1 Loewner Equations

1.1 Chordal Loewner equation

Let $T \in (0,\infty]$ and $\lambda \in C([0,T))$, the set of real valued continuous functions on $[0,T)$. The chordal Loewner equation driven by λ is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad 0 \le t < T, \quad g_0(z) = z. \tag{1.1}$$

For every $z \in \mathbb{C}$, let $\tau(z) \ge 0$ be such that $[0,\tau(z))$ is the maximal interval of the solution $t \mapsto g_t(z)$. So g_t is defined on $\{z \in \mathbb{C} : \tau(z) > t\}$. We have the following facts.

1. If $z \in \mathbb{R}$, then $g_t(z) \in \mathbb{R}$ for $0 \le t < \tau(z)$.

2. If $z \in \mathbb{H} = \{\operatorname{Im} z > 0\}$, then $g_t(z)$ stays inside $\mathbb{H}$ because it can not reach $\mathbb{R}$; and $t \mapsto \operatorname{Im} g_t(z)$ is decreasing because $\operatorname{Im} \frac{2}{g_t(z)-\lambda(t)} < 0$ if $g_t(z) \in \mathbb{H}$.

3. Each g_t commutes with the conjugate map $z \mapsto \overline{z}$ because $\overline{g_t(z)}$ satisfies the same ODE.

4. If $\tau(z) < T$, then $\lim_{t\to\tau(z)} g_t(z) - \lambda(t) = 0$. In fact, there are only two cases for the solution $t \mapsto g_t(z)$ to blow up before T: either $\lim_{t\to\tau(z)} g_t(z) - \lambda(t) = 0$ or $\lim_{t\to\tau(z)} |g_t(z)| = \infty$. If the second case happens, then $|\partial_t g_t(z)| = |\frac{2}{g_t(z)-\lambda(t)}|$ is bounded on $[0,\tau(z))$. Since $\tau(z) < \infty$, we get a contradiction.

5. For each t, $\{z \in \mathbb{C} : \tau(z) > t\}$ is open, and g_t is analytic on $\{z \in \mathbb{C} : \tau(z) > t\}$. The proof uses some standard arguments in the theory of ordinary differential equations, which says that the solution of the ODE has differentiable dependence on the parameter. Here to prove that g_t is complex differentiable at z_0, we define

$$A_t(z) = \frac{g_t(z) - g_t(z_0)}{z - z_0} - h_t(z_0),$$

   where $h_t(z)$ is the solution of $\partial_t h_t(z) = \frac{-2h_t(z)}{(g_t(z)-\lambda(t))^2}$, $h_0(z_0) = 1$. Here $h_t(z)$ is expected to be equal to $g_t'(z)$, and the ODE for h_t is obtained by differentiating (1.1) w.r.t. z. Then $A_0(z) = 0$ and $A_t(z)$ satisfies an equation like $\partial_t A_t(z) = F(t,z,z_0)A_t(z) + G(t,z,z_0)$. When $z \to z_0$, F and G both tend to 0. Then Gronwall's inequality can be applied to show that $A_t(z) \to 0$. This shows that g_t is complex differentiable at z_0, and $g_t'(z_0) = h_t(z_0)$. This argument also shows that the complex derivative of g_t commutes with the partial derivative ∂_t, and we have

$$\partial_t g_t'(z) = -\frac{2g_t'(z)}{(g_t(z) - \lambda(t))^2}, \quad g_0'(z) = 1.$$

6. Each g_t is conformal (i.e., univalent analytic) on $\{z \in \mathbb{C} : \tau(z) > t\}$. This follows from the uniqueness of the solution of ODE.

7. Each g_t maps $\{z \in \mathbb{H} : \tau(z) > t\}$ onto $\mathbb{H}$. Let $t_0 \in [0, T)$. First, we know that $g_{t_0}(\{z \in \mathbb{H} : \tau(z) > t_0\}) \subset \mathbb{H}$. Second, fix any $z_0 \in \mathbb{H}$, consider the ODE

$$h'(t) = \frac{2}{h(t) - \lambda(t)}, \quad 0 \le t \le t_0, \quad h(t_0) = z_0.$$

As t decreases from t_0 to 0, $\operatorname{Im} h(t)$ increases, so the solution will not hit the singularity, which implies that it does not blow up on $[0, t_0]$. Then we have $h(0) \in \mathbb{H}$ and $g_{t_0}(h(0)) = h(t_0) = z_0$.

Lemma 1.1 *Let $t_0 \in [0, T)$. Suppose that $|\lambda(t)| \le M$ on $[0, t_0]$. Then*

(i) $\{\tau(z) \le t_0\} \subset \{|z| \le M + 2\sqrt{2t_0}\}$.

(ii) If $|z| > M + 2\sqrt{2t_0}$, then $|g_{t_0}(z)| \ge |z| - M - \sqrt{2t_0}$.

Proof. Let $|z| > M + 2\sqrt{2t_0}$. Then $|g_0(z) - \lambda(0)| \ge |z| - M > 2\sqrt{2t_0}$. Let s_0 be the maximal number on $[0, t_0]$ such that the solution $g_t(z)$ exists on $[0, s_0)$ and $|g_t(z) - \lambda(t)| \ge \sqrt{2t_0}$ on $[0, s_0)$. Then we get $|\partial_t g_t(z)| \le \sqrt{2/t_0}$ for $0 \le t < s_0$, which implies that $|g_t(z)| \ge |z| - \sqrt{2t_0}$ for $0 \le t < s_0$. So we have $|g_t(z) - \lambda(t)| \ge |g_t(z)| - M > |z| - \sqrt{2t_0} - M > \sqrt{2t_0}$ for $0 \le t < s_0$. First, this means that $g_t(z)$ does not blow up at s_0. Second, we have $s_0 = t_0$ because if $s_0 < t_0$ then $\lim_{t \to s_0} |g_t(z) - \lambda(t)| = \sqrt{2t_0}$, which is a contradiction. So we conclude that, if $|z| > M + 2\sqrt{2t_0}$, then $\tau(z) > t_0$. This finishes the proof of (i). Since $|g_t(z) - \lambda(t)| \ge |z| - \sqrt{2t_0}$ for $0 \le t < s_0 = t_0$, we get $|g_{t_0}(z) - \lambda(t_0)| \ge |z| - \sqrt{2t_0}$. The proof of (ii) is finished since $|\lambda(t_0)| \le M$. $\square$

This lemma implies that g_t has a pole at ∞. The pole has order 1 because g_t is conformal near ∞. We write the power series expansion of g_t at ∞ as

$$g_t(z) = a_1(t) z + a_0(t) + \frac{a_{-1}(t)}{z}^{-1} + O(z^{-2}).$$

We have

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)} = \frac{2}{a_1 z + O(1)} = \frac{2}{a_1 z} \cdot \frac{1}{1 + O(z^{-1})} = \frac{2}{a_1} z^{-1} + O(z^{-2}), \quad z \to \infty.$$

Thus, $a_1'(t) = a_0'(t) = 0$ and $a_2'(t) = \frac{2}{a_1(t)}$. Since $g_0(z) = z$, $a_1 \equiv 1$, $a_0 \equiv 0$, and $a_2(t) = 2t$.

Let $K_t = \{z \in \mathbb{H} : \tau(z) \le t\}$, $0 \le t < T$. Then $K_0 = \emptyset$; $K_{t_1} \subset K_{t_2}$ if $t_1 < t_2$; each K_t is a relatively closed bounded subset of $\mathbb{H}$, $g_t : (\mathbb{H} \setminus K_t; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{H}; \infty)$, and satisfies

$$g_t(z) = z + \frac{2t}{z} + O(z^{-2}), \quad z \to \infty. \tag{1.2}$$

The g_t is uniquely determined by K_t. If $t_1 < t_2$, then $g_{t_1} \ne g_{t_2}$, so $K_{t_1} \subsetneqq K_{t_2}$.

Definition 1.1 *We call g_t and K_t the chordal Loewner maps and hulls driven by λ.*

Lemma 1.2 (Linearity) *Suppose g_t and K_t are chordal Loewner maps and hulls driven by $\lambda(t)$. Let $a > 0$ and $b \in \mathbb{R}$. Then $ag_{t/a^2}((\cdot - b)/a) + b$ and $aK_{t/a^2} + b$ are chordal Loewner maps and hulls driven by $a\lambda(t/a^2) + b$.*

Proof. The proof is straightforward. We leave it as an exercise. □

Exercise. Let $\lambda(t) = c\sqrt{t}$, $t \geq 0$. Let g_t be the chordal Loewner maps driven by λ. Since $a\lambda(t/a^2) = \lambda(t)$ for any $a > 0$, we have $ag_{t/a^2}(z/a) = g_t(z)$. Letting $a = \sqrt{t}$, we get $g_t(z) = \sqrt{t}g_1(z/\sqrt{t})$. We may derive an ODE for g_1 using the chordal Loewner equation. We can solve this ODE to get g_1.

Corollary 1.1 *If K_t are chordal Loewner maps driven by $\lambda(t)$, then $\bigcap_{t\in(0,T)} \overline{K_t} = \{\lambda(0)\}$.*

Proof. For $t \in (0, T)$, $\overline{K_t}$ is a nonempty compact set because K_t is a nonempty and bounded. Since $\overline{K_t}$ is increasing in t, we conclude that $\bigcap_{t\in(0,T)} \overline{K_t}$ is nonempty. Let z_0 lie in the intersection. From Lemma 1.2, $K_t - \lambda(0)$ are chordal Loewner hulls driven by $\lambda(t) - \lambda(0)$. Let $M_t = \sup_{s\in[0,t]} |\lambda(s) - \lambda(0)|$. Then $\lim_{t\to 0} M_t = 0$. From Lemma 1.1, we get $K_t - \lambda(0) \subset \{|z| \leq M_t + 2\sqrt{2t}\}$. Thus, $|z_0 - \lambda(0)| \leq M_t + 2\sqrt{2t}$ for any $t \in (0, T)$. So z_0 must be $\lambda(0)$. □

Lemma 1.3 *Suppose g_t and K_t are chordal Loewner maps and hulls driven by $\lambda \in C([0, T))$. Let $t_0 \in [0, T)$. Then $g_{t_0+t} \circ g_{t_0}^{-1}$ and $g_{t_0}(K_{t_0+t} \setminus K_{t_0})$, $0 \leq t < T - t_0$, are chordal Loewner maps and hulls driven by $\lambda(t_0 + t)$.*

Proof. The proof is straightforward. We leave it as an exercise. □

Lemma 1.4 *Suppose g_t and K_t are chordal Loewner maps and hulls driven by $\lambda \in C([0, T))$. Then for any $t \in [0, T)$,*

$$\{\lambda(t)\} = \bigcap_{\varepsilon\in(0,T-t)} \overline{g_t(K_{t+\varepsilon} \setminus K_t)}. \tag{1.3}$$

Proof. This follows from Corollary 1.1 and Lemma 1.3. □

Remark. This corollary says that we may recover the driving function using the maps and hulls. Since the maps are also determined by the hulls, the driving function is completely determined by the hulls.

Definition 1.2 *We say that λ generates a chordal Loewner trace β if for every t,*

$$\beta(t) := \lim_{\mathbb{H}\ni z\to\lambda(t)} g_t^{-1}(z)$$

exists, and β is a continuous curve. Such β lies on $\mathbb{H} \cup \mathbb{R}$ with $\beta(0) = \lambda(0) \in \mathbb{R}$. We call the trace β simple if it has no self intersection and intersects $\mathbb{R}$ only at $\beta(0)$.

Example. If $\lambda(t) = 0$, $0 \le t < \infty$, then $\partial_t g_t(z) = 2/g_t(z)$. So $g_t(z) = \sqrt{z^2 + 4t}$. If $g_t(z)$ blows up at some finite time t_0, then $\sqrt{z^2 + 4t_0} = 0$, which implies that $z = \pm 2i\sqrt{t_0}$. So $\{\tau(z) \le t\} = [-2i\sqrt{t_0}, 2i\sqrt{t}]$ and $K_t = (0, i\sqrt{4t}]$, $0 \le t < \infty$. We have $g_t^{-1}(z) = \sqrt{z^2 - 4t}$. We have $\beta(t) := \lim_{\mathbb{H} \ni z \to 0} g_t^{-1}(z) = i\sqrt{4t}$, $0 \le t < \infty$, is continuous, has no self-intersection, and stays in $\mathbb{H}$ for $t > 0$. So λ generates a simple trace. Note that $K_t = \beta((0, t])$ for each t.

Proposition 1.1 *If λ generates a chordal Loewner trace β, then for each t, $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \beta((0, t])$. In particular, if β is simple, then $K_t = \beta((0, t])$. Moreover, for each t, g_t^{-1} extends continuously to $\mathbb{H} \cup \mathbb{R}$.*

Remark. This proposition says that if the trace exists, then it determines the hulls, which in turn determine the driving function. The proof will be given later.

Lemma 1.5 *Let $a > 0$ and $b \in \mathbb{R}$. If $\lambda(t)$ generates a chordal Loewner trace $\beta(t)$, then $a\lambda(t/a^2) + b$ generates the chordal Loewner trace $a\beta(\cdot/a^2) + b$.*

Proof. This follows from Lemma 1.2 and some straightforward argument. □

Lemma 1.6 *Let $\lambda \in C([0, T))$, $t_0 \in [0, T)$, and $\lambda_{t_0}(t) = \lambda(t_0 + t)$, $0 \le t < T - t_0$. Let g_t be the chordal Loewner maps driven by λ. Suppose λ generates a chordal Loewner trace β and λ_{t_0} generates a chordal Loewner trace β_{t_0}. Extend $g_{t_0}^{-1}$ continuously from $\mathbb{H}$ to $\mathbb{H} \cup \mathbb{R}$. Then $\beta(t_0 + t) = g_{t_0}^{-1}(\beta_{t_0}(t) + \lambda(t_0))$ for $0 \le t < T - t_0$.*

Proof. Let $g_{t_0,t}$ be the chordal Loewner maps driven by λ_{t_0}. From Lemma 1.2 and Lemma 1.3 we get $g_{t_0,t}(z) = g_{t_0+t} \circ g_{t_0}^{-1}(z + \lambda(t_0)) - \lambda(t_0)$. So we get

$$g_{t_0+t}^{-1}(z) = g_{t_0}^{-1}(g_{t_0,t}^{-1}(z - \lambda(t_0)) + \lambda(t_0)), \quad z \in \mathbb{H}.$$

This equality still holds for $z \in \mathbb{H} \cup \mathbb{R}$ if $g_{t_0+t}^{-1}$, $g_{t_0}^{-1}$, and $g_{t_0,t}^{-1}$ extend continuously to $\mathbb{H} \cup \mathbb{R}$. Letting $z = \lambda(t_0 + t)$, we get the desired result. □

Odes Schramm introduced SLE (shorthand for stochastic Loewner evolution or Schramm-Loewner evolution) by combining Loewner equation with stochastic processes.

Definition 1.3 *For $\kappa > 0$, a standard chordal SLE(κ) is defined to be the chordal Loewner process driven by $\lambda(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$, where $B(t)$ is a standard Brownian motion.*

Note that the maps from the space of $\lambda(t)$ to space of (g_t) and the space of (K_t) are continuous or measurable if these spaces are assigned some suitable topology or σ-algebra. Here is one example. We consider the case $T = \infty$. Let the topology on the linear space $C([0, \infty))$ be generated by semi-norms: $\|\lambda\|_a = \sup_{0 \le t \le a} |\lambda(t)|$. Let the topology on the space of (g_t) be generated by $\{(g_t) : g_{t_0}^{-1}(z_0) \in U_0\}$ for $t_0 \in [0, \infty)$, $z_0 \in \mathbb{H}$, and open set $U_0 \subset \mathbb{H}$. Let the topology on the space of (K_t) be generated by $\{(K_t) : z_0 \ne K_{t_0}\}$ for $t_0 \in [0, \infty)$ and $z_0 \in \mathbb{H}$. Then the chordal Loewner maps are continuous.

This means that the distribution of SLE is a pushforward measures of the Wiener measure (the distribution of Brownian motion) under the chordal Loewner map.

Theorem 1.1 (Rohde-Schramm, Lawler-Schramm-Werner) *For any $\kappa > 0$, with probability 1 a standard chordal SLE(κ) trace exists; the trace tends to ∞ as $t \to \infty$; is simple iff $\kappa \in (0, 4]$; visits every point on $\mathbb{H} \cup \mathbb{R}$ iff $\kappa \geq 8$.*

Remark. Rohde and Schramm proved the case $\kappa \neq 8$ using Stochastic Analysis and Conformal Geometry. Lawler, Schramm and Werner proved the case $\kappa = 8$ using a different method. They showed that SLE(8) is the scaling limit of the uniform spanning tree Peano curve. We will prove Rohde and Schramm's result later.

Lemma 1.7 *Let $\beta(t)$ be a standard chordal SLE(κ) trace. Let $a > 0$. Then $a\beta(t/a^2)$ has the same distribution as $\beta(t)$.*

Proof. This follows from Lemma 1.5 with $b = 0$ and the fact that $aB(t/a^2)$ has the same distribution as $B(t)$. □

Remark. The lemma states that if we dilate a standard chordal SLE(κ) trace β by a factor a, then the new curve has the same distribution as β up to a linear time-change. If we do not care about the parametrization, then $a\beta$ has the same distribution as β.

Since a standard chordal SLE(κ) trace lies on $\overline{\mathbb{H}}$, starts from $\lambda(0) = 0$, and ends at ∞, we also view it as a chordal SLE(κ) trace in $\mathbb{H}$ from 0 to ∞.

We now define chordal SLE in a general simply connected domain. A domain in this lecture will always be a connected open subset of the extended Complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with spherical metric. A simply connected domain is a domain whose complement in $\widehat{\mathbb{C}}$ is a (nondegenerate) continuum, which is a connected compact subset with more than one point. For example, half-planes and discs are simply connected domains, but $\mathbb{C}$ and $\widehat{\mathbb{C}}$ are not. When we talk about the closure or boundary of a simply connected domain, we mean its closure or boundary in $\widehat{\mathbb{C}}$. For example, ∞ is a boundary point of $\mathbb{H}$. Riemann's mapping theorem says that any two simply connected domains are conformally equivalent.

Definition 1.4 *Let β be a standard chordal SLE(κ) trace. Let $W : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} D$. Then we call $W \circ \beta$ a chordal SLE(κ) trace in D from $W(0)$ to $W(\infty)$.*

Remarks.

1. Initially W is not defined at 0 and ∞. The values of W on $\partial\mathbb{H} = \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ should be understood as prime ends of D. If V is another conformal map from $\mathbb{H}$ onto D, then $W \circ V^{-1}$ is a Möbius transformation, which extends continuously to $\overline{\mathbb{H}}$. For $x \in \partial\mathbb{H}$, we say $W(x) = V(x)$ if the extension of $W \circ V^{-1}$ fixes x.

2. If D is bounded by a Jordan curve, then W extends continuously to $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ and induces a homeomorphism between $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and J. In this case, we may view $W(0)$ and $W(\infty)$ as two points on J.

3. In general we do not view $W(x)$ for $x \in \partial\mathbb{H}$ as boundary points of D even if W extends continuously to $\overline{\mathbb{H}}$. For example, $W(z) = z^2$ maps $\mathbb{H}$ onto $\mathbb{C} \setminus [0, \infty)$, and its continuation maps 1 and -1 to the same point 1. But we want to distinguish $W(1)$ from $W(-1)$.

4. If there is another $V : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} D$ such that $V(0) = W(0)$ and $V(\infty) = W(\infty)$. Then $V \circ W^{-1} : (\mathbb{H}; 0, \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{H}; 0, \infty)$, which implies that $V \circ W^{-1}(z) = az$ for some $a > 0$. So $V(z) = W(az)$. Thus, $V(\beta(t)) = W(a\beta(t))$. From Lemma 1.7 we see that $V(\beta(t/a^2))$ has the same distribution as $W(\beta(t))$. Thus, up to a linear time-change, the distribution of a chordal SLE(κ) trace does not depend on the choice of W.

Proposition 1.2 (Domain Markov Property for Chordal SLE) *Let K_t and $\beta(t)$, $0 \le t < \infty$, be the chordal Loewner hulls and trace driven by $\lambda(t) = \sqrt{\kappa}B(t)$. Let T_0 be a finite stopping time w.r.t. the filtration $\mathcal{F}_t$ generated by $B(t)$. Then conditioned on $\mathcal{F}_{T_0}$, $\beta(T_0 + t)$, $0 \le t < \infty$, is a chordal SLE(κ) trace in $\mathbb{H} \setminus K_{T_0}$ from $\beta(T_0)$ to ∞.*

Proof. Let g_t be the chordal Loewner maps driven by λ. Let $\lambda_{T_0}(t) = \lambda(T_0 + t) - \lambda(T_0)$. From the properties of Brownian motion, we know that $\lambda_{T_0}(t)$ has the same distribution as $\lambda(t)$, and is independent of $\mathcal{F}_{T_0}$. So λ_{T_0} generates a standard chordal SLE(κ) trace, say β_{T_0}, which is independent of $\mathcal{F}_{T_0}$. From Lemma 1.6, we see that $\beta(T_0 + t) = g_{T_0}^{-1}(\beta_{T_0}(t) + \lambda(T_0))$. The conclusion follows because $z \mapsto g_{T_0}^{-1}(z + \lambda(T_0))$ is adapted to $\mathcal{F}_{T_0}$, and maps $(\mathbb{H}; 0, \infty)$ conformally onto $(\mathbb{H} \setminus K_{T_0}; \beta(T_0), \infty)$. $\square$

1.2 Radial Loewner equation

The radial Loewner equation driven by $\lambda \in C([0, T))$ is

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad 0 \le t < T, \quad g_0(z) = z. \tag{1.4}$$

For every $z \in \mathbb{C}$, let $\tau(z) \ge 0$ be such that $[0, \tau(z))$ is the maximal interval of the solution $t \mapsto g_t(z)$. So g_t is defined on $\{z \in \mathbb{C} : \tau(z) > t\}$. We have the following facts.

1. $g_t(0) = 0$ for all $t \in [0, T)$.

2. Each g_t commutes with the map $z \mapsto \frac{1}{\bar{z}}$, which is the reflection about $\mathbb{T} = \{|z| = 1\}$. This is because $1/\overline{g_t(z)}$ satisfies the same ODE as in (1.4).

3. Each g_t is conformal on $\{z \in \mathbb{C} : \tau(z) > t\}$.

4. $$\partial_t \log(g_t(z)/z) = \frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad z \ne 0.$$

   Letting $z \to 0$, we get $\partial_t \log(g_t'(0)) = 1$. So $g_t'(0) = e^t$.

5. If $z \in \mathbb{T}$ then $g_t(z)$ stays on $\mathbb{T}$ before $\tau(z)$. This is because the real part of $\frac{e^{i\lambda(t)}+g_t(z)}{e^{i\lambda(t)}-g_t(z)}$ is 0 if $g_t(z) \in \mathbb{T}$.

6. If $z \in \mathbb{D} = \{|z| < 1\}$ then $g_t(z)$ stays inside $\mathbb{D}$ before $\tau(z)$, and $t \mapsto |g_t(z)|$ is increasing. This is because the real part of $\frac{e^{i\lambda(t)}+g_t(z)}{e^{i\lambda(t)}-g_t(z)}$ is positive if $g_t(z) \in \mathbb{D}$.

7. If $\tau(z) < T$, then $\lim_{t\to\tau(z)} g_t(z) - e^{i\lambda(t)} = 0$. If $z \in \mathbb{D} \cup \mathbb{T}$, then $g_t(z)$ stays inside the bounded set $\mathbb{D} \cup \mathbb{T}$. If the solution blows up before T, it must hit the singularity. If $z \in \{|z| > 1\}$, then the result follows from the mirror symmetry about $\mathbb{T}$.

8. Each g_t maps $\{z \in \mathbb{D} : \tau(z) > t\}$ onto $\mathbb{D}$. Let $t_0 \in [0, T)$. First, we know that $g_{t_0}(\{z \in \mathbb{D} : \tau(z) > t_0\}) \subset \mathbb{H}$. Second, fix any $z_0 \in \mathbb{H}$, consider the ODE

$$h'(t) = h(t)\frac{e^{i\lambda(t)} + h(t)}{e^{i\lambda(t)} - h(t)}, \quad 0 \le t \le t_0, \quad h(t_0) = z_0.$$

   As t decreases from t_0 to 0, $|h(t)|$ decreases, so the solution will not hit the singularity $e^{i\lambda(t)}$, which implies that it does not blow up on $[0, t_0]$. Then we have $h(0) \in \mathbb{D}$ and $g_{t_0}(h(0)) = h(t_0) = z_0$.

Remark. The radial Loewner equation is the original Loewner equation introduced by Charles Loewner. The chordal Loewner equation is in fact introduced by Oded Schramm.

Let $K_t = \{z \in \mathbb{D} : \tau(z) \le t\}$, $0 \le t < T$. Then $K_0 = \emptyset$; $K_{t_1} \subset K_{t_2}$ if $t_1 < t_2$; each K_t is a relatively closed subset of $\mathbb{H}$, $g_t : (\mathbb{D} \setminus K_t; 0) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{D}; 0)$, and satisfies $g_t'(0) = e^t$. The g_t is uniquely determined by K_t. If $t_1 < t_2$, then $g_{t_1}'(0) \ne g_{t_2}'(0)$, so $K_{t_1} \subsetneqq K_{t_2}$.

Definition 1.5 *We call g_t and K_t the radial Loewner maps and hulls driven by λ.*

Lemma 1.8 *Suppose g_t and K_t are radial Loewner maps and hulls driven by $\lambda(t)$. Let $b \in \mathbb{R}$. Then $e^{ib} g_t(\cdot/e^{ib})$ and $e^{ib} K_t$ are radial Loewner maps and hulls driven by $\lambda(t) + b$.*

Note that for any $n \in \mathbb{Z}$, $\lambda + 2n\pi$ generate the same radial Loewner maps and hulls as λ.

Lemma 1.9 *Suppose g_t and K_t are radial Loewner maps and hulls driven by $\lambda \in C([0, T))$. Let $t_0 \in [0, T)$. Then $g_{t_0+t} \circ g_{t_0}^{-1}$ and $g_{t_0}(K_{t_0+t} \setminus K_{t_0})$, $0 \le t < T - t_0$, are radial Loewner maps and hulls driven by $\lambda(t_0 + t)$.*

Lemma 1.10 *Suppose g_t and K_t are radial Loewner maps and hulls driven by $\lambda \in C([0, T))$. Then for any $t \in [0, T)$,*

$$\{e^{i\lambda(t)}\} = \bigcap_{\varepsilon \in (0, T-t)} \overline{g_t(K_{t+\varepsilon} \setminus K_t)}. \tag{1.5}$$

This lemma asserts that the radial Loewner hulls determine the driving function up to an integer multiple of 2π.

Definition 1.6 *We say that λ generates a radial Loewner trace β if*

$$\beta(t) = \lim_{\mathbb{D}\ni z\to e^{i\lambda(t)}} g_t^{-1}(z)$$

exists for $0 \le t < T$ and is a continuous curve. Such β lies on $\mathbb{D}\cup\mathbb{T}$ and $\beta(0) = e^{i\lambda(0)} \in \mathbb{T}$. We call the trace β simple if it has no self intersection and intersects $\mathbb{T}$ only at $\beta(0)$.

Proposition 1.3 *If λ generates a radial Loewner trace β, then for each t, $\mathbb{D}\backslash K_t$ is the connected component of $\mathbb{D}\setminus\beta((0,t])$ that contains 0. In particular, if β is simple, then $K_t = \beta((0,t])$. Moreover, for each t, g_t^{-1} extends continuously to $\mathbb{D}\cup\mathbb{T}$.*

Definition 1.7 *For $\kappa > 0$, a standard radial SLE(κ) is defined to be the radial Loewner process driven by $\lambda(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$.*

The distribution of radial SLE is the pushforward measures of the Wiener measure under the radial Loewner maps.

Theorem 1.2 *For any $\kappa > 0$, with probability 1 a standard radial SLE(κ) trace exists; tends to 0 as $t \to \infty$; is simple iff $\kappa \in (0,4]$; visits every point on $\mathbb{D}\cup\mathbb{T}\setminus\{0\}$ iff $\kappa \ge 8$.*

Remark. This theorem follows Theorem 1.1 and the weak equivalence between chordal SLE and radial SLE.

Since a standard radial SLE(κ) trace lies on $\overline{\mathbb{D}}$, starts from $e^{i\lambda(0)} = 1$, and ends at 0, we also view it as a radial SLE(κ) trace in $\mathbb{D}$ from 1 to 0.

Definition 1.8 *Let β be a standard radial SLE(κ) trace. Let $W : \mathbb{D} \overset{\text{Conf}}{\twoheadrightarrow} D$. Then we call $W\circ\beta$ a radial SLE(κ) trace in D from $W(1)$ to $W(0)$.*

Remark. Since W is defined on $\mathbb{D}$, $W(0)$ is well defined; while $W(1)$ should be understood as a prime end of D as in the definition of chordal SLE in a general simply connected domain.

Lemma 1.11 (Domain Markov Property of radial SLE) *Let K_t and $\beta(t)$, $0 \le t < \infty$, be the radial Loewner hulls and trace driven by $\lambda(t) = \sqrt{\kappa}B(t)$. Let T be a finite stopping time w.r.t. the filtration $\mathcal{F}_t$ generated by $B(t)$. Then conditioned on $\mathcal{F}_T$, $\beta(T+t)$, $0 \le t < \infty$, is a radial SLE(κ) trace in $\mathbb{D}\setminus K_t$ from $\beta(T)$ to 0.*

2 Conformal Mappings

2.1 Koebe's $1/4$ theorem and distortion theorem

Let $\mathcal{S}$ denote the set of maps f that maps $\mathbb{D}$ conformally into $\mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$. Any $f \in \mathcal{S}$ has expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Given $f \in \mathcal{S}$, let $F(z) = 1/f(1/z)$. Then F maps $\widehat{\mathbb{C}} \setminus (\mathbb{D} \cup \mathbb{T})$ conformally into $\widehat{\mathbb{C}} \setminus \{0\}$ with $F(\infty) = \infty$. The Laurent expansion of F at ∞ is

$$F(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}.$$

We have $b_0 = -a_2$ and $b_1 = a_2^2 - a_3$. Let $K = \widehat{\mathbb{C}} \setminus F(\widehat{\mathbb{C}})$. Then K is a compact subset of $\mathbb{C}$.

Proposition 2.1 (Area Theorem)

$$\text{area}(K) = \pi(1 - \sum_{n=1}^{\infty} n|b_n|^2).$$

In particular, we have $\sum_{n=1}^{\infty} n|b_n|^2 \le 1$.

Proof. For $r > 1$, let K_r denote the region bounded by $\gamma_r := F_K(\{|z| = r\})$. Then $\text{area}(K) = \lim_{r \to 1^+} \text{area}(K_r)$. We may calculate $\text{area}(K_r)$ using Green's Theorem. We have

$$2i\,\text{area}(K_r) = \int_{\gamma_r} \overline{z} dz = \int_{|z|=r} \overline{F_K(z)} F_K'(z) dz = \int_0^{2\pi} \overline{F_K(re^{i\theta})} F_K'(re^{i\theta}) ire^{i\theta} d\theta$$

$$= \int_0^{2\pi} (re^{-i\theta} + \overline{b_0} + \sum_{n=1}^{\infty} \overline{b_n} r^{-n} e^{in\theta})(1 - \sum_{n=1}^{\infty} n b_n r^{-n-1} e^{-i(n+1)\theta}) ire^{i\theta} d\theta$$

$$= 2\pi i (r^2 - \sum_{n=1}^{\infty} r^{-2n} |b_n|^2).$$

Thus, $\text{area}(K_r) = \pi(r^2 - \sum_{n=1}^{\infty} r^{-2n}|b_n|^2)$. The conclusion follows by letting $r \to 1$. □

Lemma 2.1 *If $f \in \mathcal{S}$, then there exists $h \in \mathcal{S}$ such that $h(z)^2 = f(z^2)$ for $z \in \mathbb{D}$.*

Proof. First, $f(z)/z$ extends to a non-zero analytic function on $\mathbb{D}$. Second, there is an analytic function g on $\mathbb{D}$ such that $g(z)^2 = f(z)/z$. Let $h(z) = zg(z^2)$. Then h is analytic, $h(0) = 0$, $h'(0) = g(0) = 1$, and $h(z)^2 = f(z^2)$. If $h(z_1) = h(z_2)$, then $f(z_1^2) = f(z_2^2)$, which implies that $z_1 = z_2$ or $z_2 = -z_2$. If $z_1 = -z_2$, then $g(z_1^2) = -g(z_2^2)$, which is a contradiction. So h is conformal. Thus, $h \in \mathcal{S}$. □

Proposition 2.2 *If $f \in \mathcal{S}$, then $|a_2| \le 2$.*

Proof. Suppose $f(z) = z + a_2 z^2 + \cdots \in \mathcal{S}$ and let h be as in the previous lemma. Then $h(z) = z + \frac{a_2}{2} z^3 + \cdots$. Let $g(z) = 1/h(1/z)$. The g has an expansion at ∞: $g(z) = z - \frac{a_2/2}{z} + \cdots$. The Area Theorem implies that $|a_2| \le 2$. $\square$

Remark. Charles Loewner introduced (radial) Loewner equation to prove $|a_3| \le 3$. Now it is known that $|a_n| \le n$ for all $n \in \mathbb{N}$.

Theorem 2.1 (Koebe's 1/4 Theorem) *1. If $f \in \mathcal{S}$, then $\operatorname{dist}(0, \partial f(\mathbb{D})) \ge 1/4$.*

2. If $f : (D_1; z_1) \overset{\text{Conf}}{\twoheadrightarrow} (D_2; z_2)$, then

$$\frac{|f'(z_1)|}{4} \le \frac{\operatorname{dist}(z_2, \partial D_2)}{\operatorname{dist}(z_1, \partial D_1)} \le 4|f'(z_1)|.$$

Proof. 1. Let $r = \operatorname{dist}(0, \partial f(\mathbb{D}))$. Suppose $z_0 \in \mathbb{C} \setminus f(\mathbb{D})$. Define $h(z) = \frac{f(z)}{1 - f(z)/z_0}$. Then $h \in \mathcal{S}$ and has expansion

$$h(z) = z + (a_2 + \frac{1}{z_0}) z^2 + \cdots.$$

From Proposition 2.2, we have $|a_2| \le 2$ and $|a_2 + 1/z_0| \le 2$. This implies $|z_0| \ge 1/4$. Since this is true for all $z_0 \in \mathbb{C} \setminus f(\mathbb{D})$, we get $r \ge 1/4$.

2. Let $r_j = \operatorname{dist}(z_j, \partial D_j)$, $j = 1, 2$. Define $h(z) = \frac{f(r_1(z_1 + z)) - z_2}{r_1 f'(z_1)}$. Then $h \in \mathcal{S}$ and $\operatorname{dist}(0, \partial h(\mathbb{D})) \le \frac{r_2}{r_1 |f'(z_1)|}$. From Part 1, we get $\frac{r_2}{r_1} \ge \frac{|f'(z_1)|}{4}$. Let $g = f^{-1}$. Then $g : (D_2; z_2) \overset{\text{Conf}}{\twoheadrightarrow} (D_1; z_1)$. So $\frac{r_1}{r_2} \ge \frac{|g'(z_2)|}{4} = \frac{1}{4|f'(z_1)|}$. $\square$

Examples.

1. 1/4 is the best possible number. The Koebe's function is $f(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$. We have
$$f(z) = \frac{1}{4}\Big(\frac{1+z}{1-z}\Big)^2 - \frac{1}{4}.$$
Since $z \mapsto \frac{1+z}{1-z}$ maps $\mathbb{D}$ conformally onto $\{\operatorname{Re} z > 0\}$ and $z \mapsto z^2$ maps $\{\operatorname{Re} z > 0\}$ conformally onto $\mathbb{C} \setminus (-\infty, 0]$, we see that f maps $\mathbb{D}$ conformally onto $\mathbb{C} \setminus (-\infty, -1/4]$. Thus, $f \in \mathcal{S}$ and $\operatorname{dist}(0, \partial f(\mathbb{D})) = 1/4$.

2. Suppose g_t and K_t, $0 \le t < \infty$, are radial Loewner maps and hulls driven by $\lambda \in C([0, \infty))$. Since $g_t : (\mathbb{D} \setminus K_t; 0) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{D}; 0)$ and $g_t'(0) = e^t$, from Koebe's 1/4 theorem, $\operatorname{dist}(0, K_t) \le 4e^{-t} \to 0$ as $t \to \infty$.

3. Suppose g_t and K_t, $0 \le t < \infty$, are chordal Loewner maps and hulls driven by λ. Since $g_t : \mathbb{H} \setminus K_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, we have $\min\{\operatorname{Im} z_0, \operatorname{dist}(z_0, K_t)\} \asymp \operatorname{Im} g_t(z_0)/|g_t'(z_0)|$ for any $z_0 \in \mathbb{H} \setminus K_t$.

This property could be used to study the phase change of SLE. Using Stochastic Analysis we can prove that for any fixed $z_0 \in \mathbb{H}$, almost surely 1) $\tau(z_0) = \infty$ for $\kappa \le 4$ and $\tau(z_0) < \infty$ for $\kappa > 4$; 2) $\lim_{t\to\tau(z_0)} \operatorname{Im} g_t(z_0)/|g_t'(z_0)| = 0$ for $\kappa \ge 8$, and > 0 for $\kappa < 8$. Assume that we have proved the existence of the chordal SLE(κ) trace β. Suppose $\kappa \in (4, 8)$. The above result implies that a.s. $\lim_{t\to\tau(z_0)^-} \operatorname{dist}(z_0, \beta((0,t])) = \lim_{t\to\tau(z_0)^-} \operatorname{dist}(z_0, K_t) = 0$. Thus, $z_0 \ne \beta((0, \tau(z_0)])$ but $z_0 \in K_{\tau(z_0)}$, which means that z_0 lies in the interior of $K_{\tau(z_0)}$. After $\tau(z_0)$, β grows in $\overline{\mathbb{H} \setminus K_{\tau(z_0)}}$. So z_0 is almost surely not visited by the trace β. Suppose $\kappa \ge 8$, then we have a.s. $\lim_{t\to\tau(z_0)^-} \operatorname{dist}(z_0, \beta((0,t])) = 0$, which implies that $z_0 = \beta(\tau(z_0))$. This can be used to show that β visits every point on $\mathbb{H}$.

Suppose $f \in \mathcal{S}$ and $w \in \mathbb{D}$. Let $T_w(z) = \frac{w+z}{1+\overline{w}z}$. Then $T_w : (\mathbb{D}; 0) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{D}; w)$, $T_w'(0) = 1-|w|^2$ and $T_w''(0) = -2\overline{w}(1-|w|^2)$. We may construct another function $h \in \mathcal{S}$ by

$$h(z) = \frac{f(T_w(z)) - f(w)}{f'(w)T_w'(0)} = \frac{f(T_w(z)) - f(w)}{f'(w)(1-|w|^2)}.$$

Then

$$h''(z) = \frac{f'(T_w(z))T_w''(z) + f''(T_w(z))T_w'(z)^2}{f'(w)(1-|w|^2)}.$$

In particular, we get

$$h''(0) = \frac{f'(w)T_w''(0) + f''(w)T_w'(0)^2}{f'(w)(1-|w|^2)} = \frac{f'(w)(-2\overline{w}(1-|w|^2) + f''(w)(1-|w|^2)^2}{f'(w)(1-|w|^2)}$$

$$= -2\overline{w} + \frac{f''(w)}{f'(w)}(1-|w|^2).$$

From Proposition 2.2 we get $|h''(0)| \le 4$. So

$$\Big|\frac{w}{|w|}\frac{f''(w)}{f'(w)} - \frac{2|w|}{1-|w|^2}\Big| \le \frac{4}{1-|w|^2}. \tag{2.1}$$

Theorem 2.2 (Distortion Theorem) *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

Proof. Let $h(z) = \log(f'(z))$. Then h is analytic on $\mathbb{D}$ with $h(0) = 0$, and $h' = f''/f'$. Suppose $z = re^{i\theta}$, $0 \le r < 1$ and $\theta \in \mathbb{R}$. Then

$$\log(f'(z)) = h(z) = \int_{[0,z]} h'(z)dz = \int_0^r h'(se^{i\theta})e^{i\theta}ds = \int_0^r \frac{f''(se^{i\theta})}{f'(se^{i\theta})}e^{i\theta}ds.$$

From (2.1) we get

$$\left|\log(f'(z)) - \int_0^r \frac{2s}{1-s^2}ds\right| \le \int_0^r \frac{4}{1-s^2}ds,$$

which is $|\log(f'(z)) + \log(1-r^2)| \le 2\log(1+r) - 2\log(1-r)$. Taking real part, we get

$$-3\log(1+r) + \log(1-r) \le \log|f'(z)| \le \log(1+r) - 3\log(1-r).$$

The proof is complete by exponentiating this inequality. □

Remark. Integrating the estimation for $|f'(z)|$ along a radial line, we can show

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

Corollary 2.1 *There is a constant $C > 1$ such that the following is true. Suppose D is a domain, f is conformal on D, and $z_0, w_0 \in D$. Suppose there is a piecewise C^1 curve γ connecting z and w. Let l be the length of γ and $r = \operatorname{dist}(\gamma, \partial D)$. Then $|f'(w_0)| \le |f'(z_0)|C^{l/r}$.*

Proof. Let $n = \lceil 2l/r \rceil$. We may find $z_1, z_2, \dots, z_n$ on γ such that $z_n = w_0$ and $|z_j - z_{j-1}| \le r/2$, $1 \le j \le n$. Construct $f_j \in \mathcal{S}$ by $f_j(z) = f(z_{j-1} + rz)/(rf'(z_{j-1}))$. Then $f_j'(z) = f'(z_{j-1} + rz)/f'(z_{j-1})$. Letting $z = (z_j - z_{j-1})/r$ and applying Distortion Theorem, we get

$$\frac{|f'(z_j)|}{|f'(z_{j-1})|} \le \frac{1+|z|}{(1-|z|)^3} \le \frac{1+1/2}{(1-1/2)^3} = 12.$$

Thus, $|f'(w_0)| = |f'(z_n)| \le 12^n|f'(z_0)| \le 12^{2l/r+1}|f'(z_0)|$. If $l \ge r/2$, then $2l/r + 1 \le 4l/r$, so $|f'(w_0)| \le (12^4)^{l/r}|f'(z_0)|$. Now suppose $l \le r/2$. Then $n = 1$ and $|z_0 - w_0| \le l \le r/2$. The above computation gives

$$\frac{|f'(w_0)|}{|f'(z_0)|} \le \frac{1+l/r}{(1-l/r)^3} \le C_0^{l/r},$$

where $C_0 = e^7$. Then $C := \max\{12^4, C_0\}$ is the constant we want. □

2.2 Extremal length

Extremal length is about some measurement of a family of curves. The value is a nonnegative real number. It is important for this course because it is conformally invariant. Let D be a domain. Let ρ be a nonnegative Borel function on D. The ρ-area of D is

$$A_\rho(D) = \int_D \rho(z)^2 dA(z).$$

Let γ be a piecewise C^1 curve in D, the ρ-length of γ is

$$L_\rho(\gamma) = \int_\gamma \rho(z) ds(z).$$

Let Γ be a family of piecewise C^1 curves in D, the ρ-length of Γ is

$$L_\rho(\Gamma) = \inf_{\gamma\in\Gamma} L_\rho(\gamma).$$

The extremal length of Γ in D is

$$L(\Gamma; D) = \sup_\rho \frac{L_\rho(\Gamma)^2}{A_\rho(D)}.$$

For two sets A and B, we say a curve γ connects A and B if one end of γ approaches to a point on A and the other end of γ approaches to a point on B. We say a curve γ separates A and B in D if γ lies in D and any curve in D connecting A and B must intersects γ. Let $\Gamma_D(A, B)$ denote the set of piecewise C^1 curves in D connecting A and B. Let $\Gamma^*_D(A, B)$ denote the set of piecewise C^1 curves in D separating A and B. Then the extremal length of $\Gamma_D(A, B)$ is called the extremal distance between A and B in D, and is denoted by $d_D(A, B)$; and the extremal length of $\Gamma^*_D(A, B)$ is called the conjugate extremal distance between A and B in D, and is denoted by $d^*_D(A, B)$

Remark The D in $L(\Gamma; D)$ is unnecessary. In fact, if $D' \supset D$, then $L(\Gamma; D') = L(\Gamma; D)$. Since Γ lie in D, to maximize $L_\rho(\Gamma)$ while keeping $A_\rho(D')$ unchanged, ρ must concentrate on D.

Examples.

1. Let D be a rectangle $\{0 < x < a, 0 < y < b\}$. Let Γ be the set of piecewise C^1 curves in D connecting the two vertical sides (of length b). Let $\rho = 1$. Then $A_\rho(D) = ab$ and $L_\rho(\Gamma) = a$. So $L(\Gamma; D) \geq \frac{a}{b}$. Now suppose ρ is any nonnegative Borel function on D. From Hölder's inequality, we have

$$A_\rho(D) = \int_0^b \int_0^a \rho(x, y)^2 dxdy \geq \int_0^b \frac{1}{a}\Big(\int_0^a \rho(x, y)dx\Big)^2 dy$$

$$\geq \int_0^b \frac{1}{a}\Big(L_\rho(\Gamma)\Big)^2 dy = \frac{b}{a} L_\rho(\Gamma)^2,$$

which gives $\frac{L_\rho(\Gamma)^2}{A_\rho(D)} \leq \frac{a}{b}$. Thus, $d_D([0, ib], [a, a+ib]) = \frac{a}{b}$. Similarly, $d_D([0, a], [a, a+ib]) = \frac{b}{a}$. We also have $d^*_D([0, ib], [a, a + ib]) = \frac{b}{a}$. Similarly, $d^*_D([0, a], [a, a + ib]) = \frac{a}{b}$.

2. Let D be an annulus $\{r_1 < |z| < r_2\}$. Let $C_j = \{|z| = r_j\}$, $j = 1, 2$, be its two boundary circles. Let Γ be the set of piecewise C^1 curves in D connecting the two boundary circles. Let $\rho(z) = \frac{1}{|z|}$. Then $A_\rho(D) = 2\pi \log(r_2/r_1)$ and $L_\rho(\Gamma) = \log(r_2/r_1)$. Thus, $L(\Gamma; D) \geq \frac{\log(r_2/r_1)}{2\pi}$. Using Hölder's inequality, we can show that $L(\Gamma) = \frac{\log(r_2/r_1)}{2\pi}$. Thus, $d_D(C_1, C_2) = \frac{\log(r_2/r_1)}{2\pi}$. Similarly, $d^*_D(C_1, C_2) = \frac{2\pi}{\log(r_2/r_1)}$.

Theorem 2.3 *Let Γ_1 be a family of piecewise C^1 curves in D_1. Suppose $f : D_1 \overset{\text{Conf}}{\twoheadrightarrow} D_2$. Let $\Gamma_2 = f(\Gamma_1) := \{f \circ \gamma : \gamma \in \Gamma_1\}$. Then $L(\Gamma_1; D_1) = L(\Gamma_2; D_2)$.*

Proof. This is because there is a one-to-one correspondence between the set of nonnegative Borel functions on D_1 and the set of nonnegative Borel functions on D_2: $\rho_1 \leftrightarrow \rho_2$ such that $A_{\rho_1}(D_1) = A_{\rho_2}(D_2)$ and $L_{\rho_1}(\gamma) = L_{\rho_2}(f\circ\gamma)$ for each $\gamma \in \Gamma_1$. In fact, given ρ_2, the corresponding ρ_1 is defined by $\rho_1(z) = |f'(z)|\rho_2(f(z))$. Then

$$A_{\rho_1}(D_1) = \int_{D_1} |f'(z)|^2 \rho_2(f(z))^2 dA(z) = \int_{D_2} \rho_2(w)^2 dA(w) = A_{\rho_2}(D_2);$$

$$L_{\rho_1}(\gamma) = \int_{\gamma} |f'(z)|\rho_2(f(z))ds(z) = \int_{f\circ\gamma} \rho_2(w)ds(w) = L_{\rho_2}(f \circ \gamma). \quad \square$$

Remark. Two rectangles or two annuli are conformally equivalent iff they have similar shapes.

Lemma 2.2 (Comparison Principle) *Let Γ_1 and Γ_2 be two families of piecewise C^1 curves. If every curve in Γ_2 contains a subcurve in Γ_1, then $L(\Gamma_2) \geq L(\Gamma_2)$.*

Proof. This is because $L_\rho(\Gamma_2) \geq L_\rho(\Gamma_1)$ for every ρ. $\square$

Example. Suppose $\text{diam}(A) = r < R = \text{dist}(A, B)$. Let Ω be the annulus $\{r < |z - z_0| < R\}$, and C_R and C_r be its boundary circles. Any curve connecting A and B must cross the annulus, so it contains a subcurve in Ω connecting C_R and C_r. Thus, for any domain D, $d_D(A, B) \geq d_\Omega(C_R, C_r) = \log(R/r)/(2\pi)$.

2.3 Boundary behaviors of conformal maps

Definition 2.1 *A topological space X is called locally connected if for every $x \in X$ and open set $U \ni x$, there exists a connected neighborhood N of x that is contained in U. A subset of a topological space X is a locally connected set if it is a locally connected space when viewed as a subspace of X.*

Remark. If X is a metric space, then X is locally connected iff for every $x \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $\text{dist}(x, y) < \delta$ then x and y lie in a connected subset of X with diameter less than ε. In addition, if X is compact, the δ can be chosen to be independent of x.

Examples.

1. Any convex set in $\mathbb{C}$ is locally connected.

2. An relatively open subset of a locally connected set is locally connected.

3. $\{x + i\sin(1/x) : x > 0\} \cup [-i, i]$ is connected but not locally connected.

Lemma 2.3 *If $f : X \to Y$ is continuous and X is compact and locally connected and Y is Hausdorff, then $f(X)$ is locally connected.*

Proof. We may assume that $Y = f(X)$. Let $y \in Y$ and V be an open subset of Y with $y \in V$. Let S be a connected component of V that contains y. Let $w \in f^{-1}(S) \subset f^{-1}(V)$. Since X is locally connected and $f^{-1}(V)$ is open, there is a connected neighborhood N of w which is contained in $f^{-1}(V)$. Then $f(N)$ is a connected subset of V which contains $f(w) \in S$. Since S is a connected component and $f(N) \cap S \neq \emptyset$, we have $f(N) \subset S$, which implies that $N \subset f^{-1}(S)$. Now for every $w \in f^{-1}(S)$, we find a neighborhood N of w which is contained in $f^{-1}(S)$. So $f^{-1}(S)$ is open. Since X is compact and Y is Hausdorff, we conclude that S is an open subset of Y. So S is a connected neighborhood of y in Y that is contained in V. □

Theorem 2.4 *Let D be a simply connected set. The followings are equivalent.*

(i) Any conformal map from $\mathbb{D}$ onto D extends continuously to $\overline{\mathbb{D}}$.

(ii) ∂D is locally connected.

(iii) There is a locally connected set K in $\widehat{\mathbb{C}}$ such that D is a connected component of $\widehat{\mathbb{C}} \setminus K$.

Proof. (i) implies (ii). Riemann's mapping theorem assures the existence of a conformal map from $\mathbb{D}$ onto $\overline{\mathbb{D}}$. Since it extends continuously to $\overline{\mathbb{D}}$, we get a continuous map from $\mathbb{T}$ onto ∂D. Since $\mathbb{T}$ is locally connected, from Lemma 2.3, ∂D is locally connected.

(ii) implies (iii). We may simply let $K = \partial D$.

(iii) implies (i). We use extremal length in the argument. We also use the fact that if the diameter of a closed set $S \subset \widehat{\mathbb{C}}$ has diameter $d < \pi/4$, then at most one component of $\widehat{\mathbb{C}} \setminus E$ has diameter greater than $2d$. Suppose $W : \mathbb{D} \overset{\text{Conf}}{\twoheadrightarrow} D$. Let $z_0 \in \mathbb{T}$. For $r > 0$, let $S_r = \{z \in \mathbb{D} : |z - z_0| < r\}$. We suffice to show that the diameter of $W(S_r)$ tends to 0 as $r \to 0$. Let E be a continuum in $\mathbb{D}$ and $R = \text{dist}(z_0, E) > 0$. For $r \in (0, R)$, let Γ_r denote the family of curves in $\mathbb{D}$ that disconnect E from S_r. Note that any curve in the annulus $\{r < |z - z_0| < R\}$ that disconnects the two boundary circle contains a subcurve which belongs to Γ_r. Thus, $L(\Gamma_r) \le 2\pi/\log(R/r)$, which tends to 0 as $r \to 0$. From the conformal invariance of extremal length, $L(W(\Gamma_r)) \to 0$ as $r \to 0$. Note that $W(\Gamma_r)$ is the family of curves that separate $W(S_r)$ from $W(E)$. Let $\rho(z) = \frac{2}{1+|z|^2}$. Then we get the spherical metric. So $A_\rho(D) \le A_\rho(\widehat{\mathbb{C}}) = 4\pi$. Thus, $L_\rho(W(\Gamma_r)) \to 0$ as $r \to 0$. In particular, this means that we may choose $\gamma_r \in W(\Gamma_r)$ such that the spherical length of γ_r tends to 0 as $r \to 0$. Since γ_r has finite spherical length, its closure has at most two points more than itself. There are three cases. Case 1. $\overline{\gamma_r}$ intersects ∂D at no more than one point. Then $W(E)$ and $W(S_r)$ lie in two components of $\widehat{\mathbb{C}} \setminus \overline{\gamma_r}$. Since the diameter of $\overline{\gamma_r}$ tends to 0 and the diameter of $W(E)$ is positive, the diameter of $W(S_r)$ should also tends to 0. Case 2. $\overline{\gamma_r}$ intersects ∂D at two points, say a_r and b_r. Then $a_r, b_r \in K$ and $\text{dist}(a, b) \le \text{diam}(\gamma_r)$. Since K is locally connected and $\text{diam}(\gamma_r) \to 0$ as $r \to 0$, K contains a connected subset $L_r \ni a_r, b_r$ with diameter tends to 0 as $r \to 0$. Now $\gamma_r \cup L_r$ has diameter tends to 0 as $r \to 0$, and separates $W(E)$ from $W(S_r)$. Again we conclude that the diameter of $W(S_r)$ tends to 0. $\square$

Remarks.

1. The lemma is still true if $\mathbb{D}$ is replaced by a Jordan domain. This implies that a conformal map from $\mathbb{D}$ onto a Jordan domain extends to a homeomorphism between the closures.

2. Suppose J is a Jordan curve. There is a conformal map W_1 from $\mathbb{D}$ onto its interior, and a conformal map W_2 from $\{|z| > 1\}$ to the exterior of J. Then we get two homeomorphism induced by W_1 and W_2 from $\mathbb{T}$ onto J. Then $W_1^{-1} \circ W_2$ is an orientation preserving automorphism of $\mathbb{T}$. The conformal welding problem is: given the homeomorphism of $\mathbb{T}$, determine wether it is induced by the above conformal maps, and find the cuve J.

3. Suppose that λ generates a chordal Loewner trace β, and we have proved that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \beta([0,t])$. From Lemma 2.3 we see that $\widehat{\mathbb{R}} \cup \beta([0,t])$ is locally connected. Since $\mathbb{H} \setminus K_t$ is one connected component of $\widehat{\mathbb{C}} \setminus (\widehat{\mathbb{R}} \cup \beta([0,t]))$, from Theorem 2.4 the conformal map g_t^{-1} from $\mathbb{H}$ onto $\mathbb{H} \setminus K_t$ extends continuously to $\overline{\mathbb{H}}$. The same argument works for the radial Loewner trace.

Theorem 2.5 *Suppose* $W : D \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{D}$. *Let* $\gamma(t)$, $0 \le t \le 1$, *be a curve with* $\gamma(0) \in \partial D$ *and* $\gamma((0,1]) \subset D$. *Then* $\lim_{t\to 0} W(\gamma(t))$ *exits. Moreover, if* β *has the same property as* γ, *and* $\beta(0) \ne \gamma(0)$, *then* $\lim_{t\to 0} W(\gamma(t)) \ne \lim_{t\to 0} W(\beta(t))$.

Proof. Let $z_0 = \gamma(0)$, E be a continuum in $\mathbb{D}$, and $R = \text{dist}(z_0, E) > 0$. For any $r \in (0, R)$, there is $\delta > 0$ such that $\gamma([0,\delta]) \subset \{|z - z_0| < r\}$. Let ρ be a curve in $\{r < |z - z_0| < R$ that separates the two boundary circle. Let t_0 be the biggest number such that $\gamma(t) \in \rho$. Then ρ contains a subcurve ρ_0 which contains $\gamma(t_0)$ and whose two ends approach two boundary points. Then ρ_0 disconnects E from $\gamma((0,\delta])$ in D. Thus, $d_D^*(E, \gamma((0,\delta]) \le 2\pi/\log(R/r)$. From conformal invariance, $d_{\mathbb{D}}^*(W(E), W \circ \gamma((0,\delta])) \le 2\pi/\log(R/r)$. Let $\rho = 1$ on $\mathbb{D}$. Then we get the Euclidean metric. Since $A_\rho(\mathbb{D}) = \text{area}(\mathbb{D}) = \pi$, this implies that there is a curve α_r with length less than $2\pi\sqrt{\log(R/r)}$ that separates $W(E)$ from $W \circ \gamma((0,\delta])$ in $\mathbb{D}$. If r is close to 0, the length of α_r is also close to 0. If r is small enough, the length of α_r is less than the diameter of $W(E)$ and the distance between $W(E)$ and $\mathbb{T}$. Then α_r must touches $\mathbb{T}$ and does not intersect $W(E)$. Since $W(\gamma((0,\delta]))$ is disconnected from $W(E)$ in $\mathbb{D}$ by α_r, we see that the diameter of $W(\gamma((0,\delta]))$ is no more than the length of α_r. Thus, the the diameter of $W(\gamma((0,\delta]))$ tends to 0 as $\delta \to 0$, which implies that $\lim_{t\to 0} W(\gamma(t))$ exists. Suppose β has the same property as γ, and $\beta(0) \ne \gamma(0)$. Then $\lim_{t\to 0} W(\beta(t))$ also exists. Since $\alpha(0) \ne \beta(0)$, we may choose $\delta > 0$ such that $d_D(\alpha((0,\delta]), \beta((0,\delta])) > 0$. Thus, $d_{\mathbb{D}}(W(\alpha((0,\delta])), W(\beta((0,\delta]))) > 0$. If $\lim_{t\to 0} W(\gamma(t)) = \lim_{t\to 0} W(\beta(t)) := w_0$, then the extremal distance is 0 because there is $r > 0$ such that any curve in $\{0 < |z - w_0| < r\}$ that surrounds 0 contains a subcurve in $\mathbb{D}$ that connects $W(\alpha((0,\delta]))$ and $W(\beta((0,\delta]))$. $\square$

Remark.

1. If $\beta(0) = \gamma(0)$, we can not conclude that $\lim_{t\to 0} W(\gamma(t)) = \lim_{t\to 0} W(\beta(t))$.

2. From Theorem 2.4, if $\mathbb{D}$ is replaced by a simply connected domain with locally connected boundary, the first statement is still true, but we may not have $\lim_{t\to 0} W(\gamma(t)) \neq \lim_{t\to 0} W(\beta(t))$. The theorem still holds if $\mathbb{D}$ is replaced by a Jordan domain

2.4 Carathéodory convergence

Definition 2.2 *Suppose D_n is a sequence of domains and D is a plane domain. We say that (D_n) converges to D, denoted by $D_n \stackrel{\text{Cara}}{\longrightarrow} D$, if for every $z \in D$, $\text{dist}(z, \partial D_n) \to \text{dist}(z, \partial D)$. This is equivalent to the followings:*
(i) every compact subset of D is contained in all but finitely many D_n's; and
(ii) for every point $z_0 \in \partial D$, $\text{dist}(z_0, \partial D_n) \to 0$ as $n \to \infty$.

Remarks.

1. The distance and boundary in the definition both refer to the spherical metric. If D_n and D are all contained in $\mathbb{C}$, then the Euclidean metric gives the same definition.

2. A sequence of domains may converge to two different domains. For example, let $D_n = \mathbb{C} \setminus ((-\infty, n])$. Then $D_n \stackrel{\text{Cara}}{\longrightarrow} \mathbb{H}$, and $D_n \stackrel{\text{Cara}}{\longrightarrow} -\mathbb{H}$ as well. But two different limit domains of the same domain sequence must be disjoint from each other, because if they have nonempty intersection, then one contains some boundary point of the other, which implies a contradiction.

Definition 2.3 *Suppose $D_n \stackrel{\text{Cara}}{\longrightarrow} D$, $f_n : D_n \to \widehat{\mathbb{C}}$, $n \in \mathbb{N}$, and $f : D \to \widehat{\mathbb{C}}$. We say that f_n converges to f locally uniformly in D, or $f_n \stackrel{\text{l.u.}}{\longrightarrow} f$ in D, if for each compact subset F of D, f_n converges to f uniformly on F in the spherical metric.*

Lemma 2.4 *Suppose $D_n \stackrel{\text{Cara}}{\longrightarrow} D$, $f_n : D_n \stackrel{\text{Conf}}{\twoheadrightarrow} E_n$, $n \in \mathbb{N}$, and $f_n \stackrel{\text{l.u.}}{\longrightarrow} f$ in D. Then either f is constant on D, or f is a conformal map on D. In the latter case, let $E = f(D)$. Then $E_n \stackrel{\text{Cara}}{\longrightarrow} E$ and $f_n^{-1} \stackrel{\text{l.u.}}{\longrightarrow} f^{-1}$ in E.*

Proof. We first prove the case that D_n an D do not contain ∞, and f_n and f do not take value ∞. It is clear that f is analytic. Suppose that f is not constant.

Let $z_1 \neq z_2 \in D$ and $w_j = f(z_j)$, $j = 1, 2$. Since f is not constant, we may choose two Jordan curves J_1 and J_2 surrounding z_1 and z_2, respectively, such that the two curves together with their interior, say Ω_j, lie in D, $(J_1 \cup \Omega_1) \cap (J_2 \cup \Omega_2) = \emptyset$, and $f(z) = w_j$ has no solution on J_j, $j = 1, 2$. Since $D_n \stackrel{\text{Cara}}{\longrightarrow} D$ and $f_n \stackrel{\text{l.u.}}{\longrightarrow} f$ in D, there is $n_0 \in \mathbb{N}$ such that $J_j \cup \Omega_j \subset D_{n_0}$ and $\max_{z\in J_j} |f_{n_0}(z) - f(z)| < \min_{z\in J_j} |f(z) - w_j|$. From Rouché's theorem, there is $z_j' \in \Omega_j$ such that $f_{n_0}(z_j') = w_j$. Since f_{n_0} is conformal and $\Omega_1 \cap \Omega_2 = \emptyset$, we have $w_1 \neq w_2$. Thus, f is conformal.

We now prove that condition (i) in Definition 2.2 holds for E_n and E. Suppose a compact ball $B_0 = \{|z-z_0| \le r_0\}$ is contained in E. We may choose $r_1 > r_0$ such that $B_1 = \{|z-z_0| \le r_1\}$ is also contained in E. Let $J = f^{-1}(\{|z-z_0| = r_1\})$ and $\Omega = f^{-1}(\{|z-z_0| < r_1\})$. For any $z \in J$ and $w \in B_0$, we have $|f(z) - w| \ge r_1 - r_0 > 0$. There is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, $\Omega \cup J \subset D_n$ and $|f_n - f| < r_1 - r_0$ on J. Rouché's theorem implies that $B_0 \subset f_n(\Omega)$ if $n \ge n_0$. Thus, $B_0 \subset E_n$ if n is big enough. This implies that for any compact set $K \subset E$, there is $n_K \in \mathbb{N}$ such that $K \subset E_n$ if $n \ge n_k$.

Now we prove that $f_n^{-1} \stackrel{\text{l.u.}}{\longrightarrow} f^{-1}$ in E. If this is not true, then there is a compact set $K \subset E$ such that f_n does not converge uniformly on K. By passing to a subsequence, we may assume that there is $a > 0$ such that $\sup_{w\in K} |f_n^{-1}(w) - f^{-1}(w)| > a$ for all $n \in \mathbb{N}$. So there is a sequence (w_n) in K such that $|f_n^{-1}(w_n) - f^{-1}(w_n)| > a$ for all $n \in \mathbb{N}$. By passing to a subsequence again, we may assume that $w_n \to w_0 \in K$. Since $f^{-1}(w_n) \to f^{-1}(w_0)$, by removing finitely many terms we may assume that $|f_n^{-1}(w_n) - f^{-1}(w_0)| > a$ for all $n \in \mathbb{N}$. Let $z_0 = f^{-1}(w_0)$. We may choose $a > 0$ small enough such that $J := \{|z - z_0| = a\}$ and $\Omega := \{|z - z_0| < a\}$ are all contained in D. Since $f(z_0) = w_0 \in \Omega$ and f is one-to-one, $f(z) - w_0$ has no root on J. Let $b = \inf_{z\in J} |f(z) - w_0| > 0$. There is $n_0 \in \mathbb{N}$ such that $\Omega \cup J \subset D_{n_0}$ and $\sup_{z\in J} |f_{n_0}(z) - f(z)| < b/2$ and $|w_{n_0} - w_0| < b/2$. Rouché's theorem implies that there is $z_{n_0} \in \Omega$ such that $f_{n_0}(z_{n_0}) = w_{n_0}$, which is a contradiction.

Now we prove that condition (ii) in Definition 2.2 holds for E_n and E. If this is not true, there is $w_0 \in \partial E$ such that $\operatorname{dist}(w_0, \partial E_n) \not\to 0$. By passing to a subsequence, we may assume that there is $a > 0$ such that $\operatorname{dist}(w_0, E_n) > a$ for all $n \in \mathbb{N}$. Since $w_0 \in \partial E$, there is $w_1 \in E$ with $|w_1 - w_0| \le a/6$. Then $\operatorname{dist}(w_1, \partial E_n) > 5/6a \ge 5 \operatorname{dist}(w_1, \partial E)$. Since $f_n^{-1} \stackrel{\text{l.u.}}{\longrightarrow} f^{-1}$ in E, $(f_n^{-1})'(w_1) \stackrel{\text{l.u.}}{\longrightarrow} (f^{-1})'(w_1)$. From Koebe 1/4 theorem, $\operatorname{dist}(f_n^{-1}(w_1), \partial D_n) > \frac{5}{4} \operatorname{dist}(f^{-1}(w_1), \partial D)$ when n is big enough. Let $z_1 = f^{-1}(w_1) \in D$. Since $f_n^{-1}(w_1) \to f^{-1}(w_1) = z_1$, we have $\operatorname{dist}(z_1, \partial D_n) > \frac{5}{4} \operatorname{dist}(z_1, \partial D)$ when n is big enough, which contradicts that $D_n \stackrel{\text{Cara}}{\longrightarrow} D$. So we conclude that $E_n \stackrel{\text{Cara}}{\longrightarrow} E$.

For the general case we may use conformal charts for the Riemann sphere $\widehat{\mathbb{C}}$. We leave this as an exercise. □

Remarks.

1. The theorem holds if the underlying space $\widehat{\mathbb{C}}$ is replaced by other Riemann surfaces.

2. To apply the theorem, we often use another theorem, which says that if $D_n \stackrel{\text{Cara}}{\longrightarrow} D$, if $f_n : D_n \to \mathbb{C}$ is analytic in D_n, $n \in \mathbb{N}$, and if the family $\{f_n\}$ are uniformly bounded, then (f_n) contains a subsequence which converges locally uniformly in D. Using Möbius transformation, we see that this is still true if $f_n : D_n \to \widehat{\mathbb{C}}$ and the images of f_n all avoid an open subset of $\widehat{\mathbb{D}}$.

3. Let K_t and g_t be chordal Loewner hulls and maps driven by $\lambda \in C([0,T))$. Let $f_t = g_t^{-1}$. Then $f_t : \mathbb{H} \stackrel{\text{Conf}}{\twoheadrightarrow} \mathbb{H} \setminus K_t$. Let (t_n) be a sequence in $[0,T)$ that converges to $t_0 \in [0,T)$.

Then $f_{t_n} \xrightarrow{\text{l.u.}} f_{t_0}$ in $\mathbb{H}$. Applying the above lemma, we get $\mathbb{H} \setminus K_{t_n} \xrightarrow{\text{Cara}} \mathbb{H} \setminus K_{t_0}$. For the radial case, we get $\mathbb{D} \setminus K_{t_n} \xrightarrow{\text{Cara}} \mathbb{D} \setminus K_{t_0}$.

4. For example, if $\beta(t)$, $0 \le t \le a$, is a simple curve with $\beta((0,a)) \subset \mathbb{H}$ and $\beta(0) \neq \beta(a) \in \mathbb{R}$, and if the chordal Loewner hulls $K_t = \beta((0,t])$ for $0 \le t < a$, then K_a equals the union of $\beta((0,a))$ with the region bounded by β and $[\beta(0), \beta(a)]$. From the view of Carathéodory topology, there is no jump from K_t, $t < a$, to K_a.

5. If $\lambda_n \to \lambda$ in the semi-norm $\|\cdot\|_a$, then $g_{n,t}^{-1} \xrightarrow{\text{l.u.}} g_t^{-1}$ for $0 \le t \le a$. We then conclude that $\mathbb{H} \setminus K_{n,t} \xrightarrow{\text{Cara}} \mathbb{H} \setminus K_t$ or $\mathbb{D} \setminus K_{n,t} \xrightarrow{\text{Cara}} \mathbb{D} \setminus K_t$ for $0 \le t \le a$.

3 Hulls and Loewner Chains

3.1 Hulls

Definition 3.1 *A hull K in $\mathbb{C}$ is a continuum in $\mathbb{C}$ such that $\widehat{\mathbb{C}} \setminus K$ is connected. Then $\widehat{\mathbb{C}} \setminus K$ is a simply connected domain. There is a unique $f_K : (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\widehat{\mathbb{C}} \setminus K; \infty)$, which satisfies*

$$f_K(z) = a_1 z + a_0 + \sum_{n=-\infty}^{-1} a_n z^n, \quad z \to \infty,$$

with $a_1 > 0$. The number a_1 is called the capacity of K, and is denoted by $\operatorname{cap}(K)$.

We have the following results.

1. $\operatorname{cap}(\mathbb{D}) = 1$.
2. $\operatorname{cap}(aK + b) = |a| \operatorname{cap}(K)$ if $a, b \in \mathbb{C}$ and $a \neq 0$.
3. The capacity of any closed disc is its radius.
4. $\operatorname{cap}([-2, 2]) = 1$, where $f_K(z) = z + \frac{1}{z}$.
5. The capacity of a line segment equals to one quarter of its length.
6. If $K_1 \subset K_2$, then $\operatorname{cap}(K_1) \le \operatorname{cap}(K_2)$. The equality holds only if $K_1 = K_2$. The proof uses Schwarz lemma.
7. $\operatorname{cap}(K) \le \operatorname{diam}(K) \le 4 \operatorname{cap}(K)$. The second inequality follows from Koebe's 1/4 theorem, and the equality holds for line segments.

Definition 3.2 *A hull K in a simply connected domain D is a relatively closed subset of D such that $D \setminus K$ is also simply connected.*

Definition 3.3 *A $\mathbb{D}$-hull is a hull in $\mathbb{D}$ that does not contain* 0. *If K is a $\mathbb{D}$-hull, there is a unique $g_K : (\mathbb{D} \setminus K; 0) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{D}; 0)$ which satisfies $g_K'(0) > 0$. Then $\log(g_K'(0))$ is called the $\mathbb{D}$-capacity of K, and is denoted by* $\text{dcap}(K)$.

We have the following results.

1. The empty set is a $\mathbb{D}$-hull, $g_\emptyset = \text{id}$, and $\text{dcap}(\emptyset) = 0$.

2. If $K_1 \subsetneqq K_2$, then $\text{cap}(K_1) < \text{cap}(K_2)$. The proof uses Schwarz lemma.

3. $\frac{1}{4}e^{-\text{dcap}(K)} \le \text{dist}(0, \mathbb{T} \cup K) \le e^{-\text{dcap}(K)}$. The two inequalities follow from Schwarz lemma and Koebe's 1/4 theorem.

4. Let K be a $\mathbb{D}$-hull. Let $K^* = \overline{\mathbb{D}} \cup \{z \in \mathbb{C} : 1/\overline{z} \in K\}$. Then K^* is a hull in $\mathbb{C}$, and $\text{cap}(\widehat{K}) = \exp(\text{dcap}(K))$.

5. If K_t are radial Loewner hulls, then each K_t is a $\mathbb{D}$-hull, and $\text{dcap}(K_t) = t$.

Definition 3.4 *An $\mathbb{H}$-hull is a bounded (from ∞) hull in $\mathbb{H}$.*

We will use $I_\mathbb{R}$ to denote the complex conjugate map $z \mapsto \overline{z}$. If K is a nonempty $\mathbb{H}$-hull, then $\overline{K} \cap \mathbb{R}$ is a nonempty compact set. Let a_K and b_K be the minimum and maximum of this set. Define

$$\widehat{K} = K \cup [a, b] \cup I_\mathbb{R}(K).$$

Then $\widehat{K}$ is a hull in $\mathbb{C}$ with $I_\mathbb{R}(\widehat{K}) = \widehat{K}$. Thus, there is a unique $f_{\widehat{K}} : (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\widehat{\mathbb{C}} \setminus \widehat{K}; \infty)$ such that in the power series expansion of $f_{\widehat{K}}$ at ∞, say $f_{\widehat{K}}(z) = a_1 z + a_0 + O(1/z)$ as $z \to \infty$, the first coefficient a_1 is positive. Let $f = I_\mathbb{R} \circ f_{\widehat{K}} \circ I_\mathbb{R}$. Since $I_\mathbb{R}(\widehat{K}) = \widehat{K}$ is symmetric about $\mathbb{R}$ and $a_1 > 0$, we have $f : (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\widehat{\mathbb{C}} \setminus \widehat{K}; \infty)$ and $f(z) = a_1 z + \overline{a_0} + O(1/z)$ as $z \to \infty$. The uniqueness of $f_{\widehat{K}}$ implies that $f = f_{\widehat{K}}$. Thus, $a_0 \in \mathbb{R}$ and $f_{\widehat{K}}$ commutes with $I_\mathbb{R}$. Let $g = W \circ f_{\widehat{K}}^{-1}$, where $W(z) = z + \frac{1}{z}$. Then $g : (\widehat{\mathbb{C}} \setminus \widehat{K}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\widehat{\mathbb{C}} \setminus [-2, 2]; \infty)$, and the power series expansion of g at ∞ is $g(z) = \frac{z}{a_1} - \frac{a_0}{a_1} + O(1/z)$. Since both $f_{\widehat{K}}$ and W commute with $I_\mathbb{R}$, the same is true for g. Let $g_K(z) = a_1 g(z) + a_0$. Set $c_K = a_0 - 2a_1$ and $d_K = a_0 + 2a_1$. Then $g_K : (\widehat{\mathbb{C}} \setminus \widehat{K}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\widehat{\mathbb{C}} \setminus [c_K, d_K]; \infty)$ and satisfies $g_K(z) = z + O(1/z)$ as $z \to \infty$. Since $a_0, a_1 \in \mathbb{R}$, g_K also commutes with $I_\mathbb{R}$. Thus, g_K maps $\widehat{\mathbb{R}} \setminus \widehat{K} = \widehat{\mathbb{R}} \setminus [a_K, b_K]$ onto $\widehat{\mathbb{R}} \setminus [c_K, d_K]$. Since $\widehat{\mathbb{R}} \setminus [a_K, b_K]$ divides $\widehat{\mathbb{C}} \setminus \widehat{K}$ into two components: $\mathbb{H} \setminus K$ and $I_\mathbb{R}(\mathbb{H} \setminus K)$, and $\widehat{\mathbb{R}} \setminus [c_K, d_K]$ divides $\mathbb{C} \setminus [c_K, d_K]$ into two components: $\mathbb{H}$ and $I_\mathbb{R}(\mathbb{H})$, we conclude that g_K maps $\mathbb{H} \setminus K$ onto $\mathbb{H}$ or $I_\mathbb{R}(\mathbb{H})$. Since $g_K(z) = z + O(z^{-1})$ as $z \to \infty$. The second case does not happen. So $g_K : (\mathbb{H} \setminus K; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{H}; \infty)$. If $K = \emptyset$, we let $g_K = \text{id}$. Then $g_K : (\mathbb{H} \setminus K; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{H}; \infty)$ and $g_K(z) = z + O(1/z)$ as $z \to \infty$ still hold. Note that such g_K is unique because if h_K also satisfies the properties of g_K, then $h_K \circ g_K^{-1} : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, and $h_K \circ g_K^{-1}(z) = z + O(z^{-1})$ as $z \to \infty$, which forces $h_K \circ g_K^{-1} = \text{id}$.

Definition 3.5 *If K is an $\mathbb{H}$-hull, let g_K denote the unique conformal map from $(\mathbb{H}\setminus K;\infty)$ onto $(\mathbb{H};\infty)$ that satisfies $g_K(z)=z+O(1/z)$ as $z\to\infty$. If the expansion of g_K at ∞ is $g_K(z)=z+\sum_{n=-\infty}^{-1} b_{-n}z^n$, we call the number b_{-1} the $\mathbb{H}$-capacity of K, and let it be denoted by $\operatorname{hcap}(K)$. In case $K\ne\emptyset$, we define $\widehat{K}$, a_K, b_K, c_K, d_K to be as in the above argument, and g_K will also be understood as a conformal map from $\widehat{\mathbb{C}}\setminus\widehat{K}$ onto $\widehat{\mathbb{C}}\setminus[c_K,d_K]$.*

Examples.

1. If $K=\emptyset$, then $g_K(z)=z$, and $\operatorname{hcap}(K)=0$.
2. If $K=\{z\in\mathbb{H}:|z-x_0|\le r\}$ for some $x_0\in\mathbb{R}$ and $r>0$, then $a_K=x_0-r$, $b_K=x_0+r$; $g_K(z)=z+\frac{r^2}{z-x_0}$; $c_K=x_0-2r$, $d_K=x_0+2r$; and $\operatorname{hcap}(K)=r^2$.
3. If $K=(0,i]$, then $a_K=b_K=0$; $g_K(z)=\sqrt{z^2+1}=z\sqrt{1+z^{-2}}$, where the branch of the square root is chosen such that $\sqrt{1+z^{-2}}\to 1$ as $z\to\infty$; $c_K=-1$, $d_K=1$. Since $g_K(z)=z(1+\frac12 z^{-2}+\cdots)$ as $z\to\infty$, $\operatorname{hcap}(K)=1/2$.
4. If K_t and g_t, $0\le t<T$, are chordal Loewner hulls and maps driven by $\lambda\in C([0,T))$, then each K_t is an $\mathbb{H}$-hull, $g_t=g_{K_t}$, and $\operatorname{hcap}(K_t)=2t$. Recall that $g_t:\mathbb{H}\setminus K_t\overset{\text{Conf}}{\twoheadrightarrow}\mathbb{H}$ and satisfies $g_t(z)=z+\frac{2t}{z}+O(z^{-2})$ as $z\to\infty$.

Lemma 3.1 *If K is an $\mathbb{H}$-hull, and $a>0$, $b\in\mathbb{R}$, then $aK+b$ is also an $\mathbb{H}$-hull, $g_{aK+b}(z)=ag_K((z-b)/a)+b$, and $\operatorname{hcap}(aK+b)=a^2\operatorname{hcap}(K)$.*

Proof. The proof is straightforward. We leave it as an exercise. $\square$

Let K be a nonempty $\mathbb{H}$-hull. Let $h(z)=g_K^{-1}(z)-z$. Then h is a $\mathbb{C}$-valued analytic function defined on $\widehat{\mathbb{C}}\setminus\widehat{K}$. In fact, $h(z)=\frac{-\operatorname{hcap}(K)}{z}+O(1/z^2)$ near ∞, so $h(\infty)=0$. Then $\operatorname{Im}h$ is a real valued harmonic function on $\widehat{\mathbb{C}}\setminus\widehat{K}$. Let $\delta>0$ be small. Since g_K^{-1} maps $i\delta+\mathbb{R}$ into $\mathbb{H}$, we have $\operatorname{Im}h(z)>-\operatorname{Im}z=-\delta$ on $i\delta+\mathbb{R}$. Since $\operatorname{Im}h(\infty)=0>-\delta$, from the Maximum principle, we have $\operatorname{Im}h(z)>-\delta$ for any $z\in\mathbb{H}$ with $\operatorname{Im}z>\delta$. Since this holds for any δ, we have $\operatorname{Im}h(z)\ge 0$ for any $z\in\mathbb{H}$. If there is $z_0\in\mathbb{H}$ with $\operatorname{Im}h(z_0)=0$, then $\operatorname{Im}h\equiv 0$ on $\mathbb{H}$, which implies that h is a real valued constant, say C. This implies that $g_K^{-1}(z)=z+C$, which contradicts that $g_K^{-1}:\mathbb{H}\overset{\text{Conf}}{\twoheadrightarrow}\mathbb{H}\setminus K$ and $K\ne\emptyset$. Thus, $\operatorname{Im}h>0$ on $\mathbb{H}$. This means that $\operatorname{Im}g_K^{-1}(z)-\operatorname{Im}z>0$ for $z\in\mathbb{H}$, and $\operatorname{Im}g_K(z)<\operatorname{Im}z$ for $z\in\mathbb{H}\setminus K$. Since $h(z)=\frac{-\operatorname{hcap}(K)}{z}+O(1/z^2)$ near ∞ and $\pm\operatorname{Im}h(z)>0$ if $\pm\operatorname{Im}z>0$, we get $\operatorname{hcap}(K)>0$. So we conclude the following lemma.

Lemma 3.2 *For any nonempty $\mathbb{H}$-hull K, $\operatorname{Im}g_K^{-1}(z)>\operatorname{Im}z$ for $z\in\mathbb{H}$, and $\operatorname{hcap}(K)>0$.*

Definition 3.6 *Let K_1,K_2 be two $\mathbb{H}$-hulls. If $K_1\subset K_2$, we say that K_1 is a sub-hull of K_2. In this case, let $K_2/K_1=g_{K_1}(K_2\setminus K_1)$. We say K_2/K_1 is a quotient-hull of K_2.*

Lemma 3.3 *If $K_1 \subset K_2$ are two $\mathbb{H}$-hulls, then K_2/K_1 is also an $\mathbb{H}$-hull, and we have*

$$g_{K_2} = g_{K_2/K_1} \circ g_{K_1} \quad on \quad \mathbb{H} \setminus K_2. \tag{3.1}$$

$$\mathrm{hcap}(K_2) = \mathrm{hcap}(K_1) + \mathrm{hcap}(K_2/K_1). \tag{3.2}$$

In particular, if L is a sub-hull or quotient-hull of K, then $\mathrm{hcap}(L) \le \mathrm{hcap}(K)$*, and the equality holds iff $L = K$.*

Proof. Since g_{K_1} maps $\mathbb{H} \setminus K_1$ onto $\mathbb{H}$, we get $K_2/K_1 \subset \mathbb{H}$. Since $K_2 \setminus K_1$ is bounded and the conformal map g_{K_1} fixes ∞, we see that K_2/K_1 is bounded. Since $g_{K_1} : \mathbb{H} \setminus K_2 \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{H} \setminus K_2/K_1$ we see that $\mathbb{H} \setminus K_2/K_1$ is simply connected. Thus, K_2/K_1 is an $\mathbb{H}$-hull. We have $g_{K_2/K_1} \circ g_{K_1}$: $\mathbb{H} \setminus K_2 \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{H}$, and $g_{K_2/K_1} \circ g_{K_1}(z) = z + \frac{\mathrm{hcap}(K_2/K_1)}{z} + \frac{\mathrm{hcap}(K_1)}{z} + O(1/z^2)$ near ∞. So we get (3.1) and (3.2). Note that $K/L = K$ implies that $L = \emptyset$, and $K/L = \emptyset$ implies that $L = K$. Using Lemma 3.2 we obtain the remaining results. □

Remark. Using the notation of quotient hulls we may rewrite (1.3) as

$$\{\lambda(t)\} = \bigcap_{\varepsilon \in (0,T-t)} \overline{K_{t+\varepsilon}/K_t}. \tag{3.3}$$

Definition 3.7 *A simple curve γ in $\mathbb{H}$ is called a crosscut if its two ends approach to two different points on $\mathbb{R}$. The closure of the bounded component of $\mathbb{H} \setminus \gamma$ in $\mathbb{H}$ is called the bubble bounded by γ.*

Remarks.

1. If K is the bubble bounded by a crosscut γ, then $\mathbb{H} \setminus K$ is a Jordan domain. Thus, g_K extends to a homeomorphism from $\overline{\mathbb{H} \setminus K}$ to $\overline{\mathbb{H}}$. Moreover, the continuation of g_K maps γ onto (c_K, d_K).
2. For any $\mathbb{H}$-hull K, there is a family of bubbles K_n such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$, and $K = \bigcap_{n \in \mathbb{N}} K_n$. We say that K is approximated by the sequence (K_n).

Lemma 3.4 *Let K be a nonempty $\mathbb{H}$-hull. Then there is a (positive) measure μ_K supported by $[c_K, d_K]$ with $|\mu_K| = \mathrm{hcap}(K)$ such that*

$$g_K^{-1}(z) - z = \int_{c_K}^{d_K} \frac{-1}{z - x} d\mu_K(x), \quad z \in \widehat{\mathbb{C}} \setminus [c_K, d_K]. \tag{3.4}$$

If K is a bubble, then $d\mu_K = \frac{1}{\pi} \mathrm{Im}\, g_K^{-1}(x) dx$, where dx is the Lebesgue measure.

Proof. We know that $h(z) := \mathrm{Im}(g_K^{-1}(z) - z)$ is a positive harmonic function in $\mathbb{H}$ and vanishes on $\widehat{\mathbb{R}} \setminus [c_K, d_K]$. In the case that K is a bubble, h is continuous on $\overline{\mathbb{H}}$ and $h(x) = \mathrm{Im}\, g_K^{-1}(x)$ on $\mathbb{R}$. Using the fact that $\frac{1}{\pi}\mathrm{Im}\frac{-1}{z-x}$ is the Poisson kernel in $\mathbb{H}$ with the pole at x, we conclude that there is a (positive) measure μ_K supported by $[c_K, d_K]$ such that

$$h(z) = \int_{c_K}^{d_K} \mathrm{Im}\,\frac{-1}{z-x} d\mu_K(x), \quad z \in \widehat{\mathbb{C}} \setminus [c_K, d_K], \tag{3.5}$$

and $d\mu_K = \frac{1}{\pi}\mathrm{Im}\, g_K^{-1}(x)dx$ if K is a bubble.

Then we conclude that the LHS of (3.4) equals to the RHS of (3.4) plus a constant $C \in \mathbb{R}$. When z is near ∞, the RHS of (3.4) equals to $-\frac{\mathrm{hcap}(K)}{z} + O(z^{-2})$, and the RHS of (3.4) equals to $-\frac{|\mu(K)|}{z} + O(z^{-2})$. Thus, $C = 0$ and $|\mu_K| = \mathrm{hcap}(K)$. So (3.4) holds. $\square$

Remarks.

1. (3.4) says that $g_K^{-1}(z) - z$ is the Stieltjes transform of μ_K.
2. If K is a $\mathbb{D}$-hull, then there is a measure μ_K supported by $\mathbb{T}$ with $|\mu_K| = \mathrm{dcap}(K)$ such that
$$\log(g_K^{-1}(z)/z) = \int_{\mathbb{T}} \frac{z+w}{z-w} d\mu_K(w).$$

Lemma 3.5 *Let γ be a crosscut in $\mathbb{H}$. Let $h = \sup \mathrm{Im}\,\gamma$. If K is the bubble bounded by γ, then*

$$\mathrm{hcap}(K) \le \frac{h}{\pi}(d_K - c_K).$$

Proof. This follows from Lemma 3.4 immediately. $\square$

Lemma 3.6 *For any nonempty $\mathbb{H}$-hull K, $[a_K, b_K] \subset [c_K, d_K]$. If $K_1 \subsetneqq K_2$ are two nonempty $\mathbb{H}$-hulls, then $[c_{K_1}, d_{K_1}] \subset [c_{K_2}, d_{K_2}]$ and $[c_{K_2/K_1}, d_{K_2/K_1}] \subset [c_{K_2}, d_{K_2}]$.*

Proof. Let K be a nonempty $\mathbb{H}$-hull. From (3.4) we conclude that

$$g_K^{-1}(x) < x, \quad x \in (d_K, \infty); \quad g_K^{-1}(x) > x, \quad x \in (-\infty, c_K). \tag{3.6}$$

Since g_K^{-1} maps $(-\infty, c_K)$ onto $(-\infty, a_K)$, we have $c_K \le a_K$. Similarly, $d_K \ge b_K$. Hence $[a_K, b_K] \subset [c_K, d_K]$.

Let $K_1 \subseteq K_2$ be two nonempty $\mathbb{H}$-hulls. Let $b \in (b_{K_2}, \infty)$. Then $\mathrm{dist}(K_2 \setminus K_1, [b, \infty]) > 0$ So $K_2/K_1 = g_{K_1}(K_2 \setminus K_1)$ is bounded away from $[g_{K_1}(b), \infty)$, which implies $b_{K_2/K_1} < g_{K_1}(b)$. Since this holds for any $b > b_{K_2}$, we have $(b_{K_2/K_1}, \infty) \supset g_{K_1}((b_{K_2}, \infty))$. Thus,

$$d_{K_2/K_1} = \inf g_{K_2/K_1}((b_{K_2/K_1}, \infty)) \le \inf g_{K_2/K_1} \circ g_{K_1}((b_{K_2}, \infty)) = g_{K_2}((b_{K_2}, \infty)) = d_{K_2}.$$

Similarly, $c_{K_2/K_1} \ge c_{K_2}$. So $[c_{K_2/K_1}, d_{K_2/K_1}] \subset [c_{K_2}, d_{K_2}]$.

If $x \in (-\infty, a_{K_2})$, then $g_{K_2}(x) \in (-\infty, c_{K_2}) \subset (-\infty, c_{K_2/K_1})$. Using (3.6) we get $g_{K_1}(x) = g_{K_2/K_1}^{-1} \circ g_{K_2}(x) > g_{K_2}(x)$. Thus,

$$c_{K_1} = \sup g_{K_1}((-\infty, a_{K_1})) \geq \sup g_{K_1}((-\infty, a_{K_2})) \geq \sup g_{K_2}((-\infty, a_{K_2})) = c_{K_2}.$$

Similarly, we have $d_{K_1} \leq d_{K_2}$. Hence $[c_{K_1}, d_{K_1}] \subset [c_{K_2}, d_{K_2}]$. $\square$

Lemma 3.7 *Let $x_0 \in \mathbb{R}$, $r > 0$. If a nonempty $\mathbb{H}$-hull K is contained in $\{|z - x_0| \leq r\}$, then $|g_K^{-1}(z) - z| \leq 15r$ for any $z \in \mathbb{C} \setminus [c_K, d_K]$, and $|g_K(z) - z| \leq 15r$ for any $z \in \mathbb{C} \setminus \widehat{K}$.*

Proof. Let $K_r = \{z \in \mathbb{H} : |z - x_0| \leq r\}$. Then $|\mu_K| = \text{hcap}(K) \leq \text{hcap}(K_r) = r^2$ and $[c_K, d_K] \subset [c_{K_r}, d_{K_r}] = [x_0 - 2r, x_0 + 2r]$. Let $\alpha = \{z \in \mathbb{C} : |z - x_0| \leq 3r\}$. Then α is a Jordan curve that encloses $[c_K, d_K]$, and $\text{dist}(\alpha, [c_K, d_K]) \geq r$. If z lies on or outside α, from equation (3.4), we get $|g_K^{-1}(z) - z| \leq |\mu_K|/r \leq r$. Since $\text{diam}(\alpha) = 6r$, we have $\text{diam}(g_K^{-1}(\alpha)) \leq 8r$. If $z \in \mathbb{C} \setminus [c_K, d_K]$ lies inside α, then $g_K^{-1}(z)$ lies inside $g_K^{-1}(\alpha)$. Choose $w \in \alpha$, then

$$\begin{aligned} |g_K^{-1}(z) - z| &\leq |z - w| + |w - g_K^{-1}(w)| + |g_K^{-1}(w) - g_K^{-1}(z)| \\ &\leq \text{diam}(\alpha) + r + \text{diam}(g_K^{-1}(\alpha)) \leq 15r. \end{aligned}$$

Since $g_K : \mathbb{C} \setminus \widehat{K} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{C} \setminus [c_K, d_K]$, we see that $|g_K(z) - z| \leq 15r$ for any $z \in \mathbb{C} \setminus \widehat{K}$. $\square$

Lemma 3.8 *Let K_n, $n \in \mathbb{N}$, be a sequence of $\mathbb{H}$-hulls with $K_{n+1} \subset K_n$ for all n. Suppose $\bigcap_{n=1}^{\infty} K_n = K$ is an $\mathbb{H}$-hull. Then $\text{hcap}(K) = \lim_{n\to\infty} \text{hcap}(K_n)$.*

Proof. Let $L_n = K_n/K$. Then $\bigcap_{n=1}^{\infty} L_n = \emptyset$. From Lemma 3.3, $\text{hcap}(L_n) = \text{hcap}(K_n) - \text{hcap}(K)$. We suffice to show that $\text{hcap}(L_n) \to 0$. The sequence of L_n is decreasing. If any L_n is empty, the result is immediate. We now suppose all L_n are nonempty. Let h_n denote the height of L_n. Then $h_n \to 0$. If L_n are all bubbles, then we have

$$\text{hcap}(L_n) \leq \frac{h_n}{\pi}(d_{L_n} - c_{L_n}) \leq \frac{h_n}{\pi}(d_{L_1} - c_{L_1}) \to 0.$$

In the general case, we may find a decreasing sequence of bubbles (L_n') such that $L_n \subset L_n'$ and $\bigcap L_n' = \emptyset$. For example, we may choose $L_n' = \{|x| \leq R, 0 < y \leq h_n\}$, where $R = \sup |\operatorname{Re} L_1|$. $\square$

Remarks.

1. For any nonempty $\mathbb{H}$-hull K, we have $\text{hcap}(K) \leq \text{diam}(K)^2$. Proof. Let $R = \text{diam}(K)$ and $x_0 \in \overline{K} \cap \mathbb{R}$. Then $K \subset \{z \in \mathbb{H} : |z - x_0| \leq R\} =: K_R$, which implies that $\text{hcap}(K) \leq \text{hcap}(K_R) = R^2$.

2. For any $M, \varepsilon > 0$, there is an $\mathbb{H}$-hull K with $\text{diam}(K) > M$ and $\text{hcap}(K) < \varepsilon$. Proof. For $n \in \mathbb{N}$, let K_n be the rectangle: $[0, M] \times (0, \frac{1}{n}]$. Then each K_n is an $\mathbb{H}$-hull with $\text{diam}(K_n) > M$. Since (K_n) is decreasing and $\bigcap_{n=1}^{\infty} K_n = \emptyset$, we have $\text{hcap}(K_n) \to 0$. So there is n_0 such that $\text{hcap}(K_{n_0}) < \varepsilon$.

Let $\mathcal{H}_*(\mathbb{H})$ denote the set of all nonempty $\mathbb{H}$-hulls. Let $\mathcal{H}_b(\mathbb{H})$ denote the set of all bubbles.

Proposition 3.1 *Suppose $x_0 \in \mathbb{R}$, I is an open real interval, Ω is a domain, and $x_0 \subset I \subset \Omega$. Suppose that W is a conformal map on Ω such that $W(I) \subset \mathbb{R}$ and $W'(x_0) > 0$. Then*

$$\lim_{\mathcal{H}_*(\mathbb{H})\ni K \to x_0} \frac{\operatorname{hcap}(W(K))}{\operatorname{hcap}(K)} = W'(x_0)^2, \tag{3.7}$$

where $K \to x_0$ means that $\operatorname{diam}(K \cup \{x_0\}) \to 0$.

Proof. Suppose $\operatorname{diam}(K \cup \{x_0\})$ is small enough such that $\widehat{K} \subset \Omega$ and $\widehat{K} \cap \mathbb{R} \subset I$. Let $\Omega_K = g_K(\Omega \setminus \widehat{K})$ and $W_K = g_{W(K)} \circ W \circ g_K^{-1}$. Then $\Omega_K \cap [c_K, d_K] = \emptyset$, $\Omega_K \cup [c_K, d_K]$ is open, and W_K is a conformal map on Ω_K. As $z \to [c_K, d_K]$ in Ω_K, $g_K^{-1}(z) \to \widehat{K}$ in $\Omega \setminus K$, $W \circ g_K^{-1}(z) \to W(\widehat{K}) = \widehat{W(K)}$, hence $W_K(z) \to [c_{W(K)}, d_{W(K)}]$. Thus, W_K extends to a conformal map defined on $\Omega_K \cup [c_K, d_K]$, and maps $[c_K, d_K]$ onto $[c_{W(K)}, d_{W(K)}]$.

Since every $\mathbb{H}$-hull can be approximated by a decreasing sequence of bubbles, from Lemma 3.8 we suffice to prove the proposition with $\mathcal{H}_*(\mathbb{H})$ replaced by $\mathcal{H}_b(\mathbb{H})$. Let $K \in \mathcal{H}_*(\mathbb{H})$. Then $W(K) \in \mathcal{H}_*(\mathbb{H})$. From Lemma 3.4 we have

$$\operatorname{hcap}(K) = \frac{1}{\pi} \int_{c_K}^{d_K} \operatorname{Im} g_K^{-1}(x) dx.$$

$$\operatorname{hcap}(W(K)) = \frac{1}{\pi} \int_{c_{W(K)}}^{d_{W(K)}} \operatorname{Im} g_{W(K)}^{-1}(x) dx.$$

$$= \frac{1}{\pi} \int_{c_K}^{d_K} W_K'(x) \operatorname{Im} g_{W(K)}^{-1} \circ W_K(x) dx = \frac{1}{\pi} \int_{c_K}^{d_K} W_K'(x) \operatorname{Im} W \circ g_K^{-1}(x) dx.$$

We suffice to show that, as $K \to x_0$, the following are true.

(L1) $\frac{\operatorname{Im} W(z)}{\operatorname{Im} z} \to W'(x_0)$ uniformly on $z \in \partial K \cap \mathbb{H}$;

(L2) $W_K'(x) \to W'(x_0)$ uniformly on $x \in [c_K, d_K]$.

Since W is analytic and takes real value on the open interval $I \ni x_0$, (L1) is clearly true. Now we prove (L2). If $K \subset K_r := \{z \in \mathbb{H} : |z - x_0| \le r\}$, then $|\mu_K| = \operatorname{hcap}(K) \le \operatorname{hcap}(K_r) = r^2$ and $[c_K, d_K] \subset [c_{K_r}, d_{K_r}] = [x_0 - 2r, x_0 + 2r]$. Let $K \to x_0$. Then $\Omega \setminus \widehat{K} \xrightarrow{\text{Cara}} \Omega \setminus \{x_0\}$ and $\inf\{r > 0 : K \subset K_r\} \to 0$, which implies that $|\mu_K| \to 0$ and $[c_K, d_K] \to x_0$. From (3.4) we have $g_K^{-1} \xrightarrow{\text{l.u.}} \text{id}$ in $\mathbb{C} \setminus \{x_0\}$, which implies that $\Omega_K \xrightarrow{\text{Cara}} \Omega \setminus \{x_0\}$ by Lemma 2.4. Similarly, since $W(K) \to W(x_0)$, we have $g_{W(K)}^{-1} \xrightarrow{\text{l.u.}} \text{id}$ in $W(\Omega \setminus \{x_0\})$, which implies that $g_{W(K)} \xrightarrow{\text{l.u.}} \text{id}$ in $W(\Omega \setminus \{x_0\})$. Since $W_K = g_{W(K)} \circ W \circ g_K^{-1}$, we have $W_K \xrightarrow{\text{l.u.}} W$ in $\Omega \setminus \{x_0\}$. From $\Omega_K \xrightarrow{\text{Cara}} \Omega \setminus \{x_0\}$ we have $\Omega_K \cup [c_K, d_K] \xrightarrow{\text{Cara}} \Omega$. Since W_k and W are analytic on Ω_K and Ω, respectively, using the Maximum principle, we conclude that $W_K \xrightarrow{\text{l.u.}} W$ in Ω. Thus, $W_K' \xrightarrow{\text{l.u.}} W'$ in Ω. Since $[c_K, d_K] \to x_0$, we conclude that (L2) is true. $\square$

Proposition 3.2 *Suppose $x_0 \in \mathbb{R}$, I is an open real interval, Ω is a domain, and $x_0 \subset I \subset \Omega$. Suppose that W is a conformal map on Ω such that $W(I) \subset \mathbb{T}$ and $W(\Omega \cap \mathbb{H}) \subset \mathbb{D}$. Then*

$$\lim_{\mathcal{H}_*(\mathbb{H}) \ni K \to x_0} \frac{\mathrm{dcap}(W(K))}{\mathrm{hcap}(K)} = \frac{1}{2}|W'(x_0)|^2.$$

Proposition 3.3 *Suppose $z_0 \in \mathbb{T}$, I is an open arc on $\mathbb{T}$, Ω is a domain, and $z_0 \subset I \subset \Omega$. Suppose that W is a conformal map on Ω such that $W(I) \subset \mathbb{T}$ and $W(\Omega \cap \mathbb{H}) \subset \mathbb{D}$. Then*

$$\lim_{\mathcal{H}_*(\mathbb{D}) \ni K \to z_0} \frac{\mathrm{dcap}(W(K))}{\mathrm{dcap}(K)} = |W'(z_0)|^2,$$

where $\mathcal{H}_*(\mathbb{D})$ denotes the space of nonempty $\mathbb{D}$-hulls.

We leave the proofs of these two propositions as exercise. Hint: First prove Proposition 3.2 in the case that W is a Möbius transform, then prove Proposition 3.2 in the general case using Proposition 3.1, and finally use Proposition 3.2 to prove Proposition 3.3.

Remarks. The factor $\frac{1}{2}$ in Proposition 3.2 somehow explains the the enumerator 2 in the chordal Loewner equations. This will be explained in more details later.

3.2 Deterministic Loewner Evolution

Definition 3.8 *Let D be a simply connected domain and $T \in (0, \infty]$. A Loewner chain in D is a family of hulls K_t, $0 \le t < T$, in D that satisfy the following conditions.*

1. *$K_0 = \emptyset$; and $K_{t_1} \subsetneqq K_{t_2}$ if $t_1 < t_2$.*
2. *for any $t_0 \in [0, T)$ and any continuum $F \subset D \setminus K_{t_0}$, $\lim_{s \to 0^+} d^*_{D \setminus K_t}(F, K_{t+s} \setminus K_t) = 0$ uniformly in $t \in [0, t_0]$. In other words, for any $\varepsilon > 0$, there is $\delta > 0$ such that if $s \in (0, \delta)$, then for any $t \in [0, t_0]$, the conjugate extremal distance between F and $K_{t+s} \setminus K_t$ in $D \setminus K_t$ is less than ε.*

Remarks. Suppose K_t, $0 \le t < T$, is a Loewner chain in D. Then we have the followings.

1. If W is a conformal map on D, then $W(K_t)$, $0 \le t < T$, is a Loewner chain in $W(D)$.
2. If u is a continuous and (strictly) increasing function on $[0, T)$ with $u(0) = 0$, then $K_{u^{-1}(t)}$, $0 \le t < u(T)$, is also a Loewner chain in D, and is called a time-changes of K_t, $0 \le t < T$.

Examples.

1. Suppose $\beta(t)$, $0 \le t < T$, is a simple curve with $\beta(0) \in \mathbb{R}$ and $\beta((0, T)) \subset \mathbb{H}$, then $K_t := \beta((0, t])$, $0 \le t < T$, is a Loewner chain in $\mathbb{H}$. We leave this as an exercise.

2. Suppose $\beta(t)$, $0 \le t < T$, is a simple curve with $\beta(0), \beta(a) \in \mathbb{R}$ and $\beta((0,a)), \beta((a,T)) \subset \mathbb{H}$. Let Ω be the bounded component of $\mathbb{H} \setminus \beta((0,a))$. Let $K_t = \beta((0,t])$, $0 \le t < a$; $K_t = \beta((0,a)) \cup \Omega \cup \beta((a,t])$, $a \le t < T$. Then K_t, $0 \le t < T$, is a Loewner chain in $\mathbb{H}$.

Proposition 3.4 [Lawler-Schramm-Werner]

(i) If K_t, $0 \le t < T$, are chordal Loewner hulls driven by some $\lambda \in C([0,T))$, then the family is a Loewner chain in $\mathbb{H}$ such that each K_t is an $\mathbb{H}$-hull and $\operatorname{hcap}(K_t) = 2t$.

(ii) If K_t, $0 \le t < T$, is a Loewner chain such that each K_t is an $\mathbb{H}$-hull, then $u(t) :=$ $\operatorname{hcap}(K_t)$ is a continuous and increasing function on $[0,T)$ with $u(0) = 0$. Moreover, if $\operatorname{hcap}(K_t) = 2t$ for each t, then K_t, $0 \le t < T$, are chordal Loewner hulls driven by some $\lambda \in C([0, u(T)))$, which is given by (3.3).

Proof. (i) We already know that each K_t is an $\mathbb{H}$-hull and $\operatorname{hcap}(K_t) = 2t$. Now we show that K_t, $0 \le t < T$, is a Loewner chain in $\mathbb{H}$. Fix $t_0 \in (0,T)$ and a continuum $F \subset \mathbb{H} \setminus K_{t_0}$. Let g_t's be the chordal Loewner maps driven by λ. Then for $0 \le t \le t_0$, g_t is well defined on F. Let $h = \inf \operatorname{Im} g_{t_0}(F)$. Then $h > 0$ because $g_{t_0}(F)$ is a compact subset of $\mathbb{H}$. Since $t \mapsto \operatorname{Im} g_t(z)$ is decreasing, we have $\operatorname{Im} g_t(z) \ge h$ for any $z \in F$ and $t \in [0, t_0]$. Fix $t \in [0, t_0]$. Then $g_t(K_{t+s} \setminus K_t) - \lambda(t)$, $0 \le s < T - t$, are chordal Loewner hulls driven by $s \mapsto \lambda(t+s) - \lambda(t)$. Let $M_s = \sqrt{8s} + \sup_{0 \le t \le t_0; 0 \le r \le s} |\lambda(t+r) - \lambda(t)|$. From Lemma 1.1, we have $g_t(K_{t+s} \setminus K_t) \subset \{z \in \mathbb{H} : |z - \lambda(t_0)| \le M_s\}$. Since λ is continuous, we have $M_s \to 0$ as $s \to 0^+$. If M_s is smaller than h, then $g_t(K_{t+s} \setminus K_t)$ can be separated from $g_t(F)$ by the annulus $\{M_s < |z - \lambda(x_0)| < h\}$, which implies that $d^*_{\mathbb{H}}(g_t(F), g_t(K_{t+s} \setminus K_t)) \le 2\pi / \log(h/M_s)$. Since $g_t : \mathbb{H} \setminus K_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, we have $d^*_{\mathbb{H} \setminus K_t}(F, K_{t+s} \setminus K_t) \le 2\pi / \log(h/M_s)$. Since M_s does not depend on t and $\lim_{s \to 0^+} M_s = 0$, we finish the proof of (i).

(ii) Fix $t_0 \in (0,T)$ and a continuum F in $\mathbb{H} \setminus K_{t_0}$. Let $d(s) = \sup_{0 \le t \le t_0} d^*_{D \setminus K_t}(F, K_{t+s} \setminus K_t)$ for $0 < s < T - t_0$. From the definition we have $\lim_{s \to 0^+} d(s) = 0$. From now on, t always ranges in $[0, t_0]$, and s ranges in $(0, T - t_0)$ or some smaller interval $(0, c)$. Since $g_{K_t} : \mathbb{H} \setminus K_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, from the conformal invariance of extremal length, we get $d^*_{\mathbb{H}}(g_t(F), K_{t+s}/K_t) \le d(s)$. Choose ρ to be the spherical metric $\frac{2}{1+|z|^2}$. Then $A_\rho(\mathbb{H}) = 2\pi$. Thus, there is a curve $\gamma_{t,s}$ in $\mathbb{H}$ disconnecting $g_t(F)$ from K_{t+s}/K_t with spherical length less than $\sqrt{7d(s)}$. We may then conclude that the Euclidean length of $\gamma_{t,s}$ tends to 0 as $s \to 0^+$, uniformly in $t \in [0, t_0]$. If s is small enough, $\gamma_{t,s}$ generates a bubble with diameter tends to 0 as $s \to 0^+$, which contains $g_t(K_{t+s} \setminus K_t)$. Thus, $u(t+s) - u(t) = \operatorname{hcap}(K_{t+s}) - \operatorname{hcap}(K_t) = \operatorname{hcap}(K_{t+s}/K_t) \to 0^+$ as $s \to 0^+$, uniformly in $t \in [0, t_0]$. This shows that u is continuous on $[0, t_0]$. Since the family K_t increases strictly an $K_0 = \emptyset$, $u(t)$ is strictly increasing with $u(0) = 0$. So we finish the proof of the first statement.

Now suppose that $\operatorname{hcap}(K_t) = 2t$, $0 \le t < T$. Let $t \in [0, t_0]$. Since $\operatorname{diam}(K_{t+s}/K_t) \le r(s)$ for $s \in (0, \delta_2)$, and $\lim_{s \to 0^+} r(s) = 0$, we see that $\bigcap_{s \in (0, T-t)} \overline{K_{t+s}/K_t}$ contains only one point. Let it be denoted by $\lambda(t)$. Suppose $t_1 < t_2 < t_3 \in [0, t_0]$ satisfy that $t_3 - t_1 < \delta_2$. Then $\lambda(t_1) \in \overline{K_{t_3}/K_{t_1}}$ and $\lambda(t_2) \in \overline{K_{t_3}/K_{t_2}}$. Choose any $z_1 \in K_{t_3}/K_{t_2}$. Then $|z_1 - \lambda(t_2)| \le r(t_3 - t_2)$. Let

$z_2 = g^{-1}_{K_{t_2}/K_{t_1}}(z_1)$. From Lemma 3.7 we have $|z_2 - z_1| \le 15r(t_2 - t_1)$. Since $g_{K_{t_2}} = g_{K_{t_2}/K_{t_1}} \circ g_{K_{t_1}}$, we have

$$z_2 = g_{K_{t_1}} \circ g^{-1}_{K_{t_2}}(z_1) \in g_{K_{t_1}}(K_{t_3} \setminus K_{t_2}) \subset g_{K_{t_1}}(K_{t_3} \setminus K_{t_1}) = K_{t_3}/K_{t_1}.$$

Thus, $|z_2 - \lambda(t_1)| \le r(t_3 - t_1)$. Thus, $|\lambda(t_2) - \lambda(t_1)| \le r(t_3 - t_2) + 15r(t_2 - t_1) + r(t_3 - t_1)$. Let $r_3 \to r_2^+$, we conclude that $|\lambda(t_2) - \lambda(t_1)| \le 16r(t_2 - t_1)$ if $t_1, t_2 \in [0, t_0]$ and $|t_2 - t_1| < \delta_2$. Since $\lim_{s \to 0^+} r(s) = 0$, we have the continuity of λ on $[0, t_0]$. Since $t_0 \in (0, T)$ is arbitrary, λ is continuous on $[0, T)$.

Let $g_t = g_{K_t}$, $0 \le t < T$. Then $g_t : \mathbb{H} \setminus K_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$. We suffice to show that (1.1) holds. Let $t \in [0, t_0]$ and $s \in (0, \delta_2)$ such that $t - s \ge 0$. From (3.4), we have

$$z - g^{-1}_{K_t/K_{t-s}}(z) = \int_{c_{K_t/K_{t-s}}}^{d_{K_t/K_{t-s}}} \frac{1}{z - x} d\mu_{K_t/K_{t-s}}(x), \quad z \in \mathbb{H}.$$

Letting $w = g_t^{-1}(z)$, we get

$$\frac{g_t(w) - g_{t-s}(w)}{s} = \frac{1}{s} \int_{c_{K_t/K_{t-s}}}^{d_{K_t/K_{t-s}}} \frac{1}{g_t(w) - x} d\mu_{K_t/K_{t-s}}(x), \quad w \in \mathbb{H} \setminus K_t.$$

We have $|\mu_{K_t/K_{t-s}}| = \text{hcap}(K_t) - \text{hcap}(K_{t-s}) = 2s$. As $s \to 0^+$, the interval $[c_{K_t/K_{t-s}}, d_{K_t/K_{t-s}}]$ converges to a single point $\lambda(t)$. So we conclude that $\partial_t^- g_t(w) = \frac{2}{g_t(w) - \lambda(t)}$, $w \in \mathbb{H} \setminus K_t$. Since λ is continuous, we see that (1.1) holds for $t \in [0, t_0)$. Since $t_0 \in (0, T)$ is arbitrary, (1.1) holds for all $t \in [0, T)$. □

Remark. Part (ii) of the proposition says that if K_t, $0 \le t < T$, is a Loewner chain in $\mathbb{H}$ composed of $\mathbb{H}$-hulls, then it is a time-change of a family of chordal Loewner hulls. The proposition mimics Pommerenke's theorem below for radial Loewner hulls.

Proposition 3.5 [Pommerenke]

(i) If K_t, $0 \le t < T$, are radial Loewner hulls driven by some $\lambda \in C([0, T))$, then the family is a Loewner chain in $\mathbb{D}$ such that each K_t is a $\mathbb{D}$-hull and $\text{dcap}(K_t) = t$.

(ii) If K_t, $0 \le t < T$, is a Loewner chain such that each K_t is a $\mathbb{D}$-hull, then $u(t) := \text{dcap}(K_t)$ is a continuous and increasing function on $[0, T)$ with $u(0) = 0$. Moreover, if $\text{dcap}(K_t) = t$ for each t, then K_t, $0 \le t < T$, are radial Loewner hulls driven by some $\lambda \in C([0, u(T)))$, which is given by (1.5) with $g_t = g_{K_t}$.

4 Stochastic Analysis

4.1 Stochastic processes

Let $(\Omega, \mathcal{F})$ be a measurable space and S be an interval of the kind $[0, \infty)$, $[0, a)$ or $[0, a]$. A filtration in $(\Omega, \mathcal{F})$ is a family of σ-algebras $(\mathcal{F}_t)_{t \in S}$ with $\mathcal{F}_t \subset \mathcal{F}$ for each t and $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ when

$t_1 \le t_2$. The filtration is called right-continuous if for each $t \in S$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$. For example, $\mathcal{F}_{t^+} = \wedge_{s>t}\mathcal{F}_s$, $t \in S$, is a right-continuous filtration. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, the filtration is called complete w.r.t. $\mathbb{P}$ if $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets. $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in S})$ is called a filtered probability space. From now on, we assume that the filtration is right-continuous and complete.

A family of measurable functions $(X_t)_{t\in S}$ on $(\Omega, \mathcal{F})$ is called adapted to $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for each t. If we are given a family of measurable functions $(X_t)_{t\in S}$ and let $\mathcal{F}^X_= \sigma(X_s, s \le t)$, then $(\mathcal{F}^X_t)_{t\in S}$ is a filtration, and (X_t) is $(\mathcal{F}^X_t)$-adapted. The $(\mathcal{F}^X_t)$ is called the natural filtration generated by (X_t). It is easy to expand $(\mathcal{F}^X_t)$ so that it is right-continuous and complete.

Definition 4.1 *A function $T : \Omega \to S \cup \{\infty\}$ is called an $(\mathcal{F}_t)$-stopping time if for any $t \in S$,*

$$\{\omega \in \Omega : T(\omega) \le t\} \in \mathcal{F}_t.$$

Given a stopping time T, the σ-algebra $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \le t\} \in \mathcal{F}_t, \quad \forall t \in S\}.$$

Remarks. A constant function $T \equiv t_0$, $t_0 \in S$, is a stopping time. In that case, $\mathcal{F}_T$ agrees with $\mathcal{F}_{t_0}$. Let T_1 and T_2 be two stopping times. Then $T_1 \vee T_2$ and $T_1 \wedge T_2$ are stopping times. This is also true for $\vee_{n=1}^\infty T_n$ and $\wedge_{n=1}^\infty T_n$. If $T_1 \le T_2$, then $\mathcal{F}_{T_1} \subset \mathcal{F}_{T_2}$. If T is a finite stopping time, then we get a new filtration $\mathcal{F}_{T+t}$, $t \ge 0$. Let (X_t) be a right-continuous or left-continuous $(\mathcal{F}_t)$-adapted process. Then for any finite $(\mathcal{F}_t)$-stopping time T, X_T is $\mathcal{F}_T$-measurable. If T is any $(\mathcal{F}_t)$-stopping time, then we get another $(\mathcal{F}_t)$-adapted process: $X^T_t := X_{T\wedge t}$, $t \in S$, the process (X) stopped at time T.

Example. Let (X_t) is a right-continuous or left-continuous adapted process, and A be an open or closed subset of $\mathbb{R}$. Let $T = \inf\{t : X_t \in A\}$ $(\inf \emptyset = \infty)$. Then T is a stopping time.

Definition 4.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in S}$. Let $(X_t)_{t\in S}$ be an $(\mathcal{F}_t)$-adapted process. If $\mathbb{E}[|X_t|] < \infty$ for each $t \in S$, and $\mathbb{E}[X_{t_2}|\mathcal{F}_{t_1}] = X_{t_1}$ a.s. for each $t_1 \le t_2 \in S$, we say that (X_t) is an $(\mathcal{F}_t)$-martingale.*

If $\mathcal{F}_1 \subset \mathcal{F}_2$ are two sub-σ-algebras of $(\Omega, \mathcal{F}, P)$, and if $X \in L^1(\Omega, \mathcal{F}_2, \mathbb{P})$, then there is $Y \in L^1(\Omega, \mathcal{F}_1, P)$ such that $\mathbb{E}[1_A Y] = \mathbb{E}[1_A X]$ for any $A \in \mathcal{F}_1$. Such Y is $\mathbb{P}$-a.s. unique, and is denoted by $\mathbb{E}[X|\mathcal{F}_1]$. If $\mathcal{F}_0 \subset \mathcal{F}_1 \subset F_2$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_0] = \mathbb{E}[X|\mathcal{F}_0]$.

Theorem 4.1 [Optional Stopping Theorem] *If (X_t) is a right-continuous $(\mathcal{F}_t)$-martingale, and T_1, T_2 are two bounded $(\mathcal{F}_t)$-stopping times, then $\mathbb{E}[X_{T_2}|\mathcal{F}_{T_1}] = X_{T_1}$.*

If (X_t) is an $(\mathcal{F}_t)$-martingale and T is an $(\mathcal{F}_t)$-stopping time, using Optional Stopping Theorem we can show that (X^T_t) is also an $(\mathcal{F}_t)$-martingale.

4.2 Brownian motion

Definition 4.3 *A standard Brownian motion is a continuous random processes B_t, $0 \le t < \infty$, such that*

1. $B_0 = 0$ *and* $t \mapsto B_t(\omega)$ *is continuous for all* ω*;*
2. *for any sequence* $0 = t_0 < t_1 < \cdots < t_n$*, the random variables* $B_{t_i} - B_{t_{i-1}}$*,* $i = 1, 2, \dots, n$ *are independent, and* $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$*, where* $N(0, t_i - t_{i-1})$ *is the normal distribution with means* 0 *and variance* $t_i - t_{i-1}$*.*

If (B_t) is a standard Brownian motion, we call $x_0 + cB_t$, where $x_0 \in \mathbb{R}$ and $c > 0$, a Brownian motion started from x_0 (rescaled by a factor c).

A standard Brownian motion grows slower than the linear function near 0 and faster than the linear function near ∞. In fact, we have

$$\limsup_{t \to 0^+} \frac{B_t}{(2t \log\log(1/t))^{1/2}} = 1, \quad \liminf_{t \to 0^+} \frac{B_t}{(2t \log\log(1/t))^{1/2}} = -1;$$

$$\limsup_{t \to \infty} \frac{B_t}{(2t \log\log(t))^{1/2}} = 1, \quad \liminf_{t \to \infty} \frac{B_t}{(2t \log\log(t))^{1/2}} = -1.$$

The second formula implies that B_t is recurrent.

If $B_t^1, B_t^2, \dots, B_t^d$ are d independent Brownian motions, then $(B_t^1, \dots, B_t^d)$ is called a Brownian motion in $\mathbb{R}^d$. We are mostly interested in the case $d = 2$. In this case (B_t^1, B_t^2) is called a planar Brownian motion or complex Brownian motion.

Definition 4.4 *Given a filtration* $(\mathcal{F}_t)$*, an* $(\mathcal{F}_t)$*-adapted process* $(B_t)_{t \ge 0}$ *is called an* $(\mathcal{F}_t)$*-Brownian motion if it is a Brownian motion, and for any* $t_0 \ge 0$*, the process* $B_{t_0+t} - B_{t_0}$*,* $t \ge 0$*, is (a Brownian motion) independent of* $\mathcal{F}_{t_0}$*.*

Remarks.

1. Let (B_t) be a Brownian motion. Let $(\mathcal{F}_t^B)$ be the filtration generated by (B_t). Then (B_t) is an $(\mathcal{F}_t^B)$-Brownian motion. Such $(\mathcal{F}_t^B)$ is called a Brownian filtration.
2. Let $(B_t^{(k)})$, $1 \le k \le n$, be n independent Brownian motions. Let $\mathcal{F}_t$ be the filtration generated by $B_s^{(k)}$, $1 \le k \le n$, $0 \le s \le t$. Then every $B_t^{(k)}$ is an $(\mathcal{F}_t)$-Brownian motion.
3. An $(\mathcal{F}_t)$-Brownian motion is a continuous $(\mathcal{F}_t)$-martingale.
4. If (B_t) is an $(\mathcal{F}_t)$-Brownian motion and T is a finite $(\mathcal{F}_t)$-stopping time, then $B_{T+t} - B_T$, $t \ge 0$, is an $(\mathcal{F}_{T+t})$ Brownian motion (independent of $\mathcal{F}_T$).

4.3 Itô's integration

Let (B_t) be an $(\mathcal{F}_t)$-Brownian motion. Let (X_t) be a left-continuous $(\mathcal{F}_t)$-adapted process. Let $a > 0$. We will define $\int_0^a X_t dB_t$. First assume that X_t is a step process on $[0, a]$, which means that there are random variables $Z_1, Z_2, \dots, Z_n$, and a partition $0 = t_0 < t_1 < \cdots t_n = a$ such that $Z_k \in \mathcal{F}_{t_k}$ and $X_t = Z_k$ when $t_k < t \le t_{k+1}$, $0 \le k \le n-1$. Then we define

$$\int_0^a X_t dB_t = \sum_{k=0}^{n-1} Z_k (B_{t_{k+1}} - B_{t_k}).$$

The value of the integration is an $\mathcal{F}_a$-measurable random variable. If $\mathbb{E}|Z_k|^2 < \infty$ for all k. then we have

$$\mathbb{E}\left[\Big(\int_0^a X_t dB_t\Big)^2\right] = \sum_{k=0}^{n-1}(t_{k+1} - t_k)\mathbb{E}[|Z_k|^2] = \int_0^a \mathbb{E}[X_t^2]dt =: \|X\|^2_{L^2[0,a]}.$$

Now we do not assume that X_t is a step function but assume that it is uniformly bounded on $[0, a]$. Then X_t can be a.s. approximated by bounded step processes (X_t^n). For example, $X_t^n = X_{\frac{k}{n}a}$ when $\frac{k}{n}a < t \le \frac{k+1}{n}a$, $0 \le k \le n-1$. Then (X_t^n) converges to (X_t) in $\|\cdot\|_{L^2[0,a]}$. For each n, we have an $\mathcal{F}_a$ measurable r.v. $\int_0^a X_t^n dB_t$. Then we get a Cauchy sequence in $L^2(\mathcal{F}_a)$. We define the limit to be $\int_0^a X_t dB_t$, which is an element in $L^2(\mathcal{F}_a)$.

Now suppose that X_t is bounded on $[0, \infty)$. For each $a \in [0, \infty)$, we have an $\mathcal{F}_a$-measurable random variable $Y_a = \int_0^a X_t dB_t$, which is unique up to a negligible event. If $a < b$ then $Y_b - Y_a$ is independent of $\mathcal{F}_a$, $\mathbb{E}[Y_b - Y_a] = 0$ and $\mathbb{E}[|Y_b - Y_a|] = \int_a^b \mathbb{E}[X_t^2]$. So (Y_t) is an $(\mathcal{F}_t)$-martingale. It is known that we may choose Y_t, $t \ge 0$, such that (Y_t) is a continuous. (The proof uses Doob's Martingale Inequality and Borel Cantelli lemma) From now on, we always assume that $t \mapsto \int_0^t X_s dB_s$ is a continuous martingale.

To extend the definition, we need the following fact. If X is a bounded left-continuous adapted process, $Y_t = \int_0^t X_s dB_s$, and T is a stopping time, then

$$\int_0^t 1_{[0,T]} X_s dB_S = Y_{t\wedge T} = Y_t^T.$$

Using this fact, we may now define $\int_0^t X_s dB_s$ for a continuous adapted process X_t which may not be bounded. Let $T_n = \inf\{t : X_t \ge n\}$. Then $1_{[0,T_n]}X_t$ is bounded. We have $Y_t^{(n)} := \int_0^t 1_{[0,T_n]} X_s dB_s$ and have the facts that $Y_{t\wedge T_n}^{(n+1)} = Y_t^{(n)}$. Then we define $Y_t = \int_0^t X_s dB_s$ to be the process such that $Y_t = Y_t^{(n)}$ on $[0, T_n]$. We find that Y_t is well defined and $Y_t^{T_n} = Y_t^{(n)}$ for each n. The process Y_t is in general not a martingale. Instead, it is a continuous local martingale. The idea in the definition is called localization.

Definition 4.5 *A process (X_t) is called a local martingale if there exists an increasing family of finite stopping times T_n, $n \in \mathbb{N}$, with $\sup T_n = \infty$ such that for each n, $X_t^{T_n}$ is a martingale.*

Remarks.

1. If (X_t) is a local martingale, and T is a stopping time, then (X_t^T) is also a local martingale.

2. The above (X_t) may not be a martingale even if X_t is integrable for each t. A theorem states that if a local martingale is uniformly bounded, then it is a martingale.

3. If M_t, $0 \le t < \infty$, is a continuous martingale, Doob's inequality implies that a.s. $\lim_{t\to\infty} M_t$ exists, which could be $\pm\infty$. We use M_∞ to denote the limit. If in addition there is a deterministic $R > 0$ such that $|M_t| \le R$ for all t, then $|M_\infty| \le R$, and from DCT we have $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$ for all t. If (X_t) is a local martingale, and if T is a stopping time such that X_t is uniformly bounded on $[0, T)$, then (X_t^T) is a uniformly bounded martingale. So $\lim_{t\to\infty} X_t^T$ exists and is bounded. In case $T < \infty$, the limit is simply X_T. If $T = \infty$, we also use X_T to denote the limit. So X_T has a well defined meaning no matter $T < \infty$ or $T = \infty$. And we have $\mathbb{E}[X_T|\mathcal{F}_t] = X_t^T = X_{T\wedge t}$ for any t.

4. Using the idea of localization, we may also define $\int_0^t X_s dB_s$ if $X_\cdot$ is a continuous adapted process defined for $0 \le t < T$, where T is a stopping time, and there exists an increasing family of stopping times T_n, $n \in \mathbb{N}$, with $T_n < T$ and $\sup T_n = T$. The resulting process $Y_t = \int_0^t X_s dB_s$ is a local martingale defined on $[0, T)$.

Definition 4.6 *A continuous semimartingale is a continuous adapted process which can be written $X = M + A$ where M is a continuous local martingale and A a continuous adapted process of finite variation.*

Example Suppose (B_t) is an $(\mathcal{F}_t)$-Brownian motion, a_t and b_t are continuous adapted processes, and $X_0 \in \mathcal{F}_0$. Then

$$X_t := X_0 + \int_0^t a_s dB_s + \int_0^t b_s ds.$$

is an $(\mathcal{F}_t)$-continuous semimartingale. We often write

$$dX_t = a_t dB_t + b_t dt.$$

We may integrate along a semimartingale. Suppose that $dX_t = a_t dB_t + b_t dt$, and (Y_t) is a continuous adapted process. Then

$$\int_0^t Y_t dX_t = \int_0^t Y_s a_s dB_s + \int_0^t Y_s b_s ds.$$

4.4 Quadratic Variation

For a $(\mathcal{F}_t)$-local martingale M_t, there is a unique adapted continuous non-decreasing process $\langle M, M\rangle_t$ with $\langle M, M\rangle_0 = 0$ such that $(M_t - M_0)^2 - \langle M, M\rangle_t$ is a local martingale. Such $\langle M, M\rangle_t$ is called the quadratic variation of M. If a semimartingale X has decomposition $M + A$, then

$\langle X, X\rangle := \langle M, M\rangle$. For two semimartingale X and Y, the bracket between X and Y is defined by

$$\langle X, Y\rangle = \frac{1}{4}\langle X+Y, X+Y\rangle - \frac{1}{4}\langle X-Y, X-Y\rangle.$$

We have the following facts.

1. For a Brownian motion B_t, $\langle B, B\rangle_t = t$.
2. If X and Y are independent, then $\langle X, Y\rangle \equiv 0$.
3. Levy's characterization Theorem states that, if a local martingale M_t, $0 \le t < \infty$, satisfies $\langle M, M\rangle_t = t$, then M_t is a Brownian motion started from some $x \in \mathbb{R}$, and if two Brownian motions B_t and B'_t satisfy $\langle B, B'\rangle = 0$, then they are independent.
4. For any stopping time T, $\langle X^T, Y^T\rangle_t = \langle X, Y\rangle_t^T$.
5. If $dX_t = a_t dB_t + b_t dt$ and $dY_t = c_t dB_t + d_t dt$, then $d\langle X, Y\rangle_t = a_t c_t dt$.
6. If $B_t^{(k)}$, $1 \le k \le n$, are independent Brownian motions, and
$$dX_t = \sum_{k=1}^n a_t^{(k)} dB_t^{(k)} + b_t dt; \quad dY_t = \sum_{k=1}^n c_t^{(k)} dB_t^{(k)} + d_t dt,$$
then $d\langle X, Y\rangle_t = \sum_{k=1}^n a_t^{(k)} c_t^{(k)} dt$.

Let $(\mathcal{F}_t)$ be a filtration and T be a stopping time. An $(\mathcal{F}_t)$-adapted process X_t, $0 \le t < T$, is called a partial $(\mathcal{F}_t)$-Brownian motion if there is another filtration $(\widetilde{\mathcal{F}}_t)$ and an $(\widetilde{F}_t)$-Brownian motion B_t such that $\mathcal{F}_t \subset \widetilde{F}_t$ for each t and $X_t = B_t$ for $0 \le t < T$. An adapted process X_t, $0 \le t < T$ is a partial Brownian motion iff it is a local martingale and $\langle X, X\rangle_t = t$ for $0 \le t < T$. The chordal or radial Loewner hulls driven by $\sqrt{\kappa}$ times a partial Brownian motion are called partial chordal or radial SLE_κ hulls.

4.5 Itô's formula

Theorem 4.2 [Itô's formula, one-dimensional] *Suppose X_t is an $(\mathcal{F}_t)$-semimartingale with $dX_t = a_t dB_t + b_t dt$. Let $f(t, x)$ be a a $C^{1,2}$ differentiable function such that $f(t, \cdot)$ is $\mathcal{F}_t$-measurable for each t. Let $Y_t = f(t, X_t)$. Then Y_t is also an $(\mathcal{F}_t)$-semimartingale, and satisfies*

$$dY_t = \frac{\partial}{\partial t} f(t, X_t) dt + \frac{\partial}{\partial x} f(t, X_t) dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t) d\langle X, X\rangle_t.$$

Theorem 4.3 [Itô's formula, multiple-dimensional] *Let $(B_t^{(k)})$, $1 \le k \le n$, be n independent $(\mathcal{F}_t)$-Brownian motions. Let $(X_t^{(j)})$, $1 \le j \le m$, be m semimartingales which satisfies*

$$dX_t^{(j)} = \sum_{k=1}^n a_t^{(j,k)} dB_t^{(k)} + b_t^{(j)} dt, \quad 1 \le j \le m.$$

Let $f(t, x_1, \dots, m)$ be a a $C^{1,2,\dots,2}$ differentiable function such that $f(t, \cdot)$ is $\mathcal{F}_t$-measurable for each t. Let $Y_t = f(t, X_t^{(1)}, \dots, X_t^{(m)})$. Then Y_t is also an $(\mathcal{F}_t)$-semimartingale, and satisfies

$$dY_t = \frac{\partial}{\partial t} f(t, X_t)dt + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f(t, X_t) dX_t^{(j)} + \frac{1}{2} \sum_{j_1, j_2=1}^{m} \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} f(t, X_t) d\langle X^{(j_1)}, X^{(j_2)} \rangle_t.$$

Corollary 4.1 [Product formula] *Let X_t and Y_t be two semimartingales. Let $Z_t = X_t Y_t$. Then Z_t is a semimartingale that satisfies*

$$dZ_t = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t.$$

4.6 Time-change

Let X_t, $0 \le t < T$, be a continuous $(\mathcal{F}_t)$-adapted process, where T is an $(\mathcal{F}_t)$-stopping time. Suppose $u(t) = u(t, \omega)$, $0 \le t < T$, is a continuous (strictly) increasing $(\mathcal{F}_t)$-adapted function, which satisfies $u(0) = 0$. Define $v(t) = v(t, \omega)$ for $0 \le t < \infty$ such that $v(t) = u^{-1}(t)$ if $t < \sup u[0, T)$; $v(t) = T$ if $t \ge \sup u[0, T)$. Then for each $t \ge 0$, $v(t)$ is an $(\mathcal{F}_t)$-stopping time. In fact,

$$\{v(t) \le a\} = \{T \le a\} \cup (\{T > a\} \cap \{u(a) \ge t\}) \in \mathcal{F}_a, \quad 0 \le a < \infty.$$

Moreover, we have $v(t_1) \le v(t_2)$ if $t_1 \le t_2$. So we get a new filtration $(\mathcal{F}_{v(t)})_{t \ge 0}$.

Let $S = \sup u[0, T)$. Then S is an $(\mathcal{F}_{v(t)})$-stopping time because

$$\{S \le a\} \cap \{v(a) \le b\} = \{S \le a\} \cap \{T \le b\} = \{T \le b\} \cap \bigcap_{q \in [0,b] \cap \mathbb{Q}} (\{T > q\} \cap \{u(q) \le a\}) \in \mathcal{F}_b.$$

We call the process $X_{v(t)}$, $0 \le t < S$, a time-change of X_t, $0 \le t < T$. Since (X) is continuous, $(X_{v(t)})$ is a continuous $(\mathcal{F}_{v(t)})$-adapted process.

We have the following facts.

1. If (X_t) is an $(\mathcal{F}_t)$-local martingale (resp. semimartingale), then $(X_{v(t)})$ is an $(\mathcal{F}_{v(t)})$-local martingale (resp. semimartingale), and $\langle X_{v(\cdot)}, X_{v(\cdot)} \rangle_t = \langle X, X \rangle_{v(t)}$.

2. If $Y_t = a_t dX_t$, then $Y_{v(t)} = a_{v(t)} dX_{v(t)}$.

3. Suppose X is a local martingale, and $\langle X, X \rangle_t$ is strictly increasing. Let $u(t) = \langle X, X \rangle_t$, then $\langle X_{v(\cdot)}, X_{v(\cdot)} \rangle_t = t$ for $0 \le t < S$. This means that $X_{v(t)}$, $0 \le t < S$, is a Brownian motion stopped at time S, or X_t is a time-change of a partial Brownian motion. This Brownian motion is called the DDS Brownian motion for X.

4. Suppose that X is a semimartingale that satisfies $dX_t = a_t dB_t + b_t dt$. Suppose c_t is a positive continuous adapted process, and $u(t) = \int_0^t c_s^2 ds$. Let $M_t = \int_0^t c_s dB_s$. Then M is a local martingale, $\langle M, M \rangle_t = u(t)$, and $dX_t = a_t / c_t dM_t + b_t dt$. Let $\widetilde{B}_t = M_{v(t)}$. Then $\widetilde{B}_t$ is an $(\mathcal{F}_{v(t)})$-Brownian motion. From $dX_t = a_t dB_t + b_t dt$, we have $dX_t = \frac{a_t}{c_t} dM_t + b_t dt$. Thus,

$$dX_{v(t)} = \frac{a_{v(t)}}{c_{v(t)}} dM_{v(t)} + b_{v(t)} dv(t) = \frac{a_{v(t)}}{c_{v(t)}} d\widetilde{B}_t + \frac{b_{v(t)}}{c_{v(t)}^2} dt.$$

4.7 Bessel process

Let $(B_t^{(1)}, \dots, B_t^{(n)})$ be an n-dimensional Brownian motion. Let $X_t = \sqrt{\sum_{j=1}^n (B_t^{(j)})^2}$. Then we find that X_t satisfies the SDE

$$dX_t = \frac{\sum_{j=1}^n B_t^{(j)} dB_t^{(j)}}{X_t} + \frac{(n-1)/2}{X_t} dt.$$

Let $B_t = \int_0^t \frac{\sum_{j=1}^n B_s^{(j)} dB_s^{(j)}}{X_s}$. Then $\widehat{B}_t$ is a local martingale with $\langle B, B \rangle_t = t$. Thus, B_t is a (partial) Brownian motion. And we have

$$dX_t = dB_t + \frac{(n-1)/2}{X_t} dt. \tag{4.1}$$

We may allow n to be any real number. The solution of the above SDE is called an n-dimensional Bessel process. The Bessel process starts from some positive number, and continues forever or stops when it hits 0.

Let $f(x) = x^{2-n}$ for $n \neq 2$ and $f(x) = \log(x)$ for $n = 2$. Itô's formula implies that $f(X_t)$ is a local martingale, i.e., a time-change of a partial Brownian motion. For $n < 2$, $X_t \to 0$ iff $f(X_t) \to 0$ and $X_t \to \infty$ iff $f(X_t) \to \infty$. For $n = 2$, $X_t \to 0$ iff $f(X_t) \to -\infty$ and $X_t \to \infty$ iff $f(X_t) \to \infty$. For $n > 2$, $X_t \to 0$ iff $f(X_t) \to \infty$ and $X_t \to \infty$ iff $f(X_t) \to 0$. From the properties of Brownian motion, we find that, for $n < 2$, X_t hits 0 in a finite time; for $n > 2$, $X_t \to \infty$ as $t \to \infty$; for $n = 2$, $\liminf X_t = 0$ and $\limsup X_t = \infty$. For $n > 2$, an n-dimensional Bessel process can be started from 0^+. This is a process X_t with $X_0 = 0$, $X_t > 0$ for $t > 0$, and satisfies (4.1) for $t > 0$.

4.8 Complex valued Itô's formula

Let D be a plane domain, and $f : D \overset{\text{Conf}}{\twoheadrightarrow} D'$. Let $B_t^{\mathbb{C}} = B_t^{(1)} + iB_t^{(2)}$ be a planar Brownian motion started from $z_0 \in D$. Let τ be the first time that $B_t^{\mathbb{C}}$ leaves D. We consider the image $f(B_t^{\mathbb{C}})$, $0 \le t < \tau$. Let $f = u + iv$. From Itô's formula and the fact that $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$, we get

$$du(B_t^{\mathbb{C}}) = u_x(B_t^{\mathbb{C}}) dB_t^{(1)} + u_y(B_t^{\mathbb{C}}) dB_t^{(2)}, \quad dv(B_t^{\mathbb{C}}) = v_x(B_t^{\mathbb{C}}) dB_t^{(1)} + v_y(B_t^{\mathbb{C}}) dB_t^{(2)}.$$

Thus, $\langle u(B), u(B) \rangle_t = \langle v(B), v(B) \rangle_t = \int_0^t |f'(B_s^{\mathbb{C}})|^2 ds$, and $\langle u(B), v(B) \rangle_t \equiv 0$. Construct a time-change using $a(t) = \int_0^t |f'(B_s^{\mathbb{C}})|^2 ds$. Let $b(t) = a^{-1}(t)$. Then we see that $u(B_{b(t)}^{\mathbb{C}})$ and $v(B_{b(t)}^{\mathbb{C}})$ are two independent Brownian motions. Thus, $f(B_t^{\mathbb{C}})$ is a time-change of a planar Brownian motion started from $f(z_0)$ stopped on leaving D'. This phenomena is called the conformal invariance of planar Brownian motion.

Let Z_t be a complex valued semimartingale which satisfies

$$dZ_t = a_t dB_t + b_t dt.$$

Here B_t is a standard real valued Brownian motion, a_t and b_t are complex valued adapted continuous process. Thus, if $Z_t = X_t + iY_t$, then $dX_t = \operatorname{Re} a_t dB_t + \operatorname{Re} b_t dt$ and $dY_t = \operatorname{Im} a_t dB_t + \operatorname{Im} b_t dt$. Suppose $f = u + iv$ is an analytic function defined in a domain which contains the range of Z_t. Let $f(Z_t) = U_t + iV_t$. Then

$$dU_t = u_x(Z_t)dX_t + u_y(Z_t)dY_t + \frac{1}{2}u_{xx}(Z_t)d\langle X,X\rangle_t + \frac{1}{2}u_{yy}(Z_t)d\langle Y,Y\rangle_t + u_{xy}(Z_t)\langle X,Y\rangle_t$$

$$= \operatorname{Re} f'(Z_t)\operatorname{Re} dZ_t - \operatorname{Im} f'(Z_t)\operatorname{Im} dZ_t + \frac{1}{2}\operatorname{Re} f''(Z_t)(\operatorname{Re} a_t)^2 dt - \frac{1}{2}\operatorname{Re} f''(Z_t)(\operatorname{Im} a_t)^2 dt$$

$$- \operatorname{Im} f''(Z_t)\operatorname{Re} a_t \operatorname{Im} a_t dt = \operatorname{Re}[f'(Z_t)dZ_t] + \operatorname{Re}[f''(Z_t)\frac{1}{2}a_t^2]dt.$$

$$dV_t = v_x(Z_t)dX_t + v_y(Z_t)dY_t + \frac{1}{2}v_{xx}(Z_t)d\langle X,X\rangle_t + \frac{1}{2}v_{yy}(Z_t)d\langle Y,Y\rangle_t + v_{xy}(Z_t)\langle X,Y\rangle_t$$

$$= \operatorname{Im} f'(Z_t)\operatorname{Re} dZ_t + \operatorname{Re} f'(Z_t)\operatorname{Im} dZ_t + \frac{1}{2}\operatorname{Im} f''(Z_t)(\operatorname{Re} a_t)^2 dt - \frac{1}{2}\operatorname{Im} f''(Z_t)(\operatorname{Im} a_t)^2 dt$$

$$+ \operatorname{Re} f''(Z_t)\operatorname{Re} a_t \operatorname{Im} a_t dt = \operatorname{Im}[f'(Z_t)dZ_t] + \operatorname{Im}[f''(Z_t)\frac{1}{2}a_t^2]dt.$$

So we have

$$df(Z_t) = f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)a_t^2 dt = f'(Z_t)a_t dB_t + f'(Z_t)b_t dt + \frac{1}{2}f''(Z_t)a_t^2 dt.$$

4.9 Girsanov Theorem

In this subsection, we will change the underlying probability measure. Let the current probability distribution be denoted by $\mathbb{P}$. Suppose that another probability distribution $\mathbb{P}_1$ satisfies $\mathbb{P}_1 \ll \mathbb{P}$ on each $\mathcal{F}_t$. It is known that the quadratic variation of a semimartingale does not change if the probability measure is changed from $\mathbb{P}$ to $\mathbb{P}_1$. Let $D_t = \frac{d\mathbb{P}_1|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}}$. Then D_t is a martingale. An $(\mathcal{F}_t)$-adapted process X_t is a martingale (resp. local martingale) under $\mathbb{P}_1$ if and only if $X_t D_t$ is a martingale (resp. local martingale) under $\mathbb{P}$. We now consider the case that D_t has an expression $dD_t = a_t D_t dB_t$ for an $(\mathcal{F}_t)$-Brownian motion B_t. Let $X_t = B_t - \int_0^t a_s ds$. Then $\langle X,X\rangle_t = t$. From the product formula,

$$dX_t D_t = X_t dD_t + D_t dX_t + \langle X,D\rangle_t = X_t dD_t + D_t dB_t - D_t a_t dt + a_t D_t dt = (X_t a_t D_t + D_t)dB_t.$$

Thus, under $\mathbb{P}_1$, X_t is a local martingale with $\langle X,X\rangle_t = t$. So $B_t - \int_0^t a_s ds$ is a Brownian motion under $\mathbb{P}_1$.

On the other hand, given a continuous adapted process a_t, we may construct a local martingale D_t with $dD_t = a_t D_t dB_t$. It is defined by

$$D_t = \exp\Big(\int_0^t a_s dB_s - \frac{1}{2}\int_0^t a_s^2 ds\Big).$$

Suppose T is a stopping time such that D_t, $0 \le t \le T$, are uniformly bounded. Then D_t^T is a bounded martingale, and $D_t^T = \mathbb{E}[D_T|\mathcal{F}_t]$ for any t. Define $\mathbb{P}_1$ such that $d\mathbb{P}_1 = D_T d\mathbb{P}$. Then $d\mathbb{P}_1|_{\mathcal{F}_t}/d\mathbb{P}|_{\mathcal{F}_t} = D_t^T$ for each t. We then can conclude that $B_t - \int_0^t a_s ds$, $0 \le t < T$, is a partial Brownian motion up to T under $\mathbb{P}_1$.

4.10 Some applications

Let g_t be chordal Loewner maps driven by $\lambda_t = \sqrt{\kappa} B_t$. Fix $x_0 > 0$. Let $Z_t = g_t(x_0) - \lambda_t$, $0 \le t < \tau = \tau_{x_0}$. Recall that $\tau < \infty$ implies that $Z_t \to 0$ as $t \to \tau$. Then Z_t stays positive and satisfies

$$dZ_t = -\sqrt{\kappa} dB_t + \frac{2}{Z_t} dt. \tag{4.2}$$

We see that $Z_t/\sqrt{\kappa}$ is a Bessel process of dimension $1 + \frac{4}{\kappa}$. Thus, if $\kappa > 4$, then $\tau < \infty$ and $Z_t \to 0$ as $t \to \tau$; if $\kappa < 4$, then $\tau = \infty$ and $Z_t \to \infty$ as $t \to \infty$; if $\kappa = 4$, then $\tau = \infty$, $\liminf_{t\to\infty} Z_t = 0$ and $\limsup_{t\to\infty} Z_t = \infty$. We have a similar result for $x_0 < 0$.

Now suppose $z_0 \in \mathbb{H}$. Let $Z_t = g_t(z_0) - \lambda_t$. Then the complex valued process Z_t also satisfies (4.2). Let $f(z) = z^{1-4/\kappa}$ for $\kappa \ne 4$ and $f(z) = \ln(z)$ for $\kappa = 4$. Since f is analytic, we find that

$$df(Z_t) = f'(Z_t)dZ_t + \frac{\kappa}{2} f''(Z_t)dt = -f'(Z_t)\sqrt{\kappa} dB_t.$$

This means that $f(Z_t)$ is a local martingale. In other words, both $\mathrm{Re}\, f(Z_t)$ and $\mathrm{Im}\, f(Z_t)$ are local martingales.

Note that Z_t stays in $\mathbb{H}$. If $\kappa = 4$, f maps $\mathbb{H}$ conformally onto $\{0 < \mathrm{Im}\, z < \pi\}$. So $\mathrm{Im}\, f(Z_t)$ is uniformly bounded, which implies that $\mathrm{Im}\, f(Z_t) = \mathrm{Im} \ln(Z_t) = \arg(Z_t)$ is a martingale. In fact, $\mathrm{Im}\, f(Z_t)/\pi$ is the probability that a planar Brownian motion started from $g_t(z_0)$ hits $(-\infty, \lambda_t)$ when exiting $\mathbb{H}$. From conformal invariance of planar Brownian motion, this is equal to the probability that a planar Brownian motion started from z_0 hits $(-\infty, 0]$ unions the "left side" of the SLE_4 trace β up to time t when it exits $\mathbb{H} \setminus \beta(0, t]$.

If $\kappa = 2$, $f(z) = 1/z$. We see that $-\frac{1}{\pi} \mathrm{Im}\, f(Z_t) = -\frac{1}{\pi} \mathrm{Im}\, \frac{1}{g_t(z_0) - \lambda_t}$ is a Poisson kernel function in $\mathbb{H}$ with pole at λ_t valued at $g_t(z_0)$. Since g_t maps the $\beta(t)$ to λ_t, this is also equal to a Poisson kernel function in $\mathbb{H} \setminus \beta(0, t]$ with pole at $\beta(t)$ valued at z_0. Here a Poisson kernel function in a simply connected domain D is a positive harmonic function in D, whose continuation vanishes on ∂D except for one point (or prime end), which is called the pole. When the domain D and the pole is given, the Poisson kernel function exists and is unique up to a positive factor.

We may also apply Itô's formula to radial Loewner equations. Recall that the radial Loewner equation driven by λ is

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\lambda_t} + g_t(z)}{e^{i\lambda_t} - g_t(z)}, \quad g_0(z) = z.$$

Let $\cot_2(z) = \cot(z/2)$. We now introduce the covering radial Loewner equation:

$$\partial_t \widetilde{g}_t(z) = \cot_2(\widetilde{g}_t(z) - \lambda_t), \quad g_0(z) = z.$$

Note that

$$i\cot_2(z-w) = i\frac{\cos_2(z-w)}{\sin_2(z-w)} = -\frac{e^{i(z-w)/2}+e^{-i(z-w)/2}}{e^{i(z-w)/2}-e^{-i(z-w)/2}} = \frac{e^{iw}+e^{iz}}{e^{iw}-e^{iz}}.$$

So we have

$$\partial_t e^{i\widetilde{g}_t(z)} = ie^{i\widetilde{g}_t(z)}\cot_2(\widetilde{g}_t(z)-\lambda_t) = e^{i\widetilde{g}_t(z)}\frac{e^{i\lambda_t}+e^{i\widetilde{g}_t(z)}}{e^{i\lambda_t}-e^{i\widetilde{g}_t(z)}}.$$

Thus, $e^{i\widetilde{g}_t(z)}$ satisfies the same ODE and initial value as $g_t(e^{iz})$. Let e^i denote the map $z \mapsto e^{iz}$. We then have $e^i \circ \widetilde{g}_t = g_t \circ e^i$. Let $\widetilde{K}_t$ denote the set of $z \in \mathbb{H}$ such that $\widetilde{g}_s(z)$ blows up before or at time t. Then we have $\widetilde{K}_t = (e^i)^{-1}(K_t)$, and $\widetilde{g}_t : \mathbb{H} \setminus \widetilde{K}_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$. We call $\widetilde{g}_t$ and $\widetilde{K}_t$ the covering radial Loewner maps and hulls driven by λ.

For every $z \in \mathbb{R}$, $\widetilde{g}_t(z)$ stays on $\mathbb{R}$ before blowing up. If $z \in \mathbb{H}$, then $\widetilde{g}_t(z)$ stays in $\mathbb{H}$, and $\operatorname{Im}\widetilde{g}_t(z)$ decreases in t. If $\tau(z) < \infty$, then $\widetilde{g}_t(z) - \lambda_t$ hits a pole of $\cot_2$ as $t \to \tau(z)$, which means that there is some $n \in \mathbb{Z}$ such that $\widetilde{g}_t(z) - \lambda_t \to 2n\pi$ as $t \to \tau(z)^-$.

Now suppose $\lambda_t = \sqrt{\kappa}B_t$. Fix $x_0 \in (0, 2\pi)$. Let $Z_t = \widetilde{g}_t(x_0) - \lambda_t$, $0 \le t < \tau = \tau(x_0)$. Then Z_t stays in $(0, 2\pi)$ and satisfies

$$dZ_t = -\sqrt{\kappa}B_t + \cot_2(Z_t)dt. \tag{4.3}$$

We may find f such that $f(Z_t)$ is a local martingale. We need that f satisfies $f'(x)\cot_2(x) + \frac{\kappa}{2}f''(x) = 0$, which implies that $f'(x) = C\sin_2(x)^{-4/\kappa}$. Let $W_t = f(Z_t)$. Then $dW_t = -f'(Z_t)\sqrt{\kappa}dB_t$. Let $u(t) = \int_0^t |f'(Z_s)|ds$. Suppose u maps $[0, \tau)$ onto $[0, T)$. Let $v(t)$, $0 \le t < T$, be the inverse of u. Then $W_{v(t)}$, $0 \le t < T$, is a Brownian motion. If $\kappa > 4$, then f maps $(0, 2\pi)$ onto a bounded interval. So we have a.s. $T < \infty$. Since $T = \int_0^\tau C^2|\sin_2(Z_s)|^{-8/\kappa}ds \ge C^2\tau$, we get $\tau < \infty$ and $\lim_{t\to\tau} Z_t = 0$ or 2π. Since $W_{v(t)}$, $0 \le t < T$, is bounded, it is a bounded martingale, and we have

$$f(x_0) = W_0 = \mathbb{E}[W_\tau] = f(0)\mathbb{P}[\lim_{t\to\tau} Z_t = 0] + f(2\pi)\mathbb{P}[\lim_{t\to\tau} Z_t = 2\pi].$$

If f has a simple formula, we may calculate the probability that $Z_t \to 0$ as $t \to \tau$. Now suppose $\kappa \le 4$. Then f maps $(0, 2\pi)$ onto $\mathbb{R}$. As a Brownian motion, $W_{v(t)}$ does not tend to $+\infty$ or $-\infty$ as $t \to T$ no matter $T = \infty$ or $T < \infty$. So Z_t does not tend to 0 or 2π as $t \to \tau$. This implies that $\tau = \infty$. Since $T \ge C^2\tau$, we have $T = \infty$. Thus, $\liminf_{t\to\infty} W_{v(t)} = -\infty$ and $\limsup_{t\to\infty} W_{v(t)} = +\infty$, which implies that $\liminf_{t\to\infty} Z_t = 0$ and $\limsup_{t\to\infty} Z_t = 2\pi$.

Fix $z_0 \in \mathbb{H}$. Let $Z_t = \widetilde{g}_t(z_0) - \lambda_t$. Then the complex valued process Z_t also satisfies (4.3). Thus, if f is an antiderivative of $C\sin_2(x)^{-4/\kappa}$, then $f(Z_t)$ is a local martingale. If $\kappa = 2$, we may choose $f(z) = \cot_2(z)$. This means that

$$\cot_2(\widetilde{g}_t(z_0) - \lambda_t) = -i\frac{e^{i\lambda_t} + g_t(e^{iz_0})}{e^{i\lambda_t} - g_t(e^{iz_0})}$$

is a local martingale. Thus, for any $w_0 \in \mathbb{D}$, $\operatorname{Re}\frac{e^{i\lambda_t}+g_t(w_0)}{e^{i\lambda_t}-g_t(w_0)}$ is a local martingale. Let $f_t(z) = \operatorname{Re}\frac{e^{i\lambda_t}+g_t(z)}{e^{i\lambda_t}-g_t(z)}$. Then f_t is a Poisson kernel function in $\mathbb{D} \setminus \beta(0, t]$ with pole at $\beta(t)$, normalized by $f_t(0) = 1$. Then for any $z \in \mathbb{D}$, $t \mapsto f_t(z)$ is a local martingale.

4.11 Phase transition

Theorem 4.4 *Let K_t be chordal Loewner hulls driven by $\lambda_t = \sqrt{\kappa}B_t$. Fix $z_0 \in \mathbb{H}$. Let $\tau = \tau(z_0)$. Then*

1. *If $\kappa \le 4$, a.s. $\tau = \infty$. If $\kappa > 4$, a.s. $\tau < \infty$.*
2. *If $\kappa < 8$, a.s. $\lim_{t\to\infty} \operatorname{dist}(z_0, K_t) > 0$. If $\kappa \ge 8$, a.s. $\lim_{t\to\infty} \operatorname{dist}(z_0, K_t) = 0$.*

Proof. Let g_t be the chordal Loewner maps. Let $Z_t = g_t(z_0) - \lambda_t$, $0 \le t < \tau$. Let $X_t = \operatorname{Re} Z_t$ and $Y_t = \operatorname{Im} Z_t$. Then X_t and Y_t satisfy

$$dX_t = -\sqrt{\kappa} dB_t + \frac{2X_t}{X_t^2 + Y_t^2}, \quad dY_t = \frac{-2Y_t}{X_t^2 + Y_t^2} dt.$$

Let $W_t = X_t/Y_t$. Then W_t satisfies

$$dW_t = \frac{-\sqrt{\kappa}}{Y_t} dB_t + \frac{4X_t/Y_t}{X_t^2 + Y_t^2} dt.$$

Let $u(t) = \frac{1}{2}(\ln(Y_0) - \ln(Y_t))$. Then $u(0) = 0$ and $u'(t) = \frac{1}{X_t^2+Y_t^2}$. Let $T = \sup u[0, \tau)$, and let $v(t)$, $0 \le t < T$, be the inverse of u. Then there is another Brownian motion $\widetilde{B}_t$ such that

$$dW_{v(t)} = \sqrt{1 + W_{v(t)}^2} \sqrt{\kappa} d\widetilde{B}_t + 4W_{v(t)} dt, \quad 0 \le t < T.$$

Let $U_t = \sinh^{-1}(W_{v(t)})$. Since $(\sinh^{-1})'(x) = \frac{1}{\sqrt{1+x^2}}$ and $(\sinh^{-1})''(x) = -\frac{x}{(1+x^2)^{3/2}}$, we have

$$dU_t = \sqrt{\kappa} d\widetilde{B}_t + (4 - \frac{\kappa}{2}) \tanh(U_t) dt, \quad 0 \le t < T. \tag{4.4}$$

Choose f on $\mathbb{R}$ such that $f'(x) = \cosh(x)^{1-8/\kappa}$. Let $V_t = f(U_t)$. Then

$$dV_t = \cosh(U_t)^{1-8/\kappa} \sqrt{\kappa} d\widetilde{B}_t, \quad 0 \le t < T.$$

So V_t is a time-change of a partial Brownian motion.

First, suppose $\kappa < 8$. Then f maps $\mathbb{R}$ onto a finite interval, which implies that $\lim_{t\to T} V_t$ a.s. exists. Thus, $\lim_{t\to T} U_t$ a.s. exists. So $\lim_{t\to\tau} W_t$ a.s. exits. We first show that a.s. $T = \infty$. If $T < \infty$, then $\lim_{t\to\tau} Y_t > 0$, and from (4.4) we see that $\lim_{t\to T} U_t$ is finite, which implies that $\lim_{t\to\tau} W_t$ is finite. Thus, $\lim_{t\to\tau} X_t$ also exists and if finite. Since $T = \int_0^\tau \frac{ds}{X_s^2+Y_s^2}$, from $T < \infty$, we have $\tau < \infty$, which implies that $\lim_{t\to\tau} Z_t = 0$ and $\lim_{t\to\tau} Y_t = 0$, so we get a contradiction. Thus, a.s. $T = \infty$. From (4.4) we see that $\lim_{t\to\infty} U_t$ can not be a finite number. Thus, a.s. $\lim_{t\to\infty} U_t = +\infty$ or $-\infty$.

From symmetry, we only need to consider the case that $\lim_{t\to\infty} U_t = +\infty$. Then $\tanh(U_t) \to 1$. From (4.4) we have $\lim_{t\to\infty} U_t/t = 4 - \kappa/2$. From $T = \int_0^\tau \frac{1}{X_s^2+Y_s^2} ds$ we get

$$\tau = \int_0^T (X_{v(s)}^2 + Y_{v(s)}^2) ds = \int_0^\infty Y_{v(s)}^2 (1 + W_{v(s)}^2) ds = Y_0^2 \int_0^\infty e^{-4s} \cosh^2(U_s) ds. \tag{4.5}$$

Suppose $\kappa \in (4,8)$. Choose $\kappa' \in (4,\kappa)$. There is some (random) $N > 0$ such that $0 < U_t < (4-\kappa'/2)t$ for $t \ge N$. So

$$\int_N^\infty e^{-4s}\cosh^2(U_s)ds \le \int_N^\infty e^{-4s}e^{2U_s}ds \le \int_N^\infty e^{(4-\kappa')s}ds < \infty,$$

which implies that $\tau < \infty$. Suppose $\kappa \in (0,4]$. Then

$$\int_0^\infty e^{-4s}\cosh^2(U_s)ds \ge \frac{1}{4}\int_0^\infty e^{2U_s-4s}ds.$$

From $\lim_{t\to\infty} U_t/t = 4-\kappa/2$ and (4.4) we see that there is some (random) $C > 0$ such that $U_t > \sqrt{\kappa}\widetilde{B}_t + (4-\kappa/2)t - C$ for all t, which implies that

$$\int_0^\infty e^{2U_s-4s}ds \ge \int_0^\infty e^{2\sqrt{\kappa}\widetilde{B}_s+(4-\kappa)s-C}ds \ge e^{-C}\int_0^\infty e^{2\sqrt{\kappa}\widetilde{B}_s}ds.$$

Since $\widetilde{B}_s$ is recurrent, we have a.s. $\int_0^\infty e^{2\sqrt{\kappa}\widetilde{B}_s}ds = \infty$. Thus, a.s. $\tau = \infty$ if $\kappa \in (4,8)$.

Next, suppose $\kappa \ge 8$. Then f maps $\mathbb{R}$ onto $\mathbb{R}$. If V_t is a time-change of an incomplete Brownian motion, then we must have (i) $\int_0^T \kappa\cosh(U_t)^{1-8/\kappa}dt < \infty$; and (ii) $\lim_{t\to T} V_t$ exists and is finite, which implies that $\lim_{t\to T} U_t$ and $\lim_{t\to\tau} W_t$ exist and are finite. Then we must have $T < \infty$. We already see that a contradiction can be obtained from $T < \infty$ and $\lim_{t\to\tau} W_t \in \mathbb{R}$. Thus, V_t is a time-change of a complete Brownian motion. So we have $\liminf_{t\to T} U_t = -\infty$ and $\limsup_{t\to T} U_t = \infty$. From (4.4) we conclude that a.s. $T = \infty$.

We will prove that a.s. $\limsup_{t\to\infty} U_t/t \le 0$. If this is not true, then there is $\delta > 0$ such that $\limsup_{t\to\infty} U_t/t > \delta$. Since $\lim_{t\to\infty} \widetilde{B}_t/t = 0$, there is some (random) $N > 0$ such that for $t \ge N$, $|\sqrt{\kappa}\widetilde{B}_t| < \frac{\delta}{2}t$. Since U_t is recurrent and $\limsup_{t\to\infty} U_t/t > \delta$, there exist $t_2 > t_1 > N$ such that $U_{t_1} = 0$, $U_{t_2} = \delta t_2$ and $U_t > 0$ for $t \in (t_1,t_2)$. From (4.4) we have

$$\delta t_2 = U_{t_2} - U_{t_1} = \sqrt{\kappa}\widetilde{B}_{t_2} - \sqrt{\kappa}\widetilde{B}_{t_1} + (4-\frac{\kappa}{2})\int_{t_1}^{t_2}\tanh_2(U_s)ds$$

$$\le \sqrt{\kappa}\widetilde{B}_{t_2} - \sqrt{\kappa}\widetilde{B}_{t_1} \le \frac{\delta}{2}t_2 + \frac{\delta}{2}t_1 < \delta t_2,$$

which is a contradiction. So a.s. $\limsup_{t\to\infty} U_t/t \le 0$. Similarly, a.s. $\liminf_{t\to\infty} U_t/t \ge 0$. Thus, $\lim_{t\to\infty} U_t/t = 0$. Thus, a.s. $\int_0^\infty e^{-4s}e^{\pm 2U_s}ds < \infty$, which implies that $\int_0^\infty e^{-4s}\cosh^2(U_s)ds < \infty$. From (4.5) we get a.s. $\tau < \infty$. This finishes the proof of (i).

Since $g_t : \mathbb{H}\setminus K_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, $\operatorname{dist}(z_0, \partial(\mathbb{H}\setminus K_t)) = \min\{\operatorname{Im} z_0, \operatorname{dist}(z_0,K_t)\}$ and $\operatorname{dist}(g_t(z_0),\partial\mathbb{H}) = \operatorname{Im} g_t(z_0)$, from Koebe's 1/4 theorem, we suffice to show that $\lim_{t\to\tau}|g_t'(z_0)|/Y_t \to \infty$ when $\kappa \ge 8$ and $\lim_{t\to\tau}|g_t'(z_0)|/Y_t < \infty$ when $\kappa < 8$. From chordal Loewner equation, we get $\partial_t g_t'(z_0) = \frac{-2g_t'(z_0)}{Z_t^2}$, which implies that $\partial_t \log|g_t'(z_0)| = \operatorname{Re}\frac{-2}{Z_t^2} = \frac{-2(X_t^2-Y_t^2)}{(X_t^2+Y_t^2)}$. Since $dY_t = \frac{-2Y_t}{X_t^2+Y_t^2}$, we get $\partial_t \log(|g_t'(z_0)|/Y_t) = \frac{4Y_t^2}{X_t^2+Y_t^2}$. Let $S = \int_0^\tau \frac{Y_s^2}{(X_s^2+Y_s^2)^2}ds$. We suffice to show that a.s. $S = \infty$ when $\kappa \ge 8$ and $S < \infty$ when $\kappa < 8$.

By changing variable we get

$$S = \int_0^\infty \frac{Y_{v(s)}^2}{X_{v(s)}^2 + Y_{v(s)}^2} ds = \int_0^\infty \frac{ds}{1 + W_{v(s)}^2} = \int_0^\infty \cosh^{-2}(U_s) ds.$$

If $\kappa < 8$, then a.s. $\lim_{t\to\infty} U_t/t = 4 - \frac{\kappa}{2}$ or $\lim_{t\to\infty} U_t/t = -(4 - \kappa/2)$. In either case we get $S < \infty$. If $\kappa \ge 8$, then U_t is a recurrent process, which implies that $S = \infty$. □

5 Locality and Restriction

5.1 Locality property

In this section, we will prove that SLE_6 satisfies locality property, and other SLE_κ satisfies weak locality property. The locality of SLE_6 means that the growth of SLE_6 does not feel the boundary before it hits it. We have the following theorem.

Theorem 5.1 *Suppose K_t, $0 \le t < \infty$, are standard chordal SLE_6 hulls. Let A be an $\mathbb{H}$-hull such that* $\text{dist}(0, A) > 0$. *Let T be the biggest time such that $K_t \cap A \neq \emptyset$ for $0 \le t < T$. Then after a time-change, K_t, $0 \le t < T$, has the same distribution as the chordal SLE_6 hulls in $\mathbb{H} \setminus A$ from 0 to ∞, stopped when touches A.*

Proof. Let $\lambda_t = \sqrt{\kappa} B_t$ be the driving function, and g_t be the chordal Loewner maps. We know that K_t, $0 \le t < \infty$, is a Loewner chain in $\mathbb{H}$. Then we easily see that K_t, $0 \le t < \infty$, is a Loewner chain in $\mathbb{H} \setminus A$. Let $W = g_A$ and $L_t = W(K_t)$, $0 \le t < T$. Then L_t, $0 \le t < T$, is a Loewner chain in $\mathbb{H}$, and each L_t is an $\mathbb{H}$-hull. Let $u(t) = \text{hcap}(L_t)/2$, $0 \le t < T$. Then u is continuous and increasing with $u(0) = 0$. Let $S = \sup u[0, T)$. Let $v = u^{-1}$. Then $L_{v(t)}$, $0 \le t < S$, is a Loewner chain in $\mathbb{H}$ with $\text{hcap}(L_{v(t)}) = 2t$ for $0 \le t < S$. Thus, $L_{v(t)}$, $0 \le t < S$, are chordal Loewner hulls driven by some $\eta \in C[0, S)$. We suffice to show that η_t, $0 \le t < S$, has the distribution as $W(0) + \sqrt{\kappa} B_t$ stopped at S. Let h_t be the chordal Loewner maps driven by η. Then $h_{u(t)} : \mathbb{H} \setminus L_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$.

For $0 \le t < T$, let $A_t = g_t(A)$ and

$$W_t = h_{u(t)} \circ W \circ g_t^{-1}.$$

Then $W_t : \mathbb{H} \setminus A_t \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$, and λ_t is bounded away from A_t. In fact, from the power series expansion of W_t at ∞, we see that $W_t = g_{A_t}$. From Schwarz reflection principle, we may extend W_t analytically across $\mathbb{R} \setminus \overline{A_t}$, and maps $\mathbb{R} \setminus \overline{A_t}$ into $\mathbb{R}$. We have $(t, z) \mapsto W_t(z)$ is continuous. Fix $t \in [0, T)$ and $s \in (0, T - t)$. we have

$$L_{t+s}/L_t = h_{u(t)}(L_{t+s} \setminus L_t) = W_t(g_t(K_{t+s} \setminus K_t)) = W_t(K_{t+s}/K_t).$$

Since $\text{hcap}(L_{t+s}/L_t) = 2u(t+s) - 2u(t)$ and $\text{hcap}(K_{t+s}/K_t) = 2s$, $\bigcap_{s>0} \overline{K_{t+s}/K_t} = \{\lambda_t\}$, and W_t is analytic at λ_t, we get $u'_+(t) = W'_t(\lambda_t)^2$, $0 \le t < T$. Since $W'_t(\lambda_t)$ is continuous in t, we have

$$u'(t) = W'_t(\lambda_t)^2, \quad 0 \le t < T. \tag{5.1}$$

Since

$$\{\lambda_t\} = \bigcap_{s>0} \overline{K_{t+s}/K_t}, \quad \{\eta_{u(t)}\} = \bigcap_{s>0} \overline{L_{t+s}/L_t},$$

we have

$$\eta_{u(t)} = W_t(\lambda_t), \quad 0 \le t < T. \tag{5.2}$$

From the definition of W_t, we get

$$W_t \circ g_t(z) = h_{u(t)} \circ W(z), \quad z \in \mathbb{H} \setminus (A \cup K_t).$$

Differentiate this equality w.r.t. t, and using (5.1) and (5.2) we get

$$\partial_t W_t(g_t(z)) + W_t'(g_t(z)) \frac{2}{g_t(z) - \lambda_t} = \frac{2W_t'(\lambda_t)^2}{h_{u(t)}(W(z)) - \eta_{u(t)}} = \frac{2W_t'(\lambda_t)^2}{W_t(g_t(z)) - W_t(\lambda_t)}.$$

Since g_t maps $\mathbb{H} \setminus (A \cup K_t)$ onto $\mathbb{H} \setminus A_t$, we conclude that

$$\partial_t W_t(w) = \frac{2W_t'(\lambda_t)^2}{W_t(w) - W_t(\lambda_t)} - \frac{2W_t'(w)}{w - \lambda_t}.$$

Let $a_j = W_t^{(j)}(\lambda_t)$, $j \in \mathbb{N}$. Let $\delta = w - \lambda_t$. Then as $\delta \to 0$,

$$\frac{2W_t'(\lambda_t)^2}{W_t(w) - W_t(\lambda_t)} - \frac{2W_t'(w)}{w - \lambda_t} = \frac{2a_1^2}{a_1\delta + \frac{a_2}{2}\delta^2 + O(\delta^3)} - \frac{2(a_1 + a_2\delta + O(\delta^2))}{\delta}$$

$$= \frac{2a_1}{\delta}(1 + \frac{a_2}{2a_1}\delta + O(\delta^2))^{-1} - \frac{2a_1}{\delta} - 2a_2 + O(\delta).$$

$$= \frac{2a_1}{\delta}(1 - \frac{a_2}{2a_1}\delta + O(\delta^2)) - \frac{2a_1}{\delta} - 2a_2 + O(\delta) = -3a_2 + O(\delta).$$

So we have

$$\partial_t W_t(\lambda_t) = -3W_t''(\lambda_t), \quad 0 \le t < T. \tag{5.3}$$

Since $\lambda_t = \sqrt{\kappa} B_t$ ($\kappa = 6$), applying Itô's formula to (5.2) we get

$$d\eta_{u(t)} = W_t'(\lambda_t) d\lambda_t + \Big(\frac{\kappa}{2} - 3\Big) W_t''(\lambda_t) dt, \quad 0 \le t < T. \tag{5.4}$$

From (5.1) we see that there is another Brownian motion $\widetilde{B}_t$ such that

$$d\eta_t = \sqrt{\kappa} d\widetilde{B}_t + \Big(\frac{\kappa}{2} - 3\Big) \frac{W_{v(t)}''(\lambda_{v(t)})}{W_{v(t)}'(\lambda_{v(t)})^2} dt, \quad 0 \le t < S.$$

If $\kappa = 6$, then η_t, $0 \le t < S$, has the same distribution as $\sqrt{\kappa} B_t$ stopped at S. So the proof is finished. $\square$

Remarks.

1. The locality property explains why the scaling limit of critical percolation is SLE_6.

2. Lawler, Schramm and Werner uses the locality of SLE_6 to compute the intersection exponent of planar Brownian motion.

3. In case $\kappa \neq 6$, from Girsanov theorem, we may find an increasing sequence stopping times (T_n) such that $T = \vee T_n$, and for each n, the distribution of K_t, $0 \le t \le T_n$, is equivalent to the distribution of a time-change of a chordal SLE_κ hulls in $\mathbb{H} \setminus A$ from 0 to ∞ stopped at some stopping time. We say that chordal SLE_κ satisfies weak locality for $\kappa \neq 6$.

4. The locality property for $\kappa = 6$ and weak locality property for $\kappa \neq 6$ are also satisfied by radial SLE. We leave this as an exercise.

5.2 Restriction property

In this subsection we will show that $\mathrm{SLE}_{8/3}$ satisfies restriction property. We have the following theorem.

Theorem 5.2 *Suppose K_t, $0 \le t < \infty$, are standard chordal $SLE_{8/3}$ hulls. Let A be an $\mathbb{H}$-hull such that* $\mathrm{dist}(0, A) > 0$. *Then conditioned on the event that $K_\infty := \bigcup K_t$ is disjoint from A, K_t, $0 \le t < \infty$, has the same distribution as the chordal $SLE_{8/3}$ hulls in $\mathbb{H} \setminus A$ from 0 to ∞.*

Proof. The initial part of the proof is the same as the proof of Theorem 5.1. Now we have derived

$$\partial_t W_t(w) = \frac{2W_t'(\lambda_t)^2}{W_t(w) - W_t(\lambda_t)} - \frac{2W_t'(w)}{w - \lambda_t}, \quad w \in \mathbb{H} \setminus A_t.$$

Differentiating this equality w.r.t. w, we get

$$\partial_t W_t'(w) = -\frac{2W_t'(\lambda_t)^2 W_t'(w)}{(W_t(w) - W_t(\lambda_t))^2} - \frac{2W_t''(w)}{w - \lambda_t} + \frac{2W_t'(w)}{(w - \lambda_t)^2}.$$

If $\delta = w - \lambda_t \to 0$, we have

$$-\frac{2W_t'(\lambda_t)^2 W_t'(w)}{(W_t(w) - W_t(\lambda_t))^2} - \frac{2W_t''(w)}{w - \lambda_t} + \frac{2W_t'(w)}{(w - \lambda_t)^2}$$

$$= -\frac{2a_1^2(a_1 + a_2\delta + \frac{a_3}{2}\delta^2 + O(\delta^3))^2}{(a_1\delta + \frac{a_2}{2}\delta^2 + \frac{a_3}{6}\delta^3 + O(\delta^4))^2} - \frac{2(a_2 + a_3\delta + O(\delta^2))}{\delta} + \frac{2(a_1 + a_2\delta + \frac{a_3}{2}\delta^2 + O(\delta^3))}{\delta^2}$$

$$= -\frac{2a_1}{\delta^2} \frac{1 + \frac{a_2}{a_1}\delta + \frac{1}{2}\frac{a_3}{a_1}\delta^2 + O(\delta^3)}{(1 + \frac{1}{2}\frac{a_2}{a_1}\delta + \frac{1}{6}\frac{a_3}{a_1}\delta^2 + O(\delta^3))^2} + \frac{2a_1}{\delta^2} - a_3 + O(\delta)$$

$$= -\frac{2a_1}{\delta^2} \frac{1 + \frac{a_2}{a_1}\delta + \frac{1}{2}\frac{a_3}{a_1}\delta^2 + O(\delta^3)}{1 + \frac{a_2}{a_1}\delta + (\frac{1}{4}\frac{a_2^2}{a_1^2} + \frac{1}{3}\frac{a_3}{a_1})\delta^2 + O(\delta^3)} + \frac{2a_1}{\delta^2} - a_3 + O(\delta)$$

$$= -\frac{2a_1}{\delta^2}(1 + (\frac{1}{6}\frac{a_3}{a_1} - \frac{1}{4}\frac{a_2^2}{a_1^2})\delta^2 + O(\delta^3)) + \frac{2a_1}{\delta^2} - a_3 + O(\delta) = \frac{1}{2}\frac{a_2^2}{a_1} - \frac{4}{3}a_3 + O(\delta).$$

Thus, we have

$$\frac{\partial_t W_t'(\lambda_t)}{W_t'(\lambda_t)} = \frac{1}{2}\Big(\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}\Big)^2 - \frac{4}{3}\frac{W_t'''(\lambda_t)}{W_t'(\lambda_t)}. \tag{5.5}$$

Since $\lambda_t = \sqrt{\kappa}B_t$, we find that $W_t'(\lambda_t)$ satisfies the SDE:

$$\frac{dW_t'(\lambda_t)}{W_t'(\lambda_t)} = \frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}d\lambda_t + \frac{1}{2}\Big(\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}\Big)^2 dt + \Big(\frac{\kappa}{2} - \frac{4}{3}\Big)\frac{W_t'''(\lambda_t)}{W_t'(\lambda_t)}dt. \tag{5.6}$$

Let $\alpha = \frac{6-\kappa}{2\kappa}$ and $\mathrm{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$. When $\kappa = \frac{8}{3}$, $\alpha = \frac{5}{8}$ and $\mathrm{c} = 0$. The c is known as the central charge of SLE_κ. Then

$$\begin{aligned}\frac{dW_t'(\lambda_t)^\alpha}{W_t'(\lambda_t)^\alpha} &= \alpha\frac{dW_t'(\lambda_t)}{W_t'(\lambda_t)} + \frac{\kappa}{2}\alpha(\alpha-1)\Big(\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}\Big)^2 dt \\ &= \alpha\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}d\lambda_t + \frac{\mathrm{c}}{6}\frac{W_t'''(\lambda_t)}{W_t'(\lambda_t)}dt - \frac{\mathrm{c}}{4}\Big(\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}\Big)^2 dt. \end{aligned} \tag{5.7}$$

If $\kappa = \frac{8}{3}$, then $W_t'(\lambda_t)^\alpha$ is a local martingale. Recall that $W_t = g_{A_t}$. Since

$$g_{A_t}^{-1}(z) - z = \int \frac{1}{x-z}d\mu_{A_t}(x),$$

we get

$$(g_{A_t}^{-1})'(z) = 1 + \int \frac{1}{(x-z)^2}d\mu_{A_t}(x).$$

Thus, for any $z \in \mathbb{R}\backslash[c_{A_t}, d_{A_t}]$, we have $(g_{A_t}^{-1})'(z) > 1$, which implies that $0 < W_t'(\lambda_t) < 1$. Thus, $W_t'(\lambda_t)^\alpha$ is a bounded martingale. Then $X := \lim_{t\to\infty} W_t'(\lambda_t)^\alpha$ exists a.s. and lies between 0 and 1. And we have $\mathbb{E}[X] = W_0'(\lambda_0)^\alpha = g_A'(0)^\alpha$. Now we define a new probability measure $\mathbb{P}_1$ such that $d\mathbb{P}_1/d\mathbb{P} = X/g_A'(0)^\alpha$. Let $D_t = \mathbb{E}[d\mathbb{P}_1/d\mathbb{P}|\mathcal{F}_t] = W_t'(\lambda_t)^\alpha/g_A'(0)^\alpha$. From (5.7) we see that, under $\mathbb{P}_1$, $\widetilde{B}_t = B_t - \alpha\sqrt{\kappa}\int_0^t \frac{W_s''(\lambda_s)}{W_s'(\lambda_s)}$ is a Brownian motion. We have

$$d\lambda_t = \sqrt{\kappa}dB_t = \sqrt{\kappa}d\widetilde{B}_t + \alpha\kappa\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}dt.$$

Formula (5.4) still holds here. So we get

$$d\eta_{u(t)} = W_t'(\lambda_t)\sqrt{\kappa}d\widetilde{B}_t.$$

From (5.1) we see that, under $\mathbb{P}_1$, there is a Brownian motion $\widehat{B}_t$ such that $d\eta_t = \sqrt{\kappa}d\widehat{B}_t$, $0 \le t < S$. This shows that, under $\mathbb{P}_1$, a time-change of $L_t = W(K_t)$, $0 \le t < T$, are partial chordal $\mathrm{SLE}_{8/3}$ hulls in $\mathbb{H}$ from $\eta_0 = W(\lambda_0)$ to ∞. Thus, under $\mathbb{P}_1$, after a time-change, K_t, $0 \le t < T$, are partial chordal $\mathrm{SLE}_{8/3}$ hulls in $\mathbb{H} \setminus A$ from 0 to ∞.

We now use the existence and properties of the chordal SLE_κ trace. We have a simple curve $\beta(t)$ such that $K_t = \beta(0,t]$ for $0 \le t < T$. Under $\mathbb{P}_1$, a time-change of $\beta(t)$, $0 \le t < T$, is a partial chordal $\text{SLE}_{8/3}$ trace in $\mathbb{H} \setminus A$ from 0 to ∞. If such trace does not finish its journey, then it ends at some interior point of $\mathbb{H} \setminus A$. From the definition of T, this is a $\mathbb{P}$-null event. So it is also a $\mathbb{P}_1$-null event. So the word "partial" can be removed.

Thus, modulo a time-change, the distribution of chordal $\text{SLE}_{8/3}$ process in $\mathbb{H} \setminus A$ from 0 to ∞ is absolutely continuous w.r.t. that of chordal $\text{SLE}_{8/3}$ process in $\mathbb{H}$ from 0 to ∞, and the Radon-Nikodym derivative is $X/\mathbb{E}[X]$. Since the trace in $\mathbb{H} \setminus A$ does not hit A, we have $\mathbb{P}_1[T < \infty] = 0$. Thus, $X = 0$ on $\{T < \infty\}$. We claim that $X = 1$ on $\{T = \infty\}$. If this is true, then $\mathbb{P}_1 = \mathbb{P}[\cdot | T = \infty] = \mathbb{P}[\cdot | K_\infty \cap A = \emptyset]$, and we are done.

Now we prove the claim in the case that $\overline{A} \cap \mathbb{R}$ lies to the right of 0. Suppose $T = \infty$, i.e., the whole trace β avoids A. As $t \to \infty$, $\beta(t) \to \infty$, so the extremal distance between $A \cup [a_A, b_A]$ and $(-\infty, 0]$ unions the "left side" of $\beta(0,t]$ in $\mathbb{H} \setminus \beta(0,t]$ tends to ∞, which implies that the extremal distance between $A_t \cup [a_{A_t}, b_{A_t}]$ and $(-\infty, \lambda_t]$ in $\mathbb{H}$ tends to ∞. This then implies that the extremal distance between $[c_{A_t}, d_{A_t}]$ and $(-\infty, g_{A_t}(\lambda_t)]$ in $\mathbb{H}$ tends to ∞ as $t \to \infty$. So we have $\frac{d_{A_t} - c_{A_t}}{c_{A_t} - g_{A_t}(\lambda_t)} \to 0$ as $t \to \infty$.

Recall that for any nonempty $\mathbb{H}$-hull K, $g_K : (\widehat{\mathbb{C}} \setminus \widehat{K}; \infty) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{C} \setminus [c_K, d_K]; \infty)$ and $g_K'(\infty) = 1$. So $\cap(\widehat{K}) = \cap([c_K, d_K]) = (d_K - c_K)/4$. Let $h(K)$ denote the height of K, then $2h(K) \le \text{diam}(\widehat{K}) \le 4 \cap(\widehat{K}) = d_K - c_K$. So $h(K) \le (d_K - c_K)/2$. If K is a bubble, then $\text{hcap}(K) \le \frac{h(K)}{\pi}(d_K - c_K) \le \frac{(d_K - c_K)^2}{2\pi}$. By approximation, this is true for any nonempty $\mathbb{H}$-hull.

Recall that $W_t = g_{A_t}$ and

$$(g_{A_t}^{-1})'(z) = 1 + \int_{c_{A_t}}^{d_{A_t}} \frac{1}{(z-x)^2} d\mu_{A_t}(x).$$

Let $z = g_{A_t}(\lambda_t)$. Since $|\mu_{A_t}| = \text{hcap}(A_t) \le \frac{(d_{A_t} - c_{A_t})^2}{2\pi}$ and $g_{A_t}(\lambda_t) < c_{A_t} < d_{A_t}$, we have

$$1 \le (g_{A_t}^{-1})'(g_{A_t}(\lambda_t)) \le 1 + \frac{1}{2\pi} \frac{(d_{A_t} - c_{A_t})^2}{(c_{A_t} - g_{A_t}(\lambda_t))^2}.$$

Since $\frac{d_{A_t} - c_{A_t}}{c_{A_t} - g_{A_t}(\lambda_t)} \to 0$, we get $W_t'(\lambda_t) \to 1$ as $t \to \infty$. So $X = 1$ on $\{T = \infty\}$.

So far, we prove the theorem in the case that $\inf(\overline{A} \cap \mathbb{R}) > 0$. Similarly, the result is true if $\sup(\overline{A} \cap \mathbb{R}) < 0$. If $\inf(\overline{A} \cap \mathbb{R}) < 0 < \sup(\overline{A} \cap \mathbb{R})$, we may divide A into the disjoint union of two $\mathbb{H}$-hulls A_+ and A_- such that $\sup(\overline{A_-} \cap \mathbb{R}) < 0$ and $\inf(\overline{A_+} \cap \mathbb{R}) > 0$. The result we obtained says that, if we condition a chordal $\text{SLE}_{8/3}$ trace in $\mathbb{H}$ from 0 to ∞ to avoid A_+, then we get a chordal $\text{SLE}_{8/3}$ trace in $\mathbb{H} \setminus A_+$ from 0 to ∞. If we further condition this trace to avoid A_-, then we get a chordal $\text{SLE}_{8/3}$ trace in $\mathbb{H} \setminus (A_+ \cup A_-) = \mathbb{H} \setminus A$ from 0 to ∞. Note that the combined effect of the two conditionings is a single conditioning: to avoid $A = A_+ \cup A_-$. So the proof is finished. $\square$

Remarks.

1. The restriction property is also satisfied by radial $\mathrm{SLE}_{8/3}$. In fact, if A is a $\mathbb{D}$-hull with $1 \notin \overline{A}$, then the probability that A is disjoint from a complete radial $\mathrm{SLE}_{8/3}$ trace is equal to $|g_A'(1)|^{5/8}|g_A'(0)|^{5/48}$. We leave this as an exercise.

2. For $\kappa \neq 8/3$, from (5.7) we may construct a local martingale M_t by
$$M_t = W_t'(\lambda_t)^\alpha \exp\Big(-\frac{\mathrm{c}}{6}\int_0^t SW_s(\lambda_s)ds\Big),$$
where $SW_s = W_s'''/W_s' - \frac{3}{2}(W_s''/W_s')^2$ is the Schwarzian derivative of W_s. Such M_t satisfies the SDE
$$\frac{dM_t}{M_t} = \alpha\frac{W_t''(\lambda_t)}{W_t'(\lambda_t)}d\lambda_t. \tag{5.8}$$
Recall that $W_s = g_{A_s}$. From the following lemma, we see that $SW_s(\lambda_s) \le 0$ for all s.

Lemma 5.1 *Let K be an $\mathbb{H}$-hull and $x \in \mathbb{R}\setminus[a_K, b_K]$. Then $Sg_K(x) \le 0$.*

Proof. We may assume that K is a bubble. We may find chordal Loewner hulls K_t, $0 \le t < T$, such that $K = K_{t_0}$ for some $t_0 \in [0, T)$. Let λ_t be the driving function. Let $x \in \mathbb{R}\setminus[a_K, b_K]$. Then $g_t(x)$ is well defined for $0 \le t \le t_0$. We have $\partial_t g_t(x) = \frac{2}{g_t(x)-\lambda_t}$, which implies that $\partial_t g_t'(x) = -\frac{2g_t'(x)}{(g_t(x)-\lambda_t)^2}$. Thus, $\partial_t \log g_t'(x) = -\frac{2}{(g_t(x)-\lambda_t)^2}$. This then implies that
$$\partial_t \frac{g_t''(x)}{g_t'(x)} = \partial_t\partial_x \log g_t'(x) = \partial_x\partial_t \log g_t'(x) = \frac{4g_t'(x)}{(g_t(x)-\lambda_t)^3}. \tag{5.9}$$
Thus,
$$\partial_t \frac{1}{2}\Big(\frac{g_t''(x)}{g_t'(x)}\Big)^2 = \frac{4g_t''(x)}{(g_t(x)-\lambda_t)^3}.$$
Differentiating (5.9) w.r.t. x, we get
$$\partial_t\Big(\frac{g_t'''(x)}{g_t'(x)} - \Big(\frac{g_t''(x)}{g_t'(x)}\Big)^2\Big) = \frac{4g_t''(x)}{(g_t(x)-\lambda_t)^3} - \frac{12g_t'(x)^2}{(g_t(x)-\lambda_t)^4}.$$
Combining the above two displayed formulas, we get
$$\partial_t Sg_t(x) = -\frac{12g_t'(x)^2}{(g_t(x)-\lambda_t)^4} \le 0.$$
Since $g_0 = \mathrm{id}$, $Sg_0(x) = 0$. So we get $Sg_{t_0}(x) \le 0$. □

Remark. If $\kappa < 8/3$, then $\mathrm{c} < 0$. So $-\frac{\mathrm{c}}{6}\int_0^t SW_s(\lambda_s)ds \le 0$. This means that $0 \le M_t \le 1$ and a.s. $X := \lim_{t\to\infty} M_t$ exists and $0 \le X \le 1$. If we define a new probability distribution $\mathbb{P}_1$ by $d\mathbb{P}_1/d\mathbb{P} = X/\mathbb{E}[X]$, then from (5.8) and Girsanov theorem, we see that, under $\mathbb{P}_1$, after a time-change, K_t are chordal SLE$_\kappa$ hulls in $\mathbb{H}\setminus A$ from 0 to ∞. Thus, for $\kappa<8/3$, modulo a time-change, the distribution of chordal SLE$_\kappa$ hulls in $\mathbb{H}\setminus A$ from 0 to ∞ is absolutely continuous w.r.t. that of chordal SLE$_\kappa$ hulls in $\mathbb{H}$ from 0 to ∞, and the Radon-Nikodym derivative is $X/\mathbb{E}[X]$. A similar argument as before shows that

$$X=\mathbf{1}_{\{K_\infty\cap A=\emptyset\}}\exp\Big(-\frac{\mathrm{c}}{6}\int_0^\infty SW_s(\lambda_s)ds\Big).$$

Lawler and Werner proved that the quantity $-\frac16\int_0^\infty SW_s(\lambda_s)ds$ can be characterized by the Brownian loop measure of the set of loops in $\mathbb{H}$ that intersect both K_∞ and A, and the quantity $\exp\Big(-\frac{\mathrm{c}}{6}\int_0^\infty SW_s(\lambda_s)ds\Big)$ can be described by the probability that, in a Brownian loop soup of density $-\mathrm{c}$ in $\mathbb{H}$ (a Poisson point process of Brownian loop measure), there exist no loops that intersect both K_∞ and A. If we attach all loops in a Brownian loops soup of density $-\mathrm{c}$ in $\mathbb{H}$ that intersect K_∞ to K_∞, we get a fat set, say F. If we condition that F avoids A, then K_t, $0\le t<\infty$, has the distribution of chordal SLE$_\kappa$ hulls in $\mathbb{H}\setminus A$ from 0 to ∞, after a time-change.

5.3 Equivalence between chordal SLE and radial SLE

Theorem 5.3 *Let K_t, $0\le t<\infty$, be standard radial SLE$_6$ hulls. Let $w_0\in\mathbb{T}\setminus\{1\}$. Let T be the biggest number such that $w_0\not\in\overline{K_t}$ for $0\le t<T$. After a time-change, K_t, $0\le t<T$, has the same distribution as chordal SLE$_\kappa$ hulls in $\mathbb{D}$ from 1 to w_0, stopped at some stopping time.*

Proof. Let $\kappa=6$. Let $\lambda_t=\sqrt{\kappa}B_t$ be the driving function for K_t, let Let g_t and $\widetilde{g}_t$ be the radial Loewner maps and covering radial Loewner maps. Let $W:(\mathbb{D};1,w_0)\overset{\mathrm{Conf}}{\twoheadrightarrow}(\mathbb{H};0,\infty)$. Let $L_t=W(K_t)$. Then L_t, $0\le t<T$, is a Leowner chain in $\mathbb{H}$ such that each L_t is an $\mathbb{H}$-hull. Let $u(t)=\mathrm{hcap}(L_t)/2$, $0\le t<T$. Then u is continuous and increasing with $u(0)=0$. Let $S=\sup u[0,T)$. Let $v=u^{-1}$. Then $L_{v(t)}$, $0\le t<S$, is a Loewner chain in $\mathbb{H}$ with $\mathrm{hcap}(L_{v(t)})=2t$ for $0\le t<S$. Thus, $L_{v(t)}$, $0\le t<S$, are chordal Loewner hulls driven by some $\eta\in C[0,S)$. We suffice to show that η_t, $0\le t<S$, has the distribution as $W(1)+\sqrt{\kappa}B_t$ stopped at S. Let h_t be the chordal Loewner maps driven by η. Then $h_{u(t)}:\mathbb{H}\setminus L_t\overset{\mathrm{Conf}}{\twoheadrightarrow}\mathbb{H}$.

For $0\le t<T$, let $W_t=h_{u(t)}\circ W\circ g_t^{-1}$. Then $W_t:\mathbb{D}\overset{\mathrm{Conf}}{\twoheadrightarrow}\mathbb{H}$. Fix $t\in[0,T)$ and $s\in(0,T-t)$. we have $L_{t+s}/L_t=W_t(K_{t+s}/K_t)$. Since $\mathrm{hcap}(L_{t+s}/L_t)=2u(t+s)-2u(t)$ and $\mathrm{dcap}(K_{t+s}/K_t)=s$, $\bigcap_{s>0}\overline{K_{t+s}/K_t}=\{e^{i\lambda_t}\}$, and W_t is analytic at λ_t, we get $u'(t)=|W_t'(e^{i\lambda_t})|^2$. Let $\widetilde{W}=W\circ e^i$ and $\widetilde{W}_t=W_t\circ e^i=h_{u(t)}\circ\widetilde{W}\circ\widetilde{g}_t^{-1}$. So we have

$$u'(t)=\widetilde{W}_t'(\lambda_t)^2. \tag{5.10}$$

From $\bigcap_{s>0}\overline{K_{t+s}/K_t}=\{e^{i\lambda_t}\}$ and $\bigcap_{s>0}\overline{L_{t+s}/L_t}=\{\eta_{u(t)}\}$ we get

$$\eta_{u(t)}=\widetilde{W}_t(\lambda_t). \tag{5.11}$$

We have

$$\widetilde{W}_t\circ\widetilde{g}_t(z)=h_{u(t)}\circ\widetilde{W}(z),\quad z\in\mathbb{H}\setminus(e^i)^{-1}(K_t).$$

Differentiate this equality w.r.t. t, and using (5.10) and (5.11) we get

$$\partial_t \widetilde{W}_t(\widetilde{g}_t(z)) + \widetilde{W}_t'(\widetilde{g}_t(z)) \cot_2(\widetilde{g}_t(z) - \lambda_t) = \frac{2\widetilde{W}_t'(\lambda_t)^2}{h_{u(t)}(\widetilde{W}(z)) - \eta_{u(t)}} = \frac{2\widetilde{W}_t'(\lambda_t)^2}{\widetilde{W}_t(\widetilde{g}_t(z)) - \widetilde{W}_t(\lambda_t)}.$$

We conclude that

$$\partial_t \widetilde{W}_t(w) = \frac{2\widetilde{W}_t'(\lambda_t)^2}{\widetilde{W}_t(w) - \widetilde{W}_t(\lambda_t)} - \widetilde{W}_t'(w) \cot_2(w - \lambda_t).$$

Letting $w \to \lambda_t$, we find that

$$\partial_t \widetilde{W}_t(\lambda_t) = -3\widetilde{W}_t''(\lambda_t), \quad 0 \le t < T. \tag{5.12}$$

Since $\lambda_t = \sqrt{\kappa} B_t$, applying Itô's formula to (5.11) we get

$$d\eta_{u(t)} = \widetilde{W}_t'(\lambda_t) d\lambda_t + \Big(\frac{\kappa}{2} - 3\Big) \widetilde{W}_t''(\lambda_t) dt, \quad 0 \le t < T. \tag{5.13}$$

From (5.1) we see that there is another Brownian motion $\widetilde{B}_t$ such that

$$d\eta_t = \sqrt{\kappa} d\widetilde{B}_t + \Big(\frac{\kappa}{2} - 3\Big) \frac{\widetilde{W}_{v(t)}''(\lambda_{v(t)})}{\widetilde{W}_{v(t)}'(\lambda_{v(t)})^2} dt, \quad 0 \le t < S.$$

If $\kappa = 6$, then η_t, $0 \le t < S$, has the same distribution as $\sqrt{\kappa} B_t$ stopped at S. So the proof is finished. □

5.4 Critical percolation and Cardy's formula

Smirnov proved that the critical site percolation on a triangular lattice contains an explorer curve which converges to SLE_6. The critical site percolation on a triangular lattice is equivalent to the critical face percolation on a hexagonal lattice. We consider a simply connected domain D. Use a hexagon lattice with small mesh to approximate D. Color all hexagon faces contained in D independently yellow or green with equal probability. Mark two points a, b on ∂D, which divide ∂D into two arcs. We assign a boundary condition to this percolation by adding a coat of hexagon faces to the above percolation, and coloring these faces such that the faces on one arc are all green and the faces on the other arc are all yellow. Then we can observe an interface curve connecting the two marked points.

Before Smirnov's work, statistical physicists observed that the explorer curve has a scaling limit when the mesh of the lattice tends to 0; and the scaling limit is invariant under conformal maps. Moreover, from the construction, the explorer curve satisfies the Domain Markov Property at the discrete level. So the scaling limit, if exists, has to be SLE with some parameter. Also note that the explorer curve does not feel the boundary before hitting it, its scaling limit must satisfies the locality property. This implies that the scaling limit should be SLE_6.

Note that the time-reversal of the explorer curve is still an explorer curve. Thus, the convergence implies that chordal SLE_6 satisfies reversibility, which means that, if $\beta(t)$, $0\le t\le\infty$, is a chordal SLE_6 trace in D from a to b, then there is a continuously decreasing function u, which maps $[0,\infty]$ onto $[0,\infty]$, such that $\beta(u(t))$, $0\le t\le\infty$, is a chordal SLE_6 trace in D from b to a.

Smirnov proved the convergence of the explorer curve by showing that Cardy's formula holds true. Cardy's formula says that, if D is a simply connected domain with four boundary points a,b,c,d lie in the ccw direction. Then the probability that there is a yellow path connecting the arc ab and the arc cd in the critical percolation on a hexagonal lattice that approximates D has a limit as the mesh tends to 0, and the limit probability depends only on the conformal type of $(D;a,b,c,d)$. It has a simple expression when D is an equilateral triangle with three vertices a,b,c. In that case, the limit probability is $|cd|/|ac|$.

We now explain the Cardy's formula by showing that chordal SLE_6 satisfies Cardy's formula. We color the faces on the arc abc yellow, and color the faces on the arc cda green. Then we study the explorer curve from a to c. If there is a yellow crossing connecting ab with cd, then the explorer curve visits cd before bc. If there is a green crossing connecting da with bc, then the explorer curve visits bc before cd. Since the explorer curve converges to chordal SLE_6 in D from a to c, the limit probability of the existence of a yellow crossing connecting ab with cd is equal to the probability that a $\mathrm{SLE}_6(D;a\to c)$ trace visits cd before bc. From conformal invariance, we may assume that $D=\mathbb{H}$, $a=0$, $c=\infty$, $b>0$, and $d<0$. The time that the trace visits $bc=(b,\infty)$ is the time that $g_t(b)$ blows up. The time that the trace visits $cd=(-\infty,d)$ is the time that $g_t(d)$ blows up. All we need is to compute $\mathbb{P}[\tau_d<\tau_b]$.

Let $\kappa=6$ and $\lambda_t=\sqrt{\kappa}B_t$ be the driving function, and g_t be the chordal Loewner maps. Since $\kappa>4$, $\tau_b,\tau_d<\infty$. Let $U_t=g_t(b)-\lambda_t$, $0\le t<\tau_b$; and $V_t=g_t(d)-\lambda_t$, $0\le t<\tau_d$. Then U_t stays positive and tends to 0^+ as $t\to\tau_b$, and V_t stays negative and tends to 0^- as $t\to\tau_d$. Since

$$\partial_t(U_t-V_t)=\partial_t g_t(b)-\partial_t g_t(d)=\frac{2}{U_t}-\frac{2}{V_t}>0,$$

we have $U_t-V_t\ge U_0-V_0=b-d>0$ for $0\le t<\tau$. Thus, it is not possible that $\tau_b=\tau_d$. Let $\tau=\tau_b\wedge\tau_d$ and $W_t=V_t/U_t$, $0\le t<\tau$. Then W_t stays negative. If $\tau_b<\tau_d$, then $\lim_{t\to\tau}W_t=-\infty$. If $\tau_d<\tau_b$, then $\lim_{t\to\tau}W_t=0$. Since U_t and V_t satisfy $dU_t=-\sqrt{\kappa}dB_t+\frac{2}{U_t}dt$ and $dV_t=-\sqrt{\kappa}dB_t+\frac{2}{V_t}dt$. We find that W_t satisfies

$$dW_t=\frac{V_t\sqrt{\kappa}}{U_t^2}dB_t-\frac{\sqrt{\kappa}}{U_t}dB_t+\frac{2}{V_tU_t}dt+\frac{(\kappa-2)V_t}{U_t^3}dt-\frac{\kappa}{U_t^2}dt,\quad 0\le t<\tau.$$

Let $u(t)=\int_0^t(\frac{1}{U_s})^2ds$ and $T=\sup u[0,\tau)$. Let $v(t)$, $0\le t<T$, be the inverse of $u(t)$, $0\le t<T$. Then $Z_t:=W_{v(t)}$ satisfies the SDE

$$dZ_t=(Z_t-1)\sqrt{\kappa}d\widetilde{B}_t+(2/Z_t+(\kappa-2)Z_t-\kappa)dt,\quad 0\le t<T.$$

We now find f defined on $(-\infty,0)$ such that $f(Z_t)$ is a local martingale. We need that

$$\frac{\kappa}{2}f''(x)(x-1)^2+f'(x)(\frac{2}{x}+(\kappa-2)x-\kappa)=0.$$

We find $\frac{f''(x)}{f'(x)} = \frac{8/\kappa-2}{x-1} + \frac{-4/\kappa}{x}$. So $f'(x) = C|x|^{-4/\kappa}(1-x)^{8/\kappa-2}$. Note that when x is close to 0^-, $f'(x) \sim |x|^{-4/\kappa}$ and $-4/\kappa > -1$; when x is close to $-\infty$, $f'(x) \sim |x|^{4/\kappa-2}$ and $4/\kappa - 2 < -1$. Thus, f maps $(-\infty, 0)$ onto a bounded interval. So $f(Z_t)$ is a bounded martingale.

We may choose f such that f is increasing and $f((-\infty,0)) = (0,1)$. If $\tau_b < \tau_d$, then $\lim_{t\to\tau} W_t = -\infty$, which implies that $\lim_{t\to T} f(Z_t) = 0$; if $\tau_d < \tau_b$, then $\lim_{t\to\tau} W_t = 0$, which implies that $\lim_{t\to T} f(Z_t) = 1$. Thus,

$$f(d/b) = f(Z_0) = \mathbb{E}[\lim_{t\to T} f(Z_t)] = \mathbb{P}[\tau_d < \tau_b].$$

So we have

$$\mathbb{P}[\tau_d < \tau_b] = \frac{\int_{-\infty}^{d/b} |x|^{-4/\kappa}(1-x)^{8/\kappa-2}dx}{\int_{-\infty}^{0} |x|^{-4/\kappa}(1-x)^{8/\kappa-2}dx}.$$

Now we give an geometric explanation. Recall that $\frac{f''(x)}{f'(x)} = \frac{8/\kappa-2}{x-1} + \frac{-4/\kappa}{x}$. Let $g(x) = f(x/b)$. Then g maps $(-\infty, 0)$ onto $(0,1)$, and satisfies $\frac{g''(x)}{g'(x)} = \frac{8/\kappa-2}{x-b} + \frac{-4/\kappa}{x}$. Moreover, we have $\mathbb{P}[\tau_d < \tau_b] = g(d)$. Now suppose h maps $\mathbb{H}$ conformally onto the interior of ΔABC with angles $p_A\pi, p_B\pi, p_C\pi$ such that $h(a) = h(0) = A$, $h(b) = B$, and $h(c) = h(\infty) = C$. From the SchwarzChristoffel mapping theorem, h satisfies $\frac{h''(z)}{h'(z)} = \frac{p_A-1}{z} + \frac{p_B-1}{z-b}$. If $p_A = 1 - 4/\kappa$ and $p_B = 8/\kappa - 1$ ($p_C = 1 - 4/\kappa = p_A$), then $\frac{h''}{h'} = \frac{g''}{g'}$ on $(-\infty, 0)$. Thus, there are $\alpha, \beta \in \mathbb{C}$ such that $h = \alpha g + \beta$. Let $D = h(d) \in [A, C]$. Then

$$\frac{|DC|}{|AC|} = \frac{D-C}{A-C} = \frac{h(d)-h(c)}{h(a)-h(c)} = \frac{g(d)-g(c)}{g(a)-g(c)} = g(d) = \mathbb{P}[\tau_d < \tau_b].$$

Finally, note that when $\kappa = 6$, ΔABC is an equilateral triangle.

Another percolation model that is expected to converge to SLE_6 is the critical bond percolation on square lattices. Let D be a simply connected domain. We use a subgraph G of $\delta\mathbb{Z}^2$ to approximate D, where $\delta > 0$ is small. We also look at the dual graph $G^\dagger$, which is a subgraph of $\delta(\mathbb{Z}+1/2)^2$. Every edge of G intersects an edge of $G^\dagger$, and vice versa. Let P denote a random subgraph of G such that P contains all vertices of G and every edge of G is contained in P with probability $1/2$ independent of each other. We may then construct a dual graph $P^\dagger$ such that an edge of G is contained in P if and only if its dual edge is not contained in $P^\dagger$. Now we mark two points a, b on ∂D, which divide ∂D into two arcs, say I_1 and I_2. Assign boundary conditions by adding all edges in $\delta\mathbb{Z}^2$ near I_1 to P, and adding all edges in $\delta(\mathbb{Z}+1/2)^2$ near I_2 to $P^\dagger$. Then there is an explorer curve connecting a and b. This curve is conjectured to converge to SLE_6. The conjecture is based on Computer simulation, the Domain Markov Property and the locality property. Smirnov's work can not be easily extended to this model because his proof essentially depends on the structure of the triangle lattice.

5.5 Self-avoiding walk and reversibility of $\text{SLE}_{8/3}$

In this subsection we talk about the scaling limits of self-avoiding walk (SAW). Most of the statements here are still conjectures. There are two meanings of SAW. The first meaning of

SAW is a simple lattice path $(X_0,\dots,X_n)$. We will focus on square lattice $\mathbb{Z}^2$ or $\delta\mathbb{Z}^2$. The points X_k are vertices. We have $X_{k-1}\sim X_k$, $1\le k\le n$; and $X_j\ne X_k$ if $j\ne k$. The number n is called the length of this path. The second meaning of SAW is a positive measure on the space of simple lattice paths.

We first consider SAW started from 0. Let C_n denote the number of SAW on $\mathbb{Z}^2$ of length n started from 0. For example, we have $C_0=1$, $C_1=4$, $C_2=12$, $C_3=36$, $C_4=100$. One may easily see that $C_{n+m}\le C_nC_m$. This implies that $\lim_{n\to\infty}\frac{1}{n}\log(C_n)$ exists. The limit β is estimated to be 2.628..., which depends on the lattice. It is conjectured that

$$C_n\sim cn^{\gamma-1}\beta^n,$$

where γ is a critical exponent independent of the lattice. It is predicted by Nienhuis that $\gamma=43/32$.

Now we define $\nu_{\rm SAW}$ to be a measure on the space of simple lattice paths on $\delta\mathbb{Z}^2$ such that each path is assigned a measure β^{-n}, where n is the length of the path. Suppose D is a simply connected domain with two boundary points z_0 and w_0. Let D^δ be an approximation of D by a subgraph of $\delta\mathbb{Z}^2$. Let z_0^δ and w_0^δ be two vertices closest to z_0 and w_0, respectively. Consider the set of all SAW connecting z_0 with w_0, which stay inside D. Let $\Gamma(D,z_0,w_0,\delta)$ denote the set of these SAW. It is conjectured that for some constant $b>0$,

$$\mu_{\rm SAW}[\Gamma(D,z_0,w_0,\delta)]\sim\delta^{-2b},$$

as $\delta\to0$. Define the probability measure $\mu^\#_{{\rm SAW},\delta}$ to be the restriction of $\mu_{\rm SAW}$ to $\Gamma(D,z_0,w_0,\delta)$ divided by the mass. It is conjectured that $\mu^\#_{\rm SAW}$ has a conformal invariant scaling limit. Note that SAW satisfies Domain Markov Property and restriction property, so the limit should be chordal $\mathrm{SLE}_{8/3}$. There is a similar conjecture about the convergence of SAW to radial $\mathrm{SLE}_{8/3}$, where z_0 is an interior point, w_0 is still a boundary point, and $\mu_{\rm SAW}[\Gamma(D,z_0,w_0,\delta)]\sim\delta^{-(a+b)}$, for some positive constants $a,b>0$.

If the convergence of SAW to $\mathrm{SLE}_{8/3}$ is true, then we immediately have the reversibility of $\mathrm{SLE}_{8/3}$. In fact, we may prove the reversibility using the restriction property. We only need to show that, if β is a chordal $\mathrm{SLE}_{8/3}$ trace in $\mathbb{H}$ from 0 to ∞, and if $W(z)=-1/z$, then the image of β has the same distribution as the image of $W(\beta)$. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ denote the distributions of the image of β and $W(\beta)$, respectively. Let S denote the set of all simple curves, which connect 0 and ∞, and stay inside $\mathbb{H}$ except for the two endpoints. Let $\mathcal{F}_S$ denote the σ-algebra on S generated by the sets $\{\beta\in S:\beta\cap F=\emptyset\}$, where F could be any relatively closed subset of $\mathbb{H}$. We need to show that $\mathbb{P}_1=\mathbb{P}_2$ on $\mathcal{F}_S$.

Let $\mathcal{A}'$ denote the family $\{\beta\in S:\beta\cap A=\emptyset\}$, where A is any $\mathbb{H}$-hull bounded away from 0. Let $\mathcal{A}=\mathcal{A}'\cup\{\emptyset\}$. First, we show that $\mathcal{A}$ is a π-system, which means that it is closed under intersection. Suppose A_1 and A_2 are two $\mathbb{H}$-hulls bounded away from 0 such that $\{\beta\in S:\beta\cap A_1=\emptyset\}\cap\{\beta\in S:\beta\cap A_2=\emptyset\}\ne\emptyset$. Then there is $\beta\in S$ disjoint from A_1 and A_2, which implies that the unbounded component of $\mathbb{H}\setminus(A_1\cup A_2)$, say H, contains a neighborhood of 0. Let $A=\mathbb{H}\setminus H$. Then A is an $\mathbb{H}$-hull bounded away from 0, and

$$\{\beta\in S:\beta\cap A_1=\emptyset\}\cap\{\beta\in S:\beta\cap A_2=\emptyset\}=\{\beta\in S:\beta\cap A=\emptyset\}.$$

So $\mathcal{A}$ is a π-system.

Second, we show that $\mathcal{F}_S$ is the σ-algebra generated by $\mathcal{A}$. First, it is clear that $\mathcal{A} \subset \mathcal{F}_S$. We suffice to show that, for every relatively closed subset F of $\mathbb{H}$, $\{\beta \in S : \beta \cap F = \emptyset\}$ can be expressed as a union of countably many elements in $\mathcal{A}$. Let $\mathcal{A}^*_+$ (resp. $\mathcal{A}^*_-$) denote the family of bubbles bounded by polygonal crosscuts in $\mathbb{H}$ with the following properties: (i) every line segment is parallel to either x or y axis; (ii) every vertex has rational coordinates; (iii) the two points on $\mathbb{R}$ are positive. (resp. negative). Let $\mathcal{A}^*$ denote the family of sets $A_+ \cup A_-$, where $A_\pm \in \mathcal{A}^*_\pm$ and $A_+ \cap A_- = \emptyset$. Then $\mathcal{A}^*$ is a countable set. Let F be a relatively closed subset of $\mathbb{H}$. Let $\mathcal{A}^*_F$ denote the set of all $A \in \mathcal{A}^*$ which contain F. We claim that

$$\{\beta \in S : \beta \cap F = \emptyset\} = \bigcup_{A \in \mathcal{A}^*_F} \{\beta \in S : \beta \cap A = \emptyset\}. \tag{5.14}$$

It is clear that the set on the right is contained in the set on the left. Now suppose β is contained in the set on the left. We may easily find $A \in \mathcal{A}^*$ such that $F \subset A$ and $A \cap \beta = \emptyset$. This means that $A \in \mathcal{A}^*_F$ and $\beta \in \{\beta \in S : \beta \cap A = \emptyset\}$. So we proved (5.14).

From Dynkin's $\pi - \lambda$ theorem, if $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{A}$, then $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{F}_S$. Let $A \in \mathcal{A}$. Then $\mathbb{P}_1[\beta \cap A = \emptyset] = g'_A(0)^{5/8}$ and $\mathbb{P}_2[\beta \cap A = \emptyset] = \mathbb{P}_1[\beta \cap W(A) = \emptyset] = g'_{W(A)}(0)^{5/8}$. Note that $W(A) \in \mathcal{A}$ and $g_{W(A)}(z) = -\frac{g'_A(0)}{g_A(W(z)) - g_A(0)} + C$ for some $C \in \mathbb{R}$. Then we have $g'_{W(A)}(0) = g'_A(0)$. Thus, $\mathbb{P}_1[\beta \cap A = \emptyset] = \mathbb{P}_2[\beta \cap A = \emptyset]$, which finishes the proof.

6 Loop-erased Random Walk and Uniform Spanning Tree

6.1 Simple random walk

Let $G = (V, E)$ be a finite connected graph without self-loops and multiple edges. For a function $f : V \to \mathbb{R}$ and any $v_0 \in V$, the discrete Laplacian of f at v_0 is defined by

$$\Delta f(v_0) = \sum_{v \sim v_0} (f(v) - f(v_0)).$$

If $\Delta f(v_0) = 0$, we say that f is harmonic at v_0. Since

$$0 = \sum_{v \sim w} (f(v) - f(w)) + (f(w) - f(v)) = \sum_{v \in V} \sum_{w \in V : w \sim v} (f(w) - f(v)),$$

we have $\sum_{v \in V} \Delta f(v) = 0$. Thus, if f is harmonic on $A \subset V$, then $\sum_{v \in V \setminus A} \Delta f(v) = 0$.

Let $v_0 \in V$. A random walk on G started from v_0 is a sequence of random vertices $(X_n)_{n=0}^\infty$ such that $X_0 = 0$ and

$$\mathbb{P}[X_{n+1} = v | X_0, \dots, X_n] = \frac{\mathbf{1}_{v \sim X_n}}{\deg(X_n)}.$$

We use $\mathbb{P}^{v_0}$ and $\mathbb{E}^{v_0}$ to denote the probability and expectation w.r.t. a random walk started from v_0. Let $A \subset V$ be nonempty. Let τ_A be the first n such that $X_n \in A$. Then τ is a stopping time and for any $v \in V$, $\mathbb{P}^v$-a.s. $\tau_A < \infty$. We call the finite random path X_n, $0 \le n \le \tau$, the random walk on G from v_0 to A, and let it be denoted by $\mathrm{RW}(v_0 \to A)$. We use $\mathbb{P}^{v_0 \to A}$ and $\mathbb{E}^{v_0 \to A}$ to denote the probability and expectation w.r.t. this stopped random walk.

If f is harmonic on $V \setminus A$, and X_n, $0 \le n \le \tau_A$, is $\mathrm{RW}(v_0 \to A)$, then $f(X_n)$, $0 \le n \le \tau_A$, is a (discrete) martingale. This means that, for any n,

$$\mathbb{E}[\mathbf{1}_{\tau_A > n} f(X_{n+1}) | X_0, \dots, X_n] = \mathbf{1}_{\tau_A > n} f(X_n).$$

This is true because $\tau_A > n$ implies that $X_n \in V \setminus A$ and $\Delta f(X_n) = 0$. So

$$\mathbb{E}[\mathbf{1}_{\tau_A > n} f(X_{n+1}) | X_0, \dots, X_n] = \mathbf{1}_{\tau_A > n} \sum_{v \sim X_n} \frac{1}{\deg(X_n)} f(v) = \mathbf{1}_{\tau_A > n} f(X_n).$$

Thus, for every $v \in V$,

$$f(v) = \mathbb{E}^v[f(X_{\tau_A})] = \sum_{w \in V_\partial} f(w) \mathbb{P}^v[X_{\tau_A} = w]. \tag{6.1}$$

This means that, given a function g on A, there exists a unique f on V, which agrees with g on A, and is harmonic on $V \setminus A$.

Let $A, B \subset V$ be such that $A \cap B = \emptyset$ and $A \cup B \ne \emptyset$. Let $h_{A|B}$ denote the unique function which equals 1 on A, equals 0 on B, and is harmonic on $V \setminus (A \cup B)$. This is called a discrete harmonic measure function. In fact, we have $h_{A|B} v) = \mathbb{P}^v[X_{\tau_{A\cup B}} = A]$. So the values of $h_{A|B}$ lie between 0 and 1. Moreover, we have $h_{B|A} = 1 - h_{A|B}$. Let $G(A,B) = \sum_{v \in B} \Delta h_{A|B}(v)$. Since $h_{A|B}$ is harmonic on $V \setminus (A \cup B)$, we have $G(A,B) = -\sum_{v \in A} \Delta h_{A|B}(v)$. Since $h_{B|A} = 1 - h_{A|B}$, we have

$$G(B,A) = \sum_{v \in A} \Delta h_{B|A}(x) = -\sum_{v \in A} \Delta h_{A|B}(x) = G(A,B).$$

Such $G(A,B)$ is called the electrical conductance between A and B. It is clear that $G(A,B) = 0$ if either A or B is empty. On the other hand, if both A and B are nonempty, then $G(A,B) > 0$. In fact, there is a path $(Z_0, \dots, Z_n)$ with $Z_0 \in A$, $Z_n \in B$, and $Z_k \in V \setminus (A \cup B)$ for $1 \le k \le n-1$. So $h_{A|B}(Z_1) = \mathbb{P}^{Z_1}[X_{\tau_{A\cup B}} \in A] > 0$, which implies that $G(A,B) \ge \Delta h_{A|B}(Z_0) \ge Z_1 - Z_0 > 0$.

Suppose $\mathbb{P}^{v_0}[X_{\tau_{A\cup B}} \in A] = h_{A|B}(v_0) > 0$. The $\mathrm{RW}(v_0 \to A \cup B)$ conditioned on the event $\{X_{\tau_{A\cup B}} \in A\}$ is called the random walk on G from v_0 to $A \cup B$ conditioned to end at A, and is denoted by $\mathrm{RW}(v_0 \to A|B)$. We use $\mathbb{P}^{v_0 \to A|B}$ and $\mathbb{E}^{v_0 \to A|B}$ to denote the probability and expectation w.r.t. this conditional stopped random walk.

6.2 Loop-erased random walk

Let $X = (X_k)_{k=0}^{\nu}$ be a finite lattice path. The loop-erasure of X is defied as follows. Let $j = 0$ and $n_0 = \max\{m : X_m = X_0\}$. Define the sequence (n_j) inductively by $n_{j+1} = \max\{m : X_m = X_{n_j+1}\}$ if n_j is defined and $n_j < \nu$. Let τ be the first j such that $n_j = n$. Let $Y_j = X_{n_j}$, $0 \le j \le \tau$. Then $Y = (Y_j)_{j=0}^{\tau}$ is a path because $Y_{j+1} = X_{n_{j+1}} = X_{n_j+1} \sim X_{n_j} = Y_j$. From the definition of n_j, we see that $X_n \ne X_{n_j}$ if $n > n_j$. Thus, $\{X_n : n > n_j\} \cap \{Y_0, \dots, Y_j\} = \emptyset$. Since $\{Y_{j+1}, \dots, Y_\tau\} \subset \{X_n : n > n_j\}$, we have $\{Y_0, \dots, Y_j\} \cap \{Y_{j+1}, \dots, Y_\tau\} = \emptyset$. So Y is a simple path. We call Y the loop-erasure of X, or $Y = LE(X)$.

If two paths $X = (X_0, \dots, X_n)$ and $Y = (Y_0, \dots, Y_m)$ satisfy $X_n = Y_0$, then we define $Z = XY$ to be a new path $Z = (X_0, \dots, X_n = Y_0, \dots, Y_m)$, and we write $X \prec Z$.

Lemma 6.1 *Let $X = (X_j)_{j=0}^{\nu}$ and $Z = (Z_j)_{j=0}^{m}$ be two paths. Then $Z \prec LE(X)$ if and only if there are paths $X^{(1)}$ and $X^{(2)}$ such that $X = X^{(1)}X^{(2)}$, $Z = LE(X^{(1)})$, and $X^{(2)}_k \notin \{Z_0, \dots, Z_m\}$ for $k > 0$. Moreover, such $X^{(1)}$ and $X^{(w)}$ are determined by these properties.*

Proof. Let n_j, $0 \le j \le \tau$, be defined as above. Since $Z \prec LE(X)$, we have $Z_j = X_{n_j}$, $0 \le j \le m$. Let $X^{(1)} = (X_0, \dots, X_{n_m})$ and $X^{(2)} = (X_{n_m}, \dots, X_\nu)$. Then $X = X^{(1)}X^{(2)}$ and $X^{(2)}_k \notin \{Z_0, \dots, Z_m\}$ for $k > 0$, which implies that the path $X^{(2)}$ has no effect on the first $m+1$ vertices of $LE(X)$. Thus, $Z = LE(X^{(1)})$. On the other hand, if $X = X^{(1)}X^{(2)}$, $Z = LE(X^{(1)})$, and $X^{(2)}_k \notin \{Z_0, \dots, Z_m\}$ for $k > 0$, then the first $m+1$ vertices of $LE(X)$ agrees with those of $LE(X^{(1)})$, i.e., $Z \prec LE(X)$.

Now we show the uniqueness of $X^{(1)}$ and $X^{(2)}$. Suppose $X^{(1)} = (X_0, \dots, X_r)$ and $X^{(2)} = (X_r, \dots, X_\nu)$. Since $X^{(2)}_k \notin \{X_{n_0}, \dots, X_{n_m}\}$ for $k > 0$, we have $r \ge n_m$. Since $Z = LE(X^{(1)})$, we have $X_{n_m} = Z_m = X_r$. From the definition of n_m, we have $r \le n_m$. So $r = n_m$. $\square$

The loop-erasure of a (stopped) random walk or conditional random walk is called a loop-erased random walk or LERW. The loop-erasure of RW($v_0 \to A$) or RW($v_0 \to A|B$) is denoted by LERW($v_0 \to A$) or LERW($v_0 \to A|B$), respectively.

Greg Lawler introduced LERW as an alternative to study SAW. Now it turns out that the two models are different. Right now, LERW has been proved to converge to SLE_2; while SAW is conjectured to converge to $\text{SLE}_{8/3}$.

For $S_1, S_2, S_3 \subset V$, let $\Gamma_{S_1,S_2}^{S_3}$ denote the finite lattice path $(X_0, \dots, X_n)$ such that $X_0 \in S_1$, $X_n \in S_2$, and $S_k \in S_3$ for $1 \le k \le n-1$. For each finite lattice path $X = (X_0, \dots, X_n)$, let

$$P_{[\cdot]}(X) = \prod_{j=0}^{n} \frac{1}{\deg(X_j)}, \quad P_{[\cdot)}(X) = \prod_{j=0}^{n-1} \frac{1}{\deg(X_j)}, \quad P_{(\cdot)}(X) = \prod_{j=1}^{n-1} \frac{1}{\deg(X_j)}.$$

If $Z = XY$, then $P_{[\cdot)}(Z) = P_{[\cdot]}(X)P_{(\cdot)}(Y)$. The distribution of RW($v_0 \to A$) is supported by $\Gamma_{v_0,A}^{V\setminus A}$ and $\mathbb{P}^{v_0 \to A}(X) = P_{[\cdot)}(X)$ for each $X \in \Gamma_{v_0,A}^{V\setminus A}$. If $A \cap B = \emptyset$, the distribution of RW($v_0 \to A|B$) is supported by $\Gamma_{v_0,A}^{V\setminus(A\cup B)}$ and $\mathbb{P}^{v_0 \to A|B}(X) = P_{[\cdot)}(X)/h_{A|B}(v_0)$ for each $X \in \Gamma_{v_0,A}^{V\setminus(A\cup B)}$.

Lemma 6.2 *Let A and B be disjoint subsets of V. Suppose $h_{A|B}(v_0) > 0$. Let $Y = (Y_0, \dots, Y_\tau)$ be LERW($v_0 \to A|B$). Let $B_n = B \cup \{Y_0, \dots, Y_n\}$ for $0 \le n \le \tau$. Then for any $n \ge 0$,*

$$\mathbb{P}[Y_{n+1} = v | Y_0, \dots, Y_n, n < \tau] = \frac{\mathbf{1}_{v \sim Y_n} h_{A|B_n}(v)}{\sum_{w \sim Y_n} h_{A|B_n}(w)}. \tag{6.2}$$

Proof. Let $W = (W_0, \ldots, W_n, W_{n+1}) \in \Gamma^{V\setminus(A\cup B)}_{v_0, V\setminus B}$ and $W' = (W_0, \ldots, W_n)$. From the previous lemma, we have

$$\mathbb{P}[Y_j = W_j, 0 \le j \le n < \tau] = \frac{1}{h_{A|B}(v_0)} \sum_{U\in\Gamma^{V\setminus(A\cup B)}_{v_0,A}, W' \prec LE(U)} P_{[\cdot)}(U)$$

$$= \frac{1}{h_{A|B}(v_0)} \sum_{U^{(1)}\in\Gamma^{V\setminus(A\cup B)}_{v_0,W_n}, W'=LE(U^{(1)})} P_{[\cdot]}(U^{(1)}) \cdot \sum_{U^{(2)}\in\Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{W_n,A}} P_{(\cdot)}(U^{(2)});$$

$$\mathbb{P}[Y_j = W_j, 0 \le j \le n+1] = \frac{1}{h_{A|B}(v_0)} \sum_{U^{(1)}\in\Gamma^{V\setminus(A\cup B)}_{v_0,W_n}, W'=LE(U^{(1)})} P_{[\cdot]}(U^{(1)}) \cdot$$

$$\cdot \sum_{U^{(2)}\in\Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{W_n,A}, U_1^{(2)}=W_{n+1}} P_{(\cdot)}(U^{(2)}).$$

Thus,

$$\mathbb{P}[Y_{n+1} = W_{n+1}|Y_j = W_j, 0 \le j \le n < \tau] = \frac{\sum\{P_{(\cdot)}(U) : U \in \Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{W_n,A}, U_1 = W_{n+1}\}}{\sum\{P_{(\cdot)}(U) : U \in \Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{W_n,A}\}}$$

$$= \frac{\sum\{P_{[\cdot)}(U') : U' \in \Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{W_{n+1},A}\}}{\sum_{w\sim W_n}\sum\{P_{[\cdot)}(U') : U' \in \Gamma^{V\setminus(A\cup B\cup\{W_j\}_{j=0}^n)}_{w,A}\}} = \frac{h_{A|B\cup\{W_j\}_{j=0}^n}(W_{n+1})}{\sum_{w\sim W_n} h_{A|B\cup\{W_j\}_{j=0}^n}(w)}.$$

So we get the desired result. □

Remarks.

1. The Laplacian random walk is defined using (6.2). So LERW is the same as the Laplacian random walk. For $p > 0$, the p-Laplacian random walk is defined using (6.2) with $h_{A|B_n}$ replaced by $h^p_{A|B_n}$. The p-Laplacian random walk is much harder to analyze.
2. From the lemma, we see that the LERW satisfies Markov property. This means that, conditioned on $n < \tau$ and $Y_0, \ldots, Y_n$, the path $(Y_n \ldots, Y_\tau)$ has the distribution of LERW$(Y_n \to A|B_{n-1})$, where $B_{n-1} = B \cup \{Y_j\}_{j=0}^{n-1}$.

6.3 Observables for LERW

Lemma 6.3 *Let A and B be disjoint subsets of V such that $A \cup B \ne \emptyset$. Let $C = V \setminus (A \cup B)$ and $x \in C$. Then*

$$\sum_{v\in A} \Delta h_{x|A\cup B}(v) = G(x, A\cup B) h_{A|B}(x).$$

Proof. We have

$$h_{A|B}(x) = \sum_{X \in \Gamma^C_{x,A}} P_{[\cdot)}(X) = \sum_{Y \in \Gamma^C_{x,x}} P_{[\cdot]}(Y) \cdot \sum_{Z \in \Gamma^{C\setminus\{x\}}_{x,A}} P_{(\cdot)}(Z) = \sum_{Y \in \Gamma^C_{x,x}} P_{[\cdot]}(Y) \cdot \sum_{v \in A} \Delta h_{x|A\cup B}(v),$$

and

$$1 = \sum_{X \in \Gamma^C_{x,A\cup B}} P_{[\cdot)}(X) = \sum_{Y \in \Gamma^C_{x,x}} P_{[\cdot]}(Y) \cdot \sum_{Z \in \Gamma^{C\setminus\{x\}}_{x,A\cup B}} P_{(\cdot)}(Z) = \sum_{Y \in \Gamma^C_{x,x}} P_{[\cdot]}(Y) \cdot G(x, A\cup B).$$

So we proved this lemma. □

Lemma 6.4 *Let A, B, C, x be as in the previous lemma. Suppose $h_{A|B}(x) > 0$. Then the function f defined by*

$$f(v) = \frac{h_{x|A\cup B}(v)}{G(x, A\cup B) h_{A|B}(x)}, \quad v \in V,$$

is the unique function on V that satisfies $f \equiv 0$ on $A\cup B$, $\Delta f \equiv 0$ on $C\setminus\{x\}$, and $\sum_{v\in A} \Delta f(v) = 1$. Moreover, such f is nonnegative and satisfies $\Delta f(x) = -1/h_{A|B}(x)$.

Proof. This follows immediately from the previous lemma. □

Lemma 6.5 *Let A, B, C, x be as in the previous lemma. Then the function f defined by*

$$f(v) = h_{A|B}(v) + \frac{G(A,B) h_{x|A\cup B}(v)}{G(x, A\cup B) h_{A|B}(x)}, \quad v \in V,$$

is the unique function on V that satisfies $f \equiv 1$ on A, $f \equiv 0$ on B, $\Delta f \equiv 0$ on $C \setminus \{x\}$, and $\sum_{v\in A} \Delta f(v) = 0$. Moreover, such f is nonnegative and $\Delta f(x) = -G(A,B)/h_{A|B}(x)$.

Proof. It is clear that $f \equiv 1$ on A, $f \equiv 0$ on B, and $\Delta f \equiv 0$ on $C \setminus \{x\}$. That $\sum_{v\in A} \Delta f(v) = 0$ follows from the Lemma 6.3. Since $h_{A|B}$ and $h_{x|A\cup B}$ are nonnegative functions, $G(A,B) \ge 0$, and $G(x, A\cup B) > 0$, f is also nonnegative. And we compute

$$\Delta f(x) = \Delta h_{A|B}(x) + \frac{G(A,B) \Delta h_{x|A\cup B}(x)}{G(x, A\cup B) h_{A|B}(x)} = -\frac{G(A,B)}{h_{A|B}(x)}.$$

Now we prove the uniqueness. Suppose g satisfies the same properties as f. Let $I = g - h_{A|B}$. Then $I \equiv 0$ on $A\cup B$ and $\Delta I \equiv 0$ on $C \setminus \{x\}$. Thus, $I = I(x) h_{x|A\cup B}$. From Lemma 6.3 we have

$$0 = \sum_{v\in A} \Delta g(v) = \sum_{v\in A} \Delta I(v) + \sum_{v\in A} \Delta h_{A|B}(v) = I(x) h_{A|B}(x) G(x, A\cup B) - G(A,B).$$

Thus, $I(x) = G(A,B)/(h_{A|B}(x) G(x, A\cup B))$. So $g = f$. □

Proposition 6.1 *Let A and B be disjoint subsets of V with $A \neq \emptyset$. Let $C = V \setminus (A \cup B)$ and $v_0 \in C$ be such that $h_{A|B}(v_0) > 0$. Let $Y = (Y_0, \dots, Y_\tau)$ be LERW($v_0 \to A|B$). Let $B_{-1} = B$ and $B_n = B \cup \{Y_0, \dots, Y_n\}$, $0 \le n \le \tau - 1$. Then for each $0 \le n \le \tau$, $h_{A|B_{n-1}}(Y_n) > 0$. For $n < \tau$, define M_n and N_n on V by*

$$M_n^{(1)}(v) = \frac{h_{Y_n|A\cup B_{n-1}}(v)}{h_{A|B_{n-1}}(Y_n)}, \quad v \in V;$$

$$M^{(2)}(v) = h_{A|B_{n-1}}(v) + \frac{G(A,B) h_{Y_n|A\cup B_{n-1}}(v)}{G(Y_n, A\cup B_{n-1}) h_{A|B_{n-1}}(Y_n)}, \quad v \in V.$$

Let $\partial A = \{v \in V \setminus A : v \sim A\}$. Fix $z \in V$. Let T_z be the first n such that $Y_n \in \partial A$ or $h_{A|B_n}(z) = 0$, which ever comes first. Then for every $z \in V$, $M_n^{(1)}(z)$ and $M_n^{(2)}(z)$ are martingales up to T_z.

Proof. Since for every $0 \le n \le \tau$, $(Y_n, \dots, Y_\tau) \in \Gamma_{Y_n,A}^{V\setminus(A\cup B_{n-1})}$, we have $h_{A|B_{n-1}}(Y_n) > 0$. For the rest of the proof, we need to show that, for any $n \ge 0$, $\mathbb{E}[M_{n+1}^{(j)}(z)|Y_0, \dots, Y_n, n < T_z] = M_n^{(j)}(z)$, $j = 1, 2$. Suppose $n < T_z$. Let $S_n = \{w \sim Y_n : h_{A|B_n}(w) > 0\}$. For each $w \in S_n$, define

$$g_{n,w}^{(1)}(v) = \frac{h_{w|A\cup B_n}(v)}{h_{A|B_n}(w)}, \quad g_{n,w}^{(2)}(v) = h_{A|B_n}(v) + \frac{G(A, B_n) h_{w|A\cup B_n}(v)}{G(w, A\cup B_n) h_{A|B_n}(w)}.$$

From Lemma 6.2 we have

$$\mathbb{E}[M_{n+1}^{(j)}(v)|Y_0, \dots, Y_n, n < T_z] = \frac{\sum_{w\in S_n} h_{A|B_n}(v) g_{n,w}^{(j)}(v)}{\sum_{w\in S_n} h_{A|B_n}(w)}, \quad j = 1, 2.$$

Let $g_n^{(j)}(v)$ denote the righthand side of the above formula. From Lemma 6.4, for each $w \in S_n$, $g_{n,w}^{(1)} \equiv 0$ on $A \cup B_n$, $\Delta g_{n,w}^{(1)} \equiv 0$ on $V \setminus (A \cup B_n \cup \{w\})$, $\sum_{v\in A} \Delta g_{n,w}^{(1)}(v) = 1$, and $\Delta g_{n,w}^{(1)}(w) = -1/h_{A|B_n}(w)$. Thus, $g_n^{(1)} \equiv 0$ on $A \cup B_n$, $\Delta g_n^{(1)} \equiv 0$ on $V \setminus (A \cup B_n \cup S_n)$, $\sum_{v\in A} \Delta g_n^{(1)}(v) = 1$, and $\Delta g_n^{(1)}(v) = -1/\sum_{w\in S_n} h_{A|B_n}(w)$ for every $v \in S_n$.

If $S_n = \{w \sim Y_n : w \in V \setminus B_n\}$, we define $\widetilde{g}_n^{(1)}$ on V such that $\widetilde{g}_n^{(1)}(Y_n) = g^{(1)}(Y_n) + 1/\sum_{w\in S_n} h_{A|B_n}(w)$, and $\widetilde{g}_n^{(1)}(v) = g_n^{(1)}(v)$ for $v \neq Y_n$. Since $Y_n \notin A$, $Y_n \not\sim A$, and $B_n \setminus \{Y_n\} = B_{n-1}$, from the previous paragraph, we have $\widetilde{g}_n^{(1)} \equiv 0$ on $A \cup B_{n-1}$, $\Delta \widetilde{g}_n^{(1)} \equiv 0$ on $V \setminus (A \cup B_n)$, and $\sum_{v\in A} \Delta \widetilde{g}_n^{(1)}(v) = 1$. This shows that $\widetilde{g}_n^{(1)} = M_{n-1}^{(1)}$. Now since $h_{A|B_n}(z) > 0$, we have $z \neq Y_n$, so $g_n^{(1)}(z) = \widetilde{g}_n^{(1)}(z) = M_{n-1}^{(1)}(z)$.

If $S_n \subsetneqq \{w \sim Y_n : w \in V \setminus B_n\}$, the situation is more complicated. We need to modify the values of $g_n^{(1)}$ at more than one point. Let V_n denote the set of vertices $v \in V \setminus B_{n-1}$ such that every $X \in \Gamma_{v,A}^{V\setminus B_{n-1}}$ must pass through Y_n. Here $Y_n \in V_n$ by definition. Then we define

$$\widetilde{g}_n^{(1)}(v) = g_n^{(1)}(v) + \mathbf{1}_{v\in Y_n} h_{Y_n,B_{n-1}}(v) / \sum_{w\in S_n} h_{A|B_n}(w).$$

One may check that $\widetilde{g}_n^{(1)} = M_{n-1}^{(1)}$. Since $h_{A|B_n}(z) > 0$, we have $z \notin V_n$, so $g_n^{(1)}(z) = \widetilde{g}_n^{(1)}(z) = M_{n-1}^{(1)}(z)$. So the proof is done for $j = 1$.

The proof for the case $j = 2$ is similar. Define $g_n^{(2)}$ similarly. Then $g_n^{(2)} \equiv 1$ on A; $g_n^{(2)} \equiv 0$ on B_n; $\Delta g_n^{(2)} \equiv 0$ on $V \setminus (A \cup B_n \cup S_n)$; and $\sum_{v\in A} \Delta g_n^{(2)}(v) = 0$. Moreover, we have $\Delta g_n^{(2)}(v) = -G(A, B_n)/\sum_{w\in S_n} h_{A|B_n}(w)$ for every $v \in S_n$. Define V_n as before. By modifying the values of $g_n^{(2)}$ on V_n, we get a new function $\widetilde{g}_n^{(2)}$, which is equal to $M_n^{(2)}$. Since $h_{A|B_n}(z) > 0$, we find that $M_n^{(2)}(z) = g_n^{(2)}(z)$. $\square$

Remark. Note that $h_{A|B_n}(z) = 0$ means that the path $X_0, \dots, X_n$ disconnects z from A.

6.4 Observables for SLE$_2$

Recall the following two statements which were proved earlier.

1. Let g_t be the chordal Loewner maps driven by $\lambda_t = \sqrt{2}B_t$. Then for every fixed $z \in \mathbb{H}$, $M_t := -\operatorname{Im} \frac{1}{g_t(z)-\lambda_t}$, $0 \le t < \tau_z$, is a local martingale.

2. Let g_t be the radial Loewner maps driven by $\lambda_t = \sqrt{2}B_t$. Then for every fixed $z \in \mathbb{D}$, $M_t := \operatorname{Re} \frac{e^{i\lambda_t}+g_t(z)}{e^{i\lambda_t}-g_(z)}$, $0 \le t < \tau_z$, is a local martingale.

Suppose $\gamma(t)$, $0 \le t < \infty$, is a radial SLE$_2$ trace in a domain D from $a \in \partial D$ to $b \in D$. Then there is $W : (\mathbb{D}; 1, 0) \overset{\text{Conf}}{\twoheadrightarrow} (D; a, b)$ and a standard radial SLE$_2$ trace β such that $\gamma = W \circ \beta$. For each $t \ge 0$, there is a unique Poisson kernel function in $D \setminus \gamma(0, t]$ with the pole at $\gamma(t)$ which is normalized by $P_t(b) = 1$. Then $Q_t := P_t \circ W$ is a Poisson kernel in $\mathbb{D} \setminus \beta(0, t]$ with the pole at $\beta(t)$ which is normalized by $Q_t(0) = 1$. So $Q_t(z) = \operatorname{Re} \frac{e^{i\lambda_t}+g_t(z)}{e^{i\lambda_t}-g_(z)}$. From the above result, for any $z \in D$, $P_t(z)$ is a local martingale up to the time that γ visits z.

Suppose $\gamma(t)$, $0 \le t < \infty$, is a chordal SLE$_2$ trace in a domain D from $a \in \partial D$ to $b \in \partial D$. Then there is $W : (\mathbb{D}; 0, \infty) \overset{\text{Conf}}{\twoheadrightarrow} (D; a, b)$ and a standard chordal SLE$_2$ trace β such that $\gamma = W \circ \beta$. Suppose that ∂D is analytic near b. Then W may extends analytically to a neighborhood of b. Suppose $W(z) = b + \frac{c}{z}$ near ∞. Let $\mathbf{n}_b$ denote the inward unit normal vector at b. Then $\mathbf{n}_b = -i\frac{c}{|c|}$. For each $t \ge 0$, there is a unique Poisson kernel function in $D \setminus \gamma(0, t]$ with the pole at $\gamma(t)$ which is normalized by $\frac{\partial}{\partial \mathbf{n}_b} P_t(b) = 1$. Then $Q_t := P_t \circ W$ is a Poisson kernel in $\mathbb{D} \setminus \beta(0, t]$ with the pole at $\beta(t)$. Moreover,

$$1 = \lim_{t\to 0} \frac{P_t(b + t\mathbf{n}_b) - P_t(b)}{t} = \lim_{t\to 0} \frac{Q_t \circ W^{-1}(b + t\mathbf{n}_b)}{t} = \lim_{t\to 0} \frac{Q_t(\frac{c}{t\mathbf{n}_b})}{t} = \lim_{t\to 0} \frac{Q_t(\frac{i|c|}{t})}{t}.$$

On the other hand, suppose λ_t is the driving function for β, and g_t are chordal Loewner maps. Then $R_t := -\operatorname{Im} \frac{1}{g_t(z)-\lambda_t}$ is a Poisson kernel in $\mathbb{H} \setminus \beta(0, t]$ with the pole at $\beta(t)$. Since $g_t(z) = z + O(1/z)$ as $z \to \infty$, we have $\lim_{t\to 0} \frac{R_t(\frac{i|c|}{t})}{t} = \frac{1}{|c|}$. Thus, $Q_t = |c| R_t$. From the above comment we know that, for any $z \in \mathbb{H}$, $R_t(z)$, $0 \le t < \tau_z$, is a local martingale. Thus, for any $z \in D$, $P_t(z) = |c| Q_t(W^{-1}(z))$ is a local martingale up to the time when z is visited by γ.

6.5 Scaling limits

Now we study the convergence of LERW to SLE_2. Let D be a simply connected domain. For simplicity, suppose that D is a lattice domain in $\mathbb{Z}^2$, which means that ∂D is a union of some edges in $\mathbb{Z}^2$. Let $\delta = 1/n$ for some $n \in \mathbb{N}$. Then D is also a lattice domain in $\delta\mathbb{Z}^2$. Let D^δ denote the subgraph of $\delta\mathbb{Z}^2$ whose vertices and edges are those of $\delta\mathbb{Z}^2$ that lie on $\overline{D}$. The vertices of D^δ that lie on ∂D are called boundary vertices, other vertices of D^δ are called interior vertices. Let ∂D^δ and $\operatorname{int} D^\delta$ denote the set of all boundary vertices and interior vertices, respectively, of D^δ.

We first construct LERW that converges to radial SLE_2. Let $a \in \partial\mathbb{D} \cap \mathbb{Z}^2$ and $b \in D \cap \mathbb{Z}^2$. Then for any $\delta \in \{1/n : n \in \mathbb{N}\}$, a is a boundary vertex of D^δ, and b is an interior vertex of D^δ. Suppose a is not a corner of D. Let $X^\delta = (X_0, \dots, X_\tau)$ be $\mathrm{LERW}(D^\delta; a \to b | \partial D^\delta \setminus \{a\})$. Extend X to be a function defined on $[0, \tau]$ by linear interpolation. So $X^\delta(t)$, $0 \le t \le \tau$, is a random simple curve with $X^\delta(0) = a$, $X^\delta(\tau) = b$, and $X^\delta(t) \in D$ for $0 < t \le \tau$.

Theorem 6.1 [Lawler-Schramm-Werner] *For every $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, there is a coupling of the LERW curve $X(t)$, $0 \le t \le \tau$, and the radial SLE_2 trace β in D from a to b, such that for some continuous increasing function $u : [0, \tau) \to [0, \infty)$,*

$$\mathbb{P}[\sup_{0 \le t < \infty} |\beta(t) - X(u^{-1}(t))| \ge \varepsilon] < \varepsilon.$$

A coupling of two random processes X and Y is a pair of random processes X' and Y' which are defined in the same probability space such that X and X' have the same distribution, and Y and Y' have the same distribution. When we say that distributions of two random processes are close, we usually mean that there exists a coupling of the two processes such that the two random processes in the coupling are close. Since the two processes in the coupling are defined in the same probability space, we may compare them pointwise. In the statement of the above theorem, we also use a time-change function u. This is because the LERW curve is not parameterized by capacities.

One of the main idea in the proof of Theorem 6.1 is to compare an observable for LERW with an observable for radial SLE_2. For any $0 \le n < \tau$, there is a positive function P_n defined on the vertices of D^δ, which satisfies the following

1. $P_n \equiv 0$ on $\partial D^\delta \cup \{X_0, \dots, X_{n-1}\}$;
2. $\Delta P_n \equiv 0$ on $\operatorname{int} D^\delta \setminus \{X_0, \dots, X_n\}$;
3. $P_n(b) = 1$.

We have proved that, for any fixed $v_0 \in \operatorname{int} D^\delta$, $P_n(v_0)$ is a discrete martingale up to the time that the LERW curve visits a neighbor of b or disconnects v_0 from b.

Then we observe that, when δ is small, P_n is close to the Poisson kernel function Q_n in $D \setminus X[0,n]$ with the pole at X_n, normalized by $Q_n(b) = 1$. In fact, the following lemma describes the closeness between P_n and Q_n. Let $\mathcal{X}^\delta$ be the family of paths on D^δ of the form $X = (X_0, \dots, X_n)$ such that $X_0 = a$ and $\bigcup_{j=1}^n (X_{j-1}, X_j] \subset D$. For each $X = (X_0, \dots, X_n) \in \mathcal{X}^\delta$, let $D_X = D \setminus \bigcup_{j=1}^n (X_{j-1}, X_j]$, which is still a simply connected domain. Let P_X denote the function on D^δ, which vanishes on $\partial D^\delta \cup \{X_0, \dots, X_{n-1}\}$, is discrete harmonic on $\operatorname{int} D \setminus \{X_0, \dots, X_n\}$, and satisfies $P_X(b) = 1$. Let Q_X denote the Poisson kernel function in D_X with the pole at X_n, normalized by $Q_X(b) = 1$. For a Jordan curve J in $\mathbb{C}$, we will use Ω_J to denote the bounded component of $\mathbb{C} \setminus J$.

Lemma 6.6 *Let J be a Jordan curve in $D \setminus \{b\}$ such that $b \in \Omega_J$. Let K be a compact subset of Ω_J. Let $\mathcal{X}_J^\delta$ be the family of $X \in \mathcal{X}^\delta$ such that $\Omega_J \subset D_X$. Then for every $\varepsilon > 0$ there is $\delta_0 > 0$ (depending on D, J, K) such that if $\delta < \delta_0$, then for every $X \in \mathcal{X}_J^\delta$ and every $v \in \operatorname{int} D^\delta \cap K$, $|P_X(v) - Q_X(v)| < \varepsilon$.*

The proof of the lemma is proceeded as follows.

1. First, assume that the conclusion is not true, then we get a sequence $\delta_n \to 0$, a sequence of paths $X^{(n)} \in \mathcal{X}_J^{\delta_n}$, and a sequence of points $v_n \in \operatorname{int} D^{\delta_n} \cap K$, such that $|P_{X_n}(v_n) - Q_{X_n}(v_n)| \geq \varepsilon_0$ for some fixed $\varepsilon_0 > 0$.

2. By passing to a subsequence, we may assume that D_{X_n} converges to some domain E in the Carathéodory topology. We must have $\Omega_J \subset E \subset D$.

3. Extend each P_{X_n} to a Lipschitz continuous function on D whose constant in each square face is bounded by a factor times the slope of P_{X_n} on the four corner vertices.

4. Some argument on discrete harmonic functions show that the Lipschitz constants of P_{X_n} are uniformly bounded on each compact subset of E.

5. Applying the Ascoli-Arzela Theorem, we find that P_{X_n} converges locally uniformly to a continuous function, say f, on E.

6. Since every P_{X_n} is discrete harmonic, we may show that f is harmonic on E.

7. Some tedious argument shows that $Q_{X_n} \xrightarrow{\text{l.u.}} f$ in E, which gives a contradiction.

One intermediate step in the proof of the theorem is to show that the driving function for a time-change of the LERW curve (via radial Loewner equation) is close to the driving function for radial SLE_2. We may find W that maps D conformally onto $\mathbb{D}$ such that a and b are mapped to 1 and 0. Let $\gamma^\delta = W \circ X$. Let $u(t) = \operatorname{dcap} \gamma(0,t]$, $0 \leq t < \tau$. Then $\gamma^\delta(u^{-1}(t))$, $0 \leq t < \infty$, is a radial Loewner trace driven by some η^δ. Let g_t^δ and $\widetilde{g}_t^\delta$ denote the radial and covering radial Loewner maps driven by η^δ. The discrete observable for LERW can then be used to show that

η^δ is close to $\sqrt{2}B_t$ on a finite time interval. Lawler-Schramm-Werner proved the following proposition.

Proposition 6.2 *Let J be a Jordan curve in $D \setminus \{b\}$ such that $b \in \Omega_J$. Let T_J be the first n such that $[X_{n-1}, X_n]$ intersects J. For every $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of η_t^δ and $\sqrt{2}B_t$ such that*

$$\mathbb{P}[\sup_{0 \le t \le u(T_J)} |\eta_t^\delta - B_{2t}| \ge \varepsilon] < \varepsilon.$$

To prove this proposition, we need the lemma below. Fix a small $d > 0$. Let $T_0 = 0$. After T_n is defined, let T_{n+1} be the smallest integer $n \ge T_n$ such that either $|\eta_{u(n)} - \eta_{u(T_n)}| \ge d$, or $u(n) - u(T_n) \ge d^2$, or $n \ge T_J$, whichever comes first. Then (T_n) is an increasing sequence of stopping times and are bounded above by T_J. Let $\Delta_n(\eta) = \eta_{u(T_{n+1})} - \eta_{u(T_n)}$ and $\Delta_n(T) = u(T_{n+1}) - u(T_n)$.

Lemma 6.7 *There is an absolute constant $C > 0$ and a constant $\delta(d) > 0$ such that if $\delta < \delta(d)$, then for any n,*

$$|\mathbb{E}[\Delta_n(\eta)|\mathcal{F}_{T_n}]| \le Cd^3,$$
$$|\mathbb{E}[\Delta_n(\eta)^2 - 2\Delta_n(T)|\mathcal{F}_{T_n}]| \le Cd^3.$$

The proof of the lemma is proceeded as follows.

1. Choose a Jordan curve $J' \subset \Omega_J \setminus \{b\}$ such that $b \in \Omega_{J'}$. Observe that if $\delta < \operatorname{dist}(J, J')$, then $X^{T_J} \in \mathcal{X}_{J'}^\delta$, where X^{T_J} is the LERW X stopped at T_J.

2. One can show that, if δ is small enough (depending on d), then $\Delta_n(T) \le 2d^2$ and $|\Delta_n(\eta)| \le 2d$. So $\Delta_n(T) = O(d^2)$ and $\Delta_n(\eta) = O(d)$.

3. Choose a compact subset K of $\Omega_{J'}$ such that $\operatorname{int} K \ne \emptyset$. The previous lemma shows that $P_n(v) - Q_n(v) \to 0$ as $\delta \to 0$ uniformly in $n \le T_J$ and $v \in K \cap D^\delta$.

4. Note that $Q_n(z) = \operatorname{Re} \frac{1 + g_{u(n)} \circ W(z)/e^{i\eta_{u(n)}}}{1 - g_{u(n)} \circ W(z)/e^{i\eta_{u(n)}}}$. So $Q_n \circ W^{-1} \circ e^i(z) = -\operatorname{Im} \cot_2(\widetilde{g}_{u(n)}(z) - \eta_{u(n)})$.

5. Let K be a compact subset of $\Omega_{J'}$. Let $L = (e^i)^{-1}(W(K))$. From the previous lemma, we find that, for any $z \in L$, $(\operatorname{Im} \cot_2(\widetilde{g}_{u(T_n)}(z) - \eta_{u(T_n)}))_{n=0}^\infty$, is close to a martingale. More specifically, we have

$$\mathbb{E}[\operatorname{Im} \cot_2(\widetilde{g}_{u(T_{n+1})}(z) - \eta_{u(T_{n+1})}) - \operatorname{Im} \cot_2(\widetilde{g}_{u(T_n)}(z) - \eta_{u(T_n)})|\mathcal{F}_{T_n}] = o_\delta(1), \tag{6.3}$$

where $o_\delta(1)$ is some quantity which tends to 0 uniformly as $\delta \to 0$.

Let $S_n = u(T_n)$, $n \geq 0$. We will estimate the quantity

$$I := \cot_2(\widetilde{g}_{S_{n+1}}(z) - \eta_{S_{n+1}}) - \cot_2(\widetilde{g}_{S_n}(z) - \eta_{S_n})$$

We have $I = I_1 + I_2 + I_3$, where $S'_n \in (S_n, S_{n+1})$, and

$$I_1 = \cot'_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot [(\widetilde{g}_{S_{n+1}}(z) - \widetilde{g}_{S_n}(z)) - (\eta_{S_{n+1}} - \eta_{S_n})];$$

$$I_2 = \frac{1}{2}\cot''_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot [(\widetilde{g}_{S_{n+1}}(z) - \widetilde{g}_{S_n}(z)) - (\eta_{S_{n+1}} - \eta_{S_n})]^2;$$

$$I_3 = \frac{1}{6}\cot'''_2(\widetilde{g}_{S'_n}(z) - \eta_{S'_n}) \cdot [(\widetilde{g}_{S_{n+1}}(z) - \widetilde{g}_{S_n}(z)) - (\eta_{S_{n+1}} - \eta_{S_n})]^3.$$

There is an uniform upper bound for $|\cot'''_2(\widetilde{g}_{S'_n}(z) - \eta_{S'_n})|$. From the ODE for $\widetilde{g}_t$, there is $S''_n \in (S_n, S_{n+1})$ such that

$$\widetilde{g}_{S_{n+1}}(z) - \widetilde{g}_{S_n}(z) = \cot_2(\widetilde{g}_{S''_n}(z) - \eta_{S''_n}) \cdot \Delta_n(T).$$

There is an uniform upper bound for $|\cot_2(\widetilde{g}_{S''_n}(z) - \eta_{S''_n})|$. Since $\Delta_n(T) = O(d^2)$ and $\Delta_n(\eta) = O(d)$, we have $I_3 = O(d^3)$. A similar argument gives

$$\cot_2(\widetilde{g}_{S''_n}(z) - \eta_{S''_n}) = \cot_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) + O(d).$$

So we have

$$\widetilde{g}_{S_{n+1}}(z) - \widetilde{g}_{S_n}(z) = \cot_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \Delta_n(T) + O(d^3).$$

Thus,

$$I_1 = \cot'_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot [\cot_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \Delta_n(T) - \Delta_n(\eta)] + O(d^3);$$

$$I_2 = \frac{1}{2}\cot''_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot [\cot_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \Delta_n(T) - \Delta_n(\eta)]^2 + O(d^3).$$

Since $\cot''_2 = -\cot_2\cot'_2$, we get

$$I = \cot''_2(\widetilde{g}_{S_n}(z) - \eta_{S_n})[\frac{1}{2}\Delta_n(\eta)^2 - \Delta_n(T)] - \cot'_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \Delta_n(\eta) + O(d^3).$$

From (6.3) we find that, for any $z \in L$, if δ is small enough (depending on d),

$$\operatorname{Im}\cot''_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \mathbb{E}[\frac{1}{2}\Delta_n(\eta)^2 - \Delta_n(T)|\mathcal{F}_{T_n}]$$

$$-\operatorname{Im}\cot'_2(\widetilde{g}_{S_n}(z) - \eta_{S_n}) \cdot \mathbb{E}[\Delta_n(\eta)|\mathcal{F}_{T_n}] = O(d^3).$$

Since $\operatorname{int} K \neq \emptyset$, we have $\operatorname{int} L \neq \emptyset$, the above formula finishes the proof of Lemma 6.7. In fact, one may prove and use the following facts:

1. $\operatorname{Im}\cot'_2(\widetilde{g}_{S_n}(z) - \eta_{S_n})$ and $\operatorname{Im}\cot''_2(\widetilde{g}_{S_n}(z) - \eta_{S_n})$ are bounded in absolute value by an absolute constant for any n and $z \in L$.

2. There is an absolute positive constant C such that for every n, we may find $z_1, z_2 \in L$, such that the absolute value of the determinant of the 2×2 matrix composed of $\operatorname{Im}\cot_2'(\widetilde{g}_{S_n}(z_j) - \eta_{S_n})$ and $\operatorname{Im}\cot_2''(\widetilde{g}_{S_n}(z_j) - \eta_{S_n})$, $j = 1, 2$, is at least C.

The next step is to apply Skorokhod's embedding theorem shown below.

Theorem 6.2 *If (M_n) is a martingale with $M_0 = 0$ and $|M_n - M_{n-1}| \le d$, then there is a standard Brownian motion B_t, and an increasing sequence of stopping times $0 = \tau_0 \le \tau_1 \le \tau_2 \le \cdots$ such that $(M_0, M_1, \dots, M_n, \dots)$ has the same joint distribution as $(B_{\tau_0}, B_{\tau_1}, \dots, B_{\tau_n}, \dots)$. Moreover, one can impose that*

$$\mathbb{E}[\tau_n - \tau_{n-1} | B[0, \tau_{n-1}]] = \mathbb{E}[(B_{\tau_n} - B_{\tau_{n-1}})^2 | B[0, \tau_{n-1}]]. \tag{6.4}$$

$$\tau_n \le \inf\{t \ge \tau_{n-1} : |B_t - B_{\tau_{n-1}}| \ge d\}. \tag{6.5}$$

Proof of Proposition 6.2. Define a martingale (M_n) by $M_0 = 0$ and

$$M_n = M_{n-1} + \Delta_{n-1}(\eta) - \mathbb{E}[\Delta_{n-1}(\eta) | \mathcal{F}_{T_{n-1}}], \quad n \ge 1.$$

Recall that $\Delta_{n-1}(\eta) = \eta_{S_n} - \eta_{S_{n-1}}$. From Lemma 6.7, by choosing δ small enough, we can ensure that $|M_n - M_{n-1}| \le 2d$. Applying Skorokhod's embedding theorem, we find a standard Brownian motion B_t and an increasing sequence of stopping times $(\tau_n)_{n=0}^{\infty}$ for B_t such that $(M_0, M_1, \dots, M_n, \dots)$ has the same joint distribution as $(B_{\tau_0}, B_{\tau_1}, \dots, B_{\tau_n}, \dots)$. Moreover, we have $|B_t - B_{\tau_{n-1}}| \le 2d$ for $t \in [\tau_{n-1}, \tau_n]$.

Let $S_J = u(T_J)$ and $N = \lceil 10 S_J / d^2 \rceil$. Then S_J is uniformly bounded above, and $S_J \asymp N d^2$. We first focus on M_n, $0 \le n \le N$. From Lemma 6.7 we have $M_n - \eta_{S_n} = O(nd^3) = O(S_J d) = O(d)$ for $0 \le n \le N$. Recall that $|\eta_t - \eta_{S_{n-1}}| \le 2d$ for $t \in [S_{n-1}, S_n]$. Using the continuity of Brownian motion, we suffice to show that when δ and d are small, with probability close to 1, $\sup_{n \le N} |\tau_n - 2S_n|$ is small and $T_N = T_J$.

Define another martingale (N_n) by $N_0 = 0$ and

$$N_n = N_{n-1} + (M_n - M_{n-1})^2 - \mathbb{E}[(M_n - M_{n-1})^2 | \mathcal{F}_{T_{n-1}}], \quad n \ge 1.$$

Since $M_n - M_{n-1} = \Delta_{n-1}(\eta) + O(d^3)$, and $\Delta_{n-1}(\eta) = O(d)$, we have $(M_n - M_{n-1})^2 = \Delta_{n-1}(\eta)^2 + O(d^4)$, which implies that

$$\mathbb{E}[(M_n - M_{n-1})^2 | \mathcal{F}_{T_{n-1}}] = \mathbb{E}[2\Delta_{n-1}(T) | \mathcal{F}_{T_{n-1}}] + O(d^3).$$

Thus, $N_n - N_{n-1} = \Delta_{n-1}(\eta)^2 - \mathbb{E}[2\Delta_{n-1}(T) | \mathcal{F}_{T_{n-1}}] + O(d^3)$.

Define another martingale (O_n) by $O_0 = 0$ and

$$O_n = O_{n-1} + 2\Delta_{n-1}(T) - \mathbb{E}[2\Delta_{n-1}(T) | \mathcal{F}_{T_{n-1}}], \quad n \ge 1. \tag{6.6}$$

Let $P_n = N_n - O_n$. Then $P_n - P_{n-1} = \Delta_{n-1}(\eta)^2 - 2\Delta_{n-1}(T) + O(d^3)$. Define another martingale (Q_n) by $Q_0 = 0$ and

$$Q_n = Q_{n-1} + (B_{\tau_n} - B_{\tau_{n-1}})^2 - (\tau_n - \tau_{n-1}), \quad n \ge 1.$$

Let $R_n = P_n - Q_n$. Then $R_n - R_{n-1} = (\tau_n - \tau_{n-1}) - 2\Delta_{n-1}(T) + O(d^3)$. Thus, $R_n = \tau_n - 2S_n + O(Nd^3)$, $n \le N$. Since $|B_{\tau_n} - B_{\tau_{n-1}}| \le 2d$, we have $\mathbb{E}[\tau_n - \tau_{n-1}|B[0,\tau_{n-1}]] = O(d^2)$. Thus, $\mathbb{E}[(R_n - R_{n-1})^2|B[0,\tau_{n-1}]] = O(d^4)$, which implies that,

$$\mathbb{E}[R_N^2] = \sum_{n=1}^{N} \mathbb{E}[\mathbb{E}[(R_n - R_{n-1})^2|B[0,\tau_{n-1}]]] = O(Nd^4).$$

Applying Doob's inequality to the martingale P_n, we get

$$\mathbb{P}[\max_{n\le N} |R_n| > d^{1/2}] \le C\mathbb{P}[|R_N|^2 > d] = O(Nd^3) = O(S_J d) = O(d).$$

This means that, with probability greater than $1 - O(d)$, $|\tau_n - 2S_n| = O(d^{1/2})$ for $n \le N$.

Suppose $T_N < T_J$. Then for $n \le N$, either $\Delta_{n-1}(\eta)^2 \ge d^2$ or $\Delta_{n-1}(T) \ge d^2$. Since $\mathbb{E}[\Delta_{n-1}(\eta)^2 - 2\Delta_{n-1}(T)|\mathcal{F}_{T_{n-1}}] = O(d^3)$, we get $\mathbb{E}[2\Delta_{n-1}(T)|\mathcal{F}_{T_{n-1}}] > d^2/2$ for $n \le N$ if d is small. From (6.6) we have $|O_N - 2S_N| \ge Nd^2 \ge 10S_J$, which implies that $O_N \ge 9S_J$. On the other hand, from (6.6) we have $O_n - O_{n-1} = O(d^2)$, which implies that

$$\mathbb{E}[O_N^2] = \sum_{n=1}^{N} \mathbb{E}[(O_n - O_{n-1})^2] = O(Nd^4) = O(S_J d^2) = O(d^2).$$

Thus, $\mathbb{P}[O_N > 9S_J] = O(d^2)$. So $\mathbb{P}[T_N = T_J] = 1 - O(d^2)$. $\square$

Proposition 6.2 implies that, when δ is small, for any $t \le S_J := u(T_J)$, under some suitable coupling, $\mathbb{D} \setminus \gamma^\delta(u^{-1}(0,t])$ is close to $\mathbb{D} \setminus \gamma(0,t]$ in the Carathéodory topology, where $\gamma(t)$ is a standard radial SLE_2 curve. To finish the proof of Theorem 6.1, one needs to use some more complicated properties of LERW. Roughly speaking, it says that LERW tends to not intersect itself uniformly in the mesh size δ. In more details, For $z \in D$ and $R > r > 0$, an $\mathcal{L}(z;r,R)$ loop on the LERW X^δ is a subcurve of X^δ, whose two end points stay within distance r from z, and which contains a point w which has distance $> R$ from z. The fact is that, for any $z \in D$ and $R > 0$, the probability that X^δ contains an $\mathcal{L}(z;r,R)$ loop tends to 0 as $r \to 0$, uniformly in δ. The proof uses relation between LERW and uniformly spanning tree, and this result can then be used to finish the proof of Theorem 6.1. Here we omit the details and refer the reader to the paper by Lawler, Schramm, and Werner.

At the end of this subsection, we briefly discuss the LERW that converges to chordal SLE_2. Let the lattice domain D and $a \in \partial D \cap \mathbb{Z}^2$ be as before. Now let $b \in \partial D \cap \mathbb{Z}^2$ be such that $b \ne a$ and b is not a corner of ∂D. Consider the $\mathrm{LERW}(D^\delta; a \to b|\partial D^\delta \setminus \{a,b\})$: $X^\delta = (X_0,\dots,X_\tau)$. The conclusion is that Theorem 6.1 still holds here if "radial" is replaced by "chordal". Let P_n denote the function on D^δ, which vanishes on $\partial D^\delta \cup \{X_0,\dots,X_{n-1}\}$, is harmonic on $\operatorname{int} D^\delta \setminus \{X_0,\dots,X_n\}$, and is normalized by $\Delta P_n(b) = 1$. Then for any $z \in \operatorname{int} D^\delta$, $(P_n(z))$ is a martingale up to a stopping time. Let b' be the unique neighbor of b in $\operatorname{int} D^\delta$. Then $\Delta P_n(b) = 1$ means that $P_n(b') - P_n(b) = 1$. One can show that, when δ is small, δP_n is close to the Poisson kernel Q_n in D_n with the pole at X_n, normalized by $\partial_{\mathbf{n_b}} Q_n(b) = 1$. The rest of the proof follows the argument for the convergence to radial SLE_2.

6.6 Uniform spanning tree and Wilson's algorithm

A tree is a connected graph without loops. For any two vertices on a tree, there is a unique simple path connecting them. Let $G = (V, E)$ be a finite connected graph. A subgraph H of G is called a spanning tree on G if H is a tree and contains all vertices of G. The total number of spanning trees on G is finite. A uniform spanning tree (UST for short) on G is a random spanning tree chosen among all the possible spanning trees on G with equal probability. UST is closely related with LERW via Wilson's algorithm.

Theorem 6.3 [Wilson's algorithm]
Let $G = (V, E)$ be a finite connected graph.

(i) Let T be a UST on G. For any $v, w \in V$, the only simple path from v to w on T has the distribution of LERW($G; v \to w$).

(ii) Suppose $V = \{v_0, \dots, v_n\}$. Let $T_0 = \{v_0\}$. When T_k is constructed for some $k < n$, we let T_{k+1} be the union of T_k and all vertices and edges on LERW($G; v_{k+1} \to T_k$). Then T_n has the distribution of a UST on G.

Note that Wilson's algorithm immediately implies that the time-reversal of LERW($v \to w$) has the same distribution as LERW($w \to v$). In fact, the following proposition is true.

Corollary 6.1 *Let $S \subset V$ and $a \neq b \in V \setminus S$. Then the time-reversal of LERW($a \to b|S$) has the same distribution as LERW($b \to a|S$).*

Proof. First we define RW'($G; v \to A|B$) to be obtained from RW($G; v \to A|B$) by removing the initial part of the path up to the last time the path visits v. So the distribution of $RW'(G; v \to A|B)$ is supported by $\Gamma_{v,A}^{V\setminus(A\cup B\cup\{v\})}$. It is clear that the loop-erasure of RW'($G; v \to A|B$) is the same as LERW($G; v \to A|B$).

Divide S into the disjoint union of two subsets A' and B'. Let $A = A' \cup \{a\}$ and $B = B' \cup \{b\}$. Let $G_{A,B}$ be obtained from G by identifying all vertices in A as a single vertex, say v_A, and identifying all vertices in B as a single vertex, say v_B. Consider the UST on $G_{A,B}$. There is a unique simple curve, say Y, connecting v_A and v_B. We order this path such that it starts from v_A and ends at v_B. From Wilson's algorithm, Y is LERW($G_{A,B}; v_A \to v_B$). Thus, $Y = LE(X)$, where X is an RW'($G_{A,B}; v_A \to v_B$). We may also view X as a random path on G, whose distribution is supported by $\Gamma_{A,B}^{V\setminus(A\cup B)}$. The probability that X follows any path $W \in \Gamma_{A,B}^{V\setminus(A\cup B)}$ is $CP_{(\cdot)}(W)$ for some constant $C > 0$. If we condition on X such that its initial vertex is a and its end vertex is b, then the resulting random path, say $X_{a,b}$, is an RW'($a \to b|S$). Thus, LERW($a \to b|S$) can be obtained by erasing loops on $X_{a,b}$. This shows that LERW($a \to b|S$) can be obtained by conditioning Y such that it starts from a and ends at b. Let Y^R denote the time-reversal of Y. Then Y^R is LERW($G_{A,B}; v_B \to v_A$). A similar argument shows that, LERW($b \to a|S$) can be obtained by conditioning Y^R such that it starts from b and ends at a. This finishes the proof. □

The above result may be applied to the LERW we studied before. Recall that the LERW that converges to chordal $\mathrm{SLE}_2(D; a \to b)$ is $\mathrm{LERW}(D^\delta; a \to b|\partial D^\delta \setminus \{a,b\}$, where $a \neq b \in \partial D \cap \mathbb{Z}^2$. From the above proposition we immediately see that the time-reversal of this LERW is $\mathrm{LERW}(D^\delta; b \to a|\partial D^\delta \setminus \{a,b\}$. From the convergence of LERW, we see that chordal SLE_2 satisfies reversibility. Also recall that the LERW that converges to radial $\mathrm{SLE}_2(D; a \to b)$ is $\mathrm{LERW}(D^\delta; a \to b|\partial D^\delta \setminus \{a\}$, where $a \in \partial D \cap \mathbb{Z}^2$ and $b \in D \cap \mathbb{Z}^2$. This LERW is the time-reversal of $\mathrm{LERW}(D^\delta; b \to a|\partial D^\delta \setminus \{a\}$, which can be obtained by conditioning $\mathrm{LERW}(D^\delta; b \to \partial D^\delta)$ on the event that the path ends at a. Note that the distribution of the end point of $\mathrm{LERW}(D^\delta; b \to \partial D^\delta)$ is the discrete harmonic measure on ∂D^δ viewed from b. As $\delta \to 0$, this distribution tends to the continuous harmonic measure on ∂D viewed from b (the distribution of the first hitting point on ∂D of a planar Brownian motion started from b). Thus, we conclude that the time-reversal of $\mathrm{LERW}(D^\delta; b \to \partial D^\delta)$ converges to radial $\mathrm{SLE}_2(D; \widetilde{a} \to b)$ up to a time-change, where $\widetilde{a}$ is a random point on ∂D, whose distribution is the harmonic measure on ∂D viewed from b. This is the exact statement in the paper by Lawler, Schramm, and Werner.

To prove Theorem 6.3, we introduce another algorithm to generate a UST on G. Fix $v_0 \in V$. Let $X = (X_0, X_1, \dots, X_n, \dots)$ be a simple random walk on G started from v_0. Construct a sequence of graphs (T_n) as follows. Let $T_0 = \{X_0\}$. Let T_{n+1} be the union of T_n and the vertex X_{n+1} and the edge (X_n, X_{n+1}) if X_{n+1} has not been visited by $X_0, \dots, X_n$; let $T_{n+1} = T_n$ if $X_{n+1} \in \{X_0, \dots, X_n\}$. Note that each T_n is a tree. Let N be the covering time for X, i.e., the first n such that X visits all vertices on V. Note that a.s. N is finite. The following theorem was discovered by A. Broder and D. J. Aldous independently.

Theorem 6.4 *T_N has the same distribution as the UST on G.*

Proof of Wilson's Algorithm using Theorem 6.4. (i) Let X be a random walk on G started from w. Let τ_v be the first time that X reaches v. Construct the family (T_n) as before the above theorem. From Theorem 6.4, T_{τ_v} is a subtree of the UST on G. Since $v, w \in T_{\tau_v}$, the only simple path on the UST connecting v and w is contained in T_{τ_v}. Let $Y = (X_{\tau_v}, X_{\tau_v - 1}, \dots, X_1, X_0)$ be the reversal of the initial part of X up to τ_v. So Y starts from v and ends at w. Let Z be the only simple path on T_{τ_v} from v to w. We claim that $Z = LE(Y)$.

Write $Z = (Z_0, \dots, Z_\nu)$. For $0 \le m \le \nu$, let τ_m denote the first n such that $X_n = Z_m$. Then $\tau_0 > \tau_1 > \cdots > \tau_\nu$. In fact, if $n < m \le \nu$, since the tree T_{τ_n} contains $X_0 = w = Z_\nu$ and $X_{\tau_n} = Z_n$, it contains the path $(Z_n, \dots, Z_\nu)$, which implies that $Z_m \in T_n$, i.e., $\tau_m < \tau_n$. Let $u_k = \tau_v - \tau_m$, $0 \le k \le \nu$. Then $u_0 < u_1 < \cdots < u_\nu$ and $Y_{u(k)} = Z_k$, $0 \le k \le m$. To prove that $Z = LE(Y)$, we suffice to show that for any j, $\{Z_0, \dots, Z_j\} \cap \{Y_n : n > u_j\} = \emptyset$. This is true because $\{Y_n : n > u_j\} = \{X_n : n < \tau_j\}$ and X does not visit $\{Z_0, \dots, Z_j\}$ before τ_j thanks to the decreasing property of (τ_j).

It remains to show that $Z = LE(Y)$ has the distribution of $\mathrm{LERW}(v \to w)$. We suffice to show that Z is a Laplacian random walk. Note that the distribution of Y is supported by $\Gamma^{V \setminus \{v\}}_{v,w}$, and for every $W \in \Gamma^{V \setminus \{v\}}_{v,w}$, $\mathbb{P}[Y = W] = P_{(\cdot]}(W)$. Let $W = (W_0, \dots, W_n, W_{n+1}) \in \Gamma^{V \setminus \{v,w\}}_{v, V \setminus \{v\}}$ and

$W' = (W_0, \dots, W_n)$. From Lemma 6.1, we have

$$\mathbb{P}[Z_j = W_j, 0 \le j \le n+1] = \sum_{U \in \Gamma_{v,w}^{V\setminus\{v\}}, W \prec LE(U)} P_{(\cdot]}(U)$$

$$= \sum_{U^{(1)} \in \Gamma_{v,W_n}^{V\setminus\{v,w\}}, W' = LE(U^{(1)})} P_{(\cdot]}(U^{(1)}) \cdot \sum_{U^{(2)} \in \Gamma_{W_n,w}^{V\setminus\{W_j\}_{j=0}^n}, U_1^{(2)} = W_{n+1}} P_{(\cdot]}(U^{(2)})$$

$$= C_n \sum_{U^{(2')} \in \Gamma_{W_{n+1},w}^{V\setminus\{W_j\}_{j=0}^n}} P_{[\cdot]}(U^{(2')})$$

$$= C_n \sum_{A \in \Gamma_{W_{n+1},w}^{V\setminus(\{W_j\}_{j=0}^n \cup\{w\})}} P_{[\cdot)}(A) \cdot \sum_{B \in \Gamma_{w,w}^{V\setminus\{W_j\}_{j=0}^n}} P_{[\cdot]}(B)$$

$$= C_n C \sum_{A \in \Gamma_{W_{n+1},w}^{V\setminus(\{W_j\}_{j=0}^n \cup\{w\})}} P_{[\cdot)}(A) = C_n C h_{w|\{W_0,\dots,W_n\}}(W_{n+1}),$$

where $C_n = \sum\{P_{(\cdot]}(U^{(1)}) : U^{(1)} \in \Gamma_{v,W_n}^{V\setminus\{v,w\}}, W' = LE(U^{(1)})\}$ depends only on $W_0, \dots, W_n$, and $C = \sum\{P_{[\cdot]}(B) : B \in \Gamma_{w,w}^{V\setminus\{W_j\}_{j=0}^n}\}$ is a constant. Thus, $\mathbb{P}[Z_j = W_j, 1 \le j \le n] = \sum_{a\sim W_n} C_n C h_{w|\{W_0,\dots,W_n\}}(a)$, which implies that

$$\mathbb{P}[Z_{n+1} = W_{n+1} | Z_j = W_j, 1 \le j \le n] = \frac{h_{w|\{W_0,\dots,W_n\}}(W_{n+1})}{\sum_{a\sim W_n} h_{w|\{W_0,\dots,W_n\}}(a)}.$$

This shows that Z is a Laplacian random walk from v to w. So (i) is proved.

One may prove (ii) using the induction on the number of vertices. Recall that $T_0 = \{v_0\}$ and T_1 is LERW$(v_1 \to v_0)$. Let $G' = G/T_1$, i.e., identifying all vertices on T_1 as a single vertex. Then the number of vertices of G' is less than that of G. Note that the UST on G conditioned to contain T_1 agrees with the UST on G', and the LERW on G whose target is $S \supset T_1$ agrees with the LERW on G' whose target is S/T_1. We leave the details to the interested readers. □

Proof of Theorem 6.4. We introduce the notation of rooted spanning trees. A rooted spanning tree on G is a spanning tree on G with a marked vertex called the root. A uniform rooted spanning tree (URST) on G is a random rooted spanning tree chosen among all the possible rooted spanning trees on G with probability proportional to the degree of the root. By forgetting the root, we get a natural map from the set of rooted spanning trees to the set of spanning trees, which maps a URST on G to a UST on G.

Let $\mathcal{P}$ denote the set of infinite paths $X = (X_0, X_1, \dots)$ on G. Let $\mathcal{P}^*$ denote the set of $X \in \mathcal{P}$ such that X visits all vertices on G. The construction before the statement of Theorem 6.4 gives a map F_T from $\mathcal{P}^*$ to the set of spanning trees on G. In fact, the construction also gives a map F_{RT} to the set of rooted spanning trees on G if we set the first vertex X_0 to be the root. Also note two facts: every rooted spanning tree can be constructed in this way; the construction depends only on $(X_0, \dots, X_N)$ if N is the covering time.

Now we construct a directed graph G_{RT} whose vertices are rooted spanning trees on G. For two rooted spanning trees (T_1, v_1) and (T_2, v_2) on G, we draw a directed edge from (T_1, v_1) to (T_2, v_2), and write $(T_1, v_1) \downarrow (T_2, v_2)$ or $(T_2, v_2) \uparrow (T_1, v_1)$, if $v_1 \sim v_2$ and $T_2 = T_1 \cup (v_1, v_2) \setminus e$, where e is the first edge on the simple path on T_1 from v_1 to v_2. Every vertex (T, v) in G_{RT} has exactly $\deg(v)$ downward neighbors and $\deg(v)$ upward neighbors. It is easy to see that if $T_1 = F_{RT}(X)$ for $X = (X_0, X_1, \dots) \in \mathcal{P}^*$, then $T_2 = F_{RT}(X^{v_2})$, where $X^{v_2} = (v_2, X_0, X_1, \dots)$. This shows that we may travel from any rooted spanning tree on G to another rooted spanning tree on G along directed edges in G_{RT}.

A time-homogeneous random walk on G is a random walk on G started from a random vertex whose distribution is proportional to the degree of the vertex. Let X be such a random walk. We claim that $F_{RT}(X)$ is a URST on G. Let $Y = (v, X_0, X_1, \dots)$, where v is chosen among neighbors of X_0 with probability $1/\deg(X_0)$ each. Then Y has the same distribution as X. So $F_{RT}(Y)$ has the same distribution as $F_{RT}(X)$. The above paragraph shows that $F_{RT}(X) \downarrow F_{RT}(Y)$ and $F_{RT}(Y)$ is chosen among all downward neighbors of $F_{RT}(X)$ in G_{RT} with equal probability $1/\deg(X_0)$.

For each rooted spanning tree (T, v) on G, let $p(T, v) = \mathbb{P}[F_{RT}(X) = (T, v)]$. Since $F_{RT}(X)$ has the same distribution as $F_{RT}(Y)$, we have

$$p(T, v) = \sum_{(S,w):(S,w)\downarrow(T,v)} \frac{p(S, w)}{\deg(w)}.$$

Let $q(T, v) = p(T, v)/\deg(v)$. Then $q(T, v) = \frac{1}{\deg(v)} \sum_{(S,w)\downarrow(T,v)} q(S, w)$. This means that the value of q at every vertex in G_{RT} is equal to the average of its upward neighbors. So q is constant on G_{RT}, which shows that $p(T, v)$ is proportional to $\deg(v)$. Thus, $F_{RT}(X)$ is a URST on G as claimed.

Finally, note that a time-homogeneous random walk conditioned to start from $v \in V$ is just a regular random walk started from v. Thus, if X is a random walk on G started from v, then $F_{RT}(X)$ is URST on G conditioned to have root v. By forgetting the root, we find that $F_T(X)$ is just a UST on G. $\square$

6.7 UST Peano curve

Let D be a rectangle with corners at $(0, 0)$, $(m_1, 0)$, (m_1, m_2), $(0, m_2)$, where $m_1, m_2 \in \mathbb{N}$. Let $\delta \in \{1/n : n \in \mathbb{N}\}$. Let D^δ as before. Let I^δ be the set of edges of D^δ on the left side and upper side. Define the dual $D^\delta_\dagger$ to be a subgraph of $(\delta/2, -\delta/2) + \delta\mathbb{Z}^2$ by shifting D^δ by $(\delta/2, -\delta/2)$. Let $I^\delta_\dagger$ be the set of edges of $D^\delta_\dagger$ on the right side and lower side. Note that every edge e of D^δ not in I^δ intersects exactly one edge, called the dual of e, of $D^\delta_\dagger$ not in $I^\delta_\dagger$, and vice versa.

There is a one-to-one correspondence between the set of spanning trees on D^δ that contain all edges in I^δ and the set of spanning trees on $D^\delta_\dagger$ that contain all edges in $I^\delta_\dagger$. If T is a spanning tree on D^δ that contains all edges in I^δ, the corresponding tree, called the dual of T, is composed of all edges in $I^\delta_\dagger$ and all edges in $D^\delta_\dagger$ whose dual edge in D^δ does not lie on T.

Let T be a UST on D^δ conditioned to contain all edges in I^δ. Let $T_\dagger$ be its dual. Then $T_\dagger$ is a UST on $D^\delta_\dagger$ conditioned to contain all edges in $I^\delta_\dagger$. Consider the graph $(\delta/4,-\delta/4)+D^{\delta/2}$. Let $a=(\delta/4,-\delta/4)$ and $b=(n+\delta/4,m-\delta/4)$ be two vertices of $(\delta/4,-\delta/4)+D^{\delta/2}$. There is a unique path, say $X=(X_0,\dots,X_k)$, on $(\delta/4,-\delta/4)+D^{\delta/2}$ from a to b, which is disjoint from all edges in T and $T_\dagger$. In fact, X visits every vertex of this graph. So $k=(2m_1+1)(2m_2+1)-1$. This path is called a UST Peano curve. As before, we extend this path to a continuous curve defined on $[0,k]$ by linear interpolation.

Theorem 6.5 [Lawler-Schramm-Werner]
For every $\varepsilon>0$, there is $\delta_0>0$ such that if $\delta<\delta_0$, there is a coupling of the UST Peano curve $X(t)$, $0\le t\le k$, and the chordal SLE$_8$ trace β in D from a to b, such that for some continuous increasing function $u:[0,k)\to[0,\infty)$,

$$\mathbb{P}[\sup_{0\le t<\infty}|\beta(t)-X(u^{-1}(t))|\ge\varepsilon]<\varepsilon.$$

Remark. The theorem implies that chordal SLE$_8$ satisfies reversibility. It together with Wilson's algorithm implies that the boundary of a chordal SLE$_8$ hull stopped at swallowing a given point is an SLE$_2$-type curve. This is one example of the duality property of SLE, which says that the boundary of an SLE$_\kappa$ ($\kappa>4$) hull is an SLE$_{16/\kappa}$ curve.

Here we are not going to give details of the proof, but only introduce the observables that are used. Let T be the UST in the setup. Let X be the Peano curve. Fix a vertex z_0 of D^δ. There is a unique simple path from z_0 to I^δ on T. Let $\mathcal{E}_{z_0,u}$ denote the event that the only simple path on T joining z_0 to I^δ has one end point that lies on the upper side of D. Then $M_n=\mathbb{E}[\mathbf{1}_{\mathcal{E}_{z_0,u}}|X_0,\dots,X_n]$ is a bounded martingale.

We will interpret M_n using discrete harmonic functions. Let V^δ_u denote the set of vertices of D^δ that lie on the upper side of D. Let V^δ_l denote the set of vertices of I^δ minus V^δ_u. From Wilson's algorithm, the simple path on T joining z_0 to I^δ is LERW$(D^\delta;z_0\to I^\delta)$. Thus, the end point of this path is the same as the end point of RW$(D^\delta;z_0\to I^\delta)$. Thus, $M_0=\mathbb{P}[\mathcal{E}_{z_0,u}]=h_{D^\delta;V^\delta_u|V^\delta_l}(z_0)$. When δ is small, M_0 is close to the bounded harmonic function h on D, which equals 1 on the upper side of D, equals to 0 on the left side of D, and whose normal derivative vanishes on the lower side and right side of D.

Suppose $X_0,\dots,X_n$ are given. Let E_n denote the set of edges of D^δ that are intersected by $[X_{j-1},X_j]$, $1\le j\le n$. Let $E^\dagger_n$ denote the set of edges of $D^\delta_\dagger$ that are intersected by $[X_{j-1},X_j]$, $1\le j\le n$. Let E^*_n denote the set of edges of D^δ that are dual of the edges in $E^\dagger_n$. Then T must not contain any edge in E_n, and $T_\dagger$ must not contain any edge in $E^\dagger_n$. So T must contain every edge in E^*_n. Let $G_0=D^\delta$ and $G_n=G_0\setminus E_n$. Let T_n denote the union of the edges in E^*_n together with those on the upper side and left side. Then T_n is a subtree of D^δ. Conditioned on $X_0,\dots,X_n$, T is a UST on G_n conditioned to contain T_n. Thus, $M_n=h_{G_n;V^\delta_u|T_n\setminus V^\delta_u}(z_0)$.

We may construct a continuous harmonic function f_n which is close to the discrete harmonic function $h_{G_n;V_u^\delta|T_n\setminus V_u^\delta}(z_0)$ when δ is small. First, let R be the open rectangle with vertices $(0,-\delta/2)$, $(m_1+\delta/2,-\delta/2)$, $(m_1+\delta/2,m_2)$, $(0,m_2)$. Remove the closed triangle with vertices $(0,0)$, $(0,-\delta/2)$, $(\delta/2,-\delta/2)$ and the closed rectangle with vertices (m_1,m_2), $(m_1+\delta/2,m_2)$, $(m_1+\delta/2,m_2-\delta/2)$, from R. Now $X_0=a$ and b are two boundary points of D_0.

Note that for every vertex v in $(\delta/4,-\delta/4)+D^{\delta/2}$, there is a unique pair (v^1,v^2) such that v^1 is a vertex in D^δ, v^2 is a vertex in $D^\delta_\dagger$, and $v=(v^1+v^2)/2$. So the path $(X_0,\dots,X_k)$ corresponds to a sequence of vertices $(X^1_0,\dots,X^1_k)$ on D^δ and a sequence of vertices $(X^2_0,\dots,X^2_k)$ on $D^\delta_\dagger$. One may notice that for each $1\le s\le k$, either $X^1_s=X^1_{s-1}$ or $X^2_s=X^2_{s-1}$. So there is a closed triangle with vertices X^1_s, X^1_{s-1}, X^2_{s-1}, X^2_s. Let $\Delta_{X,s}$ denote this triangle. Let $D_n=D_0\setminus\bigcup_{s=1}^n\Delta_{X,s}$. Then for $n<k$, D_n is a simply connected Jordan domain whose boundary contains X_n and b. Let I^u_n denote the boundary arc of D_n from X_n to b in the clockwise direction, and let I^r_n denote the other boundary arc of D_n between X_n and b.

Let T_{z_0} denote the first n such that $z_0\in\Delta_{X,n}$. Let T_u denote the first n such that $\Delta_{X,n}$ intersects the upper side of D_0. Then for $n<T_{z_0}\wedge T_u$, $z_0\in D_n$ and I^u_n contains the boundary arc I^u of D_0 from $(0,m_2)$ to b in the clockwise direction. Let h_n denote the bounded harmonic function in D_n which equals 1 on I^u, equals 0 on $I^u_n\setminus I^u$, and whose normal derivative vanishes on I^r_n. Then the value of $M_n=h_{G_n;V_u^\delta|T_n\setminus V_u^\delta}(z_0)$ is close to $h_n(z_0)$ when δ is small.

We now compare the above result on UST with the following result on chordal SLE_8.

Proposition 6.3 *Let D be a simply connected domain with three distinct boundary points a,b,c. Let $\beta(t)$, $0\le t<\infty$, be chordal SLE_8 in D from a to b. Let $D_t=D\setminus\beta(0,t]$. Let $I_{c,b}$ denote the boundary arc of D between c and b that does not contain a. Let T_1 denote the first t such that $\beta(t)\in I_{c,b}$. For $t<T_1$, let I^1_t denote the boundary arc of D_t between $\beta(t)$ and b that contains $I_{c,b}$, and let I^2_t denote the other boundary arc of D_t between $\beta(t)$ and b. For $0\le t<T_1$, let h_t be the bounded harmonic function in D_t, which equals 1 on $I_{c,b}$, equals 0 on $I^1_t\setminus I_{c,b}$, and whose normal derivative vanishes on I^2_t. Fix $z_0\in D$ and let T_{z_0} denote the first time that β visits z_0. Then $h_t(z_0)$, $0\le t<T_1\wedge T_{z_0}$ is a continuous martingale.*

Proof. We may assume that $D=\mathbb{H}$, $a=0$, $c=\infty$, and $b>0$. Suppose the driving function is $\lambda_t=\sqrt{\kappa}B_t$, and g_t are the chordal Loewner maps driven by λ. Suppose W maps $\mathbb{H}$ conformally onto the half strip $\{z\in\mathbb{C}:\mathrm{Re}\,z<0, 0<\mathrm{Im}\,z<1\}$ and maps $0,1,\infty$ to $i,-\infty,0$, respectively. Then $h_t(z)=\mathrm{Im}\,W(\frac{g_t(z)-\lambda_t}{g_t(b)-\lambda_t})$. One can show that $W(\frac{g_t(z)-\lambda_t}{g_t(b)-\lambda_t})$ is a local martingale for any $z\in\mathbb{H}$. We leave the details to the reader. □

Open problems.

1. Construct a lattice model which generates a curve that converges to radial SLE_8.

2. Let T be a UST on D^δ (without conditioning). Describe the scaling limit of the Peano curve surrounding T. Note that if we let $D^\delta_\dagger$ to be the subgraph of $(\delta/2,\delta/2)+\delta\mathbb{Z}^2$ restricted in the rectangle $\{(x,y):-\delta/2\le x\le m_1+\delta/2, -\delta/2\le y\le m_2+\delta/2\}$, then the dual of T is a UST on $D^\delta_\dagger$ with all vertices on the boundary identified as a single vertex.

3. Suppose D is a doubly connected lattice domain with boundary components C_1 and C_2. Let T be the UST on D^δ/C_1, i.e., all vertices of D^δ on C_1 are identified as a single vertex. Describe the scaling limit of the Peano curve surrounding T.

4. Let $G = D^\delta/(C_1 \cup C_2)$, i.e., all vertices of D^δ on $C_1 \cup C_2$ are identified as a single vertex. Let T be the UST on G. Since C_1 and C_2 are identified as the same vertex, there is no path on T connecting C_1 with C_2. So T has two connected components. Now the dual of T is no longer a tree. Instead, it contains a unique simple loop separating C_1 and C_2. Describe the scaling limit of this simple loop.

Remark. In the last problem, if the vertices on C_1 and the vertices on C_2 are identified as two distinct vertices, then there is a unique simple path on T connecting C_1 with C_2. The scaling limit of this path is now well understood, which is an annulus SLE_2 curve.

Because of the limited time, the following interesting topics about SLE are not covered in this course.

1. The existence and continuity of the SLE trace. S. Rhode and O. Schramm.
2. The Hausdorff dimension of the SLE trace. V. Beffara.
3. Intersection components of planar Brownian motions. G. Lawler, O. Schramm, and W. Werner.
4. Convergence of critical site percolation on triangular lattices to SLE_6. S. Smirnov.
5. Convergence of discrete Gaussian free field contour line to SLE_4. S. Sheffield and O. Schramm.
6. Convergence of critical Ising models to SLE_3 and $\text{SLE}_{16/3}$. S. Smirnov.
7. Natural parameterization of SLE. G. Lawler, S. Sheffield and W. Zhou.
8. Brownian loop soup. W. Werner and G. Lawler.
9. Conformal loop ensemble. W. Werner and S. Sheffield.
10. Extending SLE to multiply connected domains.
11. Reversibility of SLE ($\kappa \leq 4$) and duality of SLE.