Decomposition of Schramm–Loewner evolution along its curve

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Abstract

We show that, for \( \kappa \in (0, 8) \), the integral of the laws of two-sided radial SLE_{\kappa} curves through different interior points against a measure with SLE_{\kappa} Green’s function density is the law of a chordal SLE_{\kappa} curve, biased by the path’s natural length. We also show that, for \( \kappa > 0 \), the integral of the laws of extended SLE_{\kappa}(-8) curves through different interior points against a measure with a closed formula density restricted in a bounded set is the law of a chordal SLE_{\kappa} curve, biased by the path’s capacity length restricted in that set. Another result is that, for \( \kappa \in (4, 8) \), if one integrates the laws of two-sided chordal SLE_{\kappa} curves through different force points on \( \mathbb{R} \) against a measure with density on \( \mathbb{R} \), then one also gets a law that is absolutely continuous w.r.t. that of a chordal SLE_{\kappa} curve. To obtain these results, we develop a framework to study stochastic processes with random lifetime, and improve the traditional Girsanov’s Theorem.

Keywords: SLE; Girsanov’s theorem; Doob–Meyer decomposition

1. Introduction

The Schramm–Loewner evolution (SLE), first introduced by Oded Schramm in 1999 [25], is a one-parameter (\( \kappa \in (0, \infty) \)) family of measures on non-self-crossing curves, which has received a lot of attention over the past eighteen years. It has been shown that, modulo time parametrization, several discrete random paths on grids (e.g., loop-erased random walk [14], critical percolation explorer [6,27]) have SLE as a scaling limit.
SLE is defined using Loewner’s differential equation, and is originally parameterized by capacity. For the discrete random paths which converge to SLE, in order to show the convergence, people have to first reparameterize them by capacity and then prove that the reparametrized curve converges to SLE with capacity parametrization. The convergence does not take into consideration the discrete length of the path.

In order to upgrade the convergence results, Lawler and Sheffield introduced the natural parametrization of SLE in [15], and conjectured that those discrete random paths with their original length suitably rescaled, converge to the SLE with natural parametrization. Their construction used the Doob–Meyer decomposition, and they proved the existence of the natural parametrization of SLE\(_\kappa\) for \(\kappa < 5.021\).... This result was later improved by [19], where it was shown that the natural parametrization of SLE\(_\kappa\) exists for all \(\kappa \in (0, 8)\).

It was proved later in [13] that the natural parametrization agrees with the \(d\)-dimensional Minkowski content of the SLE\(_\kappa\) curve, where \(d = 1 + \frac{\kappa}{8}\) is the Hausdorff dimension of the curve (cf. [4]). It was proved recently (cf. [16–18]) that the loop-erased random walk with natural length converges to SLE\(_2\) with natural parametrization. The work uses an earlier result on the convergence of the density of the loop-erased random walk to the Green’s function for SLE\(_2\) (cf. [5]).

There are two major versions of SLE: chordal SLE and radial SLE. Most of the study focuses on chordal SLE, which describes a curve in a simply connected domain from one prime end (cf. [1]) to another prime end. Two-sided radial SLE\(_\kappa\) and SLE\(_\kappa\) Green’s function for \(\kappa \in (0, 8)\) were introduced in [15]. A two-sided radial SLE\(_\kappa\) curve has two arms: the first arm from a prime end to an interior point is a chordal SLE\(_\kappa\)\((\kappa - 8)\) process, and the second arm from the interior point to another prime end is a chordal SLE\(_\kappa\) curve conditioned on the first arm. It can be understood as a chordal SLE\(_\kappa\) curve conditioned to pass through a fixed interior point. While that event has probability zero, some limiting procedure was used to make this idea rigorous. SLE\(_\kappa\) Green’s function was defined by a closed formula, and turned out to be the density of the SLE\(_\kappa\) curve with natural parametrization. These two objects have discrete analogues. The two-sided radial SLE\(_\kappa\) curve corresponds to the discrete random path conditioned to pass through a fixed vertex, and the SLE\(_\kappa\) Green’s function corresponds to the density of the path.

Field recently proved in [8] that, for \(\kappa \in (0, 4]\), if one integrates the laws of two-sided radial SLE\(_\kappa\) in a bounded analytic domain \(D\) passing through different interior points (with the two ends fixed) against the measure with density (w.r.t. the Lebesgue measure on \(\mathbb{C}\)) being the SLE\(_\kappa\) Green’s function in \(D\), then one gets the law of a chordal SLE\(_\kappa\) curve biased by the curve’s length in the natural parametrization. This is analogous to a simple fact of discrete random paths: if one integrates the laws of the path conditioned to pass through different fixed vertices against the probability that the path passes through each fixed vertex, one should get a measure on paths, which is absolutely continuous w.r.t. the law of the original discrete random path, and the Radon–Nikodym derivative is the total number of vertices on the path, which is due to the repetition of counting.

This paper is motivated by Field’s work. We extend his result from \(\kappa \in (0, 4]\) to \(\kappa \in (0, 8)\) (see Corollary 4.3) using a new approach. We do not need to assume that the domain is bounded or analytic. The main tools used here are from Probability Theory.

Our approach is different from Field’s in that we develop a theory of stochastic processes with random lifetime, which is purely probability. The computation here is simpler because the major work: the Doob–Meyer decomposition for the definition of the natural parametrization was done earlier in [15,19]. On the other hand, Field’s proof used some SLE technique such as an escape estimate of SLE derived in his joint work [9] with Lawler. That results in the technical
assumption in his paper that $\kappa \leq 4$ and that the domain is bounded with analytic boundary. Another difference is that we study the measures on the space of curve-point pairs instead of just on curves. This makes the main theorem more convenient to be applied later in [31] to construct SLE loop measures.

After the main theorem, we find other applications of the new technique. We define the extended chordal SLE$_\kappa(\rho)$ curve in the upper half plane $\mathbb{H}$ for $\rho \leq \frac{k}{2} - 4$, which is composed of two arms: the first arm growing from 0 to $z_0 \in \mathbb{H}$ is a chordal SLE$_\kappa(\rho)$ curve, and the second arm growing from $z_0$ to $\infty$ is a chordal SLE$_\kappa$ curve conditioned on the first arm. In particular, a two-sided radial SLE$_\kappa$ curve is an extended chordal SLE$_\kappa(\kappa - 8)$ curve. We prove that, for any $\kappa > 0$, there is a positive function $G^{\kappa - 8}(z)$ with closed formula such that for any measurable $U \subset \mathbb{H}$ on which $G^{\kappa - 8}$ is integrable, if one integrates the laws of extended chordal SLE$_\kappa(\kappa - 8)$ curve through different $z$ against a measure with density $\textbf{1}_U G^{\kappa - 8}(z)$, then one gets the law of a chordal SLE$_\kappa$ curve biased by $C_{\kappa,1}$ times the total time that the curve spends in $U$ in the capacity parametrization (see Corollary 5.1), where $C_{\kappa,1}$ is a positive constant depending only on $\kappa$.

The above two main results of this paper immediately imply that, if we sample a point on a chordal SLE$_\kappa$ curve in $\mathbb{H}$ according to a law, which is absolutely continuous w.r.t. the natural parametrization (resp. capacity parametrization), and stop the curve at that point, then we get a random curve, whose law is absolutely continuous w.r.t. that of a chordal SLE$_\kappa(\kappa - 8)$ (resp. SLE$_\kappa(-8)$) curve in $\mathbb{H}$.

The connection between SLE$_\kappa$ near its tip at a fixed capacity time and SLE$_\kappa(-8)$ was derived earlier in [7, Proposition 3.10]. The tip behavior of SLE was also studied in [30] for $\kappa \in (0, 4)$. What new here is that we derive the Green’s function for the SLE in capacity parametrization up to a multiplicative constant.

Another application of the new technique is to study the intersection of SLE curve with the boundary. Using a Doob–Meyer decomposition, Alberts and Sheffield constructed in [3] a measure supported by the intersection of an SLE$_\kappa$ curve $\gamma$ in $\mathbb{H}$ for $\kappa \in (4, 8)$ with $\mathbb{R}$, and conjectured that the measure agrees with the $(2 - \frac{8}{\kappa})$-dimensional Minkowski content of $\gamma \cap \mathbb{R}$. Their work used two-sided chordal SLE$_\kappa$, which can be understood as a chordal SLE$_\kappa$ curve conditioned to pass through a fixed point on the boundary. Using their result, we prove in this paper that, if one integrates the laws of two-sided chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to $\infty$ through different $x \in \mathbb{R}$ against a measure with a suitable density function (known as boundary Green’s function for SLE) against the Lebesgue measure on a bounded interval $I \subset \mathbb{R}$, then one gets a measure on curves, which is absolutely continuous w.r.t. the law of a chordal SLE$_\kappa$ curve, and the Radon–Nikodym derivative is the measure constructed in [3] of the intersection of the curve with $I$.

The new technique has applications beyond the SLE area. For example, we use it to decompose a planar Brownian motion restricted in a simply connected domain.

In the author’s recent preprint [31], the decomposition of chordal SLE into two-sided radial SLE is used to construct SLE loop measures.

The paper is organized as follows. In Section 2, in order to study the driving functions of SLE$_\kappa(\rho)$ curves, we develop a framework on stochastic processes with random lifetime. We introduce the “local absolute continuity” between these processes, and extend the traditional Girsanov’s theorem. In Section 3, we review the definitions and basic properties of SLE$_\kappa$ and its variants, SLE$_\kappa(\rho)$ processes. In Section 4, we prove our main result about natural parametrization and two-sided radial SLE curves. In Section 5, we prove the result about capacity parametrization and extended chordal SLE$_\kappa(-8)$ curves. In Section 6, we show the application on boundary measure and two-sided chordal SLE curves. In Section 7, we use the technique to decompose
planar Brownian motions. In the Appendix, we prove the transience property of chordal SLE$_k(\rho)$ curves.

2. Stochastic processes with random lifetime

For $0 < T \leq \infty$, let $C([0, T))$ denote the space of real valued continuous functions on $[0, T)$. Let

$$\Sigma = \bigcup_{0 < T \leq \infty} C([0, T)).$$

For each $f \in \Sigma$, let $T_f$ be such that $[0, T_f)$ is the domain of $f$.

We define two basic operations on $\Sigma$: killing and continuing. For $0 < \tau \leq \infty$, we define the killing map $K_\tau : \Sigma \rightarrow \Sigma$ such that if $g = K_\tau(f)$, then $T_g = \tau \wedge T_f$ and $g = f|_{[0, T_f)}$. Let $\Sigma^\oplus = \{f \in \Sigma : T_f < \infty, f(T_f^-) := \lim_{t \uparrow T_f^-} f(t) \in \mathbb{R}\}$ and $\Sigma^\ominus = \{f \in \Sigma : f(0) = 0\}$. For example, if $T_f > \tau > 0$, then $K_\tau(f) \in \Sigma^\ominus$. For $f \in \Sigma^\oplus$ and $g \in \Sigma^\ominus$, we may continue $f$ using $g$ and get the function $f \oplus g \in \Sigma$, which is defined by $T_{f \oplus g} = T_f + T_g$

$$f \oplus g(t) = \begin{cases} f(t), & 0 \leq t < T_f; \\ f(T_f^-) + g(t - T_f), & T_f \leq t < T_f + T_g. \end{cases}$$

Sometimes we want to record the time that the two functions are joined together. For this purpose, we define $f \hat{\oplus} g = (f \oplus g, T_f)$. Then we can use $f \hat{\oplus} g$ to recover $f$ and $g$.

For $0 \leq t < \infty$, let $\mathcal{F}_t^0$ be the $\sigma$-algebra generated by

$$\{f \in \Sigma : s < T_f, f(s) \in U\}, \quad 0 \leq s \leq t, U \in \mathcal{B}(\mathbb{R}).$$

Then $(\mathcal{F}_t^0)$ is a filtration on $\Sigma$. Let $\mathcal{F}^0 = \sigma(\bigcup_{0 \leq t < \infty} \mathcal{F}_t^0)$. We will mainly work on the measurable space $(\Omega, \mathcal{F}^0)$ or its completion w.r.t. a certain measure. Every probability measure on $(\Sigma, \mathcal{F}^0)$ is the law of a continuous stochastic process with random lifetime. For any measure $\mu$ on $(\Sigma, \mathcal{F}^0)$, we use $\mathcal{F}_t^\mu$ and $\mathcal{F}_T^\mu$ to denote the $\mu$-completion of $\mathcal{F}^0$ and $\mathcal{F}_T^0$, respectively.

The continuing maps $(f, g) \mapsto f \oplus g$ and $(f, g) \mapsto f \hat{\oplus} g$ are measurable. If $\mu$ and $\nu$ are $\sigma$-finite measures supported by $\Sigma^\oplus$ and $\Sigma^\ominus$, respectively, we use $\mu \oplus \nu$ and $\mu \hat{\oplus} \nu$ to denote the pushforward measures of the product measure $\mu \otimes \nu$ under the maps $(f, g) \mapsto f \oplus g$ and $(f, g) \mapsto f \hat{\oplus} g$, respectively.

Let us recall an important notion in Probability: kernel. Suppose $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are two measurable spaces. A kernel from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is a map $\nu : U \times \mathcal{V} \rightarrow [0, \infty]$ such that (i) for every $u \in U$, $\nu(u, \cdot)$ is a measure on $(V, \mathcal{V})$, and (ii) for every $F \in \mathcal{V}$, $\nu(\cdot, U)$ is $\mathcal{U}$-measurable. Let $\mu$ be a $\sigma$-finite measure on $(U, \mathcal{U})$. Let $\mathcal{U}^\mu$ be the $\mu$-completion of $\mathcal{U}$. A $\mu$-kernel from $(U, \mathcal{U}^\mu)$ to $(V, \mathcal{V})$ is a kernel from $(U^\mu, \mathcal{U}^\mu \cap \mathcal{U}^\mu)$ to $(V, \mathcal{V})$, where $U^\mu \subset U$ is such that $U \setminus U^\mu$ is a $\mu$-null. The $\mu$-kernel is said to be finite if for $\mu$-a.s. every $u \in U$, $\nu(u, V) < \infty$; and is said to be $\sigma$-finite if there is a sequence $F_n \in \mathcal{V}$, $n \in \mathbb{N}$, with $V = \bigcup F_n$ such that for any $n \in \mathbb{N}$, and $\mu$-a.s. every $u \in U$, $\nu(u, F_n) < \infty$. If $\nu$ is a $\sigma$-finite $\mu$-kernel from $(U, \mathcal{U})$ to $(V, \mathcal{V})$, then we may define a measure $\mu \otimes \nu$ on $U \times V$ such that

$$\mu \otimes \nu(E \times F) = \int_E \nu(u, F) d\mu(u), \quad E \in \mathcal{U}, \quad F \in \mathcal{V}.$$
If $\nu$ is a $\sigma$-finite measure on $(V, \mathcal{V})$, and $\mu$ is a $\sigma$-finite $\nu$-kernel from $(V, \mathcal{V})$ to $(U, \mathcal{U})$, then we use $\mu \otimes \nu$ to denote the pushforward measure on $\mathcal{U} \times \mathcal{V}$ of $\mu \otimes \nu$ under the map $(u, v) \mapsto (u, v)$.

The killing map $K : (f, r) \mapsto K_r(f)$ is also measurable. If $\nu$ is a $\sigma$-finite $\mu$-kernel from $\Sigma$ to $(0, \infty)$, we use $K_\nu(\mu)$ to denote the pushforward measure of $\mu \otimes \nu$ under $K$.

Let $\Sigma_t = \{ f \in \Sigma : T_f > t \}, t \geq 0$. Then $(\Sigma_t)$ is a decreasing family of subspaces of $\Sigma$ with $\Sigma_0 = \Sigma$ and $\Sigma_\infty = \bigcap_{t=0}^{\infty} \Sigma_t = C([0, \infty))$. We will be interested in the restriction of $\mathcal{F}_t^0$ to $\Sigma_t$, i.e., $\mathcal{F}_t^0 \cap \Sigma_t$. Note that $\bigcup_{0 \leq t < \infty} \mathcal{F}_t^0 \cap \Sigma_t$ is a $\pi$-system, which generates the $\sigma$-algebra $\mathcal{F}_0^0$, and $\Sigma = \Sigma_0 \in \mathcal{F}_0^0 \cap \Sigma_0$. This enables us to apply Dynkin’s $\pi - \lambda$ theorem. For example, if two finite measures on $(\Sigma, \mathcal{F}_0^0)$ agree on each $\mathcal{F}_t^0 \cap \Sigma_t$, then they are equal. Let $\mu$ and $\nu$ be two measures on $(\Sigma, \mathcal{F}_0^0)$, which are $\sigma$-finite on $\mathcal{F}_0^0$ (and so are $\sigma$-finite on each $\mathcal{F}_t^0$). We say that $\nu$ is locally absolutely continuous w.r.t. $\mu$, and write $\nu < \mu$, if for any $0 \leq t < \infty$, $\nu|_{\mathcal{F}_t^0 \cap \Sigma_t}$ is absolutely continuous w.r.t. $\mu|_{\mathcal{F}_t^0 \cap \Sigma_t}$. This is certainly the case if $\nu \ll \mu$, i.e., $\nu$ is (globally) absolutely continuous w.r.t. $\mu$. The process $M_t := \frac{d\nu|_{\mathcal{F}_t^0 \cap \Sigma_t}}{d\mu|_{\mathcal{F}_t^0 \cap \Sigma_t}}, 0 \leq t < \infty$, is called the local Radon–Nikodym derivative of $\nu$ w.r.t. $\mu$. By Dynkin’s $\pi - \lambda$ theorem, we see that $\nu$ is determined by $\mu$ and $(M_t)$. Thus, we say that $\nu$ can be obtained by locally weighting $\mu$ by $(M_t)$.

**Proposition 2.1.** Let $\mu$ be a measure on $(\Sigma, \mathcal{F}_0^0)$, which is $\sigma$-finite on $\mathcal{F}_0^0$. Let $(\Upsilon, \mathcal{G})$ be a measurable space. Let $\nu : \Upsilon \times \mathcal{F}_0^0 \to [0, \infty]$ be such that for every $\nu \in \Upsilon$, $\nu(\cdot, \cdot)$ is a finite measure on $\mathcal{F}_0^0$ that is locally absolutely continuous w.r.t. $\mu$. Moreover, suppose that the local Radon–Nikodym derivatives are equal to $(M_t(\cdot, \cdot))$, where $M_t : (\Upsilon, \mathcal{G}) \times (\Sigma, \mathcal{F}_0) \to [0, \infty)$ is measurable for every $t \geq 0$. Then $\nu$ is a kernel from $(\Upsilon, \mathcal{G})$ to $(\Sigma, \mathcal{F}_0^0)$. Moreover, if $\xi$ is a $\sigma$-finite measure on $(\Upsilon, \mathcal{G})$ such that $\mu$-a.s., $\int_{\Upsilon} M_t(\nu(\cdot, \cdot))d\xi(\nu) < \infty$ for all $t \geq 0$, then $\xi \cdot \nu < \mu$, and the local Radon–Nikodym derivatives are $\int_{\Upsilon} M_t(\nu(\cdot, \cdot))d\xi(\nu), 0 \leq t < \infty$.

**Proof.** By Dynkin’s $\pi - \lambda$ theorem, to prove that $\nu(\cdot, \cdot)$ is measurable, it suffices to show that, for any $t \in [0, \infty)$ and any $A \in \mathcal{F}_t^0 \cap \Sigma_t$, $\nu(\cdot, A)$ is measurable, which easily follows from Tonelli’s theorem because $\nu(\cdot, A) = \int_A M_t(\nu(\cdot, f))d\mu(f)$. To prove that $\xi \cdot \nu < \mu$ and find the local Radon–Nikodym derivatives, we apply Tonelli’s theorem again and get

$$
\xi \cdot \nu(A) = \int_{\Upsilon} \nu(\cdot, A)d\xi(\nu) = \int_{\Upsilon} \int_A M_t(\nu(\cdot, f))d\mu(f)d\xi(\nu)
$$

for $A \in \mathcal{F}_t^0 \cap \Sigma_t$. Then we conclude that $\xi \cdot \nu < \mu$, and conclude that the local Radon–Nikodym derivatives are $\int_{\Upsilon} M_t(\nu(\cdot, \cdot))d\xi(\nu), 0 \leq t < \infty$. □

**Proposition 2.2.** Let $\mu$ be a probability measure on $(\Sigma, \mathcal{F}_0^0)$. Let $\xi$ be a $\mu$-kernel from $(\Sigma, \mathcal{F}_0^0)$ to $(0, \infty)$ that satisfies $\mathbb{E}_\mu[|\xi|] < \infty$. Then $K_\xi(\mu) < \mu$, and the local Radon–Nikodym derivatives are $\mathbb{E}_\mu[\xi((t, \infty))|\mathcal{F}_t^0], 0 \leq t < \infty$.

**Proof.** Fix $t \in [0, \infty)$, and $E \in \mathcal{F}_t^0 \cap \Sigma_t$. It is easy to see that $K_\xi(f) \in E$ iff $f \in E$ and $r > t$, which implies that

$$
K_\xi(\mu)(E) = \mu \otimes \xi(E \times (t, \infty)) = \int_E \xi(f, (t, \infty))d\mu(f) = \int_E \mathbb{E}_\mu[\xi((t, \infty))|\mathcal{F}_t^0]d\mu.
$$

Then we get the conclusion. □
Remark. Proposition 2.2 will be mainly applied to the case that $\xi = d\theta$, where $(\theta_t)$ is an $(\mathcal{F}_t^\mu)$-adapted right-continuous increasing process with $\theta_0 = 0$ and $\mathbb{E}_\mu[\theta_\infty] < \infty$. Applying the proposition, we find that $\mathcal{K}_{d\theta}(\mu) < \mu$, and

$$\frac{d\mathcal{K}_{d\theta}(\mu)_{|\mathcal{F}_t^\mu_{\cap\Sigma_t}}}{d\mu_{|\mathcal{F}_t^\mu_{\cap\Sigma_t}}} = \mathbb{E}_\mu[\theta_\infty|\mathcal{F}_t^\mu] - \theta_t. \quad (2.1)$$

Fix $\kappa > 0$. Let $\mathbb{P}_k$ be the law of $\sqrt{\kappa}$ times a standard Brownian motion. This means that $\frac{1}{\sqrt{\kappa}}$ times the coordinate process on $\Sigma$ under $\mathbb{P}_k$ is a standard Brownian motion. For this reason, we use $(B_t)$ to denote the above standard Brownian motion on $\Sigma$, i.e., $\frac{1}{\sqrt{\kappa}}$ times the coordinate process. We observe that $\mathbb{P}_k$ is supported by $\mathbb{F}_\infty \cap \Sigma_\emptyset$. Let $\mathcal{F}_t^B$ and $\mathcal{F}^B$ be the $\mathbb{P}_k$-completion of $\mathcal{F}^0_t$ and $\mathcal{F}^0$, respectively; and $\mathbb{E}_k$ denote the expectation w.r.t. $\mathbb{P}_k$.

We now use Girsanov’s theorem to derive local Radon–Nikodym derivatives. Recall that when we used Girsanov’s theorem to weight a probability measure by a positive local martingale, we had to stop the process at some stopping time to get a bounded martingale. The following proposition says that we do not need to do the stopping, and the local martingales valued at different times are just the local Radon–Nikodym derivatives.

**Proposition 2.3.** Suppose that $(X_t)_{0 \leq t < T_0}$ satisfies $X_0 = 0$ and the $(\mathcal{F}_t^B)$-adapted SDE:

$$dX_t = \sqrt{\kappa} dB_t + \sigma_t dt, \quad 0 \leq t < T_0,$$

where $T_0$ is a positive $(\mathcal{F}_t^B)$-stopping time and $(\sigma_t)_{0 \leq t < T_0}$ is a real valued $(\mathcal{F}_t^B)$-adapted continuous process. Let $\mathbb{P}^{k,\sigma}$ denote the law of $X$. Then $\mathbb{P}^{k,\sigma} < \mathbb{P}_k$. Moreover, if $M_t$, $0 \leq t < T_0$, is an $(\mathcal{F}_t^B)$-adapted continuous local martingale that satisfies $M_0 = 1$ and the SDE:

$$dM_t = M_t \frac{\sigma_t}{\sqrt{\kappa}} dB_t, \quad 0 \leq t < T_0. \quad (2.2)$$

then

$$\frac{d\mathbb{P}^{k,\sigma}_{|\mathcal{F}_t^0_{\cap\Sigma_t}}}{d\mathbb{P}_k_{|\mathcal{F}_t^0_{\cap\Sigma_t}}} = 1_{T_0 > t} M_t, \quad 0 \leq t < \infty. \quad (2.3)$$

**Proof.** Let $M_t = \exp(\int_0^t \frac{\sigma_u}{\sqrt{\kappa}} dB_u - \int_0^t \frac{\sigma_u^2}{2\kappa} ds)$, $0 \leq t < T_0$. From Itô’s formula (cf. [22]), we see that $(M_t)$ satisfies $M_0 = 1$ and (2.2). Thus, it suffices to prove (2.3).

Fix $N \in \mathbb{N}$ and let $\tau_N = \inf\{0 \leq t < T_0 : |M_t| \geq N\}$. Here we set $\inf\emptyset = T_0$. Then $\tau_N$ is a stopping time with $\tau_N \leq T_0$, and $M_t$, $0 \leq t < \tau_N$, is uniformly bounded by $N$. If $\tau_N = T_0$, then $\mathbb{P}_k$-a.s. $\lim_{t \to T_0^-} M_t$ exists. Let

$$M_t^{TN} := \begin{cases} M_t \wedge \tau_N, & t \wedge \tau_N < T_0; \\
\lim \inf_{s \to T_0^-} M_s, & t \geq \tau_N = T_0. \end{cases}$$

Then $(M_t^{TN}, 0 \leq t < \infty)$ is a uniformly bounded ($\mathbb{P}_k$-a.s.) continuous local martingale, and satisfies the SDE:

$$dM_t^{TN} = M_t^{TN} 1_{t < \tau_N} \frac{\sigma_t}{\sqrt{\kappa}} dB_t, \quad 0 \leq t < \infty. \quad (2.4)$$

Let $M^{TN}_\infty = \lim_{t \to \infty} M_t^{TN}$. Then for any $0 \leq t < \infty$, a.s. $\mathbb{E}_k[M^{TN}_\infty|\mathcal{F}^B_t] = M_t^{TN}$. In particular, since $M_0^{TN} = 1$, we have $\mathbb{E}_k[M^{TN}_\infty] = 1$. Define $\mathbb{P}^{k,\sigma}_N$ such that $\frac{d\mathbb{E}_N^{k,\sigma}}{d\mathbb{E}_k} = M^{TN}_\infty$. Then $\mathbb{P}^{N}_k$ is also a
probability measure on \((\Sigma^*, \mathcal{F}^0)\). Let
\[
B_t^N = B_t - \int_0^{\bar{\tau}^N_{\kappa, \sigma}} \frac{\sigma_s}{\sqrt{k}} ds, \quad 0 \leq t < \infty.
\]

From Girsanov’s theorem (cf. [22]) and (2.4), we know that the law of \(B^N\) under \(\mathbb{P}^{\kappa, \sigma}_N\) is also that of a standard Brownian motion. From
\[
\sqrt{k} B_t = \sqrt{k} B_t^N + \int_0^t \sigma_s ds, \quad 0 \leq t < \tau_N.
\]
we see that the law of \((\sqrt{k} B_t, 0 \leq t < \tau_N)\) under \(\mathbb{P}^{\kappa, \sigma}_N\) is the same as the law of \((X_t, 0 \leq t < \tau_N)\) under \(\mathbb{P}_\kappa\).

Fix \(t \in [0, \infty)\) and \(E \in \mathcal{F}_t^0 \cap \Sigma_t\). Since \(T_0\) is the lifetime of \(X\), and \(T_0 = \sup_{N \in \mathbb{N}} T_N\), we have
\[
X^{-1}(E) \subset X^{-1}(\Sigma_t) \subset \{T_0 > t\} = \bigcup_{N \in \mathbb{N}} \{T_N > t\}.
\]
Since the law of \((\sqrt{k} B_t, 0 \leq t < \tau_N)\) under \(\mathbb{P}^{\kappa, \sigma}_N\) is the same as the law of \((X_t, 0 \leq t < \tau_N)\) under \(\mathbb{P}_\kappa\), and \((\sqrt{k} B_t)\) is the coordinate process, we get
\[
\mathbb{P}_\kappa[X^{-1}(E) \cap \{T_N > t\}] = \mathbb{P}^{\kappa, \sigma}_N[E \cap \{T_N > t\}] = \mathbb{E}_\kappa[1_{E \cap \{T_N > t\}} M^N_t]
\]
where the third equality follows from the optional stopping theorem, and the last equality holds because \(M^N_t = M_t\) on \(\{T_N > t\}\). This together with (2.5) implies that
\[
\mathbb{P}^{\kappa, \sigma}(E) = \mathbb{P}_\kappa[X^{-1}(E)] = \lim_{N \to \infty} \mathbb{P}_\kappa[X^{-1}(E) \cap \{T_N > t\}] = \mathbb{E}_\kappa[1_{E \cap \{T_0 > t\}} M_t].
\]
So we get (2.3) and finish the proof. \(\Box\)

At the end of this section, we state and prove the following proposition, which extends the strong Markov property of Brownian motions.

**Proposition 2.4.** Let \((\theta_t)_{0 \leq t < \infty}\) be a right-continuous increasing \((\mathcal{F}^B_t)\)-adapted process that satisfies \(\theta_0 = \theta_0^+ = 0\) and \(\mathbb{E}_\kappa[\theta_\infty] < \infty\). Then
\[
\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d\theta.
\]
Thus, \(\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa \ll \mathbb{P}_\kappa\), and \(\theta_\infty\) is the Radon–Nikodym derivative.

**Remark.** If \(\theta_t = 1_{t \leq \tilde{\tau}}\), where \(\tilde{\tau}\) is a positive finite \((\mathcal{F}^B_t)\)-stopping time, then the proposition reduces to the strong Markov property of \((\sqrt{k} B_t)\), i.e., \(\mathcal{K}_{\tilde{\tau}}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa = \mathbb{P}_\kappa\).

**Proof.** First, assume that there is \(t_0 \in (0, \infty)\) and \(E \in \mathcal{F}^B_{t_0}\) such that \(\theta_{t_0}(f) = 1_{E}(f) \cdot 1_{[t_0, \infty)}(t)\). Fix \(t_0' \in (0, t_0)\), \(A \in \mathcal{F}^B_{t_0'}\), and \(B \in \mathcal{F}^B_{t_0}\). For every \(r \in [0, \infty)\), define \(S_r : \Sigma_r \to \Sigma_{r'}\) such that if \(g = S_r(f)\), then \(T_g = T_f - r\), and \(g(t) = f(r + t) - f(r), 0 \leq t < T_g\). Let \(A \oplus_{t_0} B = \{f \in A \cap \Sigma_{t_0} : S_{t_0}(f) \in B\}\). Since \(S_{t_0}(\mathcal{K}_{t_0}(f) \oplus g) = g\), and \(\mathcal{K}_{t_0}(f) \in A\) iff \(f \in A\) and \(r > t_0'\), we get
\[
\mathcal{K}_{d\theta}(\mathbb{P}_\kappa) \oplus \mathbb{P}_\kappa(A \oplus_{t_0} B) = \mathcal{K}_{d\theta}(\mathbb{P}_\kappa)(A) \mathbb{P}_\kappa(B) = \mathbb{P}_\kappa \otimes d\theta(A \times (t_0', \infty)) \mathbb{P}_\kappa(B) = \mathbb{P}_\kappa(A \cap E) \mathbb{P}_\kappa(B).
\]
From the Markov property of \((\sqrt{\kappa} B_t)\), we get
\[
\int_{A \oplus_0 B} \theta_\infty dP_k = \int_{A \oplus_0 B} 1_E dP_k = P_k((A \cap E) \oplus_0 B) = P_k(A \cap E) \cdot P_k(B).
\]

Define \(P_k^0\) such that \(dP_k^0/dP_k = \theta_\infty\). From the above two displayed formulas, we see that \(K_{d\theta}(P_k) \oplus P_k\) agrees with \(P_k^0\) on the sets \(A \oplus_0 B\), where \(A \in \mathcal{F}_{t_0}', \ t_0' \in (0, t_0),\) and \(B \in \mathcal{F}^B\). Since these sets form a \(\pi\)-system, Dynkin’s \(\pi - \lambda\) theorem implies that the two measures agree on the \(\sigma\)-algebra generated by these sets, which agrees with \(\mathcal{F}^B\) restricted to \(\Sigma_0\). Since both measures are supported by \(\Sigma_\infty \subset \Sigma_{t_0}\), we get \(K_{d\theta}(P_k) \oplus P_k = P_k^0\). Since these two measures are the projections of \(K_{d\theta}(P_k) \hat{\oplus} P_k\) and \(P_k \otimes d\theta\), respectively, to \(\Sigma\), and the projections of them to \((0, \infty)\) are both concentrated at \(t_0\), we get (2.6) in this special case.

Second, assume that \((\theta_t)\) has the form of \(\sum_{n=1}^\infty C_n \theta_t^{(n)}\), where each \(C_n\) is a nonnegative real number and each \(\theta_t^{(n)}\) satisfies the condition in the previous paragraph. In this case, we get (2.6) using the result in the above paragraph and the fact that both sides of (2.6) satisfy the countable linearity in \((\theta_t)\).

Finally, we consider the general case. Since \(E_k[\theta_\infty] < \infty\), from the linearity of both sides of (2.6) in \(d\theta\), we may assume that \(E_k[\theta_\infty] = \frac{1}{2}\). In this case both sides of (2.6) are probability measures. For \(n \in \mathbb{N}\), define \(U^{(n)}\) and \(L^{(n)}\) from \([0, \infty)\) to \([0, \infty)\) such that
\[
U^{(n)}(t) = \frac{2^n \cdot t + 1}{2^n}, \quad L^{(n)}(t) = 0 \vee \frac{2^n \cdot t - 1}{2^n}.
\]
Then \(U^{(n)}(t) \downarrow t\) and \(L^{(n)}(t) \uparrow t\) for any \(t \in [0, \infty)\). Moreover, we have
\[
L^{(n)}(t) \leq s \quad \text{if and only if} \quad t \leq U^{(n)}(s), \quad \forall t, s \in [0, \infty).
\]
(2.7)
This equivalence holds because for any \(t, s \geq 0\), both sides are equivalent to that there is an integer \(n\) such that \(2^n t \leq n \leq 2^n s + 1\). Define \((\theta^{(n)}_t)\) such that \(\theta^{(n)}_t = \theta_{U^{(n)}(t)}\). Then \((\theta^{(n)}_t)\) has the form of that in the above paragraph since it takes values only at \(k/2^n, k \in \mathbb{Z}\). Thus, \(K_{d\theta^{(n)}}(P_k) \hat{\oplus} P_k = P_k \otimes d\theta^{(n)}\) for each \(n \in \mathbb{N}\). From (2.7), we get
\[
L^{(n)}_s(d\theta(f, \cdot)) = d\theta^{(n)}(f, \cdot), \quad \forall f \in \Sigma, n \in \mathbb{N}.
\]
(2.8)
We assign \(\Sigma\) the topology of locally uniform convergence. It suffices to show that \(K_{d\theta^{(n)}}(P_k) \hat{\oplus} P_k\) and \(P_k \otimes d\theta^{(n)}\) converge weakly to \(K_{d\theta}(P_k) \hat{\oplus} P_k\) and \(P_k \otimes d\theta\), respectively. To prove that \(K_{d\theta^{(n)}}(P_k) \hat{\oplus} P_k \to K_{d\theta}(P_k) \hat{\oplus} P_k\), we define \(\Sigma \times (0, \infty)\)-valued random variables \(h^{(n)}\) and \(h\) on a probability space \((\Omega, \mathcal{F})\) such that their laws are the above measures, and a.s. \(h^{(n)} \to h\). For this purpose, we choose \(\Omega = (\Sigma_\infty \times (0, \infty)) \times \Sigma_\mathcal{F}, \ P = (P_k \otimes d\theta) \times P_k, \ h((f, t), g) = (K_t(f) \otimes g, t),\) and \(h^{(n)}((f, t), g) = h((f, L^{(n)}(t)), g).\) Using (2.8) and the convergence \(L^{(n)}(t) \uparrow t\) it is easy to check that \(h^{(n)}\) and \(h\) satisfy the desired properties. To prove that \(P_k \otimes d\theta^{(n)} \to P_k \otimes d\theta\), we choose \(\Omega = \Sigma \times (0, \infty), \ P = P_k \otimes d\theta, \ h = \text{id}_{\mathcal{F}}\), and \(h^{(n)}((f, t), g) = (f, L^{(n)}(t)).\) Then a.s. \(h^{(n)} \to h\), and from (2.8) we see that the laws of \(h^{(n)}\) and \(h\) are \(P_k \otimes d\theta^{(n)}\) and \(P_k \otimes d\theta\), respectively. So we get (2.6) in the general case.

The statement after (2.6) follows from projecting both sides of (2.6) to \(\Sigma\). □

3. Schramm–Loewner evolution

In this section, we review the Loewner equations and the Schramm–Loewner Evolution (SLE). See [11, 23] for more details. We focus on chordal SLE, and will often omit the word “chordal” before “Loewner equation” or “SLE” when there is no confusion.
The definition of SLE uses the Loewner equations. Let us first review the (chordal) Loewner equation. Let \( \lambda \in C([0, T]) \), where \( T \in (0, \infty] \). The Loewner equation driven by \( \lambda \) is the following differential equation in the complex plane:
\[
\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \lambda(t)}, \quad 0 \leq t < T; \quad g_0(z) = z.
\]

Let \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \). For \( 0 \leq t < T \), let \( K_t \) denote the set of \( z \in \mathbb{H} \) such that the solution \( s \mapsto g_s(z) \) blows up before or at \( t \). It turns out that \( g_t \) maps \( \mathbb{H} \setminus K_t \) conformally onto \( \mathbb{H} \), and satisfies \( g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}) \) as \( z \to \infty \). We call \( g_t \) and \( K_t \) the Loewner maps and hulls, respectively, driven by \( \lambda \).

Suppose for every \( t \in [0, T) \), \( g_t^{-1} \) extends continuously from \( \mathbb{H} \) to \( \mathbb{H} \). Throughout, we use \( f_t \) to denote the continuation of \( g_t^{-1} \) from \( \mathbb{H} \) into \( \mathbb{H} \). Also suppose that \( \gamma(t) := f_t(\lambda(t)) \), \( 0 \leq t < T \), is a continuous curve in \( \mathbb{H} \). Then we say that \( \gamma \) is the Loewner curve driven by \( \lambda \). In this case, for \( 0 \leq t < T \), \( \mathbb{H} \setminus K_t \) is the unbounded connected component of \( \mathbb{H} \setminus \gamma([0, t]) \). The Loewner curve driven by \( \lambda \) may not exist in general.

The Loewner equations satisfy the following scaling and translation properties. Suppose \( \lambda(t), 0 \leq t < T \), generates Loewner maps \( g_t \) and hulls \( K_t \), \( 0 \leq t < T \). Let \( a > 0 \) and \( b \in \mathbb{R} \), and \( \lambda^{a,b}(t) = b + a \cdot \lambda(t/a^2) \), \( 0 \leq t < a^2T \). Then \( \lambda^{a,b} \) generates the Loewner maps \( z \mapsto b + a \cdot g_t(z/(z-b)/a) \) and hulls \( b + a \cdot K_{t/a^2} \), \( 0 \leq t < a^2T \). If \( \lambda \) generates a Loewner curve \( \gamma \), then \( \lambda^{a,b} \) also generates a Loewner curve, which is \( b + a \cdot \gamma(\cdot/a^2) \).

Another simple and useful property of the Loewner equations is the renewal property. Suppose \( \lambda(t), 0 \leq t < T \), generates Loewner maps \( g_t \) and hulls \( K_t \), \( 0 \leq t < T \). Let \( \tau \in [0, T) \). Then \( \lambda(t+\tau), 0 \leq t < T - \tau \), generates Loewner maps \( g_t \circ g_{\tau}^{-1} \) and hulls \( g_t(K_{t+\tau} \setminus K_t) \), \( 0 \leq t < T - \tau \). If \( \lambda \) and \( \lambda(t+\cdot) \) generate Loewner curves \( \gamma \) and \( \gamma_t \), respectively, then \( \gamma(t+\tau) = \gamma_t(\tau \gamma(t)) \), \( 0 \leq t < T - \tau \).

Let \( \Sigma^C \) denote the counterpart of \( \Sigma \) with real valued continuous functions replaced by complex valued continuous functions. Let \( \Sigma^C \) denote the set of driving functions \( \lambda \in \Sigma \) that generate a Loewner curve \( \gamma \). Then \( \Sigma^C \subseteq \mathcal{F}^0 \), and the Loewner map \( \mathcal{L} : \lambda \mapsto \gamma \) from \( \Sigma^C \) to \( \Sigma^C \) is measurable. We also define here the extended Loewner map \( \hat{\mathcal{L}} \) from \( \{ \lambda, t \} : \lambda \in \Sigma^C, 0 \leq t < T_\lambda \} \) to \( \Sigma^C \times \mathbb{H} \) such that \( \hat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t)) \).

Fix \( \kappa > 0 \). Let \( B(t), 0 \leq t < \infty \), be a standard Brownian motion. The SLE\(_\kappa \) process is defined by taking \( \lambda(t) = \sqrt{\kappa} B(t) \), \( 0 \leq t < \infty \), in the Loewner equation. In this case, the Loewner curve \( \gamma \) driven by \( \lambda \) a.s. exists, and satisfies \( \gamma(0) = 0 \) and \( \lim_{t \to \infty} \gamma(t) = \infty \). Such \( \gamma \) is called a standard SLE\(_\kappa \) curve (in \( \mathbb{H} \) from 0 to \( \infty \)). In terms of measures, this means that \( \mathbb{P}_\kappa \) (the law of \( (\sqrt{\kappa} B_t) \)) is supported by \( \Sigma^C \). The pushforward measure \( \mathcal{L}_\kappa(\mathbb{P}_\kappa) \) is then the law of a standard SLE\(_\kappa \) curve.

The scaling property of Loewner equations and the scaling property of Brownian motions together imply the scaling property of the SLE curve: if \( \gamma \) is a standard SLE\(_\kappa \) curve, then \( t \mapsto a \gamma(t/a^2) \) is also a standard SLE\(_\kappa \) curve. The renewal and translation properties of Loewner equations and the strong Markov property of Brownian motions together imply the domain Markov property of SLE: if \( \gamma \) is a standard SLE\(_\kappa \) curve, and \( \tau \) is a finite stopping time, then conditioned on \( \gamma(t), t \leq \tau \), there is a standard SLE\(_\kappa \) curve \( \hat{\gamma} \) such that \( \gamma(t+\tau) = f_\tau(\hat{\gamma}(t+\lambda_t), t \geq 0 \).

The definition of SLE\(_\kappa \) extends to other simply connected domains by conformal maps. Let \( D \) be a simply connected domain with locally connected boundary. Let \( a \) and \( b \) be two distinct prime ends \([1]\) of \( D \). Let \( f \) be a conformal map from \( \mathbb{H} \) onto \( D \) such that \( f(0) = a \) and \( f(\infty) = b \). Let \( \gamma \) be a standard SLE\(_\kappa \) curve. Then \( f \circ \gamma \) is called an SLE\(_\kappa \) curve in \( D \) from \( a \) to \( b \).
From Loewner’s equation, we see that \( \gamma \) is a continuous curve in \( \overline{D} \). This condition may be weakened in some cases. Although the \( f \) is not unique, the law of \( f \circ \gamma \) is unique up to a linear time-change, thanks to the scaling property of a standard SLE\(_\kappa\) curve. From now on, an SLE\(_\kappa\) curve without the domain and two prime ends specified is always a standard SLE\(_\kappa\) curve, and the word “standard” will often be omitted.

The behavior of an SLE\(_\kappa\) curve depends on the value of \( \kappa \). Let \( \gamma \) be an SLE\(_\kappa\) curve. If \( \kappa \in (0,4] \), then \( \gamma \) is a simple curve, and does not intersect \( \mathbb{R} \) after the time 0; if \( \kappa > 4 \), then \( \gamma \) will intersect itself and \( \mathbb{R} \) after the time 0. If \( \kappa \geq 8 \), \( \gamma \) is space-filling: it visits every point in \( \mathbb{H} \); if \( \kappa < 8 \), then for every \( z_0 \in \mathbb{H} \setminus \{0\} \), the probability that \( \gamma \) visits \( z_0 \) is 0. Moreover, the Hausdorff dimension of \( \gamma \) is \( \min\{1 + \frac{\kappa}{8},2\} \) (cf. [4]).

SLE\(_\kappa(\rho)\) is a variant of SLE\(_\kappa\). Its definition involves one or more force points, which lie on the boundary or in the interior of the domain. For the purpose here, we consider the case that there is only one force point, which is an interior point. Let \( \rho \in \mathbb{R} \), \( a_0 \in \mathbb{R} \) and \( z_0 \in \mathbb{H} \). An SLE\(_\kappa(\rho)\) process started from \( a_0 \in \mathbb{R} \) with force point at \( z_0 \) is the solution of the SDE:

\[
d\lambda_t = \sqrt{\kappa} dB_t + Re \frac{\rho}{\lambda_t - g_t'(z_0)} dt, \quad \lambda_0 = a_0.
\]

Here \( g_t^\lambda \) are the Loewner maps driven by \( \lambda \), and \( \{0, T_{z_0}\} \) is the maximal solution interval. Let \( \mathbb{P}_{k,\rho}^{z_0} \) denote the law of \( \lambda_t \), \( 0 \leq t < T_{z_0} \). From Proposition 2.3, we know that \( \mathbb{P}_{0,\rho}^{z_0} \subset \mathbb{P}_{k} \). Thus, \( \mathbb{P}_{k,\rho}^{z_0} \) is supported by \( \Sigma^\rho \), i.e., the Loewner curve \( \gamma \) driven by \( \lambda \) a.s. exists, which is called a (standard) SLE\(_\kappa(\rho)\) curve started from \( a_0 \) with force point at \( z_0 \). The following proposition is a result of [21].

**Proposition 3.1.** For any \( \kappa > 0 \) and \( \rho \leq \frac{\kappa}{2} - 4 \),

(i) a.s. \( T_{z_0} < \infty \) and \( \lim_{t \to T_{z_0}} \lambda_t \in \mathbb{R} \);

(ii) a.s. \( \lim_{t \to T_{z_0}} \gamma(t) = z_0 \).

From the above proposition we see that, if \( \rho \leq \frac{\kappa}{2} - 4 \), then \( \mathbb{P}_{k,\rho}^{z_0} \) is supported by \( \Sigma^\rho \), and we may define \( \mathbb{P}_{k,\rho}^{z_0} \oplus \mathbb{P}_{k} \), which is supported by \( \Sigma^\rho \). The pushforward measure \( \mathcal{L}_\kappa(\mathbb{P}_{k,\rho}^{z_0} \oplus \mathbb{P}_{k}) \) is called the law of a standard extended SLE\(_\kappa(\rho)\) curve through \( z_0 \), which is supported by the continuous curves in \( \mathbb{H} \) from 0 to \( \infty \) that pass through \( z_0 \). In other words, a standard extended SLE\(_\kappa(\rho)\) curve through \( z_0 \) is defined by continuing an SLE\(_\kappa(\rho)\) curve started from 0 with force point at \( z_0 \) by an SLE\(_\kappa\) curve in the remaining domain from \( z_0 \) to \( \infty \).

Using conformal maps, we may define an extended SLE\(_\kappa(\rho)\) curve (for \( \rho \leq \frac{\kappa}{2} - 4 \)) in a simply connected domain from one prime end to another prime end through an interior point. We will mainly work on extended SLE\(_\kappa(\rho)\) curves in \( \mathbb{H} \) from 0 to \( \infty \) through some \( z_0 \in \mathbb{H} \), and so will omit the word “standard”.

We now derive the local Radon–Nikodym derivative of \( \mathbb{P}_{k,\rho}^{z_0} \) w.r.t. \( \mathbb{P}_{k} \). Let \( \lambda_t = \sqrt{\kappa} B_t \), and \( g_t \) be the Loewner maps driven by \( \lambda \). Fix \( z_0 = x_0 + iy_0 \in \mathbb{H} \). For \( 0 \leq t < T_{z_0} \), let

\[
Z_t = g_t(z_0) - \lambda_t, \quad X_t = Re Z_t, \quad Y_t = Im Z_t, \quad D_t = |g_t'(z_0)|.
\]

From Loewner’s equation, we see that \( X_t \) and \( Y_t \) satisfy the ODEs:

\[
\frac{dX_t}{Y_t} = \frac{-2}{X_t^2 + Y_t^2} dt, \quad \frac{dD_t}{D_t} = \frac{-2(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2} dt; \quad t \geq 0.
\]
and $X_t$ satisfies the SDE
\[ dX_t = -\sqrt{\kappa} dB_t + \frac{2X_t}{X_t^2 + Y_t^2} dt. \] (3.3)

Define
\[ M_t^{\kappa,\rho}(z_0) = |Z_t|^\rho \cdot Y_t^{\rho/8} \cdot D_t^{\frac{1-\frac{\rho}{2}}{4} + \frac{\rho}{8}}, \quad 0 \leq t < T_{z_0}, \] (3.4)
and
\[ G_t^{\kappa,\rho}(z_0) = M_0^{\kappa,\rho}(z_0) = |z_0|^\rho \cdot \text{Im}(z_0)^{\rho/8}. \] (3.5)

Using Itô’s formula and (3.2)–(3.3), it is straightforward to check that $(M_t^{\kappa,\rho}(z_0))$ is an $(F_0^t)$-adapted continuous local martingale, and satisfies the SDE:
\[ dM_t^{\kappa,\rho}(z_0) = M_t^{\kappa,\rho}(z_0) \cdot \text{Re} \frac{\rho/\sqrt{\kappa}}{\lambda_t - g_t(z_0)} dB_t, \quad 0 \leq t < T_{z_0}. \]

We further define
\[ M_t^{\kappa,\rho}(z_0) = 0, \quad T_{z_0} \leq t < \infty. \] (3.6)

For any measurable subset $U$ of $\mathbb{H}$, define
\[ \Psi_t^{\kappa,\rho}(U) = \int_{U \cap \mathbb{H}} M_t^{\kappa,\rho}(z)dA(z). \] (3.7)

Throughout this paper, we use $dA$ to denote the Lebesgue measure on $\mathbb{C}$. Since $(M_t^{\kappa,\rho}(z_0))$ is a positive local martingale, it is also a supermartingale. Thus, if $\int_U G_t^{\kappa,\rho}(z)dA(z) < \infty$, then $(\Psi_t^{\kappa,\rho}(U))$ is also a supermartingale, which has to be $\mathbb{P}_\kappa$-a.s. finite.

From Proposition 2.3 we know that
\[ \frac{d\mathbb{P}_t^{\kappa,\rho}}{d\mathbb{P}_\kappa}|_{\mathcal{F}_t = \Sigma_t} = \frac{M_t^{\kappa,\rho}(z_0)}{G_t^{\kappa,\rho}(z_0)}, \quad 0 \leq t < \infty. \] (3.8)

From Proposition 2.1, $(z, E) \mapsto \mathbb{P}_z^{\kappa,\rho}(E)$ is a probability kernel from $\mathbb{H}$ to $(\Sigma, \mathcal{F}_0)$. Thus, for any measurable subset $U$ of $\mathbb{H}$, we may define the measure
\[ \mathbb{P}_U^{\kappa,\rho} = \int_{U \cap \mathbb{H}} \mathbb{P}_z^{\kappa,\rho} G_t^{\kappa,\rho}(z)dA(z). \] (3.9)

Moreover, if $\int_U G_t^{\kappa,\rho}(z)dA(z) < \infty$, then $\mathbb{P}_U^{\kappa,\rho} \ll \mathbb{P}_\kappa$, and
\[ \frac{d\mathbb{P}_t^{\kappa,\rho}}{d\mathbb{P}_\kappa}|_{\mathcal{F}_t = \Sigma_t} = \Psi_t^{\kappa,\rho}(U), \quad 0 \leq t < \infty. \] (3.10)

4. Natural parametrization

Fix $\kappa \in (0, 8)$. Let $d = 1 + \frac{\kappa}{8}$ be the Hausdorff dimension of SLE$_\kappa$ curves. We will review the natural parametrization of SLE$_\kappa$ in this section.

First, we review the definition of the SLE Green’s function. The Green’s function for an SLE$_\kappa$ curve $\gamma$ in a simply connected domain $D$ from one prime end $a$ to another prime end $b$ is
\[ G_{(D,a,b)}(z) := \lim_{r \to 0^+} r^{d-2} \mathbb{P}[\text{dist}(z, \gamma) < r], \quad z \in D, \]
provided that the limit exists. It is clear that, if the \( SLE_\kappa \) Green’s function exists for one configuration \( (D; a, b) \), then it exists for all configurations, and it satisfies the conformal covariance, i.e., if \( f \) maps \( (D; a, b) \) conformally onto \( (D'; a', b') \), then

\[
G_{(D; a, b)}(z) = |f'(z)|^{2-d} G_{(D'; a', b')}(f(z)).
\]

It is proved in [13] that there is an unknown positive constant \( C_\kappa > 0 \) depending only on \( \kappa \) such that

\[
G_{(H; 0, \infty)}(z) = C_\kappa |z|^{d-2} \sin \frac{\kappa}{8} \pi + \frac{2}{\kappa} - 2(\arg z), \quad z \in \mathbb{H}.
\]  

(4.1)

We will write \( G(z) \) for \( G_{(H; 0, \infty)}(z) \). Define

\[
M_t(z) = |g_t'(z)|^{2-d} G(g_t(z) - \lambda(i)), \quad 0 \leq t < T_z
\]

and \( M_t(z) = 0 \) for \( t \geq T_z \). For any measurable set \( U \subset \mathbb{H} \), define \( \Psi_t(U) = \int_U G(z) dA(z) \). It is easy to check that \( G(z) \), \( M_t(z) \), \( \Psi_t(U) \) agree with \( C_\kappa \) times \( G^{\kappa, \kappa-8}(z) \), \( M_t^{\kappa, \kappa-8}(z) \), \( \Psi_t^{\kappa, \kappa-8}(U) \), respectively, defined by (3.4)–(3.7). From (3.8) we see that locally weighting \( \sqrt{\kappa} B_t \) using \( (M_t(z_0)/G(z_0)) \) generates a driving \( SLE_\kappa(\kappa - 8) \) process with force point at \( z_0 \).

A two-sided radial \( SLE_\kappa \) curve is just an extended \( SLE_\kappa(\kappa - 8) \) curve. This means that the law of a (standard) two-sided radial \( SLE_\kappa \) curve through \( z_0 \in \mathbb{H} \) can be expressed by \( \mathcal{L}_\kappa(\mathbb{H}; \kappa - 8) \oplus \mathbb{P}_\kappa \). Two-sided radial \( SLE \) is important because it can be understood as an \( SLE_\kappa \) curve conditioned to pass through an interior point. To make this rigorous, one may condition an \( SLE_\kappa \) curve \( \gamma \) on the event that \( \text{dist}(z_0, \gamma) < r \) for a fixed interior point \( z_0 \), and then pass to the limit \( r \to 0 \).

From the reversibility of chordal \( SLE_\kappa \) (cf. [20,29]) it is easy to see that two-sided radial \( SLE_\kappa \) also satisfies reversibility, i.e., the time-reversal of a two-sided radial \( SLE_\kappa \) curve in a simply connected domain \( D \) from \( a \) to \( b \) through \( z_0 \) agrees with a two-sided radial \( SLE_\kappa \) curve in \( D \) from \( b \) to \( a \) through \( z_0 \), up to a reparametrization. This property is not satisfied by extended \( SLE_\kappa(\rho) \) processes for \( \rho \neq \kappa - 8 \).

Recall that \( \mathbb{P}_\kappa^{\kappa, \kappa-8} \) is a kernel from \( \mathbb{H} \) to \( (\Sigma, \mathcal{F}_0) \). So we can define the measure \( \mathbb{P}_U = \int_{U \subset \mathbb{H}} \mathbb{P}_\kappa^{\kappa, \kappa-8}(z) G(z) dA(z) \) for any measurable set \( U \subset \mathbb{H} \). Then \( \mathbb{P}_U \) equals \( C_\kappa \) times \( \mathbb{P}_U^{\kappa, \kappa-8} \) defined by (3.9). From (3.10), if \( \int_U G(z) dA(z) < \infty \), then \( \mathbb{P}_U \ll \mathbb{P}_\kappa \) and

\[
\frac{d\mathbb{P}_U}{d\mathbb{P}_\kappa} \big|_{\mathcal{F}_t \cap \Sigma_t} = \Psi_t(U), \quad 0 \leq t < \infty.
\]  

(4.2)

It is proved by Lawler and Zhou [19] that, if \( U \) is a pre-compact measurable subset of \( \mathbb{H} \), i.e., \( \overline{U} \) is a compact subset of \( \mathbb{H} \), then (\( \Psi_t(U) \)) is of class \( \mathcal{D} \), i.e., \( \{ \Psi_T(U) : T \) is a finite stopping time\} is uniformly integrable. In this case they can apply the Doob–Meyer decomposition theorem to get a unique continuous increasing \( (\mathcal{F}_t^B) \)-adapted process \( \Theta_t(U) \) such that \( \Theta_0(U) = 0 \) and \( M_t(U) := \Psi_t(U) + \Theta_t(U) \) is uniformly integrable \( (\mathcal{F}_t^B) \)-martingale. This means that \( M_\infty(U) := \lim_{t \to \infty} M_t(U) \) a.s. exists, and \( \mathbb{E}_\kappa[M_\infty(U)|\mathcal{F}_t^B] = M_t(U), \) \( 0 \leq t < \infty \). From Lemma A.2, we know that \( \Psi_\infty(U) = 0 \). Thus,

\[
\mathbb{E}\{\Theta_\infty(U)|\mathcal{F}_\infty^B\} - \Theta_t(U) = \Psi_t(U), \quad 0 \leq t < \infty.
\]  

(4.3)

The process \( (\Theta_t(U)) \) determines a \( \mathbb{P}_\kappa \)-kernel \( d\Theta(U) \). Then we may define the killing \( \mathcal{K}_{d\Theta(U)}(\mathbb{P}_\kappa) \). From (2.1), we have

\[
\mathbb{P}_\kappa - a.s., \quad \frac{d\mathcal{K}_{d\Theta(U)}(\mathbb{P}_\kappa)|_{\mathcal{F}_t \cap \Sigma_t}}{d\mathbb{P}_\kappa|_{\mathcal{F}_t \cap \Sigma_t}} = \mathbb{E}\{\Theta_\infty(U)|\mathcal{F}_t^B\} - \Theta_t(U), \quad 0 \leq t < \infty.
\]  

(4.4)
Combining (4.2), (4.3) and (4.4), we see that,

\[ \mathbb{P}_U = \mathcal{K}_{d \Theta(U)}(\mathbb{P}_\kappa). \]

(4.5)

It is easy to check that, if \( U_1 \) and \( U_2 \) are disjoint pre-compact measurable subsets of \( \mathbb{H} \), then a.s. \( \Theta_t(U_1 \cup U_2) = \Theta_t(U_1) + \Theta_t(U_2) \) for \( 0 \leq t < \infty \). Thus, the kernel \( d \Theta_t(U) \) (and also \( \Theta_t(U) \)) is increasing in \( U \). We may extend the definition of \( \Theta_t(U) \) to any measurable subset \( U \) of \( \mathbb{H} \) as follows. Let \( (U_n) \) be a sequence of compact subsets of \( \mathbb{H} \) such that \( U_n \) is contained in the interior of \( U_{n+1} \) for every \( n \in \mathbb{N} \), and \( \mathbb{H} = \bigcup U_n \). For any measurable subset \( U \) of \( \mathbb{H} \), we first define the kernel \( \mu_U = \lim_{n \to \infty} d \Theta(U \cap U_n) \), then let \( \Theta_t(U) = \mu_U([0, t]), \ t \geq 0 \) (so \( d \Theta(U) = \mu_U \)). The definition does not depend on the choice of \( (U_n) \). In particular, \( \Theta_t := \Theta_t(\mathbb{H}) \) is called the natural parametrization of \( \gamma \).

The \( \Theta_t(U) \) is understood as the total time that the chordal \( \text{SLE}_\kappa \) curve \( \gamma \) spends in \( U \) in the natural parametrization before time \( t \). Define the kernel \( \mathcal{M}_U \) from \( \mathcal{L}(\Sigma^z) \) to \( \mathbb{H} \) by

\[ \mathcal{M}_U(\gamma, \cdot) := \gamma_*(d \Theta(U)) \]

(4.6)

This is the \( d \)-dimensional Minkowski content measure of \( \gamma \) in \( U \).

Here the Minkowski content measure of a set \( S \) in a domain \( U \) is a measure, say \( \mathcal{M} \) supported by \( S \cap U \), which satisfies the property that for any compact set \( K \subset U \), \( K \cap S \) has Minkowski content, which equals to \( \mathcal{M}(K) \) and is finite. It satisfies the property of conformal covariance. From the work of [13], we know that an \( \text{SLE}_\kappa \) curve for \( \kappa \in (0, 8) \) in \( \mathbb{H} \) from \( 0 \) to \( \infty \) possesses \( (1 + \frac{\kappa}{8}) \)-dimensional Minkowski content measure, which equals to the pushforward measure of the curve function of the natural parametrization (as a measure on the time interval). Using conformal covariance, we then know that any chordal \( \text{SLE}_\kappa \) curve possesses Minkowski content measure in the domain that it is defined. The reader is referred to [31, Section 2.3] for details of the notation.

**Theorem 4.1.** Let \( \kappa \in (0, 8) \). Let \( U \) be any measurable subset of \( \mathbb{H} \). Then we have

\[ \mathbb{P}_U \otimes \mathbb{P}_\kappa = \mathbb{P}_\kappa \otimes d \Theta(U), \]

(4.7)

\[ \mathcal{L}_*((\overset{\kappa-8}{\otimes} \mathbb{P}_\kappa) \hat{\otimes} \mathbf{1}_U G(z) dA(z)) = \mathcal{L}_* (\mathbb{P}_\kappa) \otimes \mathcal{M}_U, \]

(4.8)

where \( \mathcal{M}_U \) is the \( d \)-dimensional Minkowski content measure of \( \gamma \cap U \).

**Proof.** If \( U \) is pre-compact in \( \mathbb{H} \), then (4.7) follows from (4.5) and Proposition 2.4. For general measurable \( U \subset \mathbb{H} \), (4.7) follows from the above special case and a limiting procedure.

We now apply the extended Loewner map \( \hat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t)) \) to (4.7). We observe that \( \hat{\mathcal{L}}(\overset{\kappa-8}{\otimes} \mathbb{P}_\kappa) = \mathcal{L}_* (\overset{\kappa-8}{\otimes} \mathbb{P}_\kappa) \otimes \delta_z \), where \( \delta_z \) is the Dirac measure at \( z \). Integrating the equality over \( z \) against the measure \( \mathbf{1}_U G(z) dA(z) = \mathbf{1}_{U \cap \mathbb{H}} G(z) dA(z) \), we get

\[ \mathcal{L}_* (\overset{\kappa-8}{\otimes} \mathbb{P}_\kappa) = \mathcal{L}_* (\overset{\kappa-8}{\otimes} \mathbb{P}_\kappa) \hat{\otimes} \mathbf{1}_U G(z) dA(z). \]

(4.9)

Using (4.6) and the definition of \( \hat{\mathcal{L}} \), we find that

\[ \mathcal{L}_* (\mathbb{P}_\kappa \otimes d \Theta(U))) = \mathcal{L}_* (\mathbb{P}_\kappa) \otimes \mathcal{M}_U. \]

(4.10)

Then (4.7), (4.9) and (4.10) together imply (4.8). \( \square \)
Corollary 4.1. Let $\kappa \in (0, 8)$. Let $U$ be a measurable subset of $\mathbb{H}$ with $\int_U G(z) dA(z) < \infty$. If we integrate the laws of two-sided radial SLE$_\kappa$ curves through $z$ against the measure $\mathbf{1}_U G(z) dA(z)$, then we get a finite measure on curves, which is absolutely continuous w.r.t. the law of SLE$_\kappa$ curve, and the Radon–Nikodym derivative is the $d$-dimensional Minkowski content of $\gamma \cap U$.

Proof. This follows from projecting both sides of (4.8) to $\Sigma^C$ and that $\mathbb{E}_\kappa[\mathcal{M}_U(\gamma, \cdot)] = \mathbb{E}_\kappa[\Theta_\infty(U)] = \int_U G(z) dA(z) < \infty$. □

Corollary 4.2. Let $\kappa \in (0, 8)$. Suppose $\gamma, z$ is a $C([0, \infty), \mathbb{C}) \times \mathbb{H}$-valued random variable with the properties that $\gamma$ has the law of an SLE$_\kappa$ curve, and given $\gamma$, the law of $z$ is absolutely continuous w.r.t. the law of a two-sided radial SLE$_\kappa$ curve through $z$. Then the law of $z$ is absolutely continuous w.r.t. $\mathbf{1}_U dA(z)$, and the law of $\gamma$ given $z$ is absolutely continuous w.r.t. the law of a two-sided radial SLE$_\kappa$ curve through $z$.

Proof. From the assumption, we see that the law of $(\gamma, z)$ is absolutely continuous w.r.t. the measure in either side of (4.8) with $U = \mathbb{H}$. □

Remarks.

1. Roughly speaking, the meaning of (4.8) is that the following two methods generate the same measure on the space of curve-point pairs:
   (i) first sample a point according to the measure $\mathbf{1}_U G(z) dA(z)$, and then sample a two-sided radial SLE$_\kappa$ curve $\gamma$ through $z$;
   (ii) first sample an SLE$_\kappa$ curve $\gamma$, and then sample a point $z$ on $\gamma$ according to the $d$-dimensional Minkowski content measure of $\gamma \cap U$. Here the measure of $\gamma$ is changed after sampling $z$ because $\mathcal{M}_U$ is not a probability kernel.

2. Corollary 4.2 means that, if we sample a random point $z$ on a chordal SLE$_\kappa$ curve $\gamma$ according to a law that is absolutely continuous w.r.t. the $d$-dimensional Minkowski content measure, and then condition on $z$, then the curve $\gamma$ looks “similar” to a two-sided radial SLE$_\kappa$ curve through $z$.

Corollary 4.3. Let $D$ be a simply connected domain with two distinct prime ends $a$ and $b$. Let $\mathbb{P}^{\kappa}_{D,a,b}$ denote the law of an SLE$_\kappa$ curve in $D$ from $a$ to $b$. For any $z \in D$, let $\mathbb{P}^{\kappa}_{D,a,b;z}$ denote the law of a two-sided radial SLE$_\kappa$ curve in $D$ from $a$ to $b$ through $z$. For any measurable set $U \subset D$, let $\mathcal{M}_U(\gamma, \cdot)$ be the $d$-dimensional Minkowski content measure on $\gamma \cap U$. Then
\begin{equation}
\mathbb{P}^{\kappa}_{D,a,b;z} \otimes \mathbf{1}_U G_{D,a,b}(z) dA(z) = \mathbb{P}^{\kappa}_{D,a,b} \otimes \mathcal{M}_U.
\end{equation}
Moreover, if $\int_U G_{D,a,b}(z) dA(z) < \infty$, then $\int_U \mathbb{P}^{\kappa}_{D,a,b;z} dA(z) \ll \mathbb{P}^{\kappa}_{D,a,b}$, and the Radon–Nikodym derivative is the $d$-dimensional Minkowski content of $\gamma \cap U$.

Proof. Formula (4.11) follows immediately from (4.8), the conformal covariance of SLE$_\kappa$ Green’s function, and the transformation rule of $d$-dimensional Minkowski content measure under conformal maps. The rest follows from projecting the measures in (4.11) to $\Sigma^C$. □

Remark. The above corollary in the case $\kappa \leq 4$ was proved earlier by Laurie Field in [8], where he assumed that the domain $D$ is bounded and has analytic boundary.
Inspired by Theorem 4.1 and its corollaries, we make the following definition.

**Definition 4.1.** Let $\kappa > 0$ and $\rho \leq \frac{\kappa}{2} - 4$. We say that SLE$_{\kappa}$ admits an SLE$_{\kappa}(\rho)$ decomposition if (4.7) holds with $G(z)$ replaced by $G^{\kappa,\rho}(z)$ and $\mathbb{P}_U$ replaced by $\mathbb{P}_U^{\kappa,\rho}$, for some increasing adapted process $\Theta_t(U)$, which then implies (4.8) with $\mathbb{P}_z^{\kappa,\rho}$ replaced by $\mathbb{P}_z^{\kappa,\rho}$ and $\mathcal{M}_U$ defined by (4.6) for the new $\Theta_t(U)$.

We have shown that SLE$_{\kappa}$ admits an SLE$_{\kappa}(\kappa - 8)$ decomposition for $\kappa \in (0,8)$. From the argument, we see that, one approach to show that SLE$_{\kappa}$ admits an SLE$_{\kappa}(\rho)$ decomposition is to show that for any pre-compact measurable subset $U$ of $\mathbb{H}$, $\Psi_t^{\kappa,\rho}(U)$ defined in (3.7) is of class $\mathcal{D}$, and then apply the Doob–Meyer decomposition theorem. But this is very difficult in general (cf. [15,19]). In the next section, we will show that SLE$_{\kappa}$ admits an SLE$_{\kappa}(\kappa - 8)$ decomposition for any $\kappa > 0$ using a different approach.

5. Capacity parametrization

Fix $\kappa > 0$. We say that a chordal SLE$_{\kappa}$ curve $\gamma$ (with the reparametrization given by its definition) has capacity parametrization because the hull $K_t$ determined by $\gamma([0,t])$ has half-plane capacity $2t$ for each $t \geq 0$. In this section we consider SLE$_{\kappa}(\rho)$ processes with $\rho = -8$, which turns out to be closely related to the capacity parametrization.

For any $z \in \mathbb{H}$ and $N > 0$, let $\mathbb{P}_z^{\kappa,\rho}(E)$ be the measure on $(\Sigma,\mathcal{F}^0)$ defined by $\mathbb{P}_z^{\kappa,\rho}(E) = \mathbb{P}_z^{\kappa,\rho}(E \setminus \Sigma_N)$. Note that $\mathbb{P}_z^{\kappa,\rho}$ is not a probability measure. Then we have $\mathbb{P}_z^{\kappa,\rho} \ll \mathbb{P}_z^{\kappa,\rho}$, and so $\mathbb{P}_z^{\kappa,\rho} < \mathbb{P}_z^{\kappa,\rho}$. Since $\mathbb{P}_z^{\kappa,\rho} < \mathbb{P}_z^{\kappa,\rho}$, we get $\mathbb{P}_z^{\kappa,\rho} < \mathbb{P}_z^{\kappa,\rho}$. We now calculate the local Radon–Nikodym derivatives of $\mathbb{P}_z^{\kappa,\rho}$ w.r.t. $\mathbb{P}_z^{\kappa,\rho}$.

Let $\lambda_t = \sqrt{\kappa}B_t$, and $g_t$ the Loewner maps driven by $\lambda$. Let $M_t(z) = M^{\kappa,\rho}_t(z)$ and $G(z) = G^{\kappa,\rho}(z)$ be as defined by (3.4)–(3.6). Then we have $G(z) = (\text{Im} z/|z|)^{8/\kappa}$; $M_t(z) = G(g_t(z) - \lambda_t)|g_t'(z)|^2$, $0 \leq t < T_z$; and $M_t(z) = 0$, $T_z \leq t < \infty$.

Let $t \in [0,\infty)$ and $E \in \mathcal{F}_t \cap \Sigma_t$. If $t \geq N$, then $\mathbb{P}_z^{\kappa,\rho}(E) = 0$ because $E \subseteq \Sigma_t \subseteq \Sigma_N$. If $t < N$, from (3.8), we get

$$
\mathbb{P}_z^{\kappa,\rho}(E) = \mathbb{P}_z^{\kappa,\rho}(E \setminus \Sigma_N) = \mathbb{P}_z^{\kappa,\rho}(E) - \mathbb{P}_z^{\kappa,\rho}(E \cap \Sigma_N)
$$

$$
= \int_E \frac{M_t(z)}{G(z)} d\mathbb{P}_z - \int_{E \cap \Sigma_N} \frac{M_N(z)}{G(z)} d\mathbb{P}_z = \frac{1}{G(z)} \int_E (M_t(z) - \mathbb{E}_z[M_N(z)|\mathcal{F}_t]) d\mathbb{P}_z.
$$

Here the third “$=$” holds because $E \in \mathcal{F}_t \cap \Sigma_t$ and $E \cap \Sigma_N \in \mathcal{F}_N \cap \Sigma_N$, and the fourth “$=$” holds because $\mathbb{P}_z$ is supported by $\Sigma_N \subseteq \Sigma_N$. Thus,

$$
\frac{d\mathbb{P}_z^{\kappa,\rho}}{d\mathbb{P}_z}|_{\mathcal{F}_t \cap \Sigma_t} = \begin{cases}
\frac{1}{G(z)}(M_t(z) - \mathbb{E}_z[M_N(z)|\mathcal{F}_t]), & 0 \leq t < N; \\
0, & N \leq t < \infty.
\end{cases}
$$

For $t \geq 0$, define $G_t$ on $\mathbb{H}$ by

$$
G_t(z) = M_0(z) - \mathbb{E}_z[M_t(z)] = G(z) - \mathbb{E}_z[M_t(z)],
$$

and let $C_{\kappa,t} = \int_{\mathbb{H}} G_t(z) dA(z)$. From (3.8), we get

$$
\frac{G_t(z)}{G(z)} = 1 - \int_{\Sigma_t} \frac{M_t(z)}{G(z)} d\mathbb{P}_z = 1 - \mathbb{P}_z^{\kappa,\rho} \{ \Sigma_t \} = \mathbb{P}_z^{\kappa,\rho} \{ T_z \leq t \}.
$$

Thus, $G_0 \equiv 0$, $G_t \geq 0$ for $t > 0$, and

$$
\lim_{t \to \infty} G_t(z) = G(z) \mathbb{P}_z^{\kappa,\rho} \{ T_z < \infty \} = G(z).
$$
From the scaling property of $\text{SLE}_κ(−8)$ process, we get \( \frac{G_{az}(az)}{G_0(az)} = \frac{G(z)}{G(z)} \) for any \( a > 0 \). Since \( G(az) = G(z) \), we get \( G_{az}(az) = G(z) \). Especially, we have

\[
G_t(z) = G_1\left(\frac{z}{\sqrt{t}}\right), \quad t > 0.
\]  \( (5.4) \)

**Lemma 5.1.** We have \( C_{κ,t} = tC_{κ,1} \) for every \( t \geq 0 \), and \( C_{κ,1} \in (0, ∞) \).

**Proof.** From \( (5.4) \) and that \( G_0 ≡ 0 \), we get \( C_{κ,t} = tC_{κ,1}, t \geq 0 \). From \( (5.3) \) and \( (5.4) \), we get \( C_{κ,1} > 0 \). Now we show \( C_{κ,1} < ∞ \). Fix \( z_0 = x_0 + iy_0 \in \mathbb{H} \) and consider an $\text{SLE}_κ(−8)$ process started from 0 with force point \( z_0 \). We will use some results in the Appendix. We may express \( T_{z_0} \) by \( (A.4) \), where \( V_s \) is a diffusion process that satisfies the SDE \( (A.3) \) with initial value \( V_0 = \text{arcsinh}(x_0/y_0) \). So we immediately have \( T_{z_0} ≥ y_0^2 \int_0^∞ e^{-4s}ds = y_0^2/4 \). Thus, \( T_{z_0} > 1 \) if \( y_0 > 2 \). From \( (5.2) \) we see that \( G_1 ≡ 0 \) on \( |\text{Im} z | > 2 \). From the left–right symmetry, we see that \( G_1(x + iy) = G_1(−x + iy) \). We also know that \( |G_1(z)| ≤ |G(z)| ≤ 1 \) for all \( z \in \mathbb{H} \). Thus, to show that \( C_{κ,1} = \int_\mathbb{H} G_1(z)dA(z) < ∞ \), it suffices to prove that there is \( M > 0 \) such that \( \int_M^∞ \int_0^∞ G_1(x + iy)dxdy < ∞ \).

Suppose \( |√κB_s| ≤ 1 + bs \) for every \( s ≥ 0 \), where \( b > 0 \) is to be determined. From \( (A.3) \) and that \( |\tanh(x)| ≤ 1 \), we have \( V_t = V_0 − 1 − bs − (\frac{c}{2} + 4)s, s ≥ 0 \). Thus,

\[
cosh^2(V_2) ≥ \frac{1}{4} e^{2V_0} ≥ \frac{1}{4} e^{2V_0} \cdot e^{−(2b+κ+8)s−2} ≥ e^{−(2b+κ+8)s−2}\sinh^2(V_0)
\]

From \( (A.4) \) we then get \( T_{z_0} ≥ \frac{e^{−2x_0^2}}{1+12+2b} \), which then implies \( T_{z_0} > 1 \) if \( 2b < e^{−2x_0^2} − κ − 12 \). From \( (A.5) \) we know that \( \mathbb{P}[|√κB_s| ≤ 1 + bs, ∀s ≥ 0] ≥ 1 − 2e^{−\frac{c}{2}b} \). Thus, by taking \( 2b = e^{−2x_0^2} − κ − 12 − ε \) and letting \( ε \to 0^+ \), we get

\[
\mathbb{P}^{κ−8}_{x_0}[T_{z_0} ≤ 1] ≤ e^{−\frac{1}{8}(e^{−2x_0^2}−κ−12)}, \quad \text{if } e^{−2x_0^2} > κ + 12.
\]

Let \( M = √(2κ+12) \). From \( (5.2) \) we get

\[
G_1(z_0) ≤ G(z_0)\mathbb{P}^{κ−8}_{x_0}[T_{z_0} ≤ 1] ≤ \mathbb{P}^{κ−8}_{x_0}[T_{z_0} ≤ 1] ≤ e^{−\frac{1}{8}(e^{−2x_0^2}−κ−12)}, \quad x_0 > M.
\]

So we get \( \int_M^∞ \int_0^∞ G_1(z_0)dxdy < ∞ \), as desired. \( \square \)

Let \( m \) be the Lebesgue measure on \( \mathbb{R} \). For any measurable set \( U ⊂ \mathbb{H} \), define

\[
\Theta^{κ−8}_t(U) = C_{κ,1} m(\gamma^{−1}(U) \cap [0, t]), \quad t ≥ 0.
\]

Especially, we have \( \Theta^{κ−8}_t(\mathbb{H}) = C_{κ,1}t \). We can now state the following theorem, which is similar to Theorem 4.1.

**Theorem 5.1.** For any measurable set \( U ⊂ \mathbb{H} \),

\[
\mathbb{P}^{κ−8}_{U} \oplus \mathbb{P}_κ = \mathbb{P}_κ \otimes d\Theta^{κ−8}_t(U), \quad (5.5)
\]

\[
\mathcal{L}_s(\mathbb{P}^{κ−8}_z \oplus \mathbb{P}_κ) \otimes 1_U G^{κ−8}_s(z)dA(z) = \mathcal{L}_s(\mathbb{P}_κ) \otimes \mathcal{M}^{κ−8}_U, \quad (5.6)
\]

where \( \mathcal{M}^{κ−8}_U \) is the kernel from \( \mathcal{L}(\Sigma^L) \) to \( \mathbb{H} \) defined by \( \mathcal{M}^{κ−8}_U(γ, \cdot) = γ_*(d\Theta^{κ−8}_t(U)) \).
Proof. From the renewal property of Loewner’s equation and the Markov property of \(\sqrt{k}B_t\), we see that, if \(t \leq N\), then \(M_t(z) - \mathbb{E}_k [M_N(z)|\mathcal{F}_t^0] = 1_{T_{z>N}} |g_t'(z)|^2 G_{N-t}(g_t(z) - \lambda_t)\). Thus,

\[
\int_{\mathbb{H}} (M_t(z) - \mathbb{E}_k [M_N(z)|\mathcal{F}_t^0])dA(z) = \int_{\mathbb{H}\setminus K_t} G_{N-t}(g_t(z) - \lambda_t)|g_t'(z)|^2dA(z)
\]

\[
= \int_{\mathbb{H}} G_{N-t}(w)dA(w) = C_{k,N-t} = (N-t)C_{k,1}.
\]

Here in the second “=” we use \(w = g_t(z) - \lambda_t\) and the fact that \(g_t\) maps \(\mathbb{H}\setminus K_t\) conformally onto \(\mathbb{H}\). The above formula together with Proposition 2.1 and (5.1) implies that \(\int_{\mathbb{H}} \mathbb{P}^{\kappa,-8}_z G(z)dA(z) < \infty\), and the local Radon–Nikodym derivative is \(C_{k,1}(N-t)\) \(\forall t\). From Proposition 2.2 we get

\[
\int_{\mathbb{H}} \mathbb{P}^{\kappa,-8}_z G(z)dA(z) = \mathcal{K}_{C_{k,1}d(t\wedge N)}(\mathbb{P}_k).
\]

Letting \(N \to \infty\), we get

\[
\mathbb{P}_H^{\kappa,-8} = \mathcal{K}_{C_{k,1}d(t\wedge N)}(\mathbb{P}_k).
\]

(5.7)

Using Proposition 2.4, we get

\[
\mathbb{P}^{\kappa,-8}_t \otimes \mathbb{P}_k = \mathbb{P}_k \otimes C_{k,1} m|_{[0,\infty)}.
\]

Applying the extended Loewner map \(\widehat{\mathcal{L}}(\lambda, t) = (\mathcal{L}(\lambda), \mathcal{L}(\lambda)(t))\), we get

\[
\mathcal{L}_k(\mathbb{P}^{\kappa,-8}_t \otimes \mathbb{P}_k) \mathbb{P}^{\kappa,-8}_z G(z)dA(z) = \mathcal{L}_k(\mathbb{P}_k) \otimes M^{\kappa,-8},
\]

(5.8)

where \(M^{\kappa,-8}\) is the kernel from \(\mathcal{L}(\Sigma^C)\) to \(\mathbb{H}\) defined by \(M^{\kappa,-8}_{U}(\gamma', \cdot) = \gamma_k(C_{k,1} m|_{[0,\infty)})\). Note that \(M^{\kappa,-8}_{U}(\gamma', \cdot)\) is the restriction of \(M^{\kappa,-8}(\gamma', \cdot)\) to \(U\). So we get (5.6) by restricting both sides of (5.8) to \(\Sigma^C \times U\). Finally, we get (5.5) by applying \(\widehat{\mathcal{L}}^{-1}\). □

The following two corollaries are similar to Corollaries 4.1 and 4.2.

Corollary 5.1. Let \(U\) be a measurable subset of \(\mathbb{H}\) with \(\int_U G^{\kappa,-8}(z)dA(z) < \infty\). If we integrate the laws of extended SLE\(_{\kappa}\)\((-8)\) curve started from 0 with force point at \(z\) against the measure \(\mathbf{1}_U G^{\kappa,-8}(z)dA(z)\), then we get a finite measure on curves, which is absolutely continuous w.r.t. the capacity parametrization, and stop the SLE\(_{\kappa}\) curve according to this law, then the law of the stopped curve conditioned on that point is absolutely continuous w.r.t. the law of SLE\(_{\kappa}\)\((-8)\). The law of an extended SLE\(_{\kappa}\)\((-8)\) curve started from 0 with force point at \(z\).

Corollary 5.2. Suppose \((\gamma, t)\) is a \(C(0,\infty), \mathbb{C} \times [0,\infty)\)-valued random variable with the properties that \(\gamma\) has the law of an SLE\(_{\kappa}\) curve, and given \(\gamma\), the law of \(t\) is absolutely continuous w.r.t. the Lebesgue measure on \([0,\infty)\). Then the law of \(\gamma(t)\) is absolutely continuous w.r.t. \(\mathbf{1}_U dA(z)\), and the law of \(\gamma\) conditioned on \(z = \gamma(t)\) is absolutely continuous w.r.t. the law of an extended SLE\(_{\kappa}\)\((-8)\) curve started from 0 with force point at \(z\).

Remark. From Corollary 5.2 we see that, if we sample a point on an SLE\(_{\kappa}\) curve according to a law that is absolutely continuous w.r.t. the capacity parametrization, and stop the SLE\(_{\kappa}\) curve at that point, then the law of the stopped curve conditioned on that point is absolutely continuous w.r.t. the law of SLE\(_{\kappa}\)\((-8)\) curve. This extends a result in [30], whose argument used the symmetry of backward SLE\(_{\kappa}\) welding for \(\kappa \in (0,4)\) (cf. [24]) and the conformal removability of SLE\(_{\kappa}\) curves for \(\kappa \in (0,4)\) (cf. [10,23]).

More specifically, from [30, Remark 2 after Theorem 6.6], we know that, if \(\kappa \in (0,4)\), then the above conditional stopped curve is the conformal image of an initial segment of a whole-plane
Corollary 5.3. Let $\gamma$ be an $\text{SLE}_\kappa$ curve. Then for any measurable set $U \subset \mathbb{H}$, we have $\mathbb{E}[\mathcal{M}_U^{\gamma, -8}(z) dA(z)] = \frac{1}{C_{\kappa,1}} \int_U G^{\kappa, -8}(z) dA(z)$. Especially, we have a.s. $\mathcal{M}^{-1}(\mathbb{R}) = 0$.

Proof. The first statement follows from computing the total mass of the measures in (5.6) and the fact that $|\mathcal{M}_U^{\gamma, -8}| = C_{\kappa,1} \mathcal{M}^{-1}(U)$. The second statement follows from taking $U = \mathbb{R}$ and the fact that $\int_R G^{\kappa, -8}(z) dA(z) = 0$. \qed

Remark. The above corollary says that $\frac{1}{C_{\kappa,1}} G^{\kappa, -8}(z)$ is the density of $\text{SLE}_\kappa$ curve in capacity parametrization, and so may be called the capacity Green’s function for $\text{SLE}_\kappa$.

Corollary 5.4. Let $\gamma$ be an $\text{SLE}_\kappa$ curve. Then there is a random conformal map $W$ from $\mathbb{H}$ into $\hat{\mathbb{C}}$ such that $W(\gamma(1)) = \infty$, and the law of $W(\gamma(t))$, $0 \leq t \leq 1$, is absolutely continuous w.r.t. an end segment of a whole-plane $\text{SLE}_\kappa(\kappa + 2)$ curve, up to a reparametrization.

Proof. Let $\tilde{\gamma}$ be an $\text{SLE}_\kappa$ curve independent of $\gamma$. Let $\tilde{\gamma}_t$ and $\tilde{g}_t$ be the driving function and Loewner maps associated with $\tilde{\gamma}$. Let $\xi$ be a random variable with law $\mathcal{I}_{[0,1]} dx$ that is independent of $\tilde{\gamma}$ and $\gamma$. Define $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = \tilde{\gamma}(t)$ for $0 \leq t \leq \xi$ and $\tilde{\gamma}(t) = \tilde{f}_t(\tilde{\gamma}, \gamma(t - \xi))$ for $\xi \leq t < \infty$, where $\tilde{f}_t$ is the continuation of $\tilde{g}_\xi^{-1}$ to $\mathbb{H}$. From the domain Markov property of $\text{SLE}_\kappa$, we see that $\tilde{\gamma}$ is an $\text{SLE}_\kappa$ curve independent of $\xi$. Since $\xi + 1$ has the law $\mathcal{I}_{[1,2]} dx$, from Corollary 5.2, we see that, conditioned on $\xi = 0$, the law of $\tilde{\gamma}_1$, $0 \leq t \leq \xi + 1$, is absolutely continuous w.r.t. the law of an $\text{SLE}_\kappa(-8)$ curve started from 0 with force point at $z_0$. Let $W_1$ be a conformal map from $\mathbb{H}$ onto $\mathbb{D} := \{ |z| < 1 \}$ such that $W_1(0) = 1$ and $W_1(z_0) = 0$. From [26], we know that $W_1$ maps an $\text{SLE}_\kappa(-8)$ curve started from 0 with force point at $z_0$ to a radial $\text{SLE}_\kappa(\kappa + 2)$ curve in $\mathbb{D}$ started from 1 with force point at $W_1(\infty)$, up to a reparametrization.

By [11], there is a conformal map $W_2$ from $\mathbb{D}$ into $\hat{\mathbb{C}}$ such that $W_2(0) = \infty$, and $W_2$ maps the radial $\text{SLE}_\kappa(\kappa + 2)$ curve to an end segment of a whole-plane $\text{SLE}_\kappa(\kappa + 2)$ curve. Let $W_0(z) = \tilde{f}_t(\tilde{\gamma}, z)$, which is a conformal map from $\mathbb{H}$ into $\mathbb{H}$. Then $W := W_2 \circ W_1 \circ W_0$ is the conformal map we are looking for. \qed

Remark. Corollary 5.4 extends a weaker version of [30, Theorem 5.3], which states that, for $\kappa \in (0, 4)$, there is a random conformal map $W$ from $\mathbb{H}$ into $\hat{\mathbb{C}}$ such that $W(\gamma(1)) = 0$, and $W(\gamma(t))$, $0 \leq t \leq 1$, is an initial segment of a whole-plane $\text{SLE}_\kappa(\kappa + 2)$ curve, up to a reparametrization.

Corollary 5.5. If $\int_{\mathbb{H}} G^{\kappa, -8}(z) dA(z) < \infty$, then $M_t^{\kappa, -8}(U) := \Psi_t^{\kappa, -8}(U) + \Theta_t^{\kappa, -8}(U)$ is a uniformly integrable $(\mathcal{F}_t^\mathcal{H})$-martingale.

Proof. It suffices to show that $M_t^{\kappa, -8}(U) = \mathbb{E}_{x}[C_{\kappa,1} \mathcal{M}^{-1}(U)]_{\mathcal{F}_t^\mathcal{H}}$ for every $t \geq 0$. We have $C_{\kappa,1} \mathcal{M}^{-1}(U) = C_{\kappa,1} \mathcal{M}^{-1}(U) \cap [0, t]) + C_{\kappa,1} \mathcal{M}^{-1}(U) \cap [t, \infty])$, and $C_{\kappa,1} \mathcal{M}^{-1}(U) \cap [0, t])$, for every $t \geq 0$.
\[ [0, t]) = \Theta^\kappa_{t}^{\kappa, -8}(U) \text{ is } \mathcal{F}_t^0\text{-measurable. Thus, it remains to show that} \]
\[ \mathbb{E}_x [C_{\kappa, 1} m(\gamma^{-1}(U) \cap [t_0, \infty)) | \mathcal{F}_t^0] = \psi^{\kappa, -8}_t(U), \quad t_0 \geq 0. \]

Fix \( t_0 \geq 0 \). By the domain Markov Property of SLE_\kappa, there is an SLE_\kappa curve \( \hat{\gamma} \) independent of \( \mathcal{F}_t^0 \) such that \( \gamma(t_0 + t) = f_t(x_0 + \hat{\gamma}(t)) \) for all \( t \geq 0 \), where \( f_t \) is the continuation of \( g_t^{-1} \) to \( \mathbb{H} \), and \( \lambda_t \) and \( g_t \) are the driving function and Loewner maps associated with \( \gamma \). Let \( \tilde{g}_0(z) = g_0(z) - \lambda_{t_0} \) and \( \tilde{f}_0(z) = f_0(\lambda_{t_0} + z) \). Conditioned on \( \mathcal{F}_t^0 \), we have

\[ m(\gamma^{-1}(U) \cap [t_0, \infty)) = m(\hat{\gamma}^{-1}(\tilde{f}_0^{-1}(U))) = m(\hat{\gamma}^{-1}(\tilde{f}_0^{-1}(U) \cap \mathbb{H})) \]
\[ = m(\hat{\gamma}^{-1}(\tilde{g}_0(U) \cap (\mathbb{H} \setminus K_0))) \]

From Corollary 5.3, we get
\[ \mathbb{E}_x [C_{\kappa, 1} m(\gamma^{-1}(U) \cap [t_0, \infty)) | \mathcal{F}_t^0] = \int_{\tilde{g}_0(U \cap (\mathbb{H} \setminus K_0))} G^{\kappa, -8}(z) dA(z) \]
\[ = \int_{U \cap (\mathbb{H} \setminus K_0)} G^{\kappa, -8}(g_0(z) - \lambda_{t_0}) | g_t'(z)|^2 dA(z) = \int_U M^{\kappa, -8}_t(z) dA(z) = \psi^{\kappa, -8}_t(U). \]

This finishes the proof. \( \square \)

6. Intersection of SLE with the boundary

In this section, we decompose SLE_\kappa into SLE_\kappa(\rho) processes with the force point lying on \( \mathbb{R} \). An SLE_\kappa(\rho) process started from \( a_0 \in \mathbb{R} \) with force point at \( x_0 \in \mathbb{R} \setminus \{a_0\} \) is the solution of the Loewner equation driven by \( \lambda_t \), \( 0 \leq t < T_{x_0} \), which is the solution of the SDE:
\[ d\lambda_t = \sqrt{\kappa} dB_t + \rho \frac{\lambda_t - \lambda_0}{g_t^2(x_0)}, \quad \lambda_0 = a_0. \]

The Loewner curve driven by \( \lambda \), which a.s. exists, is called an SLE_\kappa(\rho) curve started from \( a_0 \) with force point at \( x_0 \).

Let \( \mathbb{P}^{x_0, \rho}_{x_0} \) denote the law of the driving SLE_\kappa(\rho) process started from 0 with force point at \( x_0 \in \mathbb{R} \setminus \{0\} \). From Proposition 2.3, we know that \( \mathbb{P}^{x_0, \rho}_{x_0} \subset \mathbb{P}_x \). We now derive the local Radon–Nikodym derivative. Let \( \lambda_t = \sqrt{\kappa} B_t \) and \( g_t \) be the Loewner maps driven by \( \lambda \). Define
\[ M^{\kappa, \rho}_t(x_0) = |g_t(x_0) - \lambda_t|_\rho^\frac{\rho}{2} \cdot g_t'(x_0)^\rho(1 - \frac{\rho}{2} + \frac{\rho}{2}), \quad 0 \leq t < T_{x_0}, \]
and
\[ G^{\kappa, \rho}(x) = M^{\kappa, \rho}_0(x_0) = |x_0|_\rho^\rho, \quad z \in \mathbb{H}. \]

Direct calculation using Itô’s formula shows that \( (M^{\kappa, \rho}_t(x_0)) \) is an \( \mathcal{F}_t^0 \)-adapted continuous local martingale, and satisfies the SDE:
\[ dM^{\kappa, \rho}_t(x_0) = M^{\kappa, \rho}_t(x_0) \cdot \rho \sqrt{\kappa} \frac{1}{\lambda_t - g_t(x_0)} dB_t, \quad 0 \leq t < T_{x_0}. \]

We further define \( M^{\kappa, \rho}_t(x_0) = 0 \) for \( t \in [T_{x_0}, \infty) \). From Proposition 2.3 we know that
\[ \frac{d\mathbb{P}^{x_0, \rho}_{x_0} | \mathcal{F}_t^0 \cap \Sigma_t}{d\mathbb{P}_x | \mathcal{F}_t^0 \cap \Sigma_t} = \frac{M^{\kappa, \rho}_t(x_0)}{G^{\kappa, \rho}(x_0)}, \quad 0 \leq t < \infty. \]

For a measurable subset \( U \) of \( \mathbb{R} \), define
\[ \psi^{\kappa, \rho}_t(U) = \int_U M^{\kappa, \rho}_t(x) dx, \quad \mathbb{P}^{x, \rho}_U = \int_U \mathbb{P}^{x, \rho} G^{\kappa, \rho}(x) dx. \]
For each \( x \in \mathbb{R} \setminus \{0\} \), \((M_{t}^{x,\rho}(x))\) is a supermartingale because it is a positive local martingale. Thus, if \( \int_{U} G^{x,\rho}(x)dx < \infty \), then \((\Psi_{t}^{x,\rho}(U))\) is also a supermartingale, which has to be \( \mathbb{P}_{\kappa} \)-a.s. finite. From Proposition 2.1, we have

\[
\frac{d\mathbb{P}_{t}^{x,\rho}(\xi \cap \Sigma_{t})}{d\mathbb{P}_{\kappa} | \xi \cap \Sigma_{t}} = \Psi_{t}^{x,\rho}(U), \quad 0 \leq t < \infty.
\]

It is known that (cf. [28]), if \( \rho \leq \frac{\kappa}{2} - 4 \), then for an SLE\( _{\kappa}(\rho)\) curve \( \gamma \) with force point \( x_{0} \), we have a.s.
\[ \lim_{t \to T_{x_{0}}} \gamma(t) = x_{0}, \]
which then implies that \( T_{x_{0}} < \infty \), and \( \lim_{t \to T_{x_{0}}} \lambda_{t} \) converges, where \( \lambda \) is the corresponding driving function. This means that \( \mathbb{P}_{x_{0}}^{\kappa,\rho} \) is supported by \( \Sigma^{\infty} \), and we may define \( \mathbb{P}_{x_{0}}^{\kappa,\rho} \oplus \mathbb{P}_{\kappa} \). The pushforward measure \( \mathcal{L}_{\kappa}(\mathbb{P}_{x_{0}}^{\kappa,\rho} \oplus \mathbb{P}_{\kappa}) \) is called the law of an extended SLE\( _{\kappa}(\rho)\) curve started from 0 with force point at \( x_{0} \).

For \( \kappa \in (0, 8) \) and \( \rho = \kappa - 8 \), an extended SLE\( _{\kappa}(\rho)\) curve with force point at \( x_{0} \) is also called a two-sided chordal SLE\( _{\kappa} \) curve through \( x_{0} \). It can be understood as an SLE\( _{\kappa} \) curve conditioned to pass through \( x_{0} \). To make this rigorous, one may condition an SLE\( _{\kappa}(\rho) \) curve through \( x \) against the measure \( \mathbb{P}_{\kappa} \oplus \mathbb{P}_{\kappa} \).

It is proved in [3] that, for \( \kappa \in (4, 8) \), if \( U \) is a pre-compact measurable subset of \( \mathbb{R} \setminus \{0\} \), then \((\Psi_{t}^{x,\kappa-8}(U))\) is of class \( D \) and, Doob–Meyer decomposition theorem can be applied to get a unique continuous increasing function \( \theta_{U}(t) \) such that \( \theta_{U}(0) = 0 \) and \( \Psi_{t}(U) + \theta_{U}(t) \) is a uniformly integrable martingale. We may then define \( \theta_{U}(t) \) for any measurable subset \( U \) of \( \mathbb{R} \) using a limiting procedure. It is conjectured (cf. [3] and [12]) that \( \theta_{U}(t) \) agrees up to a multiplicative constant with the \( d' \)-dimensional Minkowski content of \( \gamma([4, t]) \) for any \( d' := 2 - \frac{8}{\kappa} \) is the Hausdorff dimension of \( \gamma \cap \mathbb{R} \) (cf. [2]). Using the argument in Section 4, we can obtain the following theorem and corollaries.

**Theorem 6.1.** Let \( \kappa \in (4, 8) \). Let \( U \) be any measurable subset of \( \mathbb{R} \). Then we have

\[
\mathbb{P}_{x_{0}}^{\kappa,\infty-8} \oplus \mathbb{P}_{\kappa} = \mathbb{P}_{\kappa} \otimes d\theta_{U}(t),
\]

\[
\mathcal{L}_{\kappa}(\mathbb{P}_{x_{0}}^{\kappa,\infty-8} \oplus \mathbb{P}_{\kappa}) \mathbb{1}_{U} G^{\kappa,\infty-8}(x)dx = \mathcal{L}_{\kappa}(\mathbb{P}_{\kappa}) \otimes \mathcal{M}_{U},
\]

where \( \mathcal{M}_{U} \) is the \( \mathcal{L}_{\kappa}(\mathbb{P}_{\kappa}) \)-kernel from \( \mathcal{L}(\Sigma^{\infty}) \) to \( \mathbb{P}^{\kappa} \) defined by \( \mathcal{M}_{U}(\gamma, \cdot) = \gamma_{\kappa}(d\theta_{U}(t)) \).

**Corollary 6.1.** Let \( \kappa \in (4, 8) \). Let \( U \) be a measurable subset of \( \mathbb{R} \) with \( \int_{U} G^{\kappa,\infty-8}(x)dx < \infty \). If we integrate the laws of two-sided chordal SLE\( _{\kappa} \) through \( x \) against the measure \( \mathbb{1}_{U} G^{\kappa,\infty-8}(x)dx \), then we get a finite measure on curves, which is absolutely continuous w.r.t. the law of an SLE\( _{\kappa} \) curve, and the Radon–Nikodym derivative is \( |\mathcal{M}_{U}(\gamma, \cdot)| = \Theta_{\infty}(U)(\mathcal{L}^{\infty}(\gamma)) \).

**Corollary 6.2.** Let \( \kappa \in (4, 8) \). Suppose \( (\gamma, x) \) is a \( C([0, \infty), \mathbb{C}) \times \mathbb{R} \)-valued random variable with the properties that \( \gamma \) has the law of an SLE\( _{\kappa} \) curve, and given \( \gamma \), the law of \( x \) is absolutely continuous w.r.t. \( \mathcal{M}_{\Sigma}(\gamma, \cdot) \). Then the law of \( x \) is absolutely continuous w.r.t. \( m \), and the law of \( \gamma \) given \( x \) is absolutely continuous w.r.t. the law of a two-sided chordal SLE\( _{\kappa} \) curve through \( x \).

**Remark.** Since an SLE\( _{\kappa} \) curve for \( \kappa \in (0, 4) \) does not visit any point on \( \mathbb{R} \setminus \{0\} \), Corollary 6.1 does not hold in the case \( \kappa \in (0, 4) \), and neither does the main result in [3].

### 7. Decomposition of planar Brownian motion

We now use the argument in the proof of Theorem 4.1 to decompose a planar Brownian motion. We modify the definition of \( \Sigma \) such that the continuous functions take values in \( \mathbb{R}^{2} \). Let
$D \subset \mathbb{R}^2$ be a bounded domain that contains $0 = (0, 0)$. Let $B_t$ be a planar Brownian motion that starts from 0, and $\tau_D$ the first time that $B_t$ exists $D$. Let $G_D(\cdot, \cdot)$ denote the Dirichlet Green’s function in $D$ for Brownian motion. Fix $z_0 = (x_0, y_0) \in D \setminus \{0\}$. Consider the Doob’s $h$-transform of $B_t$ with $h = G_D(\cdot, z_0)$. This is a diffusion process that satisfies the SDE:

$$dZ_{t}^{z_0} = dB_t + \frac{\nabla G_D(\cdot, z_0)}{G_D(Z_{t}^{z_0}, \cdot)} dt, \quad Z_{0}^{z_0} = 0.$$ 

We may view $Z_{t}^{z_0}$ as $B_t$ conditioned to visit $z_0$ before exiting $D$. Let $P$ and $P_{z_0}^D$ denote the laws of $(B_t)$ and $(Z_{t}^{z_0})$, respectively. Then $P_{z_0}^D < P$, and

$$dP_{z_0}^D|_{F_{t}^{0} \cap \Sigma_t} = \frac{G_D(B_t \wedge \tau_D, z_0)}{G_D(0, z_0)}, \quad 0 \leq t < \infty.$$ 

Let $f(w) = \int_D G_D(w, z) dA(z)$. Then $f$ is the solution of the Dirichlet problem $\Delta f = -1$ in $D$ and $f \equiv 0$ on $\partial D$. Define

$$P^D = \int_D P_{z_0}^D G_D(0, z) dA(z), \quad \Psi_t = \int_D G_D(B_t \wedge \tau_D, z) dA(z) = f(B_t \wedge \tau_D).$$

From Proposition 2.1, we see that $P^D < P$ and

$$\frac{dP^D|_{F_{t}^{0} \cap \Sigma_t}}{dP|_{F_{t}^{0} \cap \Sigma_t}} = \Psi_t, \quad 0 \leq t < \infty. \tag{7.1}$$

From Itô’s formula and the PDE for $f$, we see that $M_t := \Psi_t + \frac{t}{2}, 0 \leq t < \tau_D$, is a continuous local martingale, and $\lim_{t \to \tau_D^-} M_t = \frac{1}{2} \tau_D$. We further define $M_t = \frac{1}{2} \tau_D$ for $t \geq \tau_D$. Since $D$ is bounded, the probability $P[\tau_D > N]$ decays exponentially as $N \to \infty$. Also note that $f$ is bounded. Thus, $(M_t)$ is a uniformly integrable martingale, and we have

$$\mathbb{E}[\frac{\tau_D}{2} | F_t^0] = \mathbb{E}[M_\infty | F_t^0] = M_t = \Psi_t + \frac{t \wedge \tau_D}{2}, \quad 0 \leq t < \infty.$$ 

Define a process $(\theta_t)$ such that $\theta_t = \frac{t \wedge \tau_D}{2}$. Then we have

$$\mathbb{E}[\theta_\infty | F_t^0] - \theta_t = \Psi_t, \quad 0 \leq t < \infty. \tag{7.2}$$

From Proposition 2.2, (7.1) and (7.2), we see that

$$P^D = \mathcal{K}_{d\theta}(P). \tag{7.3}$$

Apply first the operation $\Xi P$ and then the pushforward by the map $(B, t) \mapsto (B, B(t))$ to both sides of (7.3). Using a variation of Proposition 2.4, we obtain the following theorem.

**Theorem 7.1.** Let $m$ and $dA$ denote the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Then we have

$$(P^D_\infty \oplus P) \Xi G_D(z, 0) dA(z) = P \otimes \mathcal{M},$$

where $\mathcal{M}$ is a kernel from $\Sigma$ to $D$ defined by $\mathcal{M}(B, \cdot) = \frac{1}{2} B_s(m| \{0, \tau_D\})$.

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Appendix. Boundedness of SLE$_k(\rho)$

In this section, we prove Proposition 3.1(i). Fix $\kappa > 0$ and $\rho \leq \frac{\kappa}{2} - 4$. Fix $z_0 = x_0 + iy_0 \in \mathbb{H}$. Let $\gamma$ be an SLE$_k(\rho)$ curve with force point at $z_0$. Let $\lambda(t), 0 \leq t < T_{z_0}$, be the driving function, and $g_t$ be the corresponding Loewner maps. Define $X_t, Y_t, D_t$ using (3.1). Then the ODEs (3.2) still hold, and now $X_t$ satisfies the SDE:

$$dX_t = -\sqrt{\kappa} dB_t + \frac{(\rho + 2)X_t}{X_t^2 + Y_t^2} dt. \tag{A.1}$$

Let $u(t) = \int_0^t \frac{1}{X^2_s + Y^2_s} ds$. Let $\hat{X}_s = X_{u^{-1}(s)}, \hat{Y}_s = Y_{u^{-1}(s)}$, and $\hat{R}_s = \frac{\hat{X}_s}{\hat{Y}_s}$. Using (3.2) and (A.1), we find that $\hat{Y}_s = y_0 e^{-2s}$ and there is a standard Brownian motion $\hat{B}_s$ such that

$$d\hat{R}_s = \sqrt{\kappa} \sqrt{1 + \hat{R}_s^2} dB_s + (\rho + 4) \hat{R}_s ds. \tag{A.2}$$

Let $V_s = \arcsinh(\hat{R}_s)$. From Itô’s formula, we get

$$dV_s = \sqrt{\kappa} dB_s + (\rho + 4 - \frac{\kappa}{2}) \tanh(V_s) ds. \tag{A.3}$$

From $u(t) = \int_0^t \frac{1}{X^2_s + Y^2_s} ds$, $\hat{Y}_s = y_0 e^{-2s}$, and $V_s = \arcsinh(\hat{R}_s)$, we get

$$T_{z_0} = \int_0^\infty \hat{Y}_s^2 (1 + \hat{R}_s^2) ds = y_0^2 \int_0^\infty e^{-4s} \cosh^2(V_s) ds. \tag{A.4}$$

**Lemma A.1.** Suppose $(V_s)$ solves (A.3) with $\rho \leq \frac{\kappa}{2} - 4$. Then $(V_s)$ can be coupled with a standard Brownian motion $\hat{B}_s$ such that $|V_s| = |V_0 + \sqrt{\kappa} \hat{B}_s|$ for all $s$.

**Proof.** From Itô-Tanaka’s formula, $|V_s|$ satisfies the SDE:

$$d|V_s| = \text{sign}(V_s) \sqrt{\kappa} dB_s + (\rho + 4 - \frac{\kappa}{2}) \tanh(|V_s|) ds + dL^V_s,$$

where $L^V_s$ is continuous and increasing, and stays constant on the intervals on which $|V_s| > 0$. On the other hand, if $\hat{B}_s$ is a standard Brownian motion, then $|V_0 + \sqrt{\kappa} \hat{B}_s|$ satisfies the SDE:

$$d|V_0 + \sqrt{\kappa} \hat{B}_s| = \text{sign}(V_0 + \sqrt{\kappa} \hat{B}_s) \sqrt{\kappa} dB_s + dL^B_s,$$

where $L^B_s$ is continuous and increasing. Define two standard Brownian motions $B^{(1)}_t$ and $B^{(2)}_t$ such that $B^{(1)}_t = \int_0^t \text{sign}(V_s) dB_s$ and $B^{(2)}_t = \int_0^t \text{sign}(V_0 + \sqrt{\kappa} \hat{B}_s) dB_s$. We may couple $\hat{B}_s$ with $B^{(1)}_s$, and so with $V_s$, such that $B^{(1)}_s = B^{(2)}_s$. Since $\rho + 4 - \frac{\kappa}{2} \leq 0$, from the SDE for $|V_s|$ and $|V_0 + \sqrt{\kappa} \hat{B}_s|$, we find that $|V_s| - \sqrt{\kappa} B^{(1)}_s - L^V_s$ is decreasing, and $|V_0 + \sqrt{\kappa} \hat{B}_s| - \sqrt{\kappa} B^{(1)}_s$ is increasing. Suppose that there is $s_0 \geq 0$ such that $|V_{s_0}| > |V_0 + \sqrt{\kappa} \hat{B}_0|$. Then $s_0 > 0$. Let $s'_0 \in (0, s_0)$ be such that $|V_{s'_0}| = |V_0 + \sqrt{\kappa} \hat{B}_{s'_0}|$ and $|V_{s_0}| > |V_0 + \sqrt{\kappa} \hat{B}_s|$ for $s \in (s'_0, s_0)$. Then $|V_{s_0}| > 0$ on $(s'_0, s_0)$.

So $L^V_s$ stays constant on $(s'_0, s_0)$, which implies that $|V_s| - \sqrt{\kappa} B^{(1)}_s$ is decreasing on $[s'_0, s_0]$. Since $|V_0 + \sqrt{\kappa} \hat{B}_s| - \sqrt{\kappa} B^{(1)}_s$ is increasing, we conclude that $|V_s| - |V_0 + \sqrt{\kappa} \hat{B}_s|$ is decreasing on $[s'_0, s_0]$, which contradicts that $|V_{s_0}| > |V_0 + \sqrt{\kappa} \hat{B}_{s_0}|$ and $|V_{s_0}| = |V_0 + \sqrt{\kappa} \hat{B}_{s_0}|$. Thus, $|V_s| \leq |V_0 + \sqrt{\kappa} \hat{B}_s|$ for all $s \geq 0$. \qed

We will use the following well-known inequalities about Brownian motions:

$$\mathbb{P}(\sqrt{\kappa} B_t \leq at + b, \forall t \in [0, \infty)) \geq 1 - 2e^{-\frac{2ah}{\kappa}}, \quad a, b > 0. \tag{A.5}$$
**Proof of Proposition 3.1.** (i) From (A.4) we get $T_{z_0} \leq y_0^2 \int_0^\infty e^{-4s}e^{2|V_s|}ds$. From Lemma A.1, $(V_t)$ may be coupled with a standard Brownian motion $\tilde{B}_t$ such that $|V_t| \leq |V_0 + \sqrt{\kappa}\tilde{B}_t|$. Since $e^{2|V_0|} \leq 4\cosh^2(V_0) = 4(1 + \tilde{R}_0^2) = 4\frac{|z_0|^2}{y_0^2}$, we get

$$T_{z_0} \leq y_0^2 \int_0^\infty e^{-4s}e^{2|V_0|+2\sqrt{\kappa}\tilde{B}_t}ds \leq 4|z_0|^2 \int_0^\infty e^{2\sqrt{\kappa}|\tilde{B}_t|-4s}ds.$$  

From (A.5) we see that the probability that $\sqrt{\kappa}\tilde{B}_t \leq s + b$ for all $s \geq 0$ is at least $1 - 2e^{-\frac{b}{\sqrt{\kappa}}}$, and on this event, from the above formula we get $T_{z_0} \leq 2|z_0|^2e^{2b}$. Thus,

$$\mathbb{P}_0^{\kappa,\rho}[T_{z_0} \leq 2|z_0|^2e^{2b}] \geq 1 - 2e^{-\frac{b}{\sqrt{\kappa}}}, \quad b > 0.$$

(A.6)

This implies that $\mathbb{P}_0^{\kappa,\rho}$-a.s., $T_{z_0} < \infty$.

Recall that $\lambda_t = \sqrt{\kappa}B_t + \int_0^t \frac{-\rho X_s}{X_s^2 + Y_s^2}ds$, $0 \leq t < T_{z_0}$. The finiteness of $T_{z_0}$ implies that a.s. $\lim_{t \to T_{z_0}} B_t \in \mathbb{R}$. For the other term, consider

$$\int_0^{T_{z_0}} \frac{|X_t|}{X_t^2 + Y_t^2}dt = \int_0^\infty |\tilde{X}_s|ds = \int_0^\infty \tilde{Y}_s|\sinh(V_s)|ds \leq y_0 \int_0^\infty e^{V_s-2s}ds.$$  

Coupling $(V_t)$ with $(\tilde{B}_t)$ using Lemma A.1 and then using (A.5), we find that $\mathbb{P}_0^{\kappa,\rho}$-a.s., $\sup_{s \geq 0} (|V_s| - s) < \infty$. Thus, $\mathbb{P}_0^{\kappa,\rho}$-a.s., $\int_0^{T_{z_0}} \frac{|X_t|}{X_t^2 + Y_t^2}dt < \infty$, which implies that the limit $\lim_{t \to T_{z_0}} \int_0^t \frac{X_s}{X_s^2 + Y_s^2}ds$ exists and is finite. Thus, $\mathbb{P}_0^{\kappa,\rho}$-a.s., $\lim_{t \to T_{z_0}} \lambda_t \in \mathbb{R}$. □

**Corollary A.1.** Almost surely $\gamma([0, T_{z_0})]$ is bounded.

**Proof.** This follows from Proposition 3.1(i) and Lemma 4.1 in [11]. □

**Lemma A.2.** For any bounded measurable $U \subset \mathbb{R}$, $\mathbb{P}_0^{\kappa}$-a.s., $\Psi_t^{\kappa,\rho}(U) \to 0$ as $t \to \infty$.

**Proof.** Taking $e^{2b} = t/(2|z_0|^2)$ in (A.6) for some $t > 0$, we get

$$\mathbb{P}_0^{\kappa,\rho}[T_{z_0} > t] \leq 2^{1+1/\kappa}(t/|z_0|^2)^{-1/\kappa}, \quad t > 0.$$  

Using (3.8), we get

$$\mathbb{E}_0^{\kappa,M_t^{\kappa,\rho}} = G^{\kappa,\rho}(z_0)\mathbb{P}_0^{\kappa,\rho}[T_{z_0} > t] \leq 2^{1+1/\kappa}|z_0|^\frac{\rho + 2}{\kappa} + \frac{e^2}{\kappa \rho} \cdot t^{-\frac{1}{\kappa}}, \quad t > 0.$$  

Suppose $|z| \leq R$ for every $z \in U$. Then

$$\mathbb{E}_0^{\kappa,\Psi_t^{\kappa,\rho}}(U) = \int_U \mathbb{E}_0[M_t^{\kappa,\rho}(z)]dA(z) \leq \pi 2^{1+1/\kappa}R^{2+\frac{\rho + 2}{\kappa}} + \frac{e^2}{\kappa \rho} \cdot t^{-\frac{1}{\kappa}}, \quad t > 0.$$  

Thus, $\lim_{t \to \infty} \mathbb{E}_0^{\kappa,\Psi_t^{\kappa,\rho}(U)} = 0$. Since $(\Psi_t^{\kappa,\rho}(U))$ is a positive supermartingale, from Doob’s martingale convergence theorem, we see that $\mathbb{P}_0^{\kappa}$-a.s., $\lim_{t \to \infty} \Psi_t^{\kappa,\rho}(U)$ exists. From Fatou’s lemma, $\mathbb{E}_0[\lim_{t \to \infty} \Psi_t^{\kappa,\rho}(U)] \leq \lim_{t \to \infty} \mathbb{E}_0[\Psi_t^{\kappa,\rho}(U)] = 0$. So we get the conclusion. □

**Remark.** Proposition 3.1(i) and Lemma A.2 also hold if $\frac{\kappa}{2} - 4 < \rho < \frac{\kappa}{2} - 2$. In that case we may use the estimate $|V_t| \leq |V_0| + \sqrt{\kappa}|\tilde{B}_t| + (\rho + 4 - \frac{\kappa}{2})s$, which follows from (A.3) and that $|\tanh(x)| \leq 1$ for $x \in \mathbb{R}$. Then we may apply (A.5) with $a \in (0, \frac{\kappa}{2} - 2 - \rho)$. 

References