s_{13} = \frac{1}{3}, & s_{14} = \frac{1}{4}, & s_{15} = \frac{1}{5}, & \\
s_{16} = 1, & s_{17} = \frac{1}{2}, & s_{18} = \frac{1}{3}, & s_{19} = \frac{1}{4}, & s_{20} = \frac{1}{5}, & s_{21} = \frac{1}{6}, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \quad \ldots\ldots
\end{array}$$

(a) (4 points) What is the set S of subsequential limits of (s_n)? You only need to provide the answer. No justification is needed.

(b) (6 points) What are $\limsup s_n$ and $\liminf s_n$? Justify your answer. You may use the result of (a), or argue directly.

Solution. (a) The set of subsequential limits is $S \cup \{0\}$. In fact, it is easy to see that every point $s \in S$ is a subsequential limit: we may take a subsequence of (s_n) taking constant value s. It is also clear that 0 is a subsequential limit since $(\frac{1}{k} : k \in \mathbb{N})$ is a subsequence of (s_n). It takes some work to show that (s_n) has no other subsequential limits. If you want to prove this, then first observe that (s_n) is bounded, which implies that $+\infty$ and $-\infty$ are not subsequential limits. Then you can show that any $x \in \mathbb{R} \setminus (S \cup \{0\})$ is not a subsequential limit by proving that there is some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ contains no elements of (s_n). You consider three cases separately: $x < 0$, $0 < x < 1$, and $x > 1$.

(b) By the result of (a) and a theorem in the book, $\limsup s_n$ is the biggest subsequential limit of (s_n), i.e., $\max(S \cup \{0\}) = 1$; and $\liminf s_n$ is the smallest subsequential limit of (s_n), i.e., $\min(S \cup \{0\}) = 0$. You may also argue directly. Since (s_n) is a sequence in S and every element of S appears infinitely many times in (s_n), for any $N \in \mathbb{N}$, the set $\{s_n : n > N\}$ is just S. Thus, $\inf\{s_n : n > N\} = \inf S = 0$ and $\sup\{s_n : n > N\} = \sup S = 1$. So

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = 0, \quad \limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = 1.$$

□