Two-curve Green's function for 2-SLE: the boundary case

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Abstract

We prove that for a 2-SLE$_\kappa$ pair (η_1, η_2) in a simply connected domain D, whose boundary is C^1 near $z_0 \in \partial D$, there is some $\alpha > 0$ such that $\lim_{r \to 0^+} r^{-\alpha} \mathbb{P}[\text{dist}(z_0, \eta_j) < r, j = 1, 2]$ converges to a positive number, called the boundary two-curve Green's function. The exponent α equals $2(\frac{12}{\kappa} - 1)$ if z_0 is not one of the endpoints of η_1 and η_2; and otherwise equals $\frac{12}{\kappa} - 1$. We also prove the existence of the boundary (one-curve) Green's function for a single-boundary-force-point SLE$_\kappa(\rho)$ curve, for κ and ρ in some range. In addition, we find the convergence rate and the exact formula of the above Green's functions up to multiplicative constants. To derive these results, we construct a family of two-dimensional diffusion processes, and use orthogonal polynomials to obtain their transition density.

Contents

1 Introduction

1.1 Main results

This paper is the follow-up of the paper [22], in which we proved the existence of two-curve Green's function for 2-SLE$_\kappa$ at an interior point, and obtained the formula of the Green's function up to a multiplicative constant. In the present paper, we will prove the existence of the two-curve Green's function for 2-SLE$_\kappa$ at a boundary point, and also derive its formula. In addition, we will derive boundary Green's function for a chordal SLE$_\kappa(\rho)$ curve with a single boundary force point, where κ and ρ satisfy some conditions.

A 2-SLE$_\kappa$ is a particular case of multiple SLE$_\kappa$. It consists of two random curves in a simply connected domain connecting two pairs of boundary points (more precisely, prime ends), which satisfy the property that, when any one curve is given, the conditional law of the other curve is that of a chordal SLE$_\kappa$ in a complement domain of the first curve.

The two-curve Green's function of a 2-SLE$_\kappa$ is about the rescaled limit of the probability that the two curves in the 2-SLE$_\kappa$ both approach a marked point in $\overline{D}$. More specifically, it was proved in [22] that, for any $\kappa \in (0,8)$, if (η_1,η_2) is a 2-SLE$_\kappa$ in D, and $z_0 \in D$, then the limit

$$G(z_0) := \lim_{r\to 0^+} r^{-\alpha}\mathbb{P}[\text{dist}(\eta_j, z_0) < r, j = 1,2] \tag{1.1}$$

converges to a positive number, where the exponent α equals $\alpha_0 := \frac{(12-\kappa)(\kappa+4)}{8\kappa}$. The limit $G(z_0)$ is called the (interior) two-curve Green's function for (η_1,η_2). The paper [22] also derived the convergence rate and the exact formula of $G(z_0)$ up to an unknown constant.

In this paper we study the limit in the case that $z_0 \in \partial D$, assuming that ∂D is C^1 near z_0, for some suitable exponent α. Below is our first main theorem.

Theorem 1.1. *Let $\kappa \in (0,8)$. Let (η_1,η_2) be a 2-SLE$_\kappa$ in a simply connected domain D. Let $z_0 \in \partial D$. Suppose ∂D is C^1 near z_0. We have the following results.*

(A) If z_0 is not any endpoint of η_1 or η_2, then the limit in (1.1) exists and lies in $(0,\infty)$ for $\alpha=\alpha_1=\alpha_2:=2(\frac{12}{\kappa}-1)$.

(B) If z_0 is one of the endpoints of η_1 and η_2, then the limit in (1.1) exists and lies in $(0,\infty)$ for $\alpha=\alpha_3:=\frac{12}{\kappa}-1$.

Moreover, in each case we may compute $G_D(z_0)$ up to some constant $C>0$ as follows. Let $F_{\kappa,2}$ denote the hypergeometric function ${}_2F_1(\frac{4}{\kappa},1-\frac{4}{\kappa};\frac{8}{\kappa},\cdot)$. Let f map D conformally onto $\mathbb{H}$ such that $f(z_0)=\infty$. Let J denote the map $z\mapsto -1/z$.

(A1) Suppose Case (A) happens and none of η_1 and η_2 separates z_0 from the other curve. We label the f-images of the four endpoints of η_1 and η_2 by $v_-<w_-<w_+<v_+$. Then

$$G_D(z_0)=C_1|(J\circ f)'(z_0)|^{\alpha_1}G_1(\underline{w};\underline{v}),$$

where $C_1>0$ is a constant depending only on κ, and

$$G_1(\underline{w};\underline{v}):=\prod_{\sigma\in\{+,-\}}(|w_\sigma-v_\sigma|^{\frac{8}{\kappa}-1}|w_\sigma-v_{-\sigma}|^{\frac{4}{\kappa}})F_{\kappa,2}\Big(\frac{(w_+-w_-)(v_+-v_-)}{(w_+-v_-)(v_+-w_-)}\Big)^{-1}. \tag{1.2}$$

(A2) Suppose Case (A) happens and one of η_1 and η_2 separates z_0 from the other curve. We label the f-images of the four endpoints of η_1 and η_2 by $v_-<w_-<w_+<v_+$. Then

$$G_D(z_0)=C_2|(J\circ f)'(z_0)|^{\alpha_2}G_2(\underline{w};\underline{v})$$

where $C_2>0$ is a constant depending only on κ, and

$$G_2(\underline{w};\underline{v}):=\prod_{u\in\{w,v\}}|u_+-u_-|^{\frac{8}{\kappa}-1}\prod_{\sigma\in\{+,-\}}|w_\sigma-v_{-\sigma}|^{\frac{4}{\kappa}}F_{\kappa,2}\Big(\frac{(v_+-w_+)(w_--v_-)}{(w_+-v_-)(v_+-w_-)}\Big)^{-1}. \tag{1.3}$$

(B) Suppose Case (B) happens. We label the f-images of the other three endpoints of η_1 and η_2 by w_+,w_-,v_+, such that $f^{-1}(v_+)$ and z_0 are endpoints of the same curve, and w_+,v_+ lie on the same side of w_-. Then

$$G_D(z_0)=C_3|(J\circ f)'(z_0)|^{\alpha_3}G_3(\underline{w};v_+),$$

where $C_3>0$ is a constant depending only on κ, and

$$G_3(\underline{w};v_+)=|w_+-w_-|^{\frac{8}{\kappa}-1}|v_+-w_-|^{\frac{4}{\kappa}}F_{\kappa,2}\Big(\frac{v_+-w_+}{v_+-w_-}\Big)^{-1}. \tag{1.4}$$

Our second main theorem is about the boundary Green's function of a chordal $\mathrm{SLE}_\kappa(\rho)$ curve with a single boundary force point.

Theorem 1.2. *Let $\kappa\in(0,4]$ and $\rho>-2$ or $\kappa\in(4,8)$ and $\rho\ge\frac{\kappa}{2}-2$. Let $w\in\mathbb{R}$ and $v\in(\mathbb{R}\setminus\{w\})\cup\{w^-,w^+\}$. Let η be a chordal SLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w to ∞ with the force point v. Let $z_0\in\mathbb{R}\setminus\{w\}$ be such that z_0 and v lie on the same side of w, and $|z_0-w|\ge|v-w|$. Let $\alpha_2=\frac{\rho+2}{\kappa}(\rho-(\frac{\kappa}{2}-4))$, $\alpha_3=\frac{2}{\kappa}(\rho-(\frac{\kappa}{2}-4))$, $\beta_2=2\rho+6$ and $\beta_3=\rho+6$. Then*

(i) There is a positive constant C depending only on κ and ρ such that, if $z_0\ne v$, then

$$\mathbb{P}[\operatorname{dist}(\eta,z_0)<r]=Cr^{\alpha_2}|z_0-v|^{\alpha_3-\alpha_2}|z_0-w|^{-\alpha_3}\Big(1+O\Big(\frac{r}{|z_0-v|}\Big)^{\frac{\beta_2}{\beta_2+2}}\Big),\quad r\to0^+.$$

(ii) There is a positive constant C depending only on κ and ρ such that, if $z_0=v$, then

$$\mathbb{P}[\operatorname{dist}(\eta,z_0)<r]=Cr^{\alpha_3}|z_0-w|^{-\alpha_3}\Big(1+O\Big(\frac{r}{|z_0-w|}\Big)^{\frac{\beta_3}{\beta_3+2}}\Big),\quad r\to0^+.$$

For both (i) and (ii), the implicit constants depend only on κ,ρ. Moreover, if $\kappa\in(0,4]$ and $\rho\in(-2,\frac{\kappa}{2}-2)$, then (i) holds with a different constant $C>0$ if η is replaced by $\eta\cap\mathbb{R}$; if $\kappa\in(0,4]$ and $\rho\in(-2,\frac{\kappa}{2}-2)$, or $\kappa\in(4,8)$ and $\rho\ge\frac{\kappa}{2}-2$, then (ii) holds with a different constant $C>0$ if η is replaced by $\eta\cap\mathbb{R}$.

The existence of boundary Green's function for chordal SLE$_\kappa$ (without force points) was proved in [4]. It was proved in [13, Theorem 1.8] that for $\kappa>0$ and $\rho_1,\rho_2\in\mathbb{R}$ such that $\rho_1>-2$ and $\rho_1+\rho_2>\frac{\kappa}{2}-4$, if η is an SLE$_\kappa(\rho_1,\rho_2)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $(0^+,1)$, then $\mathbb{P}[\operatorname{dist}(\eta,1)<r]=r^{\alpha+o(1)}$ as $r\to0$, where $\alpha=\frac{1}{\kappa}(\rho_1+2)(\rho_1+\rho_2+4-\frac{\kappa}{2})$. Note that if $\rho_1=0$, then $\alpha=\alpha_3(\kappa,\rho_2)$; and if $\rho_2=0$, then $\alpha=\alpha_2(\kappa,\rho_1)$. This means that Theorem 1.2 improves the estimate of Theorem 1.8 of [13] in some cases.

1.2 Strategy

For the proofs of the main theorems, we use the ideas introduced in [22]. By conformal invariance of 2-SLE$_\kappa$, we may assume that $D=\mathbb{H}:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$, and $z_0=\infty$. It suffices to consider the limit $\lim_{L\to\infty}L^\alpha\mathbb{P}[\eta_j\cap\{|z|>L\}\ne\emptyset]$. In Case (A) of Theorem 1.1, we label the four endpoints of η_1 and η_2 by $v_+>w_+>w_->v_-$. There are two possible link patterns: $(w_+\leftrightarrow v_+;w_-\leftrightarrow v_-)$ and $(w_+\leftrightarrow w_-;v_+\leftrightarrow v_-)$, which respectively correspond to Case (A1) and Case (A2) of Theorem 1.1.

For the first link pattern, we label the two curves by η_+ and η_-. By translation and dilation, we may assume that $v_+=1$ and $v_-=-1$. Then we introduce a new point $v_0=0$, and make an assumption that $0\in(w_-,w_+)$. We then grow η_+ and η_- simultaneously from w_+ and w_- towards v_+ and v_-, respectively, up to the time that either curve reaches its target, or separates v_+ or v_- from ∞. The speeds of η_+ and η_- are controlled by two factors: (F1) for any t in the lifespan $[0,T^u)$, the harmonic measure of the arc between v_+ and v_- in the unbounded connected component of $\mathbb{H}\setminus(\eta_+([0,t])\cup\eta_-([0,t]))$, denoted by H_t, viewed from ∞, increases exponentially with factor 2. More specifically, if g_t maps H_t conformally onto $\mathbb{H}$, and satisfies $g_t(z)/z\to1$ as

$z \to \infty$, then $V_+(t) - V_-(t) = e^{2t}(v_+ - v_-)$, where $V_\pm(t) := g_t(v_\pm)$. (F2) the harmonic measure of $\eta_+([0,t]) \cup [v_0, v_+]$ in H_t viewed from ∞ agrees with that of $\eta_-([0,t]) \cup [v_-, v_0]$. We will see that there is a unique $V_0(t) \in (V_-(t), V_+(t))$ such that the continuous extension of g_t^{-1} on $\overline{\mathbb{H}}$ maps $[V_-(t), V_0(t)]$ into $[v_-, v_0] \cup \eta_-([0,t])$, and maps $[V_0(t), V_+(t)]$ into $[v_0, v_+] \cup \eta_+([0,t])$. The second condition means that $V_+(t) - V_0(t) = V_0(t) - V_-(t)$. In the case that $\eta_+([0,t]) \cup \eta_-([0,t])$ does not separate v_0 from ∞, $V_0(t)$ is simply $g_t(v_0)$. We will also deal with the complicated case that $\eta_+([0,t]) \cup \eta_-([0,t])$ does the separation, which may happen if $\kappa \in (4,8)$.

At the time T^u, one of the two curves, say η_+, separates v_+ or v_- from ∞. In the former case the rest of η_+ grows in a bounded connected component of $\mathbb{H} \setminus \eta_+([0,T^u))$; in the latter case, the whole η_- is disconnected from ∞ by $\eta_+([0,T^u))$. So we may focus on the parts of η_+ and η_- before T^u. Using Koebe's 1/4 theorem (applied to g_t at ∞) and Beurling's estimate (applied to a planar Brownian motion started near ∞), we find that for $0 \le t < T^u$, the diameter of both $\eta_+([0,t])$ and $\eta_-([0,t])$ are comparable to e^{2t}. Thus, there are constants $a_2 > a_1 \in \mathbb{R}$ such that for any $L > |v_+ - v_-|$,

$$\{T^u > \log(L)/2 + a_2\} \subset \{\eta_\sigma \cap \{|z| > L\} \neq \emptyset, \sigma \in \{+,-\}\} \subset \{T^u > \log(L)/2 + a_1\}. \tag{1.5}$$

We may obtain a two-dimensional diffusion process $\underline{R}(t) = (R_+(t), R_-(t)) \in [0,1]^2$, $0 \le t < T^u$, such that for every $t \in [0, T^u)$, $R_\sigma(t) = \frac{W_\sigma(t) - V_0(t)}{V_\sigma(t) - V_0(t)}$, $\sigma \in \{+,-\}$, where $W_\sigma(t) = g_t(\eta_\sigma(t)) \in [V_0(t), V_\sigma(t)]$. Note that $w_\sigma = \sigma R_\sigma(0)$, $\sigma \in \{+,-\}$. We will derive the transition density and quasi-invariant density of $(\underline{R})$ using the knowledge of 2-SLE$_\kappa$ partition function and the technique of orthogonal polynomials. The quasi-invariant density $\widetilde{p}^R$ of $(\underline{R})$ is a positive function on $(0,1)^2$, whose integral against the two-dimensional Lebesgue measure is 1, and if $\underline{R}$ starts at a random point in $(0,1)^2$, whose law has the density $\widetilde{p}^R$ against the Lebesgue measure, then $(\underline{R})$ is a quasi-stationary process with decay rate α_1 in the sense that, for any deterministic time $t > 0$, $\mathbb{P}[T^u > t] = e^{-2\alpha_1 t}$, and the law of $\underline{R}(t)$ conditional on $\{T^u > t\}$ agrees with that of $\underline{R}(0)$. From (1.5) we know that, if (η_+, η_-) has the random link pattern $(r_+ \leftrightarrow 1; -r_- \leftrightarrow -1)$ such that $(r_+, r_-) \in (0,1)^2$ follows the law with the density $\widetilde{p}^R$, then $\mathbb{P}[\eta_\sigma \cap \{|z| > L\} \neq \emptyset, \sigma \in \{+,-\}]$ is comparable to $L^{-\alpha_1}$. We will then combine this estimate with the technique introduced in [6] to prove the convergence of $\lim_{L\to\infty} L^{\alpha_1} \mathbb{P}[\eta_\sigma \cap \{|z| > L\} \neq \emptyset, \sigma \in \{+,-\}]$.

After proving the existence of the Green's function for the above random link pattern, we may then use an estimate on the convergence of the transition density of $(\underline{R})$ to its quasi-invariant density to prove the existence of the Green's function in the case that the link pattern is $(r_+ \leftrightarrow 1; -r_- \leftrightarrow -1)$, where (r_+, r_-) is a deterministic point in $(0,1)^2$. By translation and dilation, we then have the existence of Green's function in the case that $(v_+ + v_-)/2 \in (w_-, w_+)$. Finally, we will remove this assumption, and work out the general case.

The above approach, especially the transition density of $(\underline{R})$, also gives us the exact formula of the Green's function up to an unknown multiplicative constant, as well as the rate of the convergence of the rescaled probability to the Green's function. See Theorem 6.2.

For the link pattern $(w_+ \leftrightarrow w_-; v_+ \leftrightarrow v_-)$, we label the curves by η_w and η_v. We observe that η_v disconnects η_w from ∞. Thus, for $L > \max\{|v_+|, |v_-|\}$, η_w intersects $\{|z| > L\}$ implies that η_v does the intersection as well. Then the two-curve Green's function reduces to a single-curve

Green's function. We will use a similar approach as before. We still first assume that $v_+ = 1$, $v_- = -1$, and $0 \in (w_-, w_+)$, and let $v_0 = 0$. This time, we grow η_+ and η_- simultaneously along the same curve η_w such that η_σ runs from w_σ towards $w_{-\sigma}$, $\sigma \in \{+,-\}$. The growth is stopped if η_+ and η_- together exhaust the range of η_w, or any of them disconnects its target from ∞. Moreover, the speeds of the curves are controlled by two factors (F1) and (F2) as in the previous case.

We then observe that for big L, η_w intersects $\{|z| > L\}$ if and only if η_+ and η_- both intersect $\{|z| > L\}$. So we may study η_+ and η_- instead of η_w and η_v. The rest of the argument is similar to that in the previous case, except that the transition density and invariant density of the process $(\underline{R})$ will be different. We will obtain the exact formula of the Green's function up to a constant as well as the rate of convergence. See Corollary 6.5.

In Case (B), we may assume that $v_+ = 1$ and $w_+ + w_- = 0$. Let $v_0 = 0$ and $v_- = -1$. We label the curves by η_w and η_v, and grow η_+ and η_- simultaneously along the same curve η_w as in Case (A2). The rest of the proof follows the same approach in the previous cases except that the transition density and invariant density of $(\underline{R})$ will be different, and the exponent will be α_3 instead of α_1. We will obtain the exact formula of the Green's function up to a constant as well as the rate of convergence. See Corollary 6.7.

Recall that in Cases (A2) and (B), we are dealing with a single-curve Green's function about η_w. It is known that η_w is an hSLE$_\kappa$ (cf. [20, Proposition 6.10]) from w_- to w_+ with force points at v_- and v_+ (Case (A2)) or ∞ and v_+ (Case (B)). The hSLE$_\kappa$ is a special case of the intermediate SLE$_\kappa(\rho)$, abbreviated now as iSLE$_\kappa(\rho)$, in the case that $\rho = 2$. The iSLE$_\kappa(\rho)$ process was introduced in [25] for $\kappa \in (0, 4)$ and $\rho \geq \frac{\kappa}{2} - 2$ to prove the reversibility of a chordal SLE$_\kappa(\rho)$ curve with a single degenerate boundary force point. The name of intermediate SLE$_\kappa(\rho)$ comes form the fact that, for a chordal SLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from 0 to ∞ with the force point at 0^+, if one conditions on a part of the forward oriented curve up to a forward stopping time and also on a part of the backward oriented curve up to a backward stopping time, then the middle part of the curve has the law of an intermediate SLE$_\kappa(\rho)$ curve. The definition of iSLE$_\kappa(\rho)$ in [25] easily extends to all $\kappa \in (0, 8)$ and $\rho > \max\{-2, \frac{\kappa}{2} - 4\}$.

The argument in the proof of Cases (A2) and (B) of Theorem 1.1 can be used to prove a more general result. Let $\kappa \in (0, 4]$ and $\rho > -2$ or $\kappa \in (4, 8)$ and $\rho \geq \frac{\kappa}{2} - 2$. For those κ and ρ, we know that iSLE$_\kappa(\rho)$ satisfies reversibility. If η_w is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_- to w_+ with force points v_- and v_+, then the boundary Green's function for η_w at ∞ exists with the exponent being the α_2 in Theorem 1.2. See Theorem 6.4. The Green's function also exists if v_- is replaced by ∞, and the exponent is replaced by the α_3 in Theorem 1.2. See Theorem 6.6. The iSLE$_\kappa(\rho)$ curve reduces to a chordal SLE$_\kappa(\rho)$ curve if we let $v_+ \to w_+^+$, and the Green's functions still exist in the limit cases. Theorem 1.2 then follows from these results via a Möbius automorphism of $\mathbb{H}$ that maps w_+ to ∞.

1.3 Outline

Below is the outline of the paper. In Section 2, we recall definitions, notations, and some basic results that will be needed in this paper. In Section 3 we develop a framework on a commuting pair of deterministic chordal Loewner curves, which do not cross but may intersect each other. The work extends the disjoint ensemble of Loewner curves that appeared in [27, 26]. At the end of the section, we describe a way to grow the two curves simultaneously with certain properties. In Section 4, we use the results from the previous section to study a pair of multi-force-point $\mathrm{SLE}_\kappa(\underline{\rho})$ curves, which commute with each other in the sense of [2]. We obtain a two-dimensional diffusion process $\underline{R}(t) = (R_+(t), R_-(t))$, $0 \le t < \infty$, for the simultaneous growth of the two curves, and derive its transition density using orthogonal two-variable polynomials. In Section 5, we study three types of commuting pair of $\mathrm{iSLE}_\kappa(\rho)$ curves, which correspond to the three cases in Theorem 1.1. We prove that each of them is *locally* absolutely continuous w.r.t. a commuting pair of $\mathrm{SLE}_\kappa(\underline{\rho})$ curves for certain force values, and also find the Radon-Nikodym derivative at different times. For each commuting pair of $\mathrm{iSLE}_\kappa(\rho)$ curves, we obtain a two-dimensional diffusion process $\underline{R}(t) = (R_+(t), R_-(t))$ with random finite lifetime. Then we use the transition density of the $(\underline{R})$ for the commuting $\mathrm{SLE}_\kappa(\underline{\rho})$ curves to derive the transition density of the $(\underline{R})$ for the commuting $\mathrm{iSLE}_\kappa(\rho)$ curves. In addition, we find its quasi-invariant density and decay rate. In the last section we prove some important theorems, and finally prove Theorems 1.1 and 1.2.

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2 Preliminary

We first fix some notation. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $z_0 \in \mathbb{C}$ and $S \subset \mathbb{C}$, let $\mathrm{rad}_{z_0}(S) = \sup\{|z - z_0| : z \in S \cup \{z_0\}\}$. If a function f is absolutely continuous on I, and $f' = g$ a.e. on I, then we write $f' \overset{\mathrm{ae}}{=} g$ on I. This means that $f(x_2) - f(x_1) = \int_{x_1}^{x_2} g(x)dx$ for any $x_1 < x_2 \in I$. Here g may not be defined on a subset of I with Lebesgue measure zero. We will also use "$\overset{\mathrm{ae}}{=}$" for PDE or SDE in some similar sense.

2.1 $\mathbb{H}$-hulls and chordal Loewner equation

A relatively closed subset K of $\mathbb{H}$ is called an $\mathbb{H}$-hull if K is bounded and $\mathbb{H} \setminus K$ is a simply connected domain. If S is a bounded subset of $\overline{\mathbb{H}}$ such that $S \cup \mathbb{R}$ is connected and closed, then the unbounded connected component of $\mathbb{H} \setminus S$ is a simply connected domain, whose complement in $\mathbb{H}$ is an $\mathbb{H}$-hull. We call it the $\mathbb{H}$-hull generated by S, and denote it by $\mathrm{Hull}(S)$.

For an $\mathbb{H}$-hull K, there is a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ such that $g_K(z) = z + \frac{c}{z} + O(1/z^2)$ as $z \to \infty$ for some $c \ge 0$. The constant c, denoted by $\mathrm{hcap}(K)$, is called the $\mathbb{H}$-capacity of K, which is zero iff $K = \emptyset$. If $\partial(\mathbb{H} \setminus K)$ is locally connected, then g_K^{-1} extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$, which is denoted by f_K.

If $K_1 \subset K_2$ are two $\mathbb{H}$-hulls, then we define $K_2/K_1 = g_{K_1}(K_2 \setminus K_1)$, which is also an $\mathbb{H}$-hull, and we have $g_{K_2} = g_{K_2/K_1} \circ g_{K_1}$ and $\operatorname{hcap}(K_2/K_1) = \operatorname{hcap}(K_2) - \operatorname{hcap}(K_1)$. From hcap ≥ 0 we see that $\operatorname{hcap}(K_1), \operatorname{hcap}(K_2/K_1) \le \operatorname{hcap}(K_2)$ if $K_1 \subset K_2$. If $K_1 \subset K_2 \subset K_3$ are $\mathbb{H}$-hulls, then $K_2/K_1 \subset K_3/K_1$ and

$$(K_3/K_1)/(K_2/K_1) = K_3/K_2. \tag{2.1}$$

Let K be a non-empty $\mathbb{H}$-hull. Let $K^{\text{doub}} = \overline{K} \cup \{\overline{z} : z \in K\}$, where $\overline{K}$ is the closure of K, and $\overline{z}$ is the complex conjugate of z. By Schwarz reflection principle, there is a compact set $S_K \subset \mathbb{R}$ such that g_K extends to a conformal map from $\mathbb{C} \setminus K^{\text{doub}}$ onto $\mathbb{C} \setminus S_K$. Let $a_K = \min(\overline{K} \cap \mathbb{R})$, $b_K = \max(\overline{K} \cap \mathbb{R})$, $c_K = \min S_K$, $d_K = \max S_K$. Then the extended g_K maps $\mathbb{C} \setminus (K^{\text{doub}} \cup [a_K, b_K])$ conformally onto $\mathbb{C} \setminus [c_K, d_K]$. Since $g_K(z) = z + o(1)$ as $z \to \infty$, by Koebe's 1/4 theorem, $\operatorname{diam}(K) = \operatorname{diam}(K^{\text{doub}} \cup [a_K, b_K]) \asymp d_K - c_K$.

Example. Let $x_0 \in \mathbb{R}$, $r > 0$. Then $H := \{z \in \mathbb{H} : |z - x_0| \le r\}$ is an $\mathbb{H}$-hull with $g_H(z) = z + \frac{r^2}{z - x_0}$, $\operatorname{hcap}(H) = r^2$, $a_H = x_0 - r$, $b_H = x_0 + r$, $H^{\text{doub}} = \{z \in \mathbb{C} : |z - x_0| \le r\}$, $c_H = x_0 - 2r$, $d_H = x_0 + 2r$.

The next proposition combines Lemmas 5.2 and 5.3 of [28].

Proposition 2.1. *If $L \subset K$ are two non-empty $\mathbb{H}$-hulls, then $[a_K, b_K] \subset [c_K, d_K]$, $[c_L, d_L] \subset [c_K, d_K]$, and $[c_{K/L}, d_{K/L}] \subset [c_K, d_K]$.*

Proposition 2.2. *For any $x \in \mathbb{R} \setminus K^{\text{doub}}$, $0 < g_K'(x) \le 1$. Moreover, g_K' is decreasing on $(-\infty, a_K)$ and increasing on (b_K, ∞).*

Proof. By [17, Lemma C.1], there is a measure μ_K supported on S_K with $|\mu_K| = \operatorname{hcap}(K)$ such that $g_K^{-1}(z) - z = \int_{S_K} \frac{-1}{z-y} d\mu_K(y)$ for any $x \in \mathbb{R} \setminus S_K$. Differentiating this formula and letting $z = x \in \mathbb{R} \setminus S_K$, we get $(g_K^{-1})'(x) = 1 + \int_{S_K} \frac{1}{(x-y)^2} d\mu_K(y) \ge 1$. So $0 < g_K' \le 1$ on $\mathbb{R} \setminus K^{\text{doub}}$. Further differentiating the integral formula w.r.t. x, we find that $(g_K^{-1})''(x) = \int_{S_K} \frac{-2}{(x-y)^3} d\mu_K(y)$ is positive on $(-\infty, c_K)$ and negative on (d_K, ∞), which means that $(g_K^{-1})'$ is increasing on $(-\infty, c_K)$ and decreasing on (d_K, ∞). Since g_K maps $(-\infty, a_K)$ and (b_K, ∞) onto $(-\infty, c_K)$ and (d_K, ∞), respectively, we get the monotonicity of g_K' . □

Proposition 2.3. *If K is an $\mathbb{H}$-hull with $\operatorname{rad}_{x_0}(K) \le r$ for some $x_0 \in \mathbb{R}$, then $\operatorname{hcap}(K) \le r^2$, $\operatorname{rad}_{x_0}(S_K) \le 2r$, and $|g_K(z) - z| \le 3r$ for any $z \in \mathbb{C} \setminus K^{\text{doub}}$.*

Proof. We have $K \subset H := \{z \in \mathbb{H} : |z - x_0| \le r\}$. So $\operatorname{hcap}(K) \le \operatorname{hcap}(H) = r^2$. From Proposition 2.1, $S_K \subset [c_K, d_K] \subset [c_H, d_H] = [x_0 - 2r, x_0 + 2r]$. So $\operatorname{rad}_{x_0}(S_K) \le 2r$. Since $g_K(z) - z$ is analytic on $\mathbb{C} \setminus K^{\text{doub}}$ and tends to 0 as $z \to \infty$, by the maximum modulus principle,

$$\sup_{z \in \mathbb{C} \setminus K^{\text{doub}}} |g_K(z) - z| \le \limsup_{\mathbb{C} \setminus K^{\text{doub}} \ni z \to K^{\text{doub}}} |g_K(z) - z| \le r + 2r = 3r,$$

where the second inequality holds because $z \to K^{\text{doub}}$ implies that $g_K(z) \to S_K$. □

Proposition 2.4. *For two nonempty $\mathbb{H}$-hulls $K_1 \subset K_2$ such that $\overline{K_2/K_1} \cap [c_{K_1}, d_{K_1}] \neq \emptyset$, we have $|c_{K_1} - c_{K_2}|, |d_{K_1} - d_{K_2}| \leq 4 \operatorname{diam}(K_2/K_1)$.*

Proof. It suffices to estimate $|c_{K_1} - c_{K_2}|$. Let $\Delta K = K_2/K_1$. Let $c_1' = \lim_{x \uparrow a_{K_2}} g_{K_1}(x)$. Since g_{K_1} maps $\mathbb{H} \setminus K_2$ onto $\mathbb{H} \setminus \Delta K$, we have $c_1' = \min\{c_{K_1}, a_{\Delta K}\}$. Since $\overline{\Delta K} \cap [c_{K_1}, d_{K_1}] \neq \emptyset$, $c_1' \geq c_1 - \operatorname{diam}(\Delta K)$. Thus, by Proposition 2.3,

$$c_{K_2} = \lim_{x \uparrow a_{K_2}} g_{\Delta K} \circ g_{K_1}(x) = \lim_{y \uparrow c_1'} g_{\Delta K}(y) \geq c_1' - 3 \operatorname{diam}(\Delta K) \geq c_{K_1} - 4 \operatorname{diam}(\Delta K).$$

By Proposition 2.1, $c_{K_2} \leq c_{K_1}$. So we get $|c_{K_1} - c_{K_2}| \leq 4 \operatorname{diam}(\Delta K)$. $\square$

The following proposition follows immediately from Proposition 3.42 of [5].

Proposition 2.5. *Suppose K_0, K_1, K_2 are $\mathbb{H}$-hulls such that $K_0 \subset K_1 \cap K_2$. Then*

$$\operatorname{hcap}(K_1) + \operatorname{hcap}(K_2) \geq \operatorname{hcap}(\operatorname{Hull}(K_1 \cup K_2)) + \operatorname{hcap}(K_0).$$

Let $\widehat{w} \in C([0,T), \mathbb{R})$ for some $T \in (0, \infty]$. The chordal Loewner equation driven by $\widehat{w}$ is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \widehat{w}(t)}, \quad 0 \leq t < T; \quad g_0(z) = z.$$

For every $z \in \mathbb{C}$, let τ_z be the first time that the solution $g_\cdot(z)$ blows up; if such time does not exist, then set $\tau_z = \infty$. For $t \in [0,T)$, let $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$. It turns out that each K_t is an $\mathbb{H}$-hull with $\operatorname{hcap}(K_t) = 2t$, $K_t^{\text{doub}} = \{z \in \mathbb{C} : \tau_z \leq t\}$, which is connected, and each g_t agrees with g_{K_t}. We call g_t and K_t the chordal Loewner maps and hulls, respectively, driven by $\widehat{w}$. We will write $\operatorname{hcap}_2(K)$ for $\operatorname{hcap}(K)/2$. So $\operatorname{hcap}_2(K_t) = t$ for all t.

If for every $t \in [0,T)$, f_{K_t} is well defined, and $\eta(t) := f_{K_t}(\widehat{w}(t))$, $0 \leq t < T$, is continuous in t, then we say that η is the chordal Loewner curve driven by $\widehat{w}$. Such η may not exist in general. When it exists, we have $\eta(0) = \widehat{w}(0) \in \mathbb{R}$, and $K_t = \operatorname{Hull}(\eta([0,t]))$ for all t, and we say that K_t, $0 \leq t < T$, are generated by η.

Let u be a continuous and strictly increasing function on $[0,T)$. Let v be the inverse of $u - u(0)$. Suppose that g_t^u and K_t^u, $0 \leq t < T$, satisfy that $g_{v(t)}^u$ and $K_{v(t)}^u$, $0 \leq t < u(T) - u(0)$, are chordal Loewner maps and hulls, respectively, driven by $\widehat{w} \circ v$. Then we say that g_t^u and K_t^u, $0 \leq t < T$, are chordal Loewner maps and hulls, respectively, driven by $\widehat{w}$ with speed du, and call $(K_{v(t)}^u)$ the normalization of (K_t^u). If (K_t^u) are generated by a curve η^u, i.e., $K_t^u = \operatorname{Hull}(\eta^u([0,t]))$ for all t, then η^u is called a chordal Loewner curve driven by $\widehat{w}$ with speed du, and $\eta^u \circ v$ is called the normalization of η^u. If u is absolutely continuous with $u' \overset{\text{ae}}{=} q$, then we also say that the speed is q. In this case, the chordal Loewner maps satisfy the differential equation $\partial_t g_t^u(z) \overset{\text{ae}}{=} \frac{2q(t)}{g_t^u - \widehat{w}(t)}$. We omit the speed when it is constant 1.

The following proposition is straightforward.

Proposition 2.6. *Suppose K_t, $0 \le t < T$, are chordal Loewner hulls driven by $\widehat{w}(t)$, $0 \le t < T$, with speed du. Then for any $t_0 \in [0,T)$, K_{t_0+t}/K_{t_0}, $0 \le t < T-t_0$, are chordal Loewner hulls driven by $\widehat{w}(t_0+t)$, $0 \le t < T-t_0$, with speed $du(t_0+t)$. One immediate consequence is that, for any $t_1 < t_2 \in [0,T)$, $\overline{K_{t_2}/K_{t_1}}$ is connected.*

The following proposition is a slight variation of Lemma 4.13 of [5].

Proposition 2.7. *Suppose K_t, $0 \le t < T$, are chordal Loewner hulls driven by $\widehat{w}(t)$, $0 \le t < T$, with speed du. Then for any $0 \le t < T$,*

$$\mathrm{rad}_{\widehat{w}(0)}(K_t) \le 4 \max\{\sqrt{u(t)-u(0)}, \mathrm{rad}_{\widehat{w}(0)}(\widehat{w}([0,t]))\}.$$

The following proposition is a slight variation of Theorem 2.6 of [7].

Proposition 2.8. *The $\mathbb{H}$-hulls K_t, $0 \le t < T$, are chordal Loewner hulls with some speed if and only if for any fixed $a \in [0,T)$, $\lim_{\delta \downarrow 0} \sup_{0 \le t \le a} \mathrm{diam}(K_{t+\delta}/K_t) = 0$. Moreover, the driving function $\widehat{w}$ satisfies that $\{\widehat{w}(t)\} = \bigcap_{\delta>0} \overline{K_{t+\delta}/K_t}$, $0 \le t < T$; and the speed is du, where we may take $u(t) = \mathrm{hcap}_2(K_t)$, $0 \le t < T$.*

Proposition 2.9. *Suppose K_t, $0 \le t < T$, are chordal Loewner hulls driven by $\widehat{w}$ with some speed. Then for any $t_0 \in (0,T)$, $c_{K_{t_0}} \le \widehat{w}(t) \le d_{K_{t_0}}$ for all $t \in [0,t_0]$.*

Proof. Let $t_0 \in (0,T)$. If $0 \le t < t_0$, by Propositions 2.1 and 2.8, $\widehat{w}(t) \in [a_{K_{t_0}/K_t}, b_{K_{t_0}/K_t}] \subset [c_{K_{t_0}/K_t}, d_{K_{t_0}/K_t}] \subset [c_{K_{t_0}}, d_{K_{t_0}}]$. By the continuity of $\widehat{w}$, we also have $\widehat{w}(t_0) \in [c_{K_{t_0}}, d_{K_{t_0}}]$. $\square$

The following proposition combines [11, Lemma 2.5] and [10, Lemma 3.3].

Proposition 2.10. *Suppose $\widehat{w} \in C([0,T),\mathbb{R})$ generates a chordal Loewner curve η and chordal Loewner hulls K_t, $0 \le t < T$. Then the set of times $\{t \in [0,T) : \eta(t) \in \mathbb{R}\}$ has Lebesgue measure zero. Moreover, if the Lebesgue measure of $\eta([0,T)) \cap \mathbb{R}$ is zero, then the functions $c(t)$ and $d(t)$ defined by $c(t) := c_{K_t}$ and $d(t) := d_{K_t}$, $0 < t < T$, and $c(0) = d(0) := \widehat{w}(0)$ are absolutely continuous with $c'(t) \overset{\mathrm{ae}}{=} \frac{2}{c(t)-\widehat{w}(t)}$ and $d'(t) \overset{\mathrm{ae}}{=} \frac{2}{d(t)-\widehat{w}(t)}$, and are decreasing and increasing, respectively. Moreover, $c(t)$ and $d(t)$ are continuously differentiable at the set of times t such that $\eta(t) \notin \mathbb{R}$, and in that case "$\overset{\mathrm{ae}}{=}$" can be replaced by "$=$".*

Definition 2.11. (i) Modified real line: For $w \in \mathbb{R}$, we define $\mathbb{R}_w = (\mathbb{R} \setminus \{w\}) \cup \{w^-, w^+\}$, which has a total order endowed from $\mathbb{R}$ and the relation $x < w^- < w^+ < y$ for any $x, y \in \mathbb{R}$ such that $x < w$ and $y > w$. It has a topology such that $(-\infty, w) \cup \{w^-\}$ and $\{w^+\} \cup (w, \infty)$ are two connected components, and the natural projection $\pi_w : \mathbb{R}_w \to \mathbb{R}$ with $\pi_w(w^\pm) = w$ and $\pi_w(x) = x$ for $x \in \mathbb{R} \setminus \{w\}$ induces homeomorphisms between the two components and $(-\infty, w]$ and $[w, \infty)$, respectively.

(ii) Modified Loewner map: Let K be an $\mathbb{H}$-hull and $w \in \mathbb{R}$. Let $a_K^w = \min\{w, a_K\}$, $b_K^w = \max\{w, b_K\}$, $c_K^w = \lim_{x\uparrow a_K^w} g_K(x)$, and $d_K^w = \lim_{x\downarrow b_K^w} g_K(x)$. They are all equal to w if $K = \emptyset$. Define g_K^w on $\mathbb{R}_w \cup \{+\infty, -\infty\}$ such that $g_K^w(\pm\infty) = \pm\infty$, $g_K^w(x) = g_K(x)$ if $x \in \mathbb{R} \setminus [a_K^w, b_K^w]$; $g_K^w(x) = c_K^w$ if $x = w^-$ or $x \in [a_K^w, b_K^w] \cup (-\infty, w)$; and $g_K^w(x) = d_K^w$ if $x = w^+$ or $x \in [a_K^w, b_K^w] \cap (w, \infty)$. Note that g_K^w is continuous and increasing.

Proposition 2.12. *Let $K_1 \subset K_2$ be two $\mathbb{H}$-hulls. Let $w \in \mathbb{R}$ and $\widetilde{w} \in [c_{K_1}^w, d_{K_1}^w]$. Then*

$$g_{K_2/K_1}^{\widetilde{w}} \circ g_{K_1}^w(x) = g_{K_2}^w(x), \quad \forall x \in \mathbb{R}_w \cup \{+\infty, -\infty\}. \tag{2.2}$$

Here if $\widetilde{w} = g_{K_1}^w(x)$, then we understand $g_{K_2/K_1}^{\widetilde{w}} \circ g_{K_1}^w(x)$ as $g_{K_2/K_1}^{\widetilde{w}}(\widetilde{w}^+) = d_{K_2/K_1}^{\widetilde{w}}$ if $x > w$, and as $g_{K_2/K_1}^{\widetilde{w}}(\widetilde{w}^-) = c_{K_2/K_1}^{\widetilde{w}}$ if $x < w$.

Proof. By symmetry, we may assume that $x > w$. Note that both sides of (2.2) are continuous on $\{w^+\} \cup (w, \infty]$. If $x > b_{K_2}^w$, then $x > \max\{b_{K_1}^w, b_{K_2}\}$, which implies that $g_{K_1}^w(x) = g_{K_1}(x) > \max\{d_{K_1}^w, b_{K_2/K_1}\} \geq b_{K_2/K_1}^{\widetilde{w}}$. Thus, $g_{K_2/K_1}^{\widetilde{w}} \circ g_{K_1}^w(x) = g_{K_2/K_1} \circ g_{K_1}(x) = g_{K_2}(x) = g_{K_2}^w(x)$ on $(b_{K_2}^w, \infty)$. We know that $g_{K_2}^w$ is constant on $\{w^+\} \cup (w, b_{K_2}^w]$. To prove that (2.2) holds for all $x > w$, by continuity, it suffices to show that the LHS of (2.2) is constant on $\{w^+\} \cup (w, b_{K_2}^w]$. Since $g_{K_1}^w$ is constant on $\{w^+\} \cup (w, b_{K_1}^w]$, if $b_{K_1}^w = b_{K_2}^w$, then the proof is done. Suppose $b_{K_1}^w < b_{K_2}^w$. In this case, we have $b_{K_1}, w < b_{K_2}^w = b_{K_2}$. So $g_{K_1}^w$ maps $\{w^+\} \cup (w, b_{K_2}^w]$ onto $[d_{K_1}^w, b_{K_2/K_1}]$, which is in turn mapped by $g_{K_2/K_1}^{\widetilde{w}}$ to a constant because $\widetilde{w} \leq d_{K_1}^w$. □

Proposition 2.13. *Let K_t and $\eta(t)$, $0 \leq t < T$, be chordal Loewner hulls and curve driven by $\widehat{w}$ with speed q. Suppose the Lebesgue measure of $\eta([0,T)) \cap \mathbb{R}$ is 0. Let $w = \widehat{w}(0)$, and $x \in \mathbb{R}_w$. Define $X(t) = g_{K_t}^w(x)$, $0 \leq t < T$. Then X is absolutely continuous and satisfies the differential equation $X'(t) \overset{\text{ae}}{=} \frac{2q(t)}{X(t)-\widehat{w}(t)}$ on $[0,T)$; if $x > w$ (resp. $x < w$), then $X(t) \geq \widehat{w}(t)$ (resp. $X(t) \leq \widehat{w}(t)$) on $[0,T)$, and so is increasing (resp. decreasing) on $[0,T)$. Moreover, for any $0 \leq t_1 < t_2 < T$, $|X(t_1) - X(t_2)| \leq 4\,\mathrm{diam}(K_{t_2}/K_{t_1})$.*

Proof. We may assume that the speed q is constant 1. By symmetry, we may assume that $x \in (-\infty, w^-]$. If $x = w^-$, then $X(t) = c_{K_t}$ for $t > 0$ and $X(0) = \widehat{w}(0)$. Then the conclusion follows from Propositions 2.4 and 2.10. Now suppose $x \in (-\infty, w)$.

Fix $0 \leq t_1 < t_2 < T$. We first prove the upper bound for $|X(t_1) - X(t_2)|$. There are three cases. Case 1. $x \notin \overline{K_{t_j}}$, $j = 1, 2$. In this case, $X(t_2) = g_{K_{t_2}/K_{t_1}}(X(t_1))$, and the upper bound for $|X(t_1) - X(t_2)|$ follows from Proposition 2.3. Case 2. $x \in \overline{K_{t_1}} \subset \overline{K_{t_2}}$. In this case $X(t_j) = c_{K_{t_j}}$, $j = 1, 2$, and the conclusion follows from Proposition 2.4. Case 3. $x \notin \overline{K_{t_1}}$ and $x \in \overline{K_{t_2}}$. Then $X(t_1) = g_{K_{t_1}}(x_0) < c_{K_{t_1}}$ and $X(t_2) = c_{K_{t_2}}$. Moreover, we have $\tau_x \in (t_1, t_2]$, $\lim_{t\uparrow\tau_x} X(t) = \widehat{w}(\tau_x)$, and $X(t)$ satisfies the differential equation $X'(t) = \frac{2}{X(t)-\widehat{w}(t)} < 0$ on $[t_1, \tau_x)$. From Propositions 2.9 and 2.1 we know that $X(t_1) \geq \widehat{w}(\tau_x) \geq c_{K_{\tau_x}} \geq c_{K_{t_2}} = X(t_2)$. Since $c_{K_{t_1}} > X(t_1) \geq X(t_2) = c_{K_{t_2}}$, we have $|X(t_1) - X(t_2)| \leq |c_{K_{t_1}} - c_{K_{t_2}}| \leq 4\,\mathrm{diam}(K_{t_2}/K_{t_1})$ by Propositions 2.4. So X is continuous on $[0,T)$.

Since $X(t) = g_{K_t}(x)$ satisfies the chordal Loewner equation driven by $\widehat{w}$ up to τ_x, we know that $X'(t) = \frac{2}{X(t)-\widehat{w}(t)}$ on $[0,\tau_x)$. From Proposition 2.10 we know that $X'(t) \stackrel{\text{ae}}{=} \frac{2}{X(t)-\widehat{w}(t)}$ on (τ_x, T).The differential equation on $[0,T)$ then follows from the continuity of X. Since $X(t) \le c_{K(t)} \le \widehat{w}(t)$ by Proposition 2.9, it is decreasing on $[0,T)$. $\square$

2.2 Chordal SLE$_\kappa$ and 2-SLE$_\kappa$

If $\widehat{w}(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$, where $\kappa > 0$ and $B(t)$ is a standard Brownian motion, then the chordal Loewner curve η driven by $\widehat{w}$ is known to exist (cf. [16]), and is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. In fact, we have $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$. The behavior of η depends on κ: if $\kappa \in (0, 4]$, η is simple and intersects $\mathbb{R}$ at 0; if $\kappa \ge 8$, η is space-filling, i.e., $\overline{\mathbb{H}} = \eta(\mathbb{R}_+)$; if $\kappa \in (4, 8)$, η is neither simple nor space-filling. If D is a simply connected domain with two distinct marked boundary points (or more precisely, prime ends) a and b, the chordal SLE$_\kappa$ curve in D from a to b is defined to be the conformal image of a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞ under a conformal map from $(\mathbb{H}; 0, \infty)$ onto $(D; a, b)$.

For any $\kappa > 0$, chordal SLE$_\kappa$ satisfies conformal invariance and Domain Markov Property (DMP). The DMP means that if η is a chordal SLE$_\kappa$ curve in D from a to b, and T is a stopping time, then conditionally on the part of η before T and the event that η does not reach b at the time T, the part of η after T is a chordal SLE$_\kappa$ curve from $\eta(T)$ to b in the connected component of $D \setminus \eta([0, T])$ that has b on its boundary.

We will focus on the range $\kappa \in (0, 8)$ so that SLE$_\kappa$ is non-space-filling. One remarkable property of these chordal SLE$_\kappa$ is its reversibility: the time-reversal of a chordal SLE$_\kappa$ curve in D from a to b is a chordal SLE$_\kappa$ curve in D from b to a, up to a time-change ([27, 9]). Another fact that is important to us is the existence of 2-SLE$_\kappa$. Let D be a simply connected domain with distinct boundary points a_1, b_1, a_2, b_2 such that a_1 and b_1 together do not separate a_2 from b_2 on ∂D (and vice versa). A 2-SLE$_\kappa$ in D with link pattern $(a_1 \leftrightarrow b_1; a_2 \leftrightarrow b_2)$ is a pair of random curves (η_1, η_2) in $\overline{D}$ such that for $j = 1, 2$, η_j connects a_j with b_j, and conditionally on η_{3-j}, η_j is a chordal SLE$_\kappa$ curve in the connected component of $D \setminus \eta_{3-j}$ whose boundary contains a_j and b_j. Because of reversibility, we do not need to specify the orientation of η_1 and η_2. If we want to emphasize the orientation, then we use an arrow like $a_1 \to b_1$ in the link pattern. The existence of 2-SLE$_\kappa$ was proved in [3] for $\kappa \in (0, 4]$ using Brownian loop measure and in [11, 9] for $\kappa \in (4, 8)$ using flow line theory. The uniqueness of 2-SLE$_\kappa$ (for a fixed domain and link pattern) was proved in [10] (for $\kappa \in (0, 4]$) and [12] (for $\kappa \in (4, 8)$) using an ergodicity argument.

2.3 SLE$_\kappa(\underline{\rho})$ processes

First introduced in [8], SLE$_\kappa(\underline{\rho})$ processes are natural variations of SLE$_\kappa$, where one keeps track of additional marked points, often called force points, which may lie on the boundary or interior. For the generality needed here, all force points will lie on the boundary. In this subsection, we review the definition and properties of SLE$_\kappa(\underline{\rho})$ developed in [11].

Let $n \in \mathbb{N}$, $\kappa > 0$, $\underline{\rho} = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$. Let $w \in \mathbb{R}$ and $\underline{v} = (v_1, \dots, v_n) \in \mathbb{R}^n_w$. The chordal

$\mathrm{SLE}_\kappa(\underline{\rho})$ process in $\mathbb{H}$ started from w with force points $\underline{v}$ is the chordal Loewner process driven by the function $\widehat{w}(t)$, which drives chordal Loewner maps g_t and hulls K_t, and solves the SDE

$$d\widehat{w}(t) \overset{\text{ae}}{=} \sqrt{\kappa} dB(t) + \sum_{j=1}^n \frac{\rho_j}{\widehat{w}(t) - \widehat{v}_j(t)}\, dt, \quad \widehat{w}(0) = w,$$

where $B(t)$ is a standard Brownian motion, and for each j, $\widehat{v}_j(t) = g^w_{K_t}(v_j)$, $0 \le t < T$. Here we use Definition 2.11. In order for the existence of the solution, we require that for $\sigma \in \{+,-\}$, $\sum_{j:v_j = w^\sigma} \rho_j > -2$. If this holds, then the solution exists uniquely up to the first time (called a continuation threshold) that $\sum_{j:\widehat{v}_j(t) = c_{K_t}} \rho_j \le -2$ or $\sum_{j:\widehat{v}_j(t)=d_{K_t}} \rho_j \le -2$, whichever comes first. If a continuation threshold does not exist, then the lifetime is ∞. Each $\widehat{v}_j(t)$ is called the force point function started from v_j. It satisfies the differential equation $\widehat{v}_j \overset{\text{ae}}{=} \frac{2}{\widehat{v}_j - \widehat{w}}$, and is monotonically increasing or decreasing depending on whether $v_j > w$ or $v_j < w$.

Using Proposition 2.13 we easily get the following proposition.

Proposition 2.14. *The chordal Loewner process driven by $\widehat{w}$, $0 \le t < T$, with hulls K_t, is a chordal $\mathrm{SLE}_\kappa(\rho_1, \dots, \rho_n)$ process with force points $(v_1, \dots, v_n)$ if and only if $u(t) := \widehat{w}(t) + \sum_{j=1}^n \frac{\rho_j}{2} g^{\widehat{w}(0)}_{K_t}(v_j)$ is a local martingale with $\langle u \rangle_t = \kappa t$ up to T.*

A chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ process generates a chordal Loewner curve η in $\overline{\mathbb{H}}$ started from w up to the continuation threshold. If no force point is swallowed by the process at any time, this fact follows from the existence of chordal SLE_κ curve and Girsanov Theorem. The existence of the curve in the general case was proved in [11]. From Proposition 2.12 we know that the chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η satisfies the following DMP. If τ is a stopping time for η, then conditionally on the process before τ and the event that τ is less than the lifetime T, $\widehat{w}(\tau + t)$ and $\widehat{v}_j(\tau + t)$, $1 \le j \le n$, $0 \le t < T - \tau$, are the driving function and force point functions for a chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η^τ started from $\widehat{w}(\tau)$ with force points at $\widehat{v}_1(\tau), \dots, \widehat{v}_n(\tau)$, and $\eta(\tau + \cdot) = f_{K(\tau)}(\eta^\tau)$, where $K(\tau) := \mathrm{Hull}(\eta([0,\tau]))$. Here if $\widehat{v}_j(\tau) = \widehat{w}(\tau)$, then $\widehat{v}_j(\tau)$ as a force point is treated as $\widehat{w}(\tau)^+$ if $v_j \ge w^+$, or $\widehat{w}(\tau)^-$ if $v_j \le w^-$.

We now relabel the force points $v_1, \dots, v_n$ by $v^-_{n_-} \le \cdots \le v^-_1 < w < v^+_1 \le \cdots \le v^+_{n_+}$, where $n_- + n_+ = n$ (n_- or n_+ could be 0). Then for any t in the life period, $\widehat{v^-_{n_-}}(t) \le \cdots \le \widehat{v^-_1}(t) \le \widehat{w}(t) \le \widehat{v}^+_1(t) \le \cdots \le \widehat{v}^+_{n_+}(t)$. If for any $\sigma \in \{-,+\}$ and $1 \le k \le n_\sigma$, $\sum_{j=1}^k \rho^\sigma_j > -2$, then the process will never reach a continuation threshold, and so its lifetime is ∞, in which case $\lim_{t\to\infty} \eta(t) = \infty$. If for some $\sigma \in \{+,-\}$ and $1 \le k \le n_\sigma$, $\sum_{j=1}^k \rho^\sigma_j \ge \frac{\kappa}{2} - 2$, then η does not hit v^σ_k and the open interval between v^σ_k and v^σ_{k+1} ($v^\sigma_{n_\sigma+1} := \sigma \cdot \infty$). If $\kappa \in (0,8)$ and for any $\sigma \in \{+,-\}$ and $1 \le k \le n_\sigma$, $\sum_{j=1}^k \rho^\sigma_j > \frac{\kappa}{2} - 4$, then for every $x \in \mathbb{R} \setminus \{w\}$, a.s. η does not visit x, which implies by Fubini Theorem that a.s. $\eta \cap \mathbb{R}$ has Lebesgue measure zero.

2.4 Intermediate SLE$_\kappa(\rho)$ processes

For $a,b,c\in\mathbb{C}$ such that $c\not\in\{0,-1,-2,\cdots\}$, the hypergeometric function ${}_2F_1(a,b;c;z)$ (cf. [14]) is defined by the Gauss series

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}\,z^n, $$

on the disc $\{|z|<1\}$, where $(a)_n$ is rising factorial: $(a)_0=1$ and $(a)_n=a(a+1)\cdots(a+n-1)$ if $n\ge 1$. We will use the following properties in this paper.

(F1) If $\mathrm{Re}(c-a-b)>0$, then $\lim_{x\uparrow 1}{}_2F_1(a,b;c;x)=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}$.

(F2) Euler transform: ${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z)$.

(F3) Derivative: $\frac{d}{dz}\,{}_2F_1(a,b;c;z)=\frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z)$.

(F4) $F:={}_2F_1(a,b;c;\cdot)$ satisfies the hypergeometric differential equation:

$$ z(1-z)F''(z)-[(a+b+1)z-c]F'(z)-abF(z)=0. \tag{2.3} $$

Let $\kappa\in(0,8)$ and $\rho>\max\{-2,\frac{\kappa}{2}-4\}$. Let $a=\frac{2\rho}{\kappa}$, $b=1-\frac{4}{\kappa}$, $c=\frac{2\rho+4}{\kappa}$. Define

$$ F_{\kappa,\rho}(x)={}_2F_1(a,b;c;x)={}_2F_1\Big(\frac{2\rho}{\kappa},1-\frac{4}{\kappa};\frac{2\rho+4}{\kappa};x\Big). $$

Proposition 2.15. *For $\kappa\in(0,8)$ and $\rho>\max\{-2,\frac{\kappa}{2}-4\}$, $F_{\kappa,\rho}$ extends continuously to $[0,1]$ such that $F_{\kappa,\rho}$ is positive on $[0,1]$.*

Proof. The assumptions on κ and ρ imply that $c,c-a,c-b,c-a-b>0$. By Euler transform and the Gauss series for ${}_2F_1(c-a,c-b;c;x)$, $F_{\kappa,\rho}(x)=(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)>0$ on $[0,1)$. By (F1), $F_{\kappa,\rho}$ is continuous and positive on $[0,1]$. $\square$

Let $\widetilde G_{\kappa,\rho}(x)=\kappa x\frac{F_{\kappa,\rho}'(x)}{F_{\kappa,\rho}(x)}+\rho$, which is well defined on $[0,1)$.

Definition 2.16. Let $\kappa\in(0,8)$ and $\rho>\max\{-2,\frac{\kappa}{2}-4\}$. Let $w\in\mathbb{R}$, and $v_1\le v_2\in\{w^+\}\cup(w,\infty)\cup\{+\infty\}$ or $v_1\ge v_2\in\{w^-\}\cup(-\infty,w)\cup\{-\infty\}$. Suppose $\widehat w(t)$, $0\le t<\infty$, solves the following SDE with initial value $\widehat w(0)=w$:

$$ d\widehat w(t)\overset{\mathrm{ae}}{=}\sqrt{\kappa}dB(t)+\Big(\frac{1}{\widehat w(t)-\widehat v_1(t)}-\frac{1}{\widehat w(t)-\widehat v_2(t)}\Big)\widetilde G_{\kappa,\rho}\Big(\frac{\widehat w(t)-\widehat v_1(t)}{\widehat w(t)-\widehat v_2(t)}\Big)dt, $$

where $B(t)$ is a standard Brownian motion, $\widehat v_j(t)=g^w_{K_t}(v_j)$, $j=1,2$, and K_t are chordal Loewner hulls driven by $\widehat w$. Here we use the symbols in Definition 2.11. The chordal Loewner curve driven by $\widehat w$ is called an intermediate SLE$_\kappa(\rho)$, or simply iSLE$_\kappa(\rho)$, curve in $\mathbb{H}$ from w to ∞ with force points v_1,v_2. We call $v_j(t)$ the force point function started from v_j, $j=1,2$. A force point v_1 or v_2 taking value $w^\pm$ or $\pm\infty$ is called a degenerate force point. Via a conformal map, one can define an $\text{iSLE}_\kappa(\rho)$ curve in a simply connected domain D from one prime end w_1 to another prime end w_2 with two force points v_1 and v_2 such that w_1, v_1, v_2, w_2 are oriented counterclockwise or clockwise, and v_j may be immediately next to w_j, $j = 1, 2$.

Remark 2.17. There are some degenerate cases. If $v_1 = v_2$, then the $\text{iSLE}_\kappa(\rho)$ reduces to a chordal SLE_κ with no force points. If $v_2 = \pm\infty$, then the $\text{iSLE}_\kappa(\rho)$ reduces to the chordal $\text{SLE}_\kappa(\rho)$ with the force point at v_1. By (2.2) an $\text{iSLE}_\kappa(\rho)$ process also satisfies DMP as a chordal $\text{SLE}_\kappa(\underline{\rho})$ process does. If τ is a finite stopping time for an $\text{iSLE}_\kappa(\rho)$ curve η in $\mathbb{H}$ with driving function $\widehat{w}$ and force point functions $\widehat{v}_1$ and $\widehat{v}_2$, then conditionally on the part of η before τ, there is an $\text{iSLE}_\kappa(\rho)$ curve η^τ in $\mathbb{H}$ from $\widehat{w}(\tau)$ to ∞ with force points $\widehat{v}_1(\tau), \widehat{v}_2(\tau)$ such that $\eta(\tau + \cdot) = f_{K(\tau)}(\eta^\tau)$, where $K(\tau) = \text{Hull}(\eta([0, \tau]))$. Here if $\widehat{v}_j(\tau) = \widehat{w}(\tau)$, then $\widehat{v}_j(\tau)$ as a force point is treated as $\widehat{w}(\tau)^+$ if $v_j \ge w^+$, or $\widehat{w}(\tau)^-$ if $v_j \le w^-$. In the case $\kappa > 4$, η swallows v_2 at some finite time τ, at which $\widehat{v}_2(\tau) = \widehat{v}_1(\tau)$, so the DMP tells us that the remaining part of η is a chordal SLE_κ curve in the remaining domain.

Using the standard argument in [19], we obtain the following proposition describing an $\text{iSLE}_\kappa(\rho)$ curve in $\mathbb{H}$ in the chordal coordinate in the case that the target is not ∞.

Proposition 2.18. *Let $w_0 \ne w_\infty \in \mathbb{R}$. Let $v_1 \in \mathbb{R}_{w_0} \cup \{\infty\} \setminus \{w_\infty\}$ and $v_2 \in \mathbb{R}_{w_\infty} \cup \{\infty\} \setminus \{w_0\}$ be such that the cross ratio $R := \frac{(w_0 - v_1)(w_\infty - v_2)}{(w_0 - v_2)(w_\infty - v_1)} \in \{0^+\} \cup (0, 1)$. Let $\kappa \in (0, 8)$ and $\rho > \max\{-2, \frac{\kappa}{2} - 4\}$. Let $\widehat{\eta}$ be an $\text{iSLE}_\kappa(\rho)$ curve in $\mathbb{H}$ from w_0 to w_∞ with force points at v_1, v_2. Stop $\widehat{\eta}$ at the first time that it separates w_∞ from ∞, and parametrize the stopped curve by $\mathbb{H}$-capacity. Then the new curve, denoted by η, is the chordal Loewner curve driven by some function $\widehat{w}_0$, which satisfies the following SDE with initial value $\widehat{w}_0(0) = w_0$:*

$$
\begin{aligned}
d\widehat{w}_0(t) &\overset{\text{ae}}{=} \sqrt{\kappa} dB(t) + \frac{\kappa - 6}{\widehat{w}_0(t) - \widehat{w}_\infty(t)}\, dt \\
&\quad + \Big(\frac{1}{\widehat{w}(t) - \widehat{v}_1(t)} - \frac{1}{\widehat{w}(t) - \widehat{v}_2(t)} \Big) \cdot \widetilde{G}_{\kappa,\rho} \Big(\frac{(\widehat{w}_0(t) - \widehat{v}_1(t))(\widehat{v}_2(t) - \widehat{w}_\infty(t))}{(\widehat{w}_0(t) - \widehat{v}_2(t))(\widehat{v}_1(t) - \widehat{w}_\infty(t))} \Big)\, dt,
\end{aligned}
$$

where $B(t)$ is a standard Brownian motion, $\widehat{w}_\infty(t) = g_{K_t}(w_\infty)$ and $\widehat{v}_j(t) = g^{w_0}_{K_t}(v_j)$, $j = 1, 2$, and K_t are the chordal Loewner hulls driven by $\widehat{w}_0$.

Definition 2.19. We call the η in Proposition 2.18 an $\text{iSLE}_\kappa(\rho)$ curve in $\mathbb{H}$ from w_0 to w_∞ with force points at v_1, v_2, in the chordal coordinate; call $\widehat{w}_0$ the driving function; and call $\widehat{w}_\infty$, $\widehat{v}_1$ and $\widehat{v}_2$ the force point functions started from w_∞, v_1 and v_2, respectively.

Proposition 2.20. *We adopt the notation in the last proposition. Let T be the first time that w_∞ or v_2 is swallowed by the hulls. Note that $|\widehat{w}_0 - \widehat{w}_\infty|$, $|\widehat{v}_1 - \widehat{v}_2|$, $\widehat{w}_0 - \widehat{v}_2|$, and $|\widehat{w}_\infty - \widehat{v}_1|$ are all positive on $[0, T)$. We define M on $[0, T)$ by*

$$
M = |\widehat{w}_0 - \widehat{w}_\infty|^{\frac{8}{\kappa} - 1} |\widehat{v}_1 - \widehat{v}_2|^{\frac{\rho(2\rho + 4 - \kappa)}{2\kappa}} |\widehat{w}_0 - \widehat{v}_2|^{\frac{2\rho}{\kappa}} |\widehat{w}_\infty - \widehat{v}_1|^{\frac{2\rho}{\kappa}} F_{\kappa,\rho} \Big(\frac{(\widehat{w}_0 - \widehat{v}_1)(\widehat{w}_\infty - \widehat{v}_2)}{(\widehat{w}_0 - \widehat{v}_2)(\widehat{w}_\infty - \widehat{v}_1)} \Big)^{-1}.
$$

Then $(M(t))$ is a positive local martingale, and if we tilt the law of η by M, then we get the law of a chordal SLE$_\kappa(2,\rho,\rho)$ curve in $\mathbb{H}$ started from w_0 with force points w_∞, v_1 and v_2, respectively. More precisely, if $\tau < T$ is a stopping time such that M is uniformly bounded on $[0,\tau]$, then $\mathbb{E}[M(\tau)/M(0)] = 1$, and if we weight the underlying probability measure by the weight $M(\tau)/M(0)$, then the law of η stopped at the time τ under the new measure is that of a chordal SLE$_\kappa(2,\rho,\rho)$ curve in $\mathbb{H}$ started from w_0 with force points w_∞, v_1 and v_2, respectively, stopped at the time τ.

Proof. This follows from some straightforward applications of Itô's formula and Girsanov Theorem, where we use (2.3), Propositions 2.13 and 2.18. Actually, the calculation will be simpler if we tilt the law of a chordal SLE$_\kappa(2,\rho,\rho)$ curve by M^{-1} to get an iSLE$_\kappa(\rho)$ curve. □

An iSLE$_\kappa(2)$ process was called a hypergeometric SLE$_\kappa$, abbreviated as hSLE$_\kappa$, in [20]. It is important because of its connection with 2-SLE$_\kappa$: if (η_1,η_2) is a 2-SLE$_\kappa$ in D with link pattern $(a_1 \to b_1; a_2 \to b_2)$, then for $j = 1,2$, the marginal law of η_j is that of an hSLE$_\kappa$ curve in D from a_j to b_j with force points b_{3-j} and a_{3-j} (cf. [20, Proposition 6.10]). For other ρ, an iSLE$_\kappa(\rho)$ process was called hSLE$_\kappa(\nu)$ in [20], where $\nu = \rho - 2$.

It was proved in [25] that iSLE$_\kappa(\rho)$ satisfies reversibility for $\kappa \in (0,4)$ and $\rho \ge \frac{\kappa}{2} - 2$, i.e., the time-reversal of an iSLE$_\kappa(\rho)$ curve in D from w_1 to w_2 with force points v_1 and v_2 is an iSLE$_\kappa(\rho)$ curve in D from w_2 to w_1 with force points v_2 and v_1. If both v_1 and v_2 are degenerate, we get the reversibility of a chordal SLE$_\kappa(\rho)$ curve with one degenerate force point. If v_1 is non-degenerate and v_2 is degenerate, then we find that the time-reversal of a chordal SLE$_\kappa(\rho)$ curve with one non-degenerate force point, is an iSLE$_\kappa(\rho)$ curve with one degenerate force point and one non-degenerate force point. If $\kappa = 4$, since $F_{4,\rho} \equiv 1$, an iSLE$_4(\rho)$ is just a chordal SLE$_4(\rho,-\rho)$, whose reversibility was proved earlier in [26] for $\rho \ge \frac{4}{2} - 2 = 0$. Miller and Sheffield proved ([10, 9]) that chordal SLE$_\kappa(\rho)$ with one or two degenerate force point(s) satisfies reversibility for $\kappa \in (0,4)$ and $\rho > -2$, or $\kappa \in (4,8)$ and $\rho \ge \frac{\kappa}{2} - 4$. But they did not give a description of the time-reversal of a chordal SLE$_\kappa(\rho)$ with one or two non-degenerate force points. Wu recently proved ([20]) that for $\kappa \in (4,8)$ and $\rho \ge \frac{\kappa}{2} - 2$, a non-degenerate iSLE$_\kappa(\rho)$ curve also satisfies reversibility. She derived this result by showing that the law of such iSLE$_\kappa(\rho)$ can be obtained by weighting a chordal SLE$_\kappa$ by some power of the boundary excursion kernel at the two force points in one complement domain of the whole chordal SLE$_\kappa$ curve, and using the reversibility of chordal SLE$_\kappa$ derived in [9] for $\kappa \in (4,8)$. By letting the force points tend to the endpoints, one can easily obtain the reversibility of iSLE$_\kappa(\rho)$ with one or two degenerate force points. Wu conjectured that the reversibility of iSLE$_\kappa(\rho)$ also holds for $\kappa \in (0,8)$ and $\rho \in (\max\{-2, \frac{\kappa}{2} - 4\}, \frac{\kappa}{2} - 2)$. As said before, this was proved for $\kappa \in (0,4]$ and $\rho \ge \frac{\kappa}{2} - 2$. In fact, the proofs in [25] and [26] also works in the case $\kappa \in (0,4]$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$ without any modification. The proposition below combines these known results.

Proposition 2.21. *Let $\kappa \in (0,4]$ and $\rho > -2$ or $\kappa \in (4,8)$ and $\rho \ge \frac{\kappa}{2} - 2$. Let η be an iSLE$_\kappa(\rho)$ curve in a simply connected domain D from w_0 to w_∞ with force points v_1 and v_2. Then after a time change, the time-reversal of η becomes an iSLE$_\kappa(\rho)$ curve in D from w_∞ to w_0 with force point v_2 and v_1. Here if both force points are degenerate, the statement becomes* *the reversibility of a degenerate chordal $SLE_\kappa(\rho)$; when only one force point is degenerate, the statement is about the time-reversal of a non-degenerate chordal $SLE_\kappa(\rho)$.*

2.5 Two-parameter stochastic processes

In this subsection we briefly recall the framework used in [22, Section 2.3]. We assign a partial order $\le$ to $\mathbb{R}_+^2 = [0,\infty)^2$ such that $\underline{t} = (t_+, t_-) \le (s_+, s_-) = \underline{s}$ iff $t_+ \le s_+$ and $t_- \le s_-$. It has a minimal element $\underline{0} = (0,0)$. We write $\underline{t} < \underline{s}$ if $t_+ < s_+$ and $t_- < s_-$. We define $\underline{t} \wedge \underline{s} = (t_1 \wedge s_1, t_2 \wedge s_2)$. Given $\underline{t}, \underline{s} \in \mathbb{R}_+^2$, we define $[\underline{t}, \underline{s}] = \{\underline{r} \in \mathbb{R}_+^2 : \underline{t} \le \underline{r} \le \underline{s}\}$. Let $\underline{e}_+ = (1,0)$ and $\underline{e}_- = (0,1)$. So $(t_+, t_-) = t_+ \underline{e}_+ + t_- \underline{e}_-$. We introduce an extra element $\underline{\infty} = (\infty, \infty)$ and understand that $\underline{\infty} > \underline{t}$ for any $\underline{t} \in \mathbb{R}_+^2$.

Definition 2.22. Let $\mathcal{F}_{\underline{t}}$, $\underline{t} \in \mathbb{R}_+^2$, be a family of σ-algebras on a measurable space Ω such that $\mathcal{F}_{\underline{t}} \subset \mathcal{F}_{\underline{s}}$ whenever $\underline{t} \le \underline{s}$. Then we call $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$ an $\mathbb{R}_+^2$-indexed filtration on Ω. Let $\mathcal{F}_{\underline{t}}^{(+)} = \bigcap_{\underline{s} > \underline{t}} \mathcal{F}_{\underline{s}}$, $\underline{t} \in \mathbb{R}_+^2$. Then we call $(\mathcal{F}_{\underline{t}}^{(+)})_{\underline{t} \in \mathbb{R}_+^2}$ the right-continuous augmentation of $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$. We say that $(\mathcal{F}_{\underline{t}})$ is right-continuous if $\mathcal{F}_{\underline{t}}^{(+)} = \mathcal{F}_{\underline{t}}$ for all $\underline{t} \in \mathbb{R}_+^2$. A family of random variables $(X(\underline{t}))_{\underline{t} \in \mathbb{R}_+^2}$ defined on Ω is called an $(\mathcal{F}_{\underline{t}})$-adapted process if for any $\underline{t} \in \mathbb{R}_+^2$, $X(\underline{t})$ is $\mathcal{F}_{\underline{t}}$-measurable. It is called continuous if $\underline{t} \mapsto X(\underline{t})$ is sample-wise continuous.

Definition 2.23. A random map $\underline{T} : \Omega \to \mathbb{R}_+^2 \cup \{\underline{\infty}\}$ is called an extended $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$-stopping time if for any deterministic $\underline{t} \in \mathbb{R}_+^2$, $\{\underline{T} \le \underline{t}\} \in \mathcal{F}_{\underline{t}}$. If $\underline{T}$ does not take value $\underline{\infty}$, then we remove the term "extended". For an extended $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{T}$, we define a new σ-algebra $\mathcal{F}_{\underline{T}}$ by $\mathcal{F}_{\underline{T}} = \{A \in \mathcal{F} : A \cap \{\underline{T} \le \underline{t}\} \in \mathcal{F}_{\underline{t}}, \forall \underline{t} \in \mathbb{R}_+^2\}$. The stopping time $\underline{T}$ is called bounded if there is a deterministic $\underline{t} \in \mathbb{R}_+^2$ such that $\underline{T} \le \underline{t}$.

Proposition 2.24. *Let $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$ be an $\mathbb{R}_+^2$-indexed filtration with the right-continuous augmentation $(\mathcal{F}_{\underline{t}}^{(+)})$. Then the right-continuous augmentation of $(\mathcal{F}_{\underline{t}}^{(+)})$ is itself. Thus, $(\mathcal{F}_{\underline{t}}^{(+)})$ is right-continuous. A random map $\underline{T}$ is an extended $(\mathcal{F}_{\underline{t}}^{(+)})$-stopping time if and only if $\{\underline{T} < \underline{t}\} \in \mathcal{F}_{\underline{t}}$ for any $\underline{t} \in \mathbb{R}_+^2$; and for such $\underline{T}$, $A \in \mathcal{F}_{\underline{T}}^{(+)}$ if and only if $A \cap \{\underline{T} < \underline{t}\} \in \mathcal{F}_{\underline{t}}$ for any $\underline{t} \in \mathbb{R}_+^2$. If $(\underline{T}^n)_{n \in \mathbb{N}}$ is a decreasing sequence of extended $(\mathcal{F}_{\underline{t}}^{(+)})$-stopping times, then $\underline{T} := \inf_n \underline{T}^n$ is also an extended $(\mathcal{F}_{\underline{t}}^{(+)})$-stopping time, and $\mathcal{F}_{\underline{T}}^{(+)} = \bigcap_n \mathcal{F}_{\underline{T}^n}^{(+)}$.*

Proof. This follows from the same arguments that were used to prove similar statements about the right-continuous $\mathbb{R}_+$-indexed filtrations. □

Definition 2.25. A relatively open subset $\mathcal{R}$ of $\mathbb{R}_+^2$ is called a history complete region, or simply an HC region, if for any $\underline{t} \in \mathcal{R}$, we have $[\underline{0}, \underline{t}] \subset \mathcal{R}$. Given an HC region $\mathcal{R}$, for $\sigma \in \{+,-\}$, define $T_\sigma^{\mathcal{R}} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ by $T_\sigma^{\mathcal{R}}(t) = \sup\{s \ge 0 : s\underline{e}_\sigma + t\underline{e}_{-\sigma} \in \mathcal{R}\}$, where we set $\sup \emptyset = 0$.

A map $\mathcal{D}$ from Ω into the space of HC regions is called an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$-stopping region if for any $\underline{t} \in \mathbb{R}_+^2$, $\{\omega \in \Omega : \underline{t} \in \mathcal{D}(\omega)\} \in \mathcal{F}_{\underline{t}}$. A random function $X(\underline{t})$ with a random domain $\mathcal{D}$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$-adapted HC process if $\mathcal{D}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$-stopping region, and for every $\underline{t} \in \mathbb{R}_+^2$, $X_{\underline{t}}$ restricted to $\{\underline{t} \in \mathcal{D}\}$ is $\mathcal{F}_{\underline{t}}$-measurable.

The following propositions are simple extensions of Lemmas 2.7 and 2.9 of [22].

Proposition 2.26. *Let $\underline{T}$ and $\underline{S}$ be two extended $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping times. Then (i) $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{S}}$; (ii) if $\underline{S}$ is a constant $\underline{s}\in\mathbb{R}_+^2$ or $\underline{\infty}$, then $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{T}}$; and (iii) if f is an $\mathcal{F}_{\underline{T}}$-measurable function, then $\mathbf{1}_{\{\underline{T}\le\underline{S}\}}f$ is $\mathcal{F}_{\underline{S}}$-measurable. In particular, if $\underline{T}\le\underline{S}$, then $\mathcal{F}_{\underline{T}}\subset\mathcal{F}_{\underline{S}}$.*

Proposition 2.27. *Let $(X_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ be a continuous $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-adapted process. Let $\underline{T}$ be an extended $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time. Then $X_{\underline{T}}$ is $\mathcal{F}_{\underline{T}}$-measurable on $\{\underline{T}\in\mathbb{R}_+^2\}$.*

We will need the following proposition to do localization. The reader should note that for an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time $\underline{T}$ and a deterministic time $\underline{t}\in\mathbb{R}_+^2$, $\underline{T}\wedge\underline{t}$ may not be an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time. This is the reason why we introduce a more complicated stopping time.

Proposition 2.28. *Let $\underline{T}$ be an extended $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time. Fix a deterministic time $\underline{t}\in\mathbb{R}_+^2$. Define $\underline{T}^{\underline{t}}$ such that if $\underline{T}\le\underline{t}$, then $\underline{T}^{\underline{t}}=\underline{T}$; and if $\underline{T}\not\le\underline{t}$, then $\underline{T}^{\underline{t}}=\underline{t}$. Then $\underline{T}^{\underline{t}}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time bounded above by $\underline{t}$, and $\mathcal{F}_{\underline{T}^{\underline{t}}}$ agrees with $\mathcal{F}_{\underline{T}}$ on $\{\underline{T}\le\underline{t}\}$, i.e., $\{\underline{T}\le\underline{t}\}\in\mathcal{F}_{\underline{T}^{\underline{t}}}\cap\mathcal{F}_{\underline{T}}$, and for any $A\subset\{\underline{T}\le\underline{t}\}$, $A\in\mathcal{F}_{\underline{T}^{\underline{t}}}$ if and only if $A\in\mathcal{F}_{\underline{T}}$.*

Proof. Clearly $\underline{T}^{\underline{t}}\le\underline{t}$. Let $\underline{s}\in\mathbb{R}_+^2$. If $\underline{t}\le\underline{s}$, then $\{\underline{T}^{\underline{t}}\le\underline{s}\}$ is the whole space. If $\underline{t}\not\le\underline{s}$, then $\{\underline{T}^{\underline{t}}\le\underline{s}\}=\{\underline{T}\le\underline{t}\}\cap\{\underline{T}\le\underline{s}\}=\{\underline{T}\le\underline{t}\wedge\underline{s}\}\in\mathcal{F}_{\underline{t}\wedge\underline{s}}\subset\mathcal{F}_{\underline{s}}$. So $\underline{T}^{\underline{t}}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time.

By Proposition 2.26, $\{\underline{T}\le\underline{t}\}\in\mathcal{F}_{\underline{T}}$. Suppose $A\subset\{\underline{T}\le\underline{t}\}$ and $A\in\mathcal{F}_{\underline{T}}$. Let $\underline{s}\in\mathbb{R}_+^2$. If $\underline{t}\le\underline{s}$, then $A\cap\{\underline{T}^{\underline{t}}\le\underline{s}\}=A=A\cap\{\underline{T}\le\underline{t}\}\in\mathcal{F}_{\underline{t}}\subset\mathcal{F}_{\underline{s}}$. If $\underline{t}\not\le\underline{s}$, then $A\cap\{\underline{T}^{\underline{t}}\le\underline{s}\}=A\cap\{\underline{T}\le\underline{t}\wedge\underline{s}\}\in\mathcal{F}_{\underline{t}\wedge\underline{s}}\subset\mathcal{F}_{\underline{s}}$. So $A\in\mathcal{F}_{\underline{T}^{\underline{t}}}$. In particular, $\{\underline{T}\le\underline{t}\}\in\mathcal{F}_{\underline{T}^{\underline{t}}}$. On the other hand, suppose $A\subset\{\underline{T}\le\underline{t}\}$ and $A\in\mathcal{F}_{\underline{T}^{\underline{t}}}$. Let $\underline{s}\in\mathbb{R}_+^2$. If $\underline{t}\le\underline{s}$, then $A\cap\{\underline{T}\le\underline{s}\}=A=A\cap\{\underline{T}^{\underline{t}}\le\underline{t}\}\in\mathcal{F}_{\underline{t}}\subset\mathcal{F}_{\underline{s}}$. If $\underline{t}\not\le\underline{s}$, then $A\cap\{\underline{T}\le\underline{s}\}=A\cap\{\underline{T}\le\underline{t}\}\cap\{\underline{T}\le\underline{s}\}=A\cap\{\underline{T}^{\underline{t}}\le\underline{s}\}\in\mathcal{F}_{\underline{s}}$. Thus, $A\in\mathcal{F}_{\underline{T}}$. So for $A\subset\{\underline{T}\le\underline{t}\}$, $A\in\mathcal{F}_{\underline{T}^{\underline{t}}}$ if and only if $A\in\mathcal{F}_{\underline{T}}$. □

From now on, we fix a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F}:=\vee_{\underline{t}\in\mathbb{R}_+^2}\mathcal{F}_{\underline{t}})$, and let $\mathbb{E}$ denote the corresponding expectation.

Definition 2.29. An $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-adapted process $(X_{\underline{t}})$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingale (w.r.t. $\mathbb{P}$) if for any $\underline{s}\le\underline{t}\in\mathbb{R}_+^2$, a.s. $\mathbb{E}[X_{\underline{t}}|\mathcal{F}_{\underline{s}}]=X_{\underline{s}}$. If there is $X\in L^1(\Omega,\mathcal{F},\mathbb{P})$ such that $X_{\underline{t}}=\mathbb{E}[X|\mathcal{F}_{\underline{t}}]$, $\underline{t}\in\mathbb{R}_+^2$, then we call $(X_{\underline{t}})$ an X-Doob martingale w.r.t. $(\mathcal{F}_{\underline{t}})$.

Proposition 2.30. *Let $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ be an $\mathbb{R}_+^2$-indexed filtration, and $(\mathcal{F}_{\underline{t}}^{(+)})$ be its right-continuous augmentation. Then a continuous $(\mathcal{F}_{\underline{t}})$-martingale is also an $(\mathcal{F}_{\underline{t}}^{(+)})$-martingale.*

Proof. Let X be a continuous $(\mathcal{F}_{\underline{t}})$-martingale. Let $\underline{s}\le\underline{t}\in\mathbb{R}_+^2$, and $A\in\mathcal{F}_{\underline{s}}^{(+)}$. Fix $\underline{\varepsilon}\in\mathbb{R}_+^2$ with $\underline{\varepsilon}>\underline{0}$. Then $A\in\mathcal{F}_{\underline{s}+\underline{\varepsilon}}$. From $\mathbb{E}[X(\underline{t}+\underline{\varepsilon})|\mathcal{F}_{\underline{s}+\underline{\varepsilon}}]=X(\underline{s}+\underline{\varepsilon})$ we get $\mathbb{E}[\mathbf{1}_A X(\underline{t}+\underline{\varepsilon})]=\mathbb{E}[\mathbf{1}_A X(\underline{s}+\underline{\varepsilon})]$. By letting $\underline{\varepsilon}\downarrow\underline{0}$ and using uniform integrability, we get $\mathbb{E}[\mathbf{1}_A X(\underline{t})]=\mathbb{E}[\mathbf{1}_A X(\underline{s})]$. So we get $\mathbb{E}[X(\underline{t})|\mathcal{F}_{\underline{s}}^{(+)}]=X(\underline{s})$, as desired. □

The following proposition is Lemma 2.11 of [22].

Proposition 2.31 (Optional Stopping Theorem). *Suppose $(X_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ is a continuous $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingale. Then the following are true. (i) If $(X_{\underline{t}})$ is an X-Doob martingale for some $X\in L^1$, then for any $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time $\underline{T}$, $X_T=\mathbb{E}[X|\mathcal{F}_{\underline{T}}]$. (ii) If $\underline{T}\le\underline{S}$ are two bounded $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping times, then $\mathbb{E}[X_{\underline{S}}|\mathcal{F}_{\underline{T}}]=X_{\underline{T}}$.*

The following proposition about the DMP of 2-SLE is Lemma 6.1 of [22].

Proposition 2.32. *Let (η_+,η_-) be a 2-SLE$_\kappa$ in a simply connected domain D with link pattern $(a_+\to b_+;a_-\to b_-)$. Suppose, for $\sigma\in\{+,-\}$, η_σ is parametrized by the $\mathbb{H}$-capacity viewed from b_j (determined by a conformal map from D onto $\mathbb{H}$ that takes b_j to ∞), and let $(\mathcal{F}_t^\sigma)_{t\ge0}$ be the filtration generated by η_σ. Note that the lifetime of η_σ is ∞ for $\sigma\in\{+,-\}$. Let $\mathcal{F}_{(t_+,t_-)}=\mathcal{F}_{t_+}^+\vee\mathcal{F}_{t_-}^-$, $(t_+,t_-)\in\mathbb{R}_+^2$. Let $\underline{\tau}=(\tau_+,\tau_-)$ be an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time. Let $D_{\underline{\tau}}^\sigma$ denote the connected component of $D\setminus(\eta_+([0,\tau_+])\cup\eta_-([0,\tau_-]))$ whose boundary contains b_σ, $\sigma\in\{+,-\}$. Then conditionally on $\mathcal{F}_{\underline{\tau}}$ and the event that $D_{\underline{\tau}}^+=D_{\underline{\tau}}^-=:D_{\underline{\tau}}$ and that $\eta_+(\tau_+)\ne\eta_-(\tau_-)$, $\eta_+|_{[\tau_+,\infty]}$ and $\eta_-|_{[\tau_-,\infty]}$ form a 2-SLE$_\kappa$ in $D_{\underline{T}}$ with link pattern $(\eta_+(\tau_+)\to b_+;\eta_-(\tau_-)\to b_-)$.*

2.6 Jacobi polynomials

For $\alpha,\beta>-1$, Jacobi polynomials ([14, Chapter 18]) $P_n^{(\alpha,\beta)}(x)$, $n=0,1,2,3,\dots$, are a class of classical orthogonal polynomials with respect to the weight $\Psi^{(\alpha,\beta)}(x):=\mathbf{1}_{(-1,1)}(1-x)^\alpha(1+x)^\beta$. This means that each $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n, and for the inner product defined by $\langle f,g\rangle_{\Psi^{(\alpha,\beta)}}:=\int_{-1}^1 f(x)g(x)\Psi^{(\alpha,\beta)}(x)dx$, we have $\langle P_n^{(\alpha,\beta)},P_m^{(\alpha,\beta)}\rangle_{\Psi^{(\alpha,\beta)}}=0$ when $n\ne m$. The normalization is that $P_n^{(\alpha,\beta)}(1)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}$, $P_n^{(\alpha,\beta)}(-1)=(-1)^n\frac{\Gamma(\beta+n+1)}{n!\Gamma(\beta+1)}$, and

$$\|P_n^{(\alpha,\beta)}\|_{\Psi^{(\alpha,\beta)}}^2=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\cdot\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}. \tag{2.4}$$

For each $n\ge0$, $P_n^{(\alpha,\beta)}(x)$ is a solution of the second order differential equation:

$$(1-x^2)y''-[(\alpha+\beta+2)x+(\alpha-\beta)]y'+n(n+\alpha+\beta+1)y=0. \tag{2.5}$$

When $\max\{\alpha,\beta\}>-\frac12$, we have an exact value of the supernorm of $P_n^{(\alpha,\beta)}$ over $[-1,1]$:

$$\|P_n^{(\alpha,\beta)}\|_\infty=\max\{|P_n^{(\alpha,\beta)}(1)|,|P_n^{(\alpha,\beta)}(-1)|\}=\frac{\Gamma(\max\{\alpha,\beta\}+n+1)}{n!\Gamma(\max\{\alpha,\beta\}+1)}. \tag{2.6}$$

For general $\alpha,\beta>-1$, we get an upper bound of $\|P_n^{(\alpha,\beta)}\|_\infty$ using (2.6), the exact value of $P_n^{(\alpha,\beta)}(1)$, and the derivative formula $\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{\alpha+\beta+n+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x)$ for $n\ge1$:

$$\|P_n^{(\alpha,\beta)}\|_\infty\le\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}+(\alpha+\beta+n+1)\cdot\frac{\Gamma(\max\{\alpha,\beta\}+n+1)}{\Gamma(n)\Gamma(\max\{\alpha,\beta\}+2)}. \tag{2.7}$$

3 Deterministic Ensemble of Two Chordal Loewner Curves

In this section, we develop a framework about commuting pairs of deterministic chordal Loewner curves, which will be needed to study the commuting pairs of random chordal Loewner curves in the next two sections. The major length of this section is caused by the fact that we allow that the two Loewner curves have intersections. The ensemble without intersections appeared earlier in [27, 26]. For completeness, we also include a subsection about disjoint ensembles, where some similar calculation first appeared in [8]. In the last subsection, we describe a way to grow two curves simultaneously, which is important for the Green's functions.

3.1 Ensemble with possible intersections

Let $w_- < w_+ \in \mathbb{R}$. Suppose for $\sigma \in \{+,-\}$, $\eta_\sigma(t)$, $0 \le t < T_\sigma$, is a chordal Loewner curve (with speed 1) driven by $\widehat{w}_\sigma$ started from w_σ, such that η_+ does not hit $(-\infty, w_-]$, and η_- does not hit $[w_+,\infty)$. Let $K_\sigma(t_\sigma) = \text{Hull}(\eta([0,t_\sigma])$, $0 \le t_\sigma < T_\sigma$, $\sigma \in \{+,-\}$. Then $K_\sigma(\cdot)$ are chordal Loewner hulls driven by $\widehat{w}_\sigma$, $\text{hcap}_2(K_\sigma(t_\sigma)) = t_\sigma$, and by Proposition 2.8,

$$\{\widehat{w}_\sigma(t_\sigma)\} = \bigcap_{\delta>0} \overline{K_\sigma(t_\sigma+\delta)/K_\sigma(t_\sigma)}, \quad 0 \le t_\sigma < T_\sigma. \tag{3.1}$$

The corresponding chordal Loewner maps are $g_{K_\sigma(t)}$, $0 \le t < T_\sigma$, $\sigma \in \{+,-\}$. From the assumption on η_+ and η_- we get

$$a_{K_-(t_-)} \le w_- < a_{K_+(t_+)}, \quad b_{K_-(t_-)} < w_+ \le b_{K_+(t_+)}, \quad \text{for } t_+, t_- > 0. \tag{3.2}$$

Since each $K_\sigma(t)$ is generated by a curve, $f_{K_\sigma(t)}$ is well defined. Let $\mathcal{I}_\sigma = [0, T_\sigma)$, $\sigma \in \{+,-\}$, and for $(t_+,t_-) \in \mathcal{I}_+ \times \mathcal{I}_-$, define

$$K(t_+,t_-) = \text{Hull}(\eta_+([0,t_+]) \cup \eta_-([0,t_-])), \quad \text{m}(t_+,t_-) = \text{hcap}_2(K(t_+,t_-)). \tag{3.3}$$

It is obvious that $K(\cdot,\cdot)$ and $\text{m}(\cdot,\cdot)$ are increasing (may not strictly) in both variables. Let $H(t_+,t_-) = \mathbb{H} \setminus K(t_+,t_-)$. For $\sigma \in \{+,-\}$, $t_{-\sigma} \in \mathcal{I}_{-\sigma}$ and $t_\sigma \in \mathcal{I}_\sigma$, define $K_{\sigma,t_{-\sigma}}(t_\sigma) = K(t_+,t_-)/K_{-\sigma}(t_{-\sigma})$. Then we have

$$g_{K(t_+,t_-)} = g_{K_{+,t_-}(t_+)} \circ g_{K_-(t_-)} = g_{K_{-,t_+}(t_-)} \circ g_{K_+(t_+)}. \tag{3.4}$$

From (3.2) we get $a_{K(t_+,t_-)} = a_{K_-(t_-)}$ if $t_- > 0$, and $b_{K(t_+,t_-)} = b_{K_+(t_+)}$ if $t_+ > 0$. Since each $K(t_+,t_-)$ is generated by two compact curves, $f_{K(t_+,t_-)}$ is well defined.

Lemma 3.1. *For any $t_+ \le t_+' \in \mathcal{I}_+$ and $t_- \le t_-' \in \mathcal{I}_-$, we have*

$$\text{m}(t_+',t_-') - \text{m}(t_+',t_-) - \text{m}(t_+,t_-') + \text{m}(t_+,t_-) \le 0. \tag{3.5}$$

Especially, m *is Lipschitz continuous with constant* 1 *in any variable, and so is continuous on $\mathcal{I}_+ \times \mathcal{I}_-$.*

Proof. Let $t_+ \le t'_+ \in \mathcal{I}_+$ and $t_- \le t'_- \in \mathcal{I}_-$. Since $K(t'_+, t_-)$ and $K(t_+, t'_-)$ together generate the $\mathbb{H}$-hull $K(t'_+, t'_-)$, and they both contain $K(t_+, t_-)$, we obtain (3.5) from Proposition 2.5. The rest statements follow easily from (3.5), the monotonicity of m, and that $\mathrm{m}(t_+, 0) = t_+$ and $\mathrm{m}(0, t_-) = t_-$ for any $t_\pm \in \mathcal{I}_\pm$. □

Definition 3.2. Let $\eta_\pm$, $\mathcal{I}_\pm$, $K_\pm(\cdot)$, $K(\cdot,\cdot)$, $\mathrm{m}(\cdot,\cdot)$ be as above. Let $\mathcal{D} \subset \mathcal{I}_+ \times \mathcal{I}_-$ be an HC region as in Definition 2.25. Suppose that there are dense subsets $\mathcal{I}_+^*$ and $\mathcal{I}_-^*$ of $\mathcal{I}_+$ and $\mathcal{I}_-$, respectively, such that for any $\sigma \in \{+,-\}$ and $t_{-\sigma} \in \mathcal{I}_{-\sigma}^*$, the following two conditions hold:

(I) $K(t_+, t_-)/K_{-\sigma}(t_{-\sigma})$, $0 \le t_\sigma < T_\sigma^{\mathcal{D}}(t_{-\sigma})$, are chordal Loewner hulls generated by a chordal Loewner curve, denoted by $\eta_{\sigma,t_{-\sigma}}$, with some speed.

(II) $\eta_{\sigma,t_{-\sigma}}([0, T_\sigma^{\mathcal{D}}(t_{-\sigma}))) \cap \mathbb{R}$ has Lebesgue measure zero.

Then we call $(\eta_+, \eta_-; \mathcal{D})$ a commuting pair of chordal Loewner curves, and call $K(\cdot,\cdot)$ and $\mathrm{m}(\cdot,\cdot)$ the hull function and the capacity function, respectively, for this pair.

Remark 3.3. Later in Lemma 3.10 we will show that Conditions (I) and (II) hold for all $t_{-\sigma} \in \mathcal{I}_{-\sigma}$, $\sigma \in \{+,-\}$.

From now on, let $(\eta_+, \eta_-; \mathcal{D})$ be a commuting pair of chordal Loewner curves, and let $\mathcal{I}_+^*$ and $\mathcal{I}_-^*$ be as in Definition 3.2.

Lemma 3.4. *$K(\cdot,\cdot)$ and $\mathrm{m}(\cdot,\cdot)$ restricted to $\mathcal{D}$ are strictly increasing in both variables.*

Proof. By Condition (I), for any $\sigma \in \{+,-\}$ and $t_{-\sigma} \in \mathcal{I}_{-\sigma}^*$, $t \mapsto K(t_{-\sigma}\underline{e}_{-\sigma} + t\underline{e}_\sigma)$ and $t \mapsto \mathrm{m}(t_{-\sigma}\underline{e}_{-\sigma} + t\underline{e}_\sigma)$ are strictly increasing on $[0, T_\sigma^{\mathcal{D}}(t_{-\sigma}))$. By (3.5) and the denseness of $\mathcal{I}_{-\sigma}^*$ in $\mathcal{I}_{-\sigma}$, this property extends to any $t_{-\sigma} \in \mathcal{I}_{-\sigma}$. □

In the rest of the section, when we talk about $K(t_+, t_-)$, $\mathrm{m}(t_+, t_-)$, $K_{+,t_-}(t_+)$ and $K_{-,t_+}(t_-)$, it is always implicitly assumed that $(t_+, t_-) \in \mathcal{D}$. So we may now say that $K(\cdot,\cdot)$ and $\mathrm{m}(\cdot,\cdot)$ are strictly increasing in both variables.

Lemma 3.5. *(i) For $(a_+, a_-) \in \mathcal{D}$ and $\sigma \in \{+,-\}$,*

$$\lim_{\delta \downarrow 0} \sup_{0 \le t_+ \le a_+} \sup_{0 \le t_- \le a_-} \mathrm{diam}(K_{\sigma,t_{-\sigma}}(t_\sigma + \delta)/K_{\sigma,t_{-\sigma}}(t_\sigma)) = 0.$$

(ii) For any $(a_+, a_-) \in \mathcal{D}$ and $\sigma \in \{+,-\}$,

$$\lim_{\delta \downarrow 0} \sup_{0 \le t_\sigma \le a_\sigma} \sup_{t'_\sigma \in (t_\sigma, t_\sigma + \delta)} \sup_{0 \le t_{-\sigma} \le a_{-\sigma}} \sup_{z \in \mathbb{C} \setminus K_{\sigma,t_{-\sigma}}(t'_\sigma)^{\mathrm{doub}}} |g_{K_{\sigma,t_{-\sigma}}(t'_\sigma)}(z) - g_{K_{\sigma,t_{-\sigma}}(t_\sigma)}(z)| = 0.$$

(iii) The map $(t_+, t_-, z) \mapsto g_{K(t_+,t_-)}(z)$ is continuous on

$$\{(t_+, t_-, z) : (t_+, t_-) \in \mathcal{D}, z \in \mathbb{C} \setminus K(t_+, t_-)^{\mathrm{doub}}\}.$$

Proof. (i) By symmetry, it suffices to work on the case $\sigma = +$. We may assume that $a_+ \in \mathcal{I}_+^*$ and $a_- \in \mathcal{I}_-^*$. Let $r > 0$. Since η_+ is continuous, there is $\delta > 0$ such that $(a_+ + \delta, a_-) \in \mathcal{D}$, and if $t_+ \in [0, a_+]$, then $\operatorname{diam}(\eta_+([t_+, t_+ + \delta])) < r$. Fix $t_+ \in [0, a_+]$ and $t_- \in [0, a_-]$. Let $S = \{|z - \eta_+(t_+)| = r\}$ and $\Delta\eta_+ = \eta_+([t_+, t_+ + \delta])$. Then $\Delta\eta_+ \subset \{|z - \eta_+(t_+)| < r\}$. By Lemma 3.4, there is $z_* \in \Delta\eta_+ \cap H(t_+, a_-) \subset H(t_+, t_-)$. Since $z_* \in \{|z - \eta_+(t_+)| < r\}$, the set $S \cap H(t_+, t_-)$ has a connected component, denoted by J, which separates z_* from ∞ in $H(t_+, t_-)$. Such J is a crosscut of $H(t_+, t_-)$, which divides $H(t_+, t_-)$ into two domains, where the bounded domain, denoted by D_J, contains z_*.

Now $\Delta\eta_+ \cap H(t_+, a_-) \subset H(t_+, a_-) \setminus J$. We claim that there is one connected component of $H(t_+, a_-) \setminus J$, denoted by N, such that $\Delta\eta_+ \cap H(t_+, a_-) \subset N$. Note that $J \cap H(t_+, a_-)$ is a disjoint union of crosscuts, each of which divides $H(t_+, a_-)$ into two domains. To prove the claim, it suffices to show that, for each connected component J_0 of $J \cap H(t_+, a_-)$, $\Delta\eta_+ \cap H(t_+, a_-)$ is contained in exactly one connected component of $H(t_+, a_-) \setminus J_0$. Suppose that this is not true for some J_0. Let $J_0' = g_{K(t_+, a_-)}(J_0)$. Then J_0' is a crosscut of $\mathbb{H}$, which divides $\mathbb{H}$ into two domains, both of which intersect $\Delta\widehat{\eta}_+ := g_{K(t_+, a_-)}(\Delta\eta_+ \cap H(t_+, a_-))$. Since $\Delta\eta_+$ has positive distance from $S \supset J$, and $g_{K(t_+, a_-)}^{-1}|_{\mathbb{H}}$ extends continuously to $\overline{\mathbb{H}}$, $\Delta\widehat{\eta}_+$ has positive distance from J_0'. Thus, there is another crosscut J_0'' of $\mathbb{H}$, which is disjoint from and surrounded by J_0', such that the subdomain of $\mathbb{H}$ bounded by J_0'' and J_0' is disjoint from $\Delta\widehat{\eta}_+$. Let the three connected components of $\mathbb{H} \setminus (J_0' \cup J_0'')$ be denoted by D', A, D'', respectively, from outside to inside. Then $\Delta\widehat{\eta}_+$ intersects both D' and D'', but is disjoint from A.

Let $\Delta\eta_+^s = \eta_+([t_+, t_+ + s])$ and $\Delta\widehat{\eta}_+^s = g_{K(t_+, a_-)}(\Delta\eta_+^s \cap H(t_+, a_-))$, $0 \le s \le \delta$. For each $s \in [0, \delta]$, $K(t_+ + s, a_-)$ is the $\mathbb{H}$-hull generated by $K(t_+, a_-)$ and $\Delta\eta_+^s$. So $K_+'(s) := K_{+,a_-}(t_+ + s)/K_{+,a_-}(t_+) = K(t_+ + s, a_-)/K(t_+, a_-)$ (by (2.1)) is the $\mathbb{H}$-hull generated by $\Delta\widehat{\eta}_+^s$. Since A is disjoint from $\Delta\widehat{\eta}_+^s$, it is either contained in or is disjoint from $K_+'(s)$. Since $a_- \in \mathcal{I}_-^*$, by Condition (I) and Proposition 2.6, $K_+'(s)$, $0 \le s \le \delta$, are chordal Loewner hulls with some speed, and so the closure of each $K_+'(s)$ is connected. By choosing s small enough, we can make the diameter of $K_+'(s)$ less than the diameter of A. Then A is not contained in $K_+'(s)$, and so must be disjoint from $K_+'(s)$. By the connectedness of its closure, $K_+'(s)$ is then contained in either D' or D''. On the other hand, since $\widehat{\eta}_+^\delta$ intersects both D' and D'', $K_+'(\delta)$ does the same thing. Thus, there is $s_0 \in (0, \delta)$ such that for all $s \in (s_0, \delta]$, $K_+'(s)$ intersects both D' and D'', and for $s \in [0, s_0)$, $K_+'(s)$ is contained in either D' or D''. For $s > s_0$, because $\overline{K_+'(s)}$ is connected, $K_+'(s)$ intersects A, and so must contain A. Since $\mathbb{H} \setminus K_+'(s)$ is connected and unbounded, we get $A \cup D'' \subset K_+'(s)$ for $s > s_0$. The hulls $K_+'(s)$, $s \in [0, s_0)$, are either all contained in D'' or all contained in D'. In the former case, $\operatorname{hcap}(K_+'(s)) \le \operatorname{hcap}(\overline{D''})$ for $s < s_0$, and $\operatorname{hcap}(K_+'(s)) \ge \operatorname{hcap}(\overline{D'' \cup A})$ for $s > s_0$, which contradicts the continuity of $s \mapsto \operatorname{hcap}(K_+'(s))$. Suppose the latter case happens. Since $\Delta\widehat{\eta}_+^\delta$ intersects both D' and D'', there is $s_* \in (s_0, \delta]$ such that $\eta_+(t_+ + s_*) \in H(t_+, a_-)$, and $g_{K(t_+, a_-)}(\eta_+(t_+ + s_1)) \in D''$. By Lemma 3.4, there is a sequence $s_n \downarrow s_*$ such that $\eta_+(t_+ + s_n) \in H(t_+ + s_*, a_-)$. Then $\mathbb{H} \setminus K_+'(s_*) \ni g_{K(t_+, a_-)}(\eta_+(t_+ + s_n)) \to g_{K(t_+, a_-)}(\eta_+(t_+ + s_*)) \in D'' \cap K_+'(s_*)$. But this is impossible since $\mathbb{H} \setminus K_+'(s_*) \subset D'$ and $\operatorname{dist}(D', D'') > 0$. The claim is now proved.

Since $N \subset H(t_+, a_-) \setminus J \subset H(t_+, t_-) \setminus J$ and N is connected, we know that N is contained in one connected component of $H(t_+,t_-)\setminus J$. Since $N\supset \Delta\eta_+\cap H(t_+,a_-)\ni z_*$ and z_* lies in the connected component D_J of $H(t_+,t_-)\setminus J$, we get $\Delta\eta_+\cap H(t_+,a_-)\subset N\subset D_J$. Since $\Delta\eta_+\cap H(t_+,a_-)$ is dense in $\Delta\eta_+\cap H(t_+,t_-)$ (Lemma 3.4), and $\Delta\eta_+$ has positive distance from J, we get $\Delta\eta_+\cap H(t_+,t_-)\subset D_J$. Since $K(t_++\delta,t_-)$ is the $\mathbb{H}$-hull generated by $K(t_+,t_-)$ and $\Delta\eta_+\cap H(t_+,t_-)$, we get $K(t_++\delta,t_-)\setminus K(t_+,t_-)\subset D_J$.

We now write g for $g_{K(t_+,t_-)}$. From the last paragraph we know that $K'_+(\delta)$ is contained in the subdomain of $\mathbb{H}$ bounded by the crosscut $g(J)$. Thus, $\operatorname{diam}(K'_+(\delta))\le \operatorname{diam}(g(J))$. Let $L=\max\{|z|:z\in\overline{K(a_+,a_-)}\}<\infty$ and $R=2L$. From $\eta_+(t_+)\in\overline{K(a_+,a_-)}$, we get $|\eta_+(t_+)|\le L$. Suppose $r<L$. Then the arc J and the circle $\{|z-\eta_+(t_+)|=R\}$ are separated by the annulus centered at $\eta_+(t_+)$ with inner radius r and outer radius $R-L=L$. Let $J'=\{|z-\eta_+(t_+)|=R\}\cap\mathbb{H}$ and $D_{J'}=(\mathbb{H}\cap\{|z-\eta_+(t_+)|<R\})\setminus K(t_+,t_-)$. By comparison principle ([1]), the extremal length of the curves in $D_{J'}$ that separate J from J' is bounded above by $2\pi/\log(L/r)$. By conformal invariance, the extremal length of the curves in the subdomain of $\mathbb{H}$ bounded by the crosscut $g(J')$, denoted by $D_{g(J')}$, that separate $g(J)$ from $g(J')$ is also bounded above by $2\pi/\log(L/r)$. By Proposition 2.3, $g(J')\subset\{|z|\le R+3L=5L\}$. So the Euclidean area of $D_{g(J')}$ is bounded above by $25\pi L^2/2$. By the definition of extremal length, there exists a curve in Ω with Euclidean length less than

$$2[(2\pi/\log(L/r))*(25\pi L^2/2)]^{1/2}=10\pi L*(\log(L/r))^{-1/2},$$

which separates $g(J)$ from $g(J')\}$. This implies that the $\operatorname{diam}(g(J))$ is bounded above by $10\pi L*(\log(L/r))^{-1/2}$, and so is that of $K'_+(\delta)=K_{+,t_-}(t_++\delta)/K_{+,t_-}(t_+)$. For every $\varepsilon>0$, there exists $r\in(0,L)$ such that $10\pi L*(\log(L/r))^{-1/2}<\varepsilon$. Choose $\delta>0$ for such r. Then we have $\operatorname{diam}(K_{+,t_-}(t_++\delta)/K_{+,t_-}(t_+))<\varepsilon$. This finishes the proof of (i).

(ii) This follows from (i), Proposition 2.3 and $g_{K_{\pm,t_\mp}(t'_\pm)}=g_{K_{\pm,t_\mp}(t'_\pm)/K_{\pm,t_\mp}(t_\pm)}\circ g_{K_{\pm,t_\mp}(t_\pm)}$.

(iii) This follows from (ii), (3.4) and the fact that for each $(t_+,t_-)\in\mathcal{D}$, $g_{K(t_+,t_-)}$ is a conformal map defined on $\mathbb{C}\setminus K(t_+,t_-)^{\text{doub}}$. □

Remark 3.6. From the proof of Lemma 3.5 (i) we find that, for $\sigma\in\{+,-\}$, if $s_\sigma<t_\sigma\in\mathcal{I}_\sigma$ and $t_{-\sigma}\in\mathcal{I}_{-\sigma}$ satisfy that $(t_+,t_-)\in\mathcal{D}$, then

$$\operatorname{diam}(K_{\sigma,t_{-\sigma}}(t_\sigma)/K_{\sigma,t_{-\sigma}}(s_\sigma))\le 10\pi L*\log(L/r)^{-1/2},\quad \text{if } r<L,$$

where $L=\max\{|z|:z\in\eta_+([0,t_+])\cup\eta_-([0,t_-])\}$ and $r=\operatorname{diam}(\eta_\sigma([s_\sigma,t_\sigma]))$.

For a function X defined on a subset of $\mathcal{I}_+\times\mathcal{I}_-$, $\sigma\in\{+,-\}$ and $t_\sigma\in\mathcal{I}_\sigma$, we use $X|^\sigma_{t_\sigma}(t)$ to denote the function $X(t_\sigma\underline{e}_\sigma+t\underline{e}_{-\sigma})$, which depends on only one variable; and use ∂_+X (resp. ∂_-X) to denote the partial derivative of X w.r.t. the first (resp. second) variable.

Lemma 3.7. *There are two functions $W_+,W_-\in C(\mathcal{D},\mathbb{R})$ such that for any $\sigma\in\{+,-\}$ and $t_{-\sigma}\in\mathcal{I}_{-\sigma}$, $K_{\sigma,t_{-\sigma}}(t_\sigma)$, $0\le t_\sigma<T^{\mathcal{D}}_\sigma(t_{-\sigma})$, are chordal Loewner hulls driven by $W_\sigma|^{-\sigma}_{t_{-\sigma}}$ with speed $d\,\mathrm{m}|^{-\sigma}_{t_{-\sigma}}$, and for any $(t_+,t_-)\in\mathcal{D}$, $\eta_\sigma(t_\sigma)=f_{K(t_+,t_-)}(W_\sigma(t_+,t_-))$.*

Proof. By symmetry, we only need to prove the case that $\sigma = +$. Since

$$\text{hcap}_2(K_{+,t_-}(t_+ + \delta)) - \text{hcap}_2(K_{+,t_-}(t_+)) = \text{m}(t_+ + \delta, t_-) - \text{m}(t_+, t_-),$$

by Lemma 3.5 (i) and Proposition we know that, for every $t_- \in \mathcal{I}_-$, $K_{+,t_-}(t_+)$, $0 \le t_+ < T_+^{\mathcal{D}}(t_-)$, are chordal Loewner hulls with speed $d\,\text{m}(\cdot, t_-)$, and the driving function, denoted by $W_+(\cdot, t_-)$, satisfies that

$$\bigcap_{\delta>0} \overline{K_{+,t_-}(t_+ + \delta)/K_{+,t_-}(t_+)} = \{W_+(t_+, t_-)\}, \quad \forall (t_+, t_-) \in \mathcal{D}. \tag{3.6}$$

We now show that $f_{K(t_+,t_-)}(W_+(t_+, t_-)) = \eta_+(t_+)$. Fix $(t_+, t_-) \in \mathcal{D}$. From Lemma 3.4, we may find a sequence $t_+^n \downarrow t_+$ such that $\eta_+(t_+^n) \in K(t_+^n, t_-) \setminus K(t_+, t_-)$ for all n. Then we get $g_{K(t_+,t_-)}(\eta_+(t_+^n)) \in K(t_+^n, t_-)/K(t_+, t_-) = K_{+,t_-}(t_+^n)/K_{+,t_-}(t_+)$. From (3.6) we get $g_{K(t_+,t_-)}(\eta_+(t_+^n)) \to W_+(t_+, t_-)$. From the continuity of $f_{K(t_+,t_-)}$ and η_+, we then get

$$\eta_+(t_+) = \lim_{n\to\infty} \eta_+(t_+^n) = \lim_{n\to\infty} f_{K(t_+,t_-)}(g_{K(t_+,t_-)}(\eta_+(t_+^n))) = f_{K(t_+,t_-)}(W_+(t_+, t_-)).$$

It remains to show that W_+ is continuous on $\mathcal{D}$. As a driving function, it is continuous in t_+. It now suffices to show that for any $(a_+, a_-) \in \mathcal{D}$, the family of functions $[0, a_-] \ni t_- \mapsto W_+(t_+, \cdot)$, $0 \le t_+ \le a_+$, are equicontinuous. Fix $(a_+, a_-) \in \mathcal{D}$, $t_+ \in [0, a_+]$ and $t_-^1 < t_-^2 \in [0, a_-]$. By Lemma 3.4, there is a sequence $\delta_n \downarrow 0$ such that $z_n := \eta_+(t_+ + \delta_n) \in H(t_+, t_-^2)$. Then $g_{K(t_+,t_-^j)}(z_n) \in K(t_+ + \delta_n, t_-^j)/K(t_+, t_-^j) = K_{+,t_-^j}(t_+ + \delta_n)/K_{+,t_-^j}(t_+)$, $j = 1, 2$. From (3.6) we get

$$|W_+(t_+, t_-^j) - g_{K(t_+,t_-^j)}(z_n)| \le \text{diam}(K_{+,t_-^j}(t_+ + \delta_n)/K_{+,t_-^j}(t_+)), \quad j = 1, 2.$$

Since $g_{K(t_+,t_-^2)}(z_n) = g_{K(t_+,t_-^2)/K(t_+,t_-^1)} \circ g_{K(t_+,t_-^1)}(z_n)$, by Proposition 2.3 we get

$$|g_{K(t_+,t_-^2)}(z_n) - g_{K(t_+,t_-^1)}(z_n)| \le 3\,\text{diam}(K(t_+, t_-^2)/K(t_+, t_-^1)) = 3\,\text{diam}(K_{-,t_+}(t_-^2)/K_{-,t_+}(t_-^1)).$$

Combining the above displayed formulas and letting $n \to \infty$, we get

$$|W_+(t_+, t_-^2) - W_+(t_+, t_-^1)| \le 3\,\text{diam}(K_{-,t_+}(t_-^2)/K_{-,t_+}(t_-^1)),$$

which together with Lemma 3.4 (i) implies the equicontinuity that we need. □

Definition 3.8. We call W_+ and W_- the driving functions for the commuting pair $(\eta_+, \eta_-; \mathcal{D})$.

Remark 3.9. By (3.6) and Propositions 2.6 and 2.9, for $t_+^1 < t_+^2 \in \mathcal{I}_+$ and $t_- \in \mathcal{I}_-$ such that $(t_+^2, t_-) \in \mathcal{D}$,

$$|W_+(t_+^2, t_-) - W_+(t_+^1, t_-)| \le 4\,\text{diam}(K_{+,t_-}(t_+^2)/K_{+,t_-}(t_+^1)).$$

This combined with the last displayed formula in the above proof and Remark 3.6 implies that, if η_+ extends continuously to $[0, T_+]$ and η_- extends continuously to $[0, T_-]$, then W_+ and W_- are uniformly continuous on $\mathcal{D}$, and so extend continuously to $\overline{\mathcal{D}}$.

Lemma 3.10. *For any $\sigma \in \{+,-\}$ and $t_{-\sigma} \in \mathcal{I}_{-\sigma}$, the chordal Loewner hulls $K_{\sigma,t_{-\sigma}}(t_\sigma) = K(t_+,t_-)/K_{-\sigma}(t_{-\sigma})$, $0 \le t_\sigma < T^{\mathcal{D}}_\sigma(t_{-\sigma})$, are generated by a chordal Loewner curve, denoted by $\eta_{\sigma,t_{-\sigma}}$, which intersects $\mathbb{R}$ at a set with Lebesgue measure zero such that $\eta_\sigma|_{[0,T^{\mathcal{D}}_\sigma(t_{-\sigma}))} = f_{K_{-\sigma}(t_{-\sigma})} \circ \eta_{\sigma,t_{-\sigma}}$. Moreover, for $\sigma \in \{+,-\}$, $(t_+,t_-) \mapsto \eta_{\sigma,t_{-\sigma}}(t_\sigma)$ is continuous on $\mathcal{D}$.*

Proof. It suffices to work on the case that $\sigma = +$. First we show that there exists a continuous function $(t_+,t_-) \mapsto \eta_{+,t_-}(t_+)$ from $\mathcal{D}$ into $\overline{\mathbb{H}}$ such that

$$\eta_+(t_+) = f_{K_-(t_-)}(\eta_{+,t_-}(t_+)), \quad \forall (t_+,t_-) \in \mathcal{D}. \tag{3.7}$$

Let $t_- \in \mathcal{I}^*_-$ and $(t_+,t_-) \in \mathcal{D}$. By Lemma 3.4, there is a sequence $t^n_+ \downarrow t_+$ such that for all n, $(t^n_+,t_-) \in \mathcal{D}$ and $\eta_+(t^n_+) \in \mathbb{H} \setminus K(t_+,t_-)$. Then we get $g_{K_-(t_-)}(\eta_+(t^n_+)) \in g_{K_-(t_-)}(K(t^n_+,t_-) \setminus K(t_+,t_-)) = K_{+,t_-}(t^n_+) \setminus K_{+,t_-}(t_+)$. By Condition (I), $\bigcap_n \overline{K_{+,t_-}(t^n_+) \setminus K_{+,t_-}(t_+)} = \{\eta_{+,t_-}(t_+)\}$. Thus, $g_{K_-(t_-)}(\eta_+(t^n_+)) \to \eta_{+,t_-}(t_+)$. From the continuity of $f_{K_-(t_-)}$ and η_+, we find that (3.7) holds if $t_- \in \mathcal{I}^*_-$. Thus,

$$\eta_{+,t_-}(t_+) = g_{K_-(t_-)}(\eta_+(t_+)), \quad \text{if } (t_+,t_-) \in \mathcal{D},\ t_- \in \mathcal{I}^*_- \text{ and } \eta_+(t_+) \in \mathbb{H} \setminus K_-(t_-). \tag{3.8}$$

Fix $a_- \in \mathcal{I}^*_-$. Let $\mathcal{R} = \{t_+ \in \mathcal{I}_+ : (t_+,a_-) \in \mathcal{D}, \eta_+(t_+) \in \mathbb{H} \setminus K_-(a_-)\}$, which by Lemma 3.4 is dense in $[0,T^{\mathcal{D}}_+(a_-))$. By Propositions 2.3 and 2.8,

$$\lim_{\delta\to 0^+} \sup_{t_- \in [0,a_-]} \sup_{t'_- \in [0,a_-]\cap(t_- - \delta, t_- + \delta)} \sup_{t_+ \in \mathcal{R}} |g_{K_-(t_-)}(\eta_+(t_+)) - g_{K_-(t'_-)}(\eta_+(t_+))| = 0. \tag{3.9}$$

This combined with (3.8) implies that

$$\lim_{\delta\to 0^+} \sup_{t_- \in [0,a_-]\cap\mathcal{I}^*_-} \sup_{t'_- \in [0,a_-]\cap\mathcal{I}^*_-\cap(t_- - \delta, t_- + \delta)} \sup_{t_+ \in \mathcal{R}} |\eta_{+,t_-}(t_+) - \eta_{+,t'_-}(t_+)| = 0. \tag{3.10}$$

By the denseness of $\mathcal{R}$ in $[0,T^{\mathcal{D}}_+(a_-))$ and the continuity of each η_{+,t_-}, $t_- \in \mathcal{I}^*_-$, we know that (3.10) still holds if $\sup_{t_+ \in \mathcal{R}}$ is replaced by $\sup_{t_+ \in [0,T^{\mathcal{D}}_+(a_-))}$. Since $\mathcal{I}^*_-$ is dense in $\mathcal{I}_-$, the continuity of each η_{+,t_-}, $t_- \in \mathcal{I}^*_-$, together with (3.10) implies that there exists a continuous function $[0,T^{\mathcal{D}}_+(a_-)) \times [0,a_-] \ni (t_+,t_-) \mapsto \eta_{+,t_-}(t_+) \in \overline{\mathbb{H}}$, which extends those $\eta_{+,t_-}|_{[0,T^{\mathcal{D}}_+(a_-))}$, $t_- \in \mathcal{I}^*_- \cap [0,a_-]$. Running a_- from 0 to T_-, we get a continuous function $\mathcal{D} \ni (t_+,t_-) \mapsto \eta_{+,t_-}(t_+) \in \overline{\mathbb{H}}$, which extends those η_{+,t_-}, $t_- \in \mathcal{I}^*_-$. Since $\eta_{+,t_-}(t_+) = g_{K_-(t_-)}(\eta_+(t_+))$ for all $t_+ \in \mathcal{R}$ and $t_- \in [0,a_-] \cap \mathcal{I}^*_-$, from (3.8,3.9) we know that it is also true for any $t_- \in [0,a_-]$. Thus, $\eta_+(t_+) = f_{K_-(t_-)}(\eta_{+,t_-}(t_+))$ for all $t_+ \in \mathcal{R}$ and $t_- \in [0,a]$. By the denseness of $\mathcal{R}$ in $[0,T^{\mathcal{D}}_+(a_-))$ and the continuity of η_+, $f_{K_-(t_-)}$ and η_{+,t_-}, we get (3.7) for all $t_- \in [0,a_-]$ and $t_+ \in [0,T^{\mathcal{D}}_+(a_-))$. Running a_- from 0 to T_- we then get (3.7) for all $(t_+,t_-) \in \mathcal{D}$.

For $(t_+,t_-) \in \mathcal{D}$, since $K(t_+,t_-)$ is the $\mathbb{H}$-hull generated by $K_-(t_-)$ and $\eta_+([0,t_+]) \cap (\mathbb{H} \setminus K_-(t_-))$, we see that $K_{+,t_-}(t_+) = g_{K_-(t_-)}(K(t_+,t_-) \setminus K_-(t_-))$ is the $\mathbb{H}$-hull generated by $g_{K_-(t_-)}(\eta_+([0,t_+]) \cap (\mathbb{H} \setminus K_-(t_-))) = \eta_{+,t_-}([0,t_+]) \cap \mathbb{H}$. So $K_{+,t_-}(t_+) = \mathrm{Hull}(\eta_{+,t_-}([0,t_+]))$. By Lemma 3.7, for any $t_- \in [0,T_-)$, $\eta_{+,t_-}(t_+)$, $0 \le t_+ < T^{\mathcal{D}}_+(t_-)$, is the chordal Loewner curve driven by $W_+(\cdot,t_-)$ with speed $d\,\mathrm{m}(\cdot,t_-)$. So we have $\eta_{+,t_-}(t_+) = f_{K_{+,t_-}(t_+)}(W_+(t_+,t_-))$, which together with $\eta_+(t_+) = f_{K(t_+,t_-)}(W_+(t_+,t_-))$ implies that $\eta_+(t_+) = f_{K_-(t_-)}(\eta_{+,t_-}(t_+))$.

Finally, we show that $\eta_{+,t_-}\cap\mathbb{R}$ has Lebesgue measure zero for all $t_-\in\mathcal{I}_-$. Fix $t_-\in\mathcal{I}_-$ and $\widehat{t}_+\in\mathcal{I}_+$ such that $(\widehat{t}_+,t_-)\in\mathcal{D}$. It suffices to show that $\eta_{+,t_-}([0,\widehat{t}_+])\cap\mathbb{R}$ has Lebesgue measure zero. There exists a sequence $\mathcal{I}_-^*\ni t_-^n\downarrow t_-$ such that $(\widehat{t}_+,t_-^n)\in\mathcal{D}$ for all n. Let $K_n = K_-(t_-^n)/K_-(t_-)$, $g_n = g_{K_n}$, and $f_n = g_n^{-1}$. Then $f_{K_-(t_-)} = f_{K_-(t_-^n)}\circ g_n$ on $\mathbb{H}\setminus K_n$, which together with $f_{K_-(t_-)}(\eta_{+,t_-}(t_+)) = \eta_+(t_+) = f_{K_-(t_-^n)}(\eta_{+,t_-^n}(t_+))$ implies that $\eta_{+,t_-^n}(t_+) = g_n(\eta_{+,t_-}(t_+))$ if $\eta_{+,t_-}(t_+)\in\mathbb{H}\setminus K_n$. By continuity we get $\eta_{+,t_-^n}(t_+) = g_n(\eta_{+,t_-}(t_+))$ if $\eta_{+,t_-^n}(t_+)\in\overline{\mathbb{H}}\setminus\overline{K_n}$, $0\le t_+\le\widehat{t}_+$. Thus, $\eta_{+,t_-}([0,\widehat{t}_+])\cap(\mathbb{R}\setminus[a_{K_n},b_{K_n}])\subset f_n(\eta_{+,t_-^n}([0,\widehat{t}_+])\cap(\mathbb{R}\setminus[c_{K_n},d_{K_n}]))$. By Condition (II), $\eta_{+,t_-^n}([0,\widehat{t}_+])\cap\mathbb{R}$ has Lebesgue measure zero for all n. From the analyticity of f_n on $\mathbb{R}\setminus[c_{K_n},d_{K_n}]$ we know that $\eta_{+,t_-}([0,\widehat{t}_+])\cap(\mathbb{R}\setminus[a_{K_n},b_{K_n}])$ has Lebesgue measure zero. Sending $n\to\infty$ and using the fact that $[a_{K_n},b_{K_n}]\downarrow\{\widehat{w}_-(t_-)\}$ (by (3.1)), we see that $\eta_{+,t_-}([0,\widehat{t}_+])\cap\mathbb{R}$ also has Lebesgue measure zero. $\square$

Lemma 3.11. *For any $\sigma\in\{+,-\}$ and $(t_+,t_-)\in\mathcal{D}$, $\widehat{w}_\sigma(t_\sigma) = f_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(W_\sigma(t_+,t_-))\in\partial(\mathbb{H}\setminus K_{-\sigma,t_\sigma}(t_{-\sigma}))$.*

Proof. By symmetry, it suffices to work on the case $\sigma=+$. For any $(t_+,t_-)\in\mathcal{D}$, by Lemma 3.4 there is a sequence $t_+^n\downarrow t_+$ such that $\eta_+(t_+^n)$ lies in $K(t_+^n,t_-)\setminus K(t_+,t_-)$ for all n. From (3.1) and Lemma 3.7 we get $g_{K_+(t_+)}(\eta_+(t_+^n))\to\widehat{w}_+(t_+)$ and $g_{K(t_+,t_-)}(\eta_+(t_+^n))\to W_+(t_+,t_-)$. From (3.4) we get $g_{K_+(t_+)} = f_{K_{-,t_+}(t_-)}\circ g_{K(t_+,t_-)}$. From the continuity of $f_{K_{-,t_+}(t_-)}$ on $\overline{\mathbb{H}}$, we then get $\widehat{w}_+(t_+) = f_{K_{-,t_+}(t_-)}(W_+(t_+,t_-))$. Finally, $\widehat{w}_+(t_+)\in\partial(\mathbb{H}\setminus K_{-,t_+}(t_-))$ because $W_+(t_+,t_-)\in\partial\mathbb{H}$ and $f_{K_{-,t_+}(t_-)}$ is conformal in $\mathbb{H}$ and continuous on $\overline{\mathbb{H}}$. $\square$

3.2 Force point functions

For $\sigma\in\{+,-\}$, define C_σ and D_σ on $\mathcal{D}$ such that if $t_\sigma>0$, $C_\sigma(t_+,t_-) = c_{K_{\sigma,t_{-\sigma}}(t_\sigma)}$ and $D_\sigma(t_+,t_-) = d_{K_{\sigma,t_{-\sigma}}(t_\sigma)}$; and if $t_\sigma=0$, then $C_\sigma = D_\sigma = W_\sigma$ at $t_{-\sigma}\underline{e}_{-\sigma}$. Since $K_{\sigma,t_{-\sigma}}(\cdot)$ are chordal Loewner hulls driven by $W_\sigma|_{t_{-\sigma}}^{-\sigma}$ with some speed, by Proposition 2.9 we get

$$C_\sigma\le W_\sigma\le D_\sigma,\quad \text{on } \mathcal{D},\quad \sigma\in\{+,-\}. \tag{3.11}$$

Since $K_{\sigma,t_{-\sigma}}(t_\sigma)$ is the $\mathbb{H}$-hull generated by $\eta_{\sigma,t_{-\sigma}}([0,t_\sigma])$, we get

$$f_{K_{\sigma,t_{-\sigma}}(t_\sigma)}([C_\sigma(t_+,t_-),D_\sigma(t_+,t_-)])\subset\eta_{\sigma,t_{-\sigma}}([0,t_\sigma]). \tag{3.12}$$

Lemma 3.12. *Let $I_0=(w_-,w_+)\cup\{w_-,w_+\}$, $I_+=(w_+,\infty)\cup\{w_+\}$, $I_-=(-\infty,w_-)\cup\{w_-\}$, and $\mathbb{R}_{\underline{w}} = I_0\cup I_+\cup I_-$. Assign the obvious order to $\mathbb{R}_{\underline{w}}$ endowed from $\mathbb{R}$; and assign the topology to $\mathbb{R}_{\underline{w}}$ such that I_-,I_0,I_+ are three connected components of $\mathbb{R}_{\underline{w}}$, which are homeomorphic to $(-\infty,w_-],[w_-,w_+],[w_+,\infty)$, respectively. Then for any $\underline{t}=(t_+,t_-)\in\mathcal{D}$, $g^{W_+(0,t_-)}_{K_{+,t_-}(t_+)}\circ g^{w_-}_{K_-(t_-)}$ and $g^{W_-(t_+,0)}_{K_{-,t_+}(t_-)}\circ g^{w_+}_{K_+(t_+)}$ agree on $\mathbb{R}_{\underline{w}}$, and the common function, denoted by $g^{\underline{w}}_{K(\underline{t})}$, satisfies the following properties.*

(i) $g^{\underline{w}}_{K(\underline{t})}$ *is increasing and continuous on* $\mathbb{R}_{\underline{w}}$*, and agrees with* $g_{K(\underline{t})}$ *on* $\mathbb{R}_{\underline{w}} \setminus \overline{K(\underline{t})}$*.*

(iii) $g^{\underline{w}}_{K(\underline{t})}$ *maps* $I_+ \cap (\overline{K(\underline{t})} \cup \{w_+^+\})$ *and* $I_- \cap (\overline{K(\underline{t})} \cup \{w_-^-\})$ *to* $\{D_+(\underline{t})\}$ *and* $\{C_-(\underline{t})\}$*, respectively.*

(iv) If $\overline{K_+(t_+)} \cap \overline{K_-(t_-)} = \emptyset$*,* $g^{\underline{w}}_{K(\underline{t})}$ *maps* $I_0 \cap (\overline{K_+(t_+)} \cup \{w_+^-\}$ *and* $I_0 \cap (\overline{K_-(t_-)} \cup \{w_-^+\})$ *to* $\{C_+(\underline{t})\}$ *and* $\{D_-(\underline{t})\}$*, respectively.*

(v) If $\overline{K_+(t_+)} \cap \overline{K_-(t_-)} \neq \emptyset$*,* $g^{\underline{w}}_{K(\underline{t})}$ *maps* I_0 *to* $\{C_+(\underline{t})\} = \{D_-(\underline{t})\}$*.*

(vi) The map $(\underline{t}, v) \mapsto g^{\underline{w}}_{K(\underline{t})}(v)$ *from* $\mathcal{D} \times \mathbb{R}_{\underline{w}}$ *to* $\mathbb{R}$ *is jointly continuous.*

Here we are using Definition 2.11, and understand $g^{w_{-\sigma}}_{K_{-\sigma}(t_{-\sigma})}(w_\sigma^\pm)$ *as* $g_{K_{-\sigma}(t_{-\sigma})}(w_\sigma)^\pm$*.*

Proof. Fix $\underline{t} = (t_+, t_-) \in \mathcal{D}$. For $\sigma \in \{+,-\}$, we write K for $K(\underline{t})$, K_σ for $K_\sigma(t_\sigma)$, $\widetilde{K}_\sigma$ for $K_{\sigma,t_{-\sigma}}(t_\sigma)$, $\widetilde{w}_\sigma$ for $W_\sigma(t_{-\sigma}\underline{e}_{-\sigma})$, C_σ for $C_\sigma(\underline{t})$, and D_σ for $D_\sigma(\underline{t})$. The equality now reads $g^{\widetilde{w}_+}_{\widetilde{K}_+} \circ g^{w_-}_{K_-} = g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$. We are going to show that both sides are well defined and satisfy (i-iv) with a slight modification in (iv) (see below). First consider $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$.

(i) From Lemma 3.7, $w_- = f_{K_+}(\widetilde{w}_-)$. Since η_+ starts from w_+, which is $> w_-$, and does not hit $(-\infty, w_-]$, we have $w_- \notin \overline{K_+}$. So $\widetilde{w}_- = g_{K_+}(w_-)$. Thus, $g^{w_+}_{K_+}$ maps $I_+ \cup I_0$ and I_- respectively into $\{\widetilde{w}_-^+\} \cup (\widetilde{w}_-, \infty)$ and $(-\infty, \widetilde{w}_-) \cup \{\widetilde{w}_-^-\}$, which are all contained in $\mathbb{R}_{\widetilde{w}_-}$. So $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ is well defined. The continuity and monotonicity of the composition follows from the continuity and monotonicity of both $g^{\widetilde{w}_-}_{\widetilde{K}_-}$ and $g^{w_+}_{K_+}$.

Let $v \in \mathbb{R}_{\underline{w}} \setminus \overline{K}$. Then $v \notin \overline{K_+}$, and $g^{w_+}_{K_+}(v) = g_{K_+}(v)$. We claim that $g_{K_+}(v) \notin \overline{\widetilde{K}_-}$. If this is not true, there exists a sequence (z_n) in K_- such that $z_n \to g_{K_+}(v)$, which implies that $f_{K_+}(z_n) \to v$. Since $\widetilde{K}_- = K/K_+$, $f_{K_+}(z_n) \in f_{K_+}(K/K_+) = K \setminus K_+$, which implies that $v \in \overline{K}$, a contradiction. From the claim we get $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}(v) = g_{\widetilde{K}_-} \circ g_{K_+}(v) = g_K(v)$.

We now write η_σ for $\eta_\sigma([0, t_\sigma])$ and $\widetilde{\eta}_\sigma$ for $\eta_{\sigma,t_{-\sigma}}([0, t_\sigma])$. In the proof of (ii,iii) below, when $t_\sigma = 0$, i.e., $K_\sigma = \widetilde{K}_\sigma = \emptyset$, we understand $a_{K_\sigma} = b_{K_\sigma} = c_{K_\sigma} = d_{K_\sigma} = w_\sigma$, and $a_{\widetilde{K}_\sigma} = b_{\widetilde{K}_\sigma} = c_{\widetilde{K}_\sigma} = d_{\widetilde{K}_\sigma} = \widetilde{w}_\sigma$. Then it is always true that $a_{K_\sigma} = \min\{\eta_\sigma \cap \mathbb{R}\}$, $b_{K_\sigma} = \max\{\eta_\sigma \cap \mathbb{R}\}$, $a_{\widetilde{K}_\sigma} = \min\{\widetilde{\eta}_\sigma \cap \mathbb{R}\}$, $b_{\widetilde{K}_\sigma} = \max\{\widetilde{\eta}_\sigma \cap \mathbb{R}\}$, $c_{\widetilde{K}_\sigma} = C_\sigma$, and $d_{\widetilde{K}_\sigma} = D_\sigma$. Since $\eta_\pm = f_{K_\mp}(\widetilde{\eta}_\pm)$, we get $b_{\widetilde{K}_+} = g_{K_-}(b_{K_+})$, $a_{\widetilde{K}_-} = g_{K_+}(a_{K_-})$; and if $\overline{K_+} \cap \overline{K_-} = \emptyset$, $a_{\widetilde{K}_+} = g_{K_-}(a_{K_+})$, $b_{\widetilde{K}_-} = g_{K_+}(b_{K_-})$.

(ii) Since $I_+ \cap (\overline{K} \cup \{w_+^+\}) = \{w_+^+\} \cup (w_+, b_K] = \{w_+^+\} \cup (w_+, b_{K_+}]$ is mapped by $g^{w_+}_{K_+}$ to a single point, it is also mapped by $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ to a single point, which by (i) is equal to

$$\lim_{x \downarrow b_K} g_K(x) = \lim_{x \downarrow b_{K_+}} g_{\widetilde{K}_+} \circ g_{K_-}(x) = \lim_{y \downarrow b_{\widetilde{K}_+}} g_{\widetilde{K}_+}(y) = d_{\widetilde{K}_+} = D_+.$$

To show that $I_- \cap \overline{K}$ is mapped by $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ to C_-, by (i) it suffices to show that

$\lim_{x\uparrow a_K} g_K(x) = g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}(w^-_-) = c_{\widetilde{K}_-}$. This holds because

$$g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}(w^-_-) = g^{\widetilde{w}_-}_{\widetilde{K}_-}(\widetilde{w}^-_-) = c_{\widetilde{K}_-} = \lim_{x\uparrow a_{\widetilde{K}_-}} g_{\widetilde{K}_-}(x) = \lim_{x\uparrow a_{K_-}} g_{\widetilde{K}_-} \circ g_{K_+}(x) = \lim_{x\uparrow a_K} g_K(x).$$

(iii) Suppose $\overline{K_+} \cap \overline{K_-} = \emptyset$. Then $I_0 \cap (\overline{K_+} \cup \{w^-_+\}) = [a_{K_+}, w_+) \cup \{w^-_+\}$ is mapped by $g^{w_+}_{K_+}$ to a single point, so is also mapped by $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ to a single point. By (i) the latter point is

$$\lim_{x\uparrow a_{K_+}} g_K(x) = \lim_{x\uparrow a_{K_+}} g_{\widetilde{K}_+} \circ g_{K_-}(x) = \lim_{y\uparrow a_{\widetilde{K}_+}} g_{\widetilde{K}_+}(y) = c_{\widetilde{K}_+} = C_+.$$

Since $I_0 \cap (\overline{K_-} \cup \{w^+_-\}) = (w_-, b_{K_-}] \cup \{w^+_-\}$ is mapped by $g^{w_+}_{K_+}$ to $\{\widetilde{w}^+_-\} \cup (\widetilde{w}_-, b_{\widetilde{K}_-}]$, it is mapped by $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ to $\{d_{\widetilde{K}_-}\} = \{D_-\}$.

(iv) Suppose $\overline{K_+} \cap \overline{K_-} \neq \emptyset$. For now, we only show that I_0 is mapped to $\{D_-\}$. Then $t_+, t_- > 0$, and $[c_{K_+}, d_{K_+}] \cap \overline{\widetilde{K}_-} \neq \emptyset$, which implies that $c_{K_+} \le b_{\widetilde{K}_-}$. Thus, $g^{w_+}_{K_+}(I_0) \subset \{\widetilde{w}^+_-\} \cup (\widetilde{w}_-, b_{\widetilde{K}_-}]$, from which follows that $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}(I_0) = \{d_{\widetilde{K}_-}\} = \{D_-\}$.

Now $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ satisfy (i-iv). By symmetry, this is also true for $g^{\widetilde{w}_+}_{\widetilde{K}_+} \circ g^{w_-}_{K_-}$, where for (iv), I_0 is mapped to $\{C_+\}$. It remains to show that the two functions agree on $\mathbb{R}_{\underline{w}}$. From (ii) we know that $g^{\widetilde{w}_+}_{\widetilde{K}_+} \circ g^{w_-}_{K_-}$ and $g^{\widetilde{w}_-}_{\widetilde{K}_-} \circ g^{w_+}_{K_+}$ agree on $\mathbb{R}_{\underline{w}} \setminus \overline{K}$. By (i,ii) the two functions also agree on $I_+ \cap \overline{K}$ and $I_- \cap \overline{K}$. Thus they agree on both I_+ and I_-. By (i,iii) they agree on I_0 when $\overline{K_+} \cap \overline{K_-} = \emptyset$. To prove that they agree on I_0 when $\overline{K_+} \cap \overline{K_-} \neq \emptyset$, by (iv) we only need to show that $c_{\widetilde{K}_+} = d_{\widetilde{K}_-}$ in that case.

First, we show that $d_{\widetilde{K}_-} \le c_{\widetilde{K}_+}$. Suppose $d_{\widetilde{K}_-} > c_{\widetilde{K}_+}$. Then $J := (c_{\widetilde{K}_+}, d_{\widetilde{K}_-}) \subset [c_{\widetilde{K}_-}, d_{\widetilde{K}_-}] \cap [c_{\widetilde{K}_+}, d_{\widetilde{K}_+}]$. So $f_{\widetilde{K}_+}(J) \subset \partial(\mathbb{H} \setminus \widetilde{K}_+)$. If $f_{\widetilde{K}_+}(J) \subset \mathbb{R}$, then it is disjoint from $\overline{\widetilde{K}_+}$. Since $\widetilde{K}_+$ is generated by $\widetilde{\eta}_+$, which does not spend any nonempty interval of time on $\mathbb{R}$, we see that $f_{\widetilde{K}_+}(J)$ is disjoint from $[a_{\widetilde{K}_+}, b_{\widetilde{K}_+}]$, which then implies that J is disjoint from $[c_{\widetilde{K}_+}, d_{\widetilde{K}_+}]$, a contradiction. So there is $x_0 \in J$ such that $f_{\widetilde{K}_+}(x_0) \subset \mathbb{H}$. This then implies that $f_K(x_0) = f_{K_-} \circ f_{\widetilde{K}_+}(x_0) \in \mathbb{H} \setminus K_-$. But on the other hand, since $x_0 \in [c_{\widetilde{K}_-}, d_{\widetilde{K}_-}]$, $f_K(x_0) = f_{K_+} \circ f_{\widetilde{K}_-}(x_0) \subset f_{K_+}(\widetilde{\eta}_-) = \eta_-$, which contradicts that $f_K(x_0) \in \mathbb{H} \setminus K_-$. So $d_{\widetilde{K}_-} \le c_{\widetilde{K}_+}$.

Second, we show that $d_{\widetilde{K}_-} \ge c_{\widetilde{K}_+}$. Suppose $d_{\widetilde{K}_-} < c_{\widetilde{K}_+}$. Let $J = (d_{\widetilde{K}_-}, c_{\widetilde{K}_+})$. Then $f_{\widetilde{K}_+}(J) = (f_{\widetilde{K}_+}(d_{\widetilde{K}_-}), a_{\widetilde{K}_+})$. From $\overline{K_+} \cap \overline{K_-} \neq \emptyset$ we know $a_{\widetilde{K}_+} \le d_{K_-}$. From $a_{\widetilde{K}_-} = g_{K_+}(a_{K_-})$ and $a_K = a_{K_-}$ we get

$$d_{\widetilde{K}_-} \ge c_{\widetilde{K}_-} = \lim_{x\uparrow a_{\widetilde{K}_-}} g_{\widetilde{K}_-}(x) = \lim_{y\uparrow a_{K_-}} g_{\widetilde{K}_-} \circ g_{K_+}(y) = \lim_{y\uparrow a_{K_-}} g_{\widetilde{K}_+} \circ g_{K_-}(y).$$

Thus, $f_{\widetilde{K}_+}(d_{\widetilde{K}_-}) \ge \lim_{y\uparrow a_{K_-}} g_{K_-}(y) = c_{K_-}$. So we get $f_{\widetilde{K}_+}(J) \subset [c_{K_-}, a_{\widetilde{K}_+}] \subset [c_{K_-}, d_{K_-}]$, which is mapped into η_- by f_{K_-}. Thus, $f_K(J) \subset \eta_-$. Symmetrically, $f_K(J) \subset \eta_+$. Since $\eta_- = f_{K_+}(\widetilde{\eta}_-)$ and $f_K(J) \subset \partial(\mathbb{H} \setminus K)$, for every $x \in J$, there is $z_- \in \widetilde{\eta}_- \cap \partial(\mathbb{H} \setminus \widetilde{K}_-)$ such that

$f_K(x) = f_{K_+}(z_-)$. Then there is $y_- \in [c_{\widetilde{K}_-}, d_{\widetilde{K}_-}]$ such that $z_- = f_{\widetilde{K}_-}(y_-)$. So $f_K(x) = f_K(y_-)$. Similarly, for every $x \in J$, there is $y_+ \in [c_{\widetilde{K}_-}, d_{\widetilde{K}_-}]$ such that $f_K(x) = f_K(y_+)$. Here y_+, y_- depend on x. Pick $x^1 < x^2 \in J$ such that $f_K(x^1) \neq f_K(x^2)$. This is possible because $f_K(J)$ has positive harmonic measure in $\mathbb{H}\setminus K$. Then there exist $y_+^1 \in [c_{\widetilde{K}_+}, d_{\widetilde{K}_+}]$ and $y_-^2 \in [c_{\widetilde{K}_-}, d_{\widetilde{K}_-}]$ such that $f_K(x^1) = f_K(y_1^+)$ and $f_K(x^2) = f_K(y_-^2)$. This is impossible because $y_+^1 > x^2 > x^1 > y_-^2$. So $d_{\widetilde{K}_-} \geq c_{\widetilde{K}_+}$. Combining the last two paragraphs, we get $c_{\widetilde{K}_+} = d_{\widetilde{K}_-}$, as desired.

(v) From (i) we know that $g^{\underline{w}}_{K(\underline{t})}$ is continuous on $\mathbb{R}_{\underline{w}}$ for any $\underline{t} \in \mathcal{D}$. It suffices to show that, for any $(a_+, a_-) \in \mathcal{D}$, the family of maps $[0, a_+] \ni t_+ \mapsto g^{\underline{w}}_{K(\underline{t})}(v)$, $(t_-, v) \in [0, a_-] \times \mathbb{R}_{\underline{w}}$, are equicontinuous, and the family of maps $[0, a_-] \ni t_- \mapsto g^{\underline{w}}_{K(\underline{t})}(v)$, $(t_+, v) \in [0, a_+] \times \mathbb{R}_{\underline{w}}$, are equicontinuous. The first statement follows from the expression $g^{\underline{w}}_{K(\underline{t})} = g^{W_+(0,t_-)}_{K_{+,t_-}(t_+)} \circ g^{w_-}_{K_-(t_-)}$, Proposition 2.13 and Lemma 3.5 (i). The second is symmetric. □

Lemma 3.13. *For any* $(t_+, t_-) \in \mathcal{D}$ *and* $\sigma \in \{+, -\}$, $W_\sigma(t_+, t_-) = g^{W_{-\sigma}(t_\sigma \underline{e}_\sigma)}_{K_{-\sigma, t_\sigma}(t_{-\sigma})}(\widehat{w}_\sigma(t_\sigma))$.

Proof. Fix $\underline{t} = (t_+, t_-) \in \mathcal{D}$. By symmetry, we may assume that $\sigma = +$. If $t_- = 0$, it is obvious since $W_+(\cdot, 0) = \widehat{w}_+$ and $K_{-,t_+}(0) = \emptyset$. Suppose $t_- > 0$. From (3.11) and Lemma 3.12 (iii,iv) we know that $W_+(\underline{t}) \geq C_+(\underline{t}) \geq D_-(\underline{t}) = d_{K_{-,t_+}(t_-)}$. Since $\widehat{w}_+(t_+) = f_{K_{-,t_+}(t_-)}(W_+(\underline{t}))$ by Lemma 3.11, we find that either $W_+(\underline{t}) = d_{K_{-,t_+}(t_-)}$ and $\widehat{w}_+(t_+) = b_{K_{-,t_+}(t_-)}$, or $W_+(\underline{t}) > d_{K_{-,t_+}(t_-)}$ and $W_+(\underline{t}) = g_{K_{-,t_+}(t_-)}(\widehat{w}_+(t_+))$. In either case, we get the equality. □

Definition 3.14. For $v \in \mathbb{R}_{\underline{w}}$, we call $V(\underline{t}) := g^{\underline{w}}_{K(\underline{t})}(v)$, $\underline{t} \in \mathcal{D}$, the force point function (for the commuting pair $(\eta_+, \eta_-; \mathcal{D})$) started from v, which is continuous by Lemma 3.12.

Remark 3.15. Suppose for $\sigma \in \{+, -\}$, $\eta_\sigma(t)$, $0 \leq t_\sigma < T_\sigma$, is a chordal Loewner curve with speed du_σ, where $u_\sigma(0) = 0$, and $\mathcal{D} \subset [0, T_+) \times [0, T_-)$. Let $u_\oplus(t_+, t_-) = (u_+(t_+), u_-(t_-))$. If $(\eta_+ \circ u_+^{-1}, \eta_- \circ u_-^{-1}; u_\oplus(\mathcal{D}))$ is a commuting pair of chordal Loewner curves, then we call $(\eta_+, \eta_-; \mathcal{D})$ a commuting pair of chordal Loewner curves with speeds (du_+, du_-), and call $(\eta_+ \circ u_+^{-1}, \eta_- \circ u_-^{-1}; u_\oplus(\mathcal{D}))$ its normalization. For such $(\eta_+, \eta_-; \mathcal{D})$, most lemmas in this section still hold (except that m may not be Lipschitz continuous), and we may still define the hull function $K(\cdot, \cdot)$ and the capacity function $\mathrm{m}(\cdot, \cdot)$ using (3.3), define the driving functions W_+ and W_- using Lemma 3.7, and define the force point functions by $V(\underline{t}) = g^{\underline{w}}_{K(\underline{t})}(v)$.

Definition 3.16. Let $(\eta_+, \eta_-; \mathcal{D})$ and $(\widetilde{\eta}_+, \widetilde{\eta}_-; \widetilde{\mathcal{D}})$ be two commuting pairs of chordal Loewner curves with some speeds. Let $K(\cdot, \cdot)$ be the hull function for $(\eta_+, \eta_-; \mathcal{D})$. Let $\underline{\tau} = (\tau_+, \tau_-) \in \mathcal{D}$. We say that, up to a conformal map, $(\widetilde{\eta}_+, \widetilde{\eta}_-; \widetilde{\mathcal{D}})$ agrees with $(\eta_+, \eta_-; \mathcal{D})$ after $\underline{\tau}$, if $\widetilde{\mathcal{D}} = \{\underline{t} - \underline{\tau} : \underline{t} \in \mathcal{D}, \underline{t} \geq \underline{\tau}\}$ and $\eta_\sigma(\tau_\sigma + t) = f_{K(\underline{\tau})} \circ \widetilde{\eta}_\sigma(t)$, $0 \leq t < T^{\mathcal{D}}_\sigma(\tau_{-\sigma}) - t$, $\sigma \in \{+, -\}$.

Lemma 3.17. *Let* $(\eta_+, \eta_-; \mathcal{D})$ *be a commuting pair of chordal Loewner curves with some speeds. Let* $K, \mathrm{m}, W_\pm$ *be its hull function, capacity function, and driving functions, respectively. Let* $\tau \in \mathcal{D}$. *Suppose for* $\sigma \in \{+, -\}$, *there is a dense subset* $\widetilde{\mathcal{I}}^*_\sigma$ *of* $\widetilde{\mathcal{I}}_\sigma := [0, T_\sigma - \tau_\sigma)$, *such that* $\widetilde{\mathcal{I}}^*_\sigma \ni 0$, *and for every* $t_{-\sigma} \in \widetilde{\mathcal{I}}^*_{-\sigma}$, *the* $\mathbb{H}$*-hulls* $K(\underline{\tau} + t_{-\sigma}\underline{e}_{-\sigma} + t_\sigma \underline{e}_\sigma)/K(\underline{\tau} + t_{-\sigma}\underline{e}_{-\sigma})$,

$0\le t_\sigma<T^{\mathcal D}_\sigma(\tau_{-\sigma}+t_{-\sigma})-\tau_\sigma$, *are generated by a chordal Loewner curve with some speed, which intersects* $\mathbb R$ *at a set of Lebesgue measure zero. For* $\sigma\in\{+,-\}$, *let* $\widetilde\eta_\sigma$ *be the chordal Loewner curve that generates* $K(\underline\tau+t_\sigma\underline e_\sigma)/K(\underline\tau)$, $0\le t_\sigma<T^{\mathcal D}_\sigma(\tau_{-\sigma})-\tau_\sigma$. *Let* $\widetilde{\mathcal D}=\{\underline t-\underline\tau:\underline t\in\mathcal D,\underline t\ge\underline\tau\}$. *Then* $(\widetilde\eta_+,\widetilde\eta_-;\widetilde{\mathcal D})$ *is a commuting pair of chordal Loewner curves with some speeds, which up to a conformal map agrees with the part of* $(\eta_+,\eta_-;\mathcal D)$ *after* $\underline\tau$.

Proof. Fix $\sigma\in\{+,-\}$. Since $K(\underline\tau+t\underline e_\sigma)/K(\underline\tau)$ is the $\mathbb H$-hull generated by $\widetilde\eta_\sigma([0,t])$, $K(\underline\tau+t\underline e_\sigma)$ is the $\mathbb H$-hull generated by $K(\underline\tau)$ and $f_{K(\underline\tau)}\circ\widetilde\eta_\sigma([0,t])$ for each $0\le t<\widetilde T_\sigma:=T^{\mathcal D}(\tau_{-\sigma})-\tau_\sigma$. Since $K(\underline\tau+t\underline e_\sigma)$ is the $\mathbb H$-hull generated by $K(\underline\tau)$ and $\eta_\sigma([\tau_\sigma,\tau_\sigma+t])$ for all $0\le t<\widetilde T_\sigma$, we get $\eta_\sigma(\tau_\sigma+t)=f_{K(\underline\tau)}\circ\widetilde\eta_\sigma(t)$, $0\le t<\widetilde T_\sigma$.

It remains to show that $(\widetilde\eta_+,\widetilde\eta_-;\widetilde{\mathcal D})$ is a commuting pair of chordal Loewner curves. Note that $T^{\widetilde{\mathcal D}}_\sigma(t)=T^{\mathcal D}_\sigma(\tau_{-\sigma}+t)-\tau_\sigma$, $\sigma\in\{+,-\}$. Define $\widetilde K$ on $\widetilde{\mathcal D}$ using (3.3) with $\widetilde\eta_\pm$ in place of $\eta_\pm$. For $\underline t=(t_+,t_-)\in\widetilde{\mathcal D}$, since $\widetilde K(\underline t)$ is the $\mathbb H$-hull generated by $\widetilde\eta_+([0,t_+])$ and $\widetilde\eta_-([0,t_-])$, $K(\underline\tau)\cup f_{K(\underline\tau)}(\widetilde K(\underline t))$ is the $\mathbb H$-hull generated by $K(\underline\tau)$ and $f_{K(\underline\tau)}\circ\widetilde\eta_\sigma([0,t_\sigma])=\eta_\sigma([\tau_\sigma,\tau_\sigma+t_\sigma])$, $\sigma\in\{+,-\}$, which is $K(\underline\tau+\underline t)$. So for $\underline t\in\widetilde{\mathcal D}$, $K(\underline\tau+\underline t)/K(\underline\tau)=\widetilde K(\underline t)$. By assumption, for every $\sigma\in\{+,-\}$ and $t_{-\sigma}\in\widetilde{\mathcal I}^*_{-\sigma}$, $\widetilde K_{\sigma,t_{-\sigma}}(t_\sigma):=\widetilde K(\underline t)/\widetilde K_{-\sigma}(t_{-\sigma})=K(\underline\tau+t_{-\sigma}\underline e_{-\sigma}+t_\sigma\underline e_\sigma)/K(\underline\tau+t_{-\sigma}\underline e_{-\sigma})$, $0\le t_\sigma<T^{\widetilde{\mathcal D}}_\sigma(t_{-\sigma})$, are generated by a chordal Loewner curve with some speed. □

Lemma 3.18. *Suppose up to a conformal map,* $(\widetilde\eta_+,\widetilde\eta_-;\widetilde{\mathcal D})$ *agrees with the part of* $(\eta_+,\eta_-;\mathcal D)$ *after* $\underline\tau$. *Then the following hold.*

(i) Let $K,\mathrm m,W_\pm$ *and* $\widetilde K,\widetilde{\mathrm m},\widetilde{W_\pm}$ *be the hull function, capacity function, and driving functions for* $(\eta_+,\eta_-;\mathcal D)$ *and* $(\widetilde\eta_+,\widetilde\eta_-;\widetilde{\mathcal D})$, *respectively. Then for any* $\underline t\in\widetilde{\mathcal D}$, $\widetilde K(\underline t)=K(\underline\tau+\underline t)/K(\underline\tau)$, $\widetilde{\mathrm m}(\underline t)=\mathrm m(\underline\tau+\underline t)-\mathrm m(\underline\tau)$, *and* $\widetilde W_\sigma(\underline t)=W_\sigma(\underline\tau+\underline t)$, $\sigma\in\{+,-\}$.

(ii) Let $w_\sigma=W_\sigma(\underline 0)$ *and* $\widetilde w_\sigma=\widetilde W_\sigma(\underline 0)$, $\sigma\in\{+,-\}$. *Let* $v\in\mathbb R_{\underline w}$ *and* $V(\underline t)$ *be the force point function for* $(\eta_+,\eta_-;\mathcal D)$ *started from* v. *Define* $\widetilde v\in\mathbb R_{\underline{\widetilde w}}$ *such that if* $V(\underline\tau)\not\in\{\widetilde w_+,\widetilde w_-\}$, *then* $\widetilde v=V(\underline\tau)$; *and if* $V(\underline\tau)=\widetilde w_\sigma$ *and* $\nu\cdot(v-w_\sigma)>0$, *then* $\widetilde v=\widetilde w^\nu_{\underline\sigma}$, $\sigma,\nu\in\{+,-\}$. *Let* $\widetilde V$ *be the force point function for* $(\widetilde\eta_+,\widetilde\eta_-;\widetilde{\mathcal D})$ *started from* $\widetilde v$. *Then* $\widetilde V=V(\underline\tau+\cdot)$ *on* $\widetilde{\mathcal D}$.

Proof. (i) The formula $\widetilde K(\underline t)=K(\underline\tau+\underline t)/K(\underline\tau)$ follows from the argument in the second paragraph of the previous proof. It then implies that $\widetilde{\mathrm m}(\underline t)=\mathrm m(\underline\tau+\underline t)-\mathrm m(\underline\tau)$. The formula $\widetilde W_\sigma(\underline t)=W_\sigma(\underline\tau+\underline t)$ then follows from (3.6), (2.1), and that $\widetilde K(\underline t)=K(\underline\tau+\underline t)/K(\underline\tau)$.

(ii) For $\underline t=(t_+,t_-)\in\widetilde{\mathcal D}$, by (i), Proposition 2.12 and Lemma 3.12, if $V(\underline\tau)\not\in\{\widetilde w_+,\widetilde w_-\}$,

$$\begin{aligned}\widetilde V(\underline t)&=g^{\widetilde W_+(0,t_-)}_{\widetilde K(\underline t)/\widetilde K_-(t_-)}\circ g^{\widetilde w_-}_{\widetilde K_-(t_-)}(\widetilde v)=g^{W_+(\tau_+,\tau_-+t_-)}_{K(\underline\tau+\underline t)/K(\tau_+,\tau_-+t_-)}\circ g^{W_-(\underline\tau)}_{K(\tau_+,\tau_-+t_-)/K(\underline\tau)}(\widetilde v)\\&=g^{W_+(\tau_+,\tau_-+t_-)}_{K(\underline\tau+\underline t)/K(\tau_+,\tau_-+t_-)}\circ g^{W_-(\underline\tau)}_{K(\tau_+,\tau_-+t_-)/K(\underline\tau)}\circ g^{W_-(\tau_+,0)}_{K(\underline\tau)/K(\tau_+,0)}\circ g^{w_+}_{K(\tau_+,0)}(v)\\&=g^{W_+(\tau_+,\tau_-+t_-)}_{K(\underline\tau+\underline t)/K(\tau_+,\tau_-+t_-)}\circ g^{W_-(\tau_+,0)}_{K(\tau_+,\tau_-+t_-)/K(\tau_+,0)}\circ g^{w_+}_{K(\tau_+,0)}(v)\\&=g^{W_+(\tau_+,\tau_-+t_-)}_{K(\underline\tau+\underline t)/K(\tau_+,\tau_-+t_-)}\circ g^{W_+(0,\tau_-+t_-)}_{K(\tau_+,\tau_-+t_-)/K(0,\tau_-+t_-)}\circ g^{w_-}_{K(0,\tau_-+t_-)}(v)\end{aligned}$$

$$= g^{W_+(0,\tau_-+t_-)}_{K(\underline{\tau}+\underline{t})} \circ g^{w_-}_{K(0,\tau_-+t_-)}(v) = g^{(w_+,w_-)}_{K(\underline{\tau}+\underline{t})}(v) = V(\underline{\tau}+\underline{t}).$$

Here we used Proposition 2.12 in the 3rd and the 5th lines and Lemma 3.12 in the 4th line. We now consider the case that $V(\underline{\tau}) \in \{\widetilde{w}_+, \widetilde{w}_-\}$. By symmetry, we may assume that $V(\underline{\tau}) = \widetilde{w}_- = W_-(\underline{\tau})$. Suppose $v > w_-$. In the second line of the displayed formula, we will encounter $g^{W_-(\underline{\tau})}_{K(\tau_+,\tau_-+t_-)/K(\underline{\tau})}(W_-(\underline{\tau}))$, which is not defined. However, we now understand it as $g^{W_-(\underline{\tau})}_{K(\tau_+,\tau_-+t_-)/K(\underline{\tau})}(W_-(\underline{\tau})^+)$, which is consistent with our definition of $\widetilde{v}$ in this case. With this understanding, the equality in the third line still holds by Proposition 2.12. In fact, we have $x := g^{w_+}_{K(\tau_+,0)}(v) > g^{w_+}_{K(\tau_+,0)}(w_-) = W_-(\tau_+,0)$, and $g^{W_-(\tau_+,0)}_{K(\underline{\tau})/K(\tau_+,0)}(x) = V(\underline{\tau}) = W_-(\underline{\tau})$, so by Proposition 2.12, $g^{W_-(\underline{\tau})}_{K(\tau_+,\tau_-+t_-)/K(\underline{\tau})}(W_-(\tau_+,0)^+) = g^{W_-(\tau_+,0)}_{K(\tau_+,\tau_-+t_-)/K(\tau_+,0)}(x)$. So the displayed formula holds with this explanation. The case that $v < w_-$ is similar. $\square$

From now on, we fix $v_0 \in (w_-, w_+) \cup \{w_-^+, w_+^-\}$, $v_+ \in (w_+, \infty) \cup \{w_+^+\}$, and $v_- \in (-\infty, w_-) \cup \{w_-^-\}$, and let $V_\nu(\underline{t})$, $\underline{t} \in \mathcal{D}$, be the force point function started from v_ν, $\nu \in \{0,+,-\}$. By Lemma 3.12, $V_- \le C_- \le D_- \le V_0 \le C_+ \le D_+ \le V_+$, which combined with (3.11) implies

$$V_- \le C_- \le W_- \le D_- \le V_0 \le C_+ \le W_+ \le D_+ \le V_+. \tag{3.13}$$

The following Lemma describes some connections between V_0, V_+, V_- and η_+, η_-.

Lemma 3.19. *For any $\underline{t} = (t_+, t_-) \in \mathcal{D}$, we have*

$$|V_+(\underline{t}) - V_-(\underline{t})|/4 \le \operatorname{diam}(K(\underline{t}) \cup [v_-, v_+]) \le |V_+(\underline{t}) - V_-(\underline{t})|. \tag{3.14}$$

$$f_{K(\underline{t})}([V_0(\underline{t}), V_\nu(\underline{t})]) \subset \eta_\nu([0,t_\nu]) \cup [v_0, v_\nu], \quad \nu \in \{+,-\} \tag{3.15}$$

Here for $x,y \in \mathbb{R}$, the $[x,y]$ in (3.15) is the line segment connecting x with y, which is the same as $[y,x]$; and if any v_ν, $\nu \in \{0,+,-\}$, takes value w_σ^+ or w_σ^- for some $\sigma \in \{+,-\}$, then its appearance in (3.14,3.15) is understood as w_σ.

Proof. Fix $\underline{t} = (t_+, t_-) \in \mathcal{D}$. We write K for $K(\underline{t})$, $K_\pm$ for $K_\pm(t_\pm)$, $\widetilde{K}_\pm$ for $K_{\pm,t_\mp}(t_\pm)$, $\eta_\pm$ for $\eta_\pm([0,t_\pm])$, $\widetilde{\eta}_\pm$ for $\eta_{\pm,t_\mp}([0,t_\pm])$, and X for $X(\underline{t})$, $X \in \{V_0, V_+, V_-, C_+, C_-, D_+, D_-\}$.

Since g_K maps $\mathbb{C} \setminus (K^{\rm doub} \cup [v_-, v_+])$ conformally onto $\mathbb{C} \setminus [V_-, V_+]$, fixes ∞, and has derivative 1 at ∞, by Koebe's 1/4 theorem, we get (3.14). For (3.15) by symmetry we only need to prove the case $\nu = +$. From (3.13) we have $V_0 \le C_+ \le D_+ \le V_+$. By (3.12) and Lemma 3.5 we get $f_K([C_+, D_+]) \subset f_{K_-}(\widetilde{\eta}_+) = \eta_+$. It remains to show that $f_K((D_+, V_+]) \subset [w_0, v_+]$ and $f_K([V_0, C_+)) \subset [v_0, w_0]$. If $V_+ = D_+$, then $(D_+, V_+] = \emptyset$, and $f_K((D_+, V_+]) \subset [w_0, v_+]$ holds trivially. Suppose $V_+ > D_+$. By Lemma 3.12, $D_+ = \lim_{x \downarrow \max((\overline{K} \cap \mathbb{R}) \cup \{w_+\})} g_K(x)$, and $V_+ = g_K(v_+)$. So f_K maps $(D_+, V_+]$ onto $(\max((\overline{K} \cap \mathbb{R}) \cup \{w_+\}), v_+] \subset [w_+, v_+]$. If $V_0 = C_+$, then $[V_0, C_+) = \emptyset$, so $f_K([V_0, C_+)) \subset [v_0, w_0]$ holds trivially. If $V_0 < C_+$, by Lemma 3.12 (iii,iv), $\overline{K_+} \cap \overline{K_-} = \emptyset$, $v_0 \ne \overline{K_+}$, and $C_+ = \lim_{x \uparrow \min((\overline{K_+} \cap \mathbb{R}) \cup \{w_+\})} g_K(x)$. Now either $v_0 \notin \overline{K} \cup \{w_-^+\}$ and $V_0 = g_K(v_0)$, or $v_0 \in \overline{K_-} \cup \{w_-^+\}$ and $V_0 = D_-$. In the first case, we have $f_K([V_0, C_+)) \subset [v_0, \min((\overline{K_+} \cap \mathbb{R}) \cup \{w_+\})) \subset [v_0, w_+]$. In the second case, we have $f_K([V_0, C_+)) = [\max((\overline{K_-} \cap \mathbb{R}) \cup \{w_-\}), \min((\overline{K_+} \cap \mathbb{R}) \cup \{w_+\})) \subset [v_0, w_+]$. $\square$

3.3 Ensemble without intersections

We say that the commuting pair $(\eta_+,\eta_-;\mathcal{D})$ is disjoint, if $\eta_+([0,t_+])\cap\eta_-([0,t_-])=\emptyset$ for any $(t_+,t_-)\in\mathcal{D}$. If $\eta_\sigma(t)$, $0\le t<T_\sigma$, $\sigma\in\{+,-\}$, are two chordal Loewner curves that intersect $\mathbb{R}$ at a Lebesgue measure zero set, then we can obtain a disjoint commuting par $(\eta_+,\eta_-;\mathcal{D}^{\rm disj})$ by defining $\mathcal{D}^{\rm disj}=\{(t_+,t_-)\in[0,T_+)\times[0,T_-):\eta_+([0,t_+])\cap\eta_-([0,t_-])=\emptyset\}$.

In this subsection, we assume that $(\eta_+,\eta_-;\mathcal{D})$ is disjoint. From Lemma 3.11 we know that for any $\sigma\in\{+,-\}$ and $(t_+,t_-)\in\mathcal{D}$, $\mathrm{dist}(\widehat{w}_\sigma(t_\sigma),K_{-\sigma,t_\sigma}(t_\sigma))>0$. So $g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}$ is analytic at $\widehat{w}_\sigma(t_\sigma)=W_\sigma(t_\sigma\underline{e}_\sigma)$. By Lemma 3.13, $W_\sigma(t_+,t_-)=g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{w}_\sigma(t_\sigma))$. We further define $W_{\sigma,j}$, $j=1,2,3$, and $W_{\sigma,S}$ on $\mathcal{D}$ by

$$W_{\sigma,j}(t_+,t_-)=g^{(j)}_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{w}_\sigma(t_\sigma)),\quad W_{\sigma,S}=\frac{W_{\sigma,3}}{W_{\sigma,1}}-\frac{3}{2}\Big(\frac{W_{\sigma,2}}{W_{\sigma,1}}\Big)^2,\quad \sigma\in\{+,-\}. \tag{3.16}$$

They are all continuous on $\mathcal{D}$ because $(t_+,t_-,z)\mapsto g^{(j)}_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(z)$ is continuous by Lemma 3.5. Note that $W_{\sigma,S}(t_+,t_-)$ is the Schwarzian derivative of $g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}$ at $\widehat{w}_\sigma(t_\sigma)$.

Lemma 3.20. m *is continuously differentiable with* $\partial_\sigma\mathrm{m}=W_{\sigma,1}^2$, $\sigma\in\{+,-\}$.

Proof. This follows from a standard argument, which first appeared in [7, Lemma 2.8]. The statement for ensemble of chordal Loewner curves first appeared in [27, Formula (3.7)]. □

So for any $\sigma\in\{+,-\}$ and $t_{-\sigma}\in\mathcal{I}_{-\sigma}$, $K_{\sigma,t_{-\sigma}}(t_\sigma)$, $0\le t_\sigma<T^{\mathcal{D}}_\sigma(t_{-\sigma})$, are chordal Loewner hulls driven by $W_\sigma|^{-\sigma}_{t_{-\sigma}}$ with speed $(W_{\sigma,1}|^{-\sigma}_{t_{-\sigma}})^2$, and we get the differential equation for $g_{K_{\sigma,t_{-\sigma}}(t_\sigma)}$:

$$\partial_{t_\sigma}g_{K_{\sigma,t_{-\sigma}}(t_\sigma)}(z)=\frac{2(W_{\sigma,1}(t_+,t_-)^2}{g_{K_{\sigma,t_{-\sigma}}(t_\sigma)}(z)-W_\sigma(t_+,t_-)}, \tag{3.17}$$

which together with Lemmas 3.13 and 3.12 implies the differential equations for V_0,V_+,V_-:

$$\partial_\sigma V_\nu\overset{\rm ae}{=}\frac{2W_{\sigma,1}^2}{V_\nu-W_\sigma},\quad \nu\in\{0,+,-\}, \tag{3.18}$$

and the differential equations for W_σ, $W_{\sigma,1}$ and $W_{\sigma,S}$:

$$\partial_{-\sigma}W_\sigma=\frac{2W_{-\sigma,1}^2}{W_\sigma-W_{-\sigma}},\quad \frac{\partial_{-\sigma}W_{\sigma,1}}{W_{\sigma,1}}=\frac{-2W_{-\sigma,1}^2}{(W_+-W_-)^2},\quad \partial_{-\sigma}W_{\sigma,S}=-\frac{12W_{+,1}^2W_{-,1}^2}{(W_+-W_-)^4}. \tag{3.19}$$

Define F on $\mathcal{D}$ by

$$F(t_+,t_-)=\exp\Big(\int_0^{t_+}\int_0^{t_-}-\frac{12W_{+,1}(s_+,s_-)^2W_{-,1}(s_+,s_-)^2}{(W_+(s_+,s_-)-W_-(s_+,s_-))^4}ds_-ds_+\Big). \tag{3.20}$$

Then F is continuous and positive with $F(t_+,t_-)=1$ when $t_+\cdot t_-=0$. From (3.19) we get

$$\frac{\partial_\sigma F}{F}=W_{\sigma,S},\quad \sigma\in\{+,-\}. \tag{3.21}$$

By (3.13), $V_+ \ge W_+ \ge C_+ \ge V_0 \ge D_- \ge W_- \ge V_-$ on $\mathcal{D}$. For disjoint commuting pair, we further have $C_+ > D_-$. To see this, let $\underline{t} \in \mathcal{D}$. We may choose $v_0^1 < v_0^2 \in (w_-, w_+) \setminus K(\underline{t})$ and let V_0^j be the force point function started from v_0^j, $j = 1, 2$. Then we have $C_+(\underline{t}) \ge V_0^2(\underline{t}) > V_0^1(\underline{t}) \ge D_-(\underline{t})$ and $V_0^2(t_+, t_-) = g_{K(t_+,t_-)}(v_0^2) > g_{K(t_+,t_-)}(v_0^1) = V_0^1(t_+, t_-)$, where the strict inequality holds because $V_0^j(\underline{t}) = g_{K(\underline{t})}(v_j)$, $j = 1, 2$. By Lemma 3.12,

$$V_\sigma(t_+, t_-) = g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(V_\sigma(t_\sigma \underline{e}_\sigma)); \tag{3.22}$$

$$V_0(t_+, t_-) = g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(V_0(t_\sigma \underline{e}_\sigma)), \quad \text{if } v_0 \notin \overline{K_{-\sigma}(t_{-\sigma})}. \tag{3.23}$$

We emphasize that each "g functions" in the formulas is not a modified Loewner map, i.e., it is analytic at the point at which it is evaluated on the RHS.

Let $\underline{t} = (t_+, t_-) \in \mathcal{D}$. For $\sigma \in \{+, -\}$, differentiating (3.4) w.r.t. t_σ, letting $\widehat{z} = g_{K_\sigma(t_\sigma)}(z)$, and using Lemma 3.13 and (3.17,3.16) we get

$$\partial_{t_\sigma} g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z}) = \frac{2g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{w}_\sigma(t_\sigma))^2}{g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z}) - g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{w}_\sigma(t_\sigma))} - \frac{2g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z})}{\widehat{z} - \widehat{w}_\sigma(t_\sigma)}. \tag{3.24}$$

Letting $\mathbb{H} \setminus K_{-\sigma,t_\sigma}(t_{-\sigma}) \ni \widehat{z} \to \widehat{w}_\sigma(t_\sigma)$ and using the power series expansion of $g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}$ at $\widehat{w}_\sigma(t_\sigma)$, we get

$$\partial_{t_\sigma} g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z})|_{\widehat{z}=\widehat{w}_\sigma(t_\sigma)} = -3W_{\sigma,2}(\underline{t}), \quad \sigma \in \{+, -\}. \tag{3.25}$$

Differentiating (3.24) w.r.t. $\widehat{z}$ and letting $\widehat{z} \to \widehat{w}_\sigma(t_\sigma)$, we get

$$\frac{\partial_{t_\sigma} g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z})}{g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(\widehat{z})}\Big|_{\widehat{z}=\widehat{w}_\sigma(t_\sigma)} = \frac{1}{2}\Big(\frac{W_{\sigma,2}(\underline{t})}{W_{\sigma,1}(\underline{t})}\Big)^2 - \frac{4}{3}\frac{W_{\sigma,3}(\underline{t})}{W_{\sigma,1}(\underline{t})}, \quad \sigma \in \{+, -\}. \tag{3.26}$$

For $\sigma \in \{+, -\}$, define $W_{\sigma,N}$ on $\mathcal{D}$ by $W_{\sigma,N} = \frac{W_{\sigma,1}}{W_{\sigma,1}|_0^\sigma}$. Since $W_{\sigma,1}|_0^{-\sigma} \equiv 1$, we get $W_{\sigma,N}(t_+, t_-) = 1$ when $t_+ t_- = 0$. From (3.19) we get

$$\frac{\partial_\sigma W_{-\sigma,N}}{W_{-\sigma,N}} = \frac{-2W_{\sigma,1}^2}{(W_{-\sigma} - W_\sigma)^2}\,\partial t_\sigma - \frac{-2W_{\sigma,1}^2}{(W_{-\sigma} - W_\sigma)^2}\Big|_0^{-\sigma}\,\partial t_\sigma, \quad \sigma \in \{+, -\}. \tag{3.27}$$

We now define $V_{0,N}, V_{+,N}, V_{-,N}$ on $\mathcal{D}$ by

$$V_{\mu,N}(\underline{t}) = g'_{K_{-\mu,t_\mu}(t_{-\mu})}(V_\mu(t_\mu \underline{e}_\mu))/g'_{K_{-\mu}(t_{-\mu})}(v_\mu), \quad \mu \in \{+, -\};$$

$$V_{0,N}(\underline{t}) = g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(V_0(t_\sigma \underline{e}_\sigma))/g'_{K_{-\sigma}(t_{-\sigma})}(v_0), \quad \text{if } v_0 \notin \overline{K_{-\sigma}(t_{-\sigma})}, \quad \sigma \in \{+, -\}. \tag{3.28}$$

By (3.22-3.23), the RHS of these two formulas are well defined. There is no contradiction in (3.28) because when $v_0 \notin \overline{K_-(t_-)}$ and $v_0 \notin \overline{K_+(t_+)}$ both hold, for either $\sigma = +$ or $-$, the RHS of (3.28) equals $g'_{K(t_+,t_-)}(v_0)/(g'_{K_+(t_+)}(v_0) g'_{K_-(t_-)}(v_0))$ by (3.4).

Note that $V_{\nu,N}(t_+,t_-)=1$ if $t_+t_-=0$ for $\nu\in\{0,+,-\}$. From (3.22-3.23) and (3.4,3.17) we find that these functions satisfy the following differential equations on $\mathcal{D}$:

$$\frac{\partial_\sigma V_{\nu,N}}{V_{\nu,N}}=\frac{-2W_{\sigma,1}^2}{(V_\nu-W_\sigma)^2}\,\partial t_\sigma-\frac{-2W_{\sigma,1}^2}{(V_\nu-W_\sigma)^2}\Big|_0^{-\sigma}\partial t_\sigma,\quad \sigma\in\{+,-\},\quad \nu\in\{0,-\sigma\},\quad \text{if } v_\nu\not\in\overline{K_\sigma(t_\sigma)}. \tag{3.29}$$

We now define $E_{X,Y}$ on $\mathcal{D}$ for $X\ne Y\in\{W_+,W_-,V_0,V_+,V_-\}$ as follows. First, let

$$E_{X,Y}(t_+,t_-)=\frac{(X(t_+,t_-)-Y(t_+,t_-))(X(0,0)-Y(0,0))}{(X(t_+,0)-Y(t_+,0))(X(0,t_-)-Y(0,t_-))}, \tag{3.30}$$

if the denominator is not 0. If the denominator is 0, then since $V_+\ge W_+\ge V_0\ge W_-\ge V_-$ and $W_+>W_-$, there are two cases. Case 1. $\{X,Y\}\subset\{W_+,V_+,V_0\}$. Case 2. $\{X,Y\}\subset\{W_-,V_-,V_0\}$. By symmetry, we will only describe the definition of $E_{X,Y}$ in Case 1. If $X(t_+,0)=Y(t_+,0)$, by Lemmas 3.12 and 3.13, $X(t_+,\cdot)\equiv Y(t_+,\cdot)$. If $X(0,t_-)=Y(0,t_-)$, then we must have $X(\underline{0})=Y(\underline{0})$, and so $X(0,\cdot)\equiv Y(0,\cdot)$. For the definition of $E_{X,Y}$ in Case 1, we modify (3.30) by writing the RHS as $\frac{X(t_+,t_-)-Y(t_+,t_-)}{X(t_+,0)-Y(t_+,0)}:\frac{X(0,t_-)-Y(0,t_-)}{X(0,0)-Y(0,0)}$, replacing the numerator (before ":") by $g'_{K_{-,t_+}(t_-)}(X(t_+,0))$ when $X(t_+,0)=Y(t_+,0)$, replacing the denominator (after ":") by $g'_{K_-(t_-)}(X(0,0))$ when $X(0,t_-)=Y(0,t_-)$; and do both replacements when both $X(t_+,0)=Y(t_+,0)$ and $X(0,t_-)=Y(0,t_-)$. Then all $E_{X,Y}$ are continuous and positive on $\mathcal{D}$, and $E_{X,Y}(t_+,t_-)=1$ if $t_+\cdot t_-=0$. By (3.18,3.19), for $\sigma\in\{+,-\}$, if $X,Y\ne W_\sigma$, then

$$\frac{\partial_\sigma E_{X,Y}}{E_{X,Y}}\stackrel{\text{ae}}{=}\frac{-2W_{\sigma,1}^2}{(X-W_\sigma)(Y-W_\sigma)}\,\partial t_\sigma-\frac{-2W_{\sigma,1}^2}{(X-W_\sigma)(Y-W_\sigma)}\Big|_0^{-\sigma}\partial t_\sigma. \tag{3.31}$$

3.4 A time curve in the time region

In this subsection we do not assume that $(\eta_+,\eta_-;\mathcal{D})$ is disjoint. Let v_ν and V_ν, $\nu\in\{0,+,-\}$, be as before. We assume in this subsection that $v_+-v_0=v_0-v_-=:I>0$.

Lemma 3.21. *There exists a unique continuous and strictly increasing function $\underline{u}:[0,T^u)\to\mathcal{D}$, for some $T^u\in(0,\infty]$, with $\underline{u}(0)=\underline{0}$, such that for any $0\le t<T^u$ and $\sigma\in\{+,-\}$, $|V_\sigma(\underline{u}(t))-V_0(\underline{u}(t))|=e^{2t}|v_\sigma-v_0|$; and $\underline{u}$ can not be extended beyond T^u with such property.*

Sketch of the proof. We use an argument that is similar to Section 4 of [22]. Define Λ and Υ on $\mathcal{D}$ by $\Lambda=\frac12\log\frac{V_+-V_0}{V_0-V_-}$ and $\Upsilon=\frac12\log\frac{V_+-V_-}{v_+-v_-}$. By assumption, $\Lambda(\underline{0})=\Upsilon(\underline{0})=0$. Since $V_+\ge W_+\ge V_0\ge W_-\ge V_-$, by the definition of V_ν, Proposition 2.13 and Lemma 3.12, for $\sigma\in\{+,-\}$, $|V_\sigma-V_0|$ and $|V_\sigma-V_{-\sigma}|$ are strictly increasing in t_σ, and $|V_0-V_{-\sigma}|$ is strictly decreasing in t_σ. Thus, Λ is strictly increasing in t_+ and strictly decreasing in t_-, and Υ is strictly increasing in both t_+ and t_-. These monotone properties guarantee the existence and uniqueness of $\underline{u}:[0,T^u)\to\mathcal{D}$ with $\Lambda(\underline{u}(t))=0$ and $\Upsilon(\underline{u}(t))=t$ for all t. □

Lemma 3.22. *For any $t\in[0,T^u)$,*

$$e^{2t}|v_+-v_-|/128\le \operatorname{rad}_{v_0}(\eta_\sigma([0,u_\sigma(t)])\cup[v_0,v_\sigma])\le e^{2t}|v_+-v_-|,\quad \sigma\in\{+,-\}. \tag{3.32}$$

If $T^u < \infty$, then $\lim_{t \uparrow T^u} \underline{u}(t)$ converges to a point in $\partial\mathcal{D} \cap (0,\infty)^2$. If $\mathcal{D} = \mathbb{R}_+^2$, then $T^u = \infty$. If $T^u = \infty$, then $\operatorname{diam}(\eta_+) = \operatorname{diam}(\eta_-) = \infty$.

Proof. Let $t \in [0, T^u)$ and $L_\sigma = \operatorname{rad}_{v_0}(\eta_\sigma([0, u_\sigma(t)]) \cup [v_0, v_\sigma])$. From (3.14) and that $|V_+(\underline{u}(t)) - V_-(\underline{u}(t))| = e^{2t}|v_+ - v_-|$, we get $e^{2t}|v_+ - v_-|/8 \le \max\{L_+, L_-\} \le e^{2t}|v_+ - v_-|$. Since $V_+(\underline{u}(t)) - V_0(\underline{u}(t)) = V_0(\underline{u}(t)) - V_-(\underline{u}(t))$, from Lemma 3.19 and Beurling's estimate (applied to a Brownian motion started from ∞), we see that $\max\{L_+, L_-\} \le 16 \min\{L_+, L_-\}$. So we get (3.32). Since η_+ and η_- are parametrized by $\mathbb{H}$-capacity, for any $\sigma \in \{+,-\}$, $u_\sigma(t) = \operatorname{hcap}_2(\operatorname{Hull}(\eta_\sigma([0, u_\sigma(t)]))) \le L_\sigma^2 \le e^{4t}|v_+ - v_-|^2$. Suppose $T^u < \infty$. Then u_+ and u_- are bounded on $[0, T^u)$. Since $\underline{u}$ is increasing, $\lim_{t \uparrow T^u} \underline{u}(t)$ converges to a point in $(0,\infty)^2$, which must lie on $\partial\mathcal{D}$ because otherwise $\underline{u}$ may be further extended, which contradicts that $\underline{u}$ cannot be extended beyond T^u. If $\mathcal{D} = \mathbb{R}_+^2$, then $\partial\mathcal{D} \cap (0,\infty)^2 = \emptyset$, so $T^u = \infty$. Finally, if $T^u = \infty$, then by letting $t \uparrow \infty$ in (3.32), we get $\operatorname{diam}(\eta_\sigma) = \infty$, $\sigma \in \{+,-\}$. □

For a function X defined on $\mathcal{D}$ or a subset of $\mathcal{D}$, we define $X^u = X \circ \underline{u}$. From the definition of $\underline{u}$, we have $|V_+^u(t) - V_0^u(t)| = |V_-^u(t) - V_0^u(t)| = e^{2t} I$ for any $t \ge 0$. Let $R_\sigma = \frac{W_\sigma^u - V_0^u}{V_\sigma^u - V_0^u} \in [0,1]$, $\sigma \in \{+,-\}$, and $\underline{R} = (R_+, R_-)$. Let $e^{c\cdot}$ denote the function $t \mapsto e^{ct}$ for $c \in \mathbb{R}$.

Lemma 3.23. *Let $\mathcal{D}^{\mathrm{disj}} = \{(t_+, t_-) \in \mathcal{D} : \eta_+([0,t_+]) \cap \eta_-([0,t_-]) = \emptyset\}$. Let $T^u_{\mathrm{disj}} \in (0, T^u]$ be such that $\underline{u}(t) \in \mathcal{D}^{\mathrm{disj}}$ for $0 \le t < T^u_{\mathrm{disj}}$. Then $\underline{u}$ is continuously differentiable on $[0, T^u_{\mathrm{disj}})$, and*

$$(W_{\sigma,1}^u)^2 u_\sigma' = \frac{R_\sigma(1 - R_\sigma^2)}{R_+ + R_-} e^{4\cdot} I^2 \text{ on } [0, T^u_{\mathrm{disj}}), \quad \sigma \in \{+,-\}. \tag{3.33}$$

Proof. From (3.18) we find that the Λ and Υ introduced in the proof of Lemma 3.21 satisfy the following differential equations on $\mathcal{D}^{\mathrm{disj}}$:

$$\partial_\sigma \Lambda \overset{\mathrm{ae}}{=} \frac{(V_+ - V_-) W_{\sigma,1}^2}{\prod_{\nu \in \{0,+,-\}} (V_\nu^u - W_\sigma^u)} \text{ and } \partial_\sigma \Upsilon \overset{\mathrm{ae}}{=} \frac{-W_{\sigma,1}^2}{\prod_{\nu \in \{0,+,-\}} (V_\nu^u - W_\sigma^u)}.$$

From $\Lambda^u(t) = 0$ and $\Psi^u(t) = t$, we get

$$\sum_{\sigma \in \{+,-\}} \frac{(W_{\sigma,1}^u)^2 u_\sigma'}{\prod_{\nu \in \{0,+,-\}} (V_\nu^u - W_\sigma^u)} \equiv 0 \text{ and } \sum_{\sigma \in \{+,-\}} \frac{-(W_{\sigma,1}^u)^2 u_\sigma'}{\prod_{\nu \in \{+,-\}} (V_\nu^u - W_\sigma^u)} \equiv 1.$$

Solving the system of equations, we get $(W_{\sigma,1}^u)^2 u_\sigma' \overset{\mathrm{ae}}{=} (\prod_{\nu \in \{0,+,-\}} (V_\nu^u - W_\sigma^u))/(W_\sigma - W_{-\sigma})$, $\sigma \in \{+,-\}$. Using $V_\sigma^u - V_0^u = \sigma e^{2\cdot} I$ and $W_\sigma^u - V_0^u = R_\sigma (V_\sigma^u - V_0^u)$, we find that (3.33) holds with "$\overset{\mathrm{ae}}{=}$" in place of "$=$". Since $W_+ > W_-$ on $\mathcal{D}^{\mathrm{disj}}$, we get $R_+ + R_- > 0$ on $[0, T^u_{\mathrm{disj}})$. So the original (3.33) holds by the continuity of its RHS. □

Now suppose that η_+ and η_- are random curves, and $\mathcal{D}$ is a random region. Then $\underline{u}$ and T^u are also random. Suppose that there is an $\mathbb{R}_+^2$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathbb{R}_+^2}$ such that $\mathcal{D}$ is an

$(\mathcal{F}_{\underline{t}})$-stopping region, and V_0, V_+, V_- are all $(\mathcal{F}_{\underline{t}})$-adapted. Now we extend $\underline{u}$ to $\mathbb{R}_+$ such that if $T^u < \infty$, then $\underline{u}(s) = \lim_{t\uparrow T^u} \underline{u}(t)$ for $s \in [T^u, \infty)$. The following proposition has the same form as [22, Lemma 4.1], whose proof can also be used here.

Proposition 3.24. *For every $t \in \mathbb{R}_+$, the extended $\underline{u}(t)$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time.*

Since $\underline{u}$ is non-decreasing, we get a new filtration $(\mathcal{F}_{\underline{u}(t)})_{t\ge 0}$ by Propositions 2.26 and 3.24.

4 Commuting Pair of SLE$_\kappa(\underline{\rho})$ Curves

In this section, we apply the results from the previous section to study a pair of commuting SLE$_\kappa(\underline{\rho})$ curves, which arise as flow lines of a GFF with piecewise constant boundary data (cf. [11]). For a particular way of growing two curves simultaneously, we will obtain a two-dimensional diffusion process, derive its SDE, and calculate its transition density using orthogonal polynomials. The results of this section will be used in the next section to study 2-SLE$_\kappa$ and iSLE$_\kappa(\rho)$ that we are mostly interested in.

4.1 Martingale and domain Markov property

Throughout this section, we fix $\kappa, \rho_0, \rho_+, \rho_-$ such that $\kappa \in (0, 8)$, $\rho_+, \rho_- > \max\{-2, \frac{\kappa}{2} - 4\}$, $\rho_0 \ge \frac{\kappa}{4} - 2$ (see Remark 4.14), and $\rho_0 + \rho_\sigma \ge \frac{\kappa}{2} - 4$, $\sigma \in \{+, -\}$. Let $w_- < w_+ \in \mathbb{R}$. Let $v_+ \in (w_+, \infty) \cup \{w_+^+\}$, $v_- \in (-\infty, w_-) \cup \{w_-^-\}$, and $v_0 \in (w_-, w_+) \cup \{w_-^+, w_+^-\}$. Write $\underline{\rho}$ for (ρ_0, ρ_+, ρ_-). From ([11]) we know that there is a coupling of two chordal Loewner curves $\eta_+(t_+)$, $0 \le t_+ < \infty$, and $\eta_-(t_-)$, $0 \le t_- < \infty$, driven by $\widehat{w}_+$ and $\widehat{w}_-$ (with speed 1), respectively, such that

(A) For $\sigma \in \{+, -\}$, η_σ is a chordal SLE$_\kappa(2, \underline{\rho})$ curve in $\mathbb{H}$ started from w_σ with force points at $w_{-\sigma}$ and v_ν, $\nu \in \{0, +, -\}$. Here any v_ν equals $w_{-\sigma}^\pm$, then we treat it as $w_{-\sigma}$. Let $\widehat{w}_\sigma$ denote the driving function for η_σ. Let $\widehat{w}_{-\sigma}^\sigma, \widehat{v}_\nu^\sigma$, $\nu \in \{0, +, -\}$, denote the force point functions for $\eta_\pm$ started from $w_\mp, v_\nu$, $\nu \in \{0, +, -\}$, respectively.

(B) η_+ and η_- satisfy the following commutation relation: Let $\sigma \in \{+, -\}$. If $\tau_{-\sigma}$ is a finite stopping time w.r.t. the filtration $(\mathcal{F}_t^{-\sigma})_{t\ge 0}$ generated by $\eta_{-\sigma}$, then a.s. there is a chordal Loewner curve $\eta_{\sigma, t_{-\sigma}}(t)$, $0 \le t < \infty$, with some speed such that $\eta_\sigma = f_{K_{-\sigma}(\tau_{-\sigma})} \circ \eta_{\sigma, \tau_{-\sigma}}$, where $K_{-\sigma}(\tau_{-\sigma}) = \text{Hull}(\eta_{-\sigma}([0, \tau_{-\sigma}]))$. Moreover, the conditional law of the normalization of $\eta_{\sigma, \tau_{-\sigma}}$ given $\mathcal{F}_{\tau_{-\sigma}}^{-\sigma}$ is that of a chordal SLE$_\kappa(2, \underline{\rho})$ curve in $\mathbb{H}$ started from $\widehat{w}_\sigma^{-\sigma}(\tau_{-\sigma})$ with force points at $\widehat{w}_{-\sigma}(\tau_{-\sigma}), \widehat{v}_\nu^{-\sigma}(\tau_{-\sigma})$, $\nu \in \{0, +, -\}$, respectively.

In fact, one may construct η_+ and η_- as flow lines of a GFF on $\mathbb{H}$ with some piecewise boundary conditions (cf. [11]). The conditions on κ and $\underline{\rho}$ ensure that (i) there is no continuation threshold for either η_+ or η_-, and so η_+ and η_- both have lifetime ∞ and $\eta_\pm(t) \to \infty$ as $t \to \infty$; and (ii) η_+ does not hit $(-\infty, w_-]$, and η_- does not hit $[w_+, \infty)$. If $\rho_0 \ge \frac{\kappa}{2} - 2$, η_+ and η_- are disjoint; otherwise they do touch but not cross each other. We call the above (η_+, η_-) a commuting pair of chordal SLE$_\kappa(2, \underline{\rho})$ curves in $\mathbb{H}$ started from $(w_+, w_-; v_0, v_+, v_-)$.

We may take $\tau_{-\sigma}$ in (B) to be a deterministic time. So for each $t_{-\sigma}\in\mathbb{R}_+$, a.s. there is an SLE_κ-type curve $\eta_{\sigma,t_{-\sigma}}$ defined on $\mathbb{R}_+$ such that $\eta_\sigma=f_{K_{-\sigma}(t_{-\sigma})}\circ\eta_{\sigma,t_{-\sigma}}$. The conditions on κ and $\underline{\rho}$ implies that the Lebesgue measure of $\eta_{\sigma,t_{-\sigma}}\cap\mathbb{R}$ is 0. By setting $\mathcal{I}_+=\mathcal{I}_-=\mathbb{R}_+$, $\mathcal{I}_+^*=\mathcal{I}_-^*=\mathbb{Q}_+$, we can now say that a.s. for every $t_{-\sigma}\in\mathcal{I}_\mp^*$, there is a chordal Loewner curve $\eta_{\sigma,t_{-\sigma}}(t)$, $0\le t<\infty$, with some speed defined on $\mathbb{R}_+$ such that $\eta_\sigma=f_{K_{-\sigma}(t_{-\sigma})}\circ\eta_{\sigma,t_{-\sigma}}$ and the Lebesgue measure of $\eta_{\sigma,t_{-\sigma}}\cap\mathbb{R}$ is 0. This implies that a.s. η_+ and η_- satisfy the conditions in Definition 3.2 with $\mathcal{D}=\mathbb{R}_+^2$. So (η_+,η_-) is a.s. a commuting pair of chordal Loewner curves. Here we omit $\mathcal{D}$ when it is $\mathbb{R}_+^2$. Let K and m be the hull function and the capacity function, W_+,W_- be the driving functions, and V_0,V_+,V_- be the force point functions started from v_0,v_+,v_-, respectively. Then $\widehat{w}_\sigma=W_\sigma|_0^{-\sigma}$, $\widehat{w}^\sigma_{-\sigma}=W_{-\sigma}|_0^{-\sigma}$, and $\widehat{v}^\sigma_\nu=V_\nu|_0^{-\sigma}$, $\nu\in\{0,+,-\}$. For each $(\mathcal{F}^{-\sigma}_t)$-stopping time $\tau_{-\sigma}$, $\eta_{\sigma,\tau_{-\sigma}}$ is the chordal Loewner curve driven by $W_\sigma|^{-\sigma}_{\tau_{-\sigma}}$ with speed $d\,\mathrm{m}|^{-\sigma}_{\tau_{-\sigma}}$, and the force point functions are $W_{-\sigma}|^{-\sigma}_{\tau_{-\sigma}}$ and $V_\nu|^{-\sigma}_{\tau_{-\sigma}}$, $\nu\in\{0,+,-\}$.

Now we deal with the randomness. Let $(\mathcal{F}^\pm_t)_{t\ge0}$ be as in (B). Define the $\mathbb{R}_+^2$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ by $\mathcal{F}_{(t_+,t_-)}=\mathcal{F}^+_{t_+}\vee\mathcal{F}^-_{t_-}$. From (A) we know that, for $\sigma\in\{+,-\}$, there exists a standard $(\mathcal{F}^\sigma_t)$-Brownian motions B_σ such that the driving functions $\widehat{w}_\sigma$ satisfies the SDE

$$d\widehat{w}_\sigma\overset{\mathrm{ae}}{=}\sqrt{\kappa}dB_\sigma+\Big[\frac{2}{\widehat{w}_\sigma-\widehat{w}^\sigma_{-\sigma}}+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu}{\widehat{w}_\sigma-\widehat{v}^\sigma_\nu}\Big]dt_\sigma. \tag{4.1}$$

Here we note that B_+ and B_- are not independent.

Lemma 4.1. *Let (η_+,η_-) be a random commuting pair of chordal Loewner curves with driving functions W_+ and W_- started from w_+,w_-. Let V_ν be force point functions for this pair started from v_ν, $\nu\in\{0,+,-\}$, respectively. Define $U=W_++W_-+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu}{2}V_\nu$ on $\mathbb{R}_+^2$. Then η_+ and η_- is a commuting pair of chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curves in $\mathbb{H}$ started from $(w_+,w_-;v_0,v_+,v_-)$ if and only if U and $U^2-\kappa\,\mathrm{m}$ are $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingales.*

Proof. (i) The "only if" part. Fix $t_-\ge0$. From (B) and Proposition 2.14, conditional on $\mathcal{F}^-_{t_-}$, $U(\cdot,t_-)$ is a local martingale with quadratic variation $\langle U(\cdot,t_-)\rangle_t=\kappa\,\mathrm{m}(t,t_-)-\kappa\,\mathrm{m}(0,t_-)$. Since m is Lipschitz continuous, $U(\cdot,t_-)$ and $U(\cdot,t_-)^2-\kappa\,\mathrm{m}(\cdot,t_-)$ are true martingales. Symmetrically, $U(0,\cdot)$ and $U(0,\cdot)^2-\kappa\,\mathrm{m}(0,\cdot)$ are martingales. The two statements together imply that U and $U^2-\kappa\,\mathrm{m}$ are $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingales.

(ii) The "if" part. Fix a finite $(\mathcal{F}^-_{t_-})$-stopping time τ_-. By Proposition 2.31, $U(\cdot,\tau_-)$ and $U(\cdot,\tau_-)^2-\kappa\,\mathrm{m}(\cdot,\tau_-)$ are $(\mathcal{F}_{(t_+,\tau_-)})_{t_+\ge0}$-martingales. So $\langle U(\cdot,\tau_-)\rangle_t=\mathrm{m}(t,\tau_-)-\mathrm{m}(0,\tau_-)$. Using Proposition 2.14, we see that (B) holds for $\sigma=+$. Symmetrically, (B) also holds for $\sigma=-$. Setting $\tau_{-\sigma}\equiv0$, $\sigma\in\{+,-\}$, in (B) we find that (A) also holds. □

Remark 4.2. From the proof of Lemma 4.1 we see that Condition (B) is equivalent to a seemingly weaker condition, in which $\tau_{-\sigma}$ is only assumed to be a deterministic time.

Lemma 4.3. *Let $\underline{\tau}=(\tau_+,\tau_-)$ be an extended stopping time with respect to the right-continuous augmentation $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ of $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$. Let $\sigma\in\{+,-\}$. Then on the event that $\underline{\tau}\in\mathbb{R}_+^2$, a.s. $K(\underline{\tau}+t\underline{e}_\sigma)/K(\underline{\tau})$, $t\ge0$, are generated by a chordal Loewner curve $\widehat{\eta}_\sigma$ with some speed such that*

$\eta_\sigma(\tau_\sigma+\cdot)=f_{K(\underline{\tau})}\circ\widehat\eta_\sigma$. Let $h_\sigma(t)=\mathrm{m}(\underline{\tau}+t\underline{e}_\sigma)-\mathrm{m}(\underline{\tau})$ and $\widetilde\eta_\sigma=\widehat\eta_\sigma\circ h_\sigma^{-1}$. Then the conditional law of $\widetilde\eta_\sigma$ given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is that of a chordal SLE$_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau})$ with force points $W_{-\sigma}(\underline{\tau})$ and $V_\nu(\underline{\tau})$, $\nu\in\{0,+,-\}$, where if $V_\sigma(\underline{\tau})$ equals $W_\sigma(\underline{\tau})$, then as a force point it is treated as $W_\sigma(\underline{\tau})^\sigma$, and if $W_{-\sigma}(\underline{\tau})$, V_0, or $V_{-\sigma}$ equals $W_\sigma(\underline{\tau})$, then it is treated as $W_\sigma(\underline{\tau})^{-\sigma}$. Moreover, the driving function for $\widetilde\eta_\sigma$ is $W_\sigma(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$, and the force point functions are $W_{-\sigma}(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$ and $V_\nu(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$, $\nu\in\{0,+,-\}$.

Proof. Let U be as in Lemma 4.1. For $X\in\{\mathrm{m},W_+,W_-,V_0,V_+,V_-,U\}$, we write $X^{\underline{\tau},\sigma}(t)$ for $X(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$. We write $K^{\underline{\tau}}_\sigma(t)$ for $K(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)/K(\underline{\tau})$. By Lemma 3.7 and Proposition 2.8, when $\underline{\tau}$ is finite, $K(\underline{\tau}+t\underline{e}_\sigma)/K(\underline{\tau})$, $t\ge 0$, are chordal Loewner hulls driven by $W_\sigma(\underline{\tau}+t\underline{e}_\sigma)$, $t\ge0$, with speed $d\,\mathrm{m}(\underline{\tau}+t\underline{e}_\sigma)$. So $K^{\underline{\tau}}_\sigma(t)$, $t\ge0$, are chordal Loewner hulls driven by $W^{\underline{\tau},\sigma}_\sigma(t)$, $t\ge0$ (with speed $d\,\mathrm{m}(\underline{\tau}+h_\sigma^{-1}(t))=1$). By Lemmas 3.12 and 3.13, and Propositions 2.12 and 2.13, we find that, if $X\in\{W_{-\sigma},V_0,V_+,V_-\}$, then

$$X^{\underline{\tau},\sigma}(t)=g^{W_\sigma(\underline{\tau})}_{K^{\underline{\tau}}_\sigma(t)}(X(\underline{\tau})),\quad \frac{d}{dt}X^{\underline{\tau},\sigma}(t)\overset{\mathrm{ae}}{=}\frac{2}{X^{\underline{\tau},\sigma}(t)-W^{\underline{\tau},\sigma}_\sigma(t)},\quad t\ge 0. \tag{4.2}$$

We first assume that $\underline{\tau}$ is bounded. Then for any $t\ge0$, $\underline{\tau}+t\underline{e}_\sigma$ is a bounded stopping time. By Lemma 4.1, Propositions 2.31 and 2.30, if X is U or $U^2-\kappa\,\mathrm{m}$, then $X(\underline{\tau}+t\underline{e}_\sigma)$, $t\ge0$, is a continuous $(\mathcal{F}^{(+)}_{\underline{\tau}+t\underline{e}_\sigma})_{t\ge0}$-martingale. Since $(h_\sigma(t))$ is $(\mathcal{F}^{(+)}_{\underline{\tau}+t\underline{e}_\sigma})$-adapted, for each $t\ge0$, $h_\sigma^{-1}(t)$ is an $(\mathcal{F}^{(+)}_{\underline{\tau}+t\underline{e}_\sigma})$-stopping time. Since $\mathrm{m}^{\underline{\tau},\sigma}(t)=t$, we see that $U^{\underline{\tau},\sigma}(t)$ and $U^{\underline{\tau},\sigma}(t)^2-\kappa t$, $t\ge0$, are continuous $(\mathcal{F}^{(+)}_{\underline{\tau}+\cdot\underline{e}_\sigma}|_{h_\sigma^{-1}(t)})_{t\ge0}$-local martingales. By Levy's characterization of Brownian motion, we see that $(U^{\underline{\tau},\sigma}(t)-U(\underline{\tau}))/\sqrt\kappa$ is a Brownian motion, say $B^{\underline{\tau}}_\sigma(t)$, independent of $\mathcal{F}^{(+)}_{\underline{\tau}}$. By the definition of U and (4.2), $W^{\underline{\tau},\sigma}_\sigma(t)$ satisfies the SDE:

$$dW^{\underline{\tau},\sigma}_\sigma(t)=\sqrt\kappa dB^{\underline{\tau}}_\sigma(t)+\frac{2dt}{W^{\underline{\tau},\sigma}_\sigma(t)-g^{W_\sigma(\underline{\tau})}_{K^{\underline{\tau}}_\sigma(t)}(W_{-\sigma}(\underline{\tau}))}+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu dt}{W^{\underline{\tau},\sigma}_\sigma(t)-g^{W_\sigma(\underline{\tau})}_{K^{\underline{\tau}}_\sigma(t)}(V_\nu(\underline{\tau}))}.$$

Since $W^{\underline{\tau},\sigma}_\sigma(0)=W_\sigma(\underline{\tau})$ and $K^{\underline{\tau}}_\sigma(t)$, $t\ge0$, are chordal Loewner hulls driven by $W^{\underline{\tau},\sigma}_\sigma(t)$, we conclude that $K^{\underline{\tau}}_\sigma(t)$, $t\ge0$, are a.s. generated by a chordal Loewner curve, say $\widetilde\eta_\sigma$, whose conditional law given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is that of a chordal SLE$_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau})$ with force points $W_{-\sigma}(\underline{\tau})$ and $V_\nu(\underline{\tau})$, $\nu\in\{0,+,-\}$. We also easily see that the driving function for $\widetilde\eta_\sigma$ is $W_\sigma(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$, and the force point functions are $W_{-\sigma}(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$ and $V_\nu(\underline{\tau}+h_\sigma^{-1}(t)\underline{e}_\sigma)$, $\nu\in\{0,+,-\}$. Let $\widehat\eta_\sigma=\widetilde\eta_\sigma\circ h_\sigma$. Then $\widehat\eta_\sigma$ is a chordal Loewner curve with some speed, which generates $K(\underline{\tau}+t\underline{e}_\sigma)/K(\underline{\tau})$, $t\ge0$. Since $K(\underline{\tau}+t\underline{e}_\sigma)$ is the $\mathbb{H}$-hull generated by $K(\underline{\tau})$ and $\eta_\sigma([\tau_\sigma,\tau_\sigma+t])$, we get $\eta_\sigma(\tau_\sigma+\cdot)=f_{K(\underline{\tau})}\circ\widehat\eta_\sigma$.

We now consider the general case. We use Proposition 2.28 to do localization. Fix $\underline{N}=(N_+,N_-)\in\mathbb{R}_+^2$. Then $\underline{\tau}^{\underline{N}}$ is a bounded $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time. By the last paragraph, $K(\underline{\tau}^{\underline{N}}+t\underline{e}_\sigma)/K(\underline{\tau}^{\underline{N}})$, $t\ge0$, are a.s. generated by a chordal Loewner curve, say $\widehat\eta^{\underline{N}}_\sigma$, with some speed such that $\eta_\sigma(\tau^{\underline{N}}_\sigma+\cdot)=f_{K(\underline{\tau}^{\underline{N}})}\circ\widehat\eta^{\underline{N}}_\sigma$. Let $h^{\underline{N}}_\sigma(t)=\mathrm{m}(\underline{\tau}^{\underline{N}}+t\underline{e}_\sigma)-\mathrm{m}(\underline{\tau}^{\underline{N}})$ and $\widetilde\eta^{\underline{N}}_\sigma=\widehat\eta^{\underline{N}}_\sigma\circ(h^{\underline{N}}_\sigma)^{-1}$. Then the conditional law of $\widetilde\eta^{\underline{N}}_\sigma$ given $\mathcal{F}^{(+)}_{\underline{\tau}^{\underline{N}}}$ is that of a chordal SLE$_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau}^{\underline{N}})$ with force points $W_{-\sigma}(\underline{\tau}^{\underline{N}})$ and $V_\nu(\underline{\tau}^{\underline{N}})$, $\nu \in \{0,+,-\}$. On the event $\{\underline{\tau} \le \underline{N}\}$, since $\underline{\tau}^{\underline{N}} = \underline{\tau}$ and $\mathcal{F}^{(+)}_{\underline{\tau}^{\underline{N}}}$ agrees with $\mathcal{F}^{(+)}_{\underline{\tau}}$, we see that $K(\underline{\tau}+t\underline{e}_\sigma)/K(\underline{\tau})$, $t \ge 0$, are a.s. generated by $\widehat{\eta}^{\underline{N}}_\sigma$, $\eta_\sigma(\tau^{\underline{N}}_\sigma + \cdot) = f_{K(\underline{\tau})} \circ \widehat{\eta}^{\underline{N}}_\sigma$, $\widetilde{\eta}^{\underline{N}}_\sigma = \widehat{\eta}^{\underline{N}}_\sigma \circ h_\sigma^{-1}$, and the conditional law of $\widetilde{\eta}^{\underline{N}}_\sigma$ given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is that of a chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau})$ with force points $W_{-\sigma}(\underline{\tau})$ and $V_\nu(\underline{\tau})$, $\nu \in \{0,+,-\}$. This means that $\widetilde{\eta}^{\underline{N}}_\sigma$ and $\widehat{\eta}^{\underline{N}}_\sigma$ are the curves $\widetilde{\eta}_\sigma$ and $\widehat{\eta}_\sigma$ we want on the event $\{\underline{\tau} \le \underline{N}\}$. We then may complete the proof by letting $N_+, N_- \to \infty$. □

The following lemma describes the DMP of a commuting pair of chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curves.

Lemma 4.4. *Let $w_- < w_+$, $v_0 \in (w_-, w_+) \cup \{w_-^+, w_+^-\}$, $v_+ \in (w_+, \infty) \cup \{w_+^+\}$ and $v_- \in (-\infty, w_-) \cup \{w_-^-\}$. Suppose (η_+, η_-) is a commuting pair of chordal $SLE_\kappa(2,\underline{\rho})$ curves started from $(w_+, w_-; v_0, v_+, v_-)$. Let $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ be the right-continuous augmentation of the $\mathbb{R}_2$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ generated by η_+ and η_-. Let $\underline{\tau} = (\tau_+, \tau_-)$ be an extended $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time. Then on the event that $\underline{\tau} \in \mathbb{R}_+^2$ and $W_+(\underline{\tau}) > W_-(\underline{\tau})$, there a.s. exists a random commuting pair of chordal Loewner curves $(\widehat{\eta}_+, \widehat{\eta}_-)$ with some speeds, which up to a conformal map agrees with the part of (η_+, η_-) after $\underline{\tau}$. Moreover, the conditional law of the normalization of $(\widehat{\eta}_+, \widehat{\eta}_-)$ given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is that of a commuting pair of chordal $SLE_\kappa(2,\underline{\rho})$ curves started from $(W_+, W_-; V_0, V_+, V_-)|_{\underline{\tau}}$, where if $V_\sigma(\underline{\tau}) = W_\sigma(\underline{\tau})$ for some $\sigma \in \{+,-\}$, then $V_\sigma(\underline{\tau})$ is treated as $W_\sigma(\underline{\tau})^\sigma$, and if $V_0(\underline{\tau}) = W_\sigma(\underline{\tau})$ for some $\sigma \in \{+,-\}$, then $V_0(\underline{\tau})$ is treated as $W_\sigma(\underline{\tau})^{-\sigma}$.*

Proof. Let $\sigma \in \{+,-\}$. Assume that the event that $\underline{\tau} \in \mathbb{R}_+^2$ and $W_+(\underline{\tau}) > W_-(\underline{\tau})$ happens. Applying Lemma 4.3, we get a pair of chordal Loewner curves with speeds $\widehat{\eta}_+$ and $\widehat{\eta}_-$ such that for $\sigma \in \{+,-\}$, $\eta_\sigma(\tau_\sigma + \cdot) = f_{K(\underline{\tau})} \circ \widehat{\eta}_\sigma$. Let $h_\sigma(t) = \mathrm{m}(\underline{\tau} + t\underline{e}_\sigma) - \mathrm{m}(\underline{\tau})$ and $\widetilde{\eta}_\sigma = \widehat{\eta}_\sigma \circ h_\sigma^{-1}$. Then $\widetilde{\eta}_\sigma$ is the normalization of $\widehat{\eta}_\sigma$, and the conditional law of $\widetilde{\eta}_\sigma$ given $(\mathcal{F}^{(+)}_{\underline{\tau}})$ is that of a chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau})$ with force points $W_{-\sigma}(\underline{\tau})$ and $V_\nu(\underline{\tau})$, $\nu \in \{0,+,-\}$. Moreover, the driving function for $\widetilde{\eta}_\sigma$ is $W_\sigma(\underline{\tau} + h_\sigma^{-1}(t)\underline{e}_\sigma)$, and the force point functions are $W_{-\sigma}(\underline{\tau} + h_\sigma^{-1}(t)\underline{e}_\sigma)$ $V_\nu(\underline{\tau} + h_\sigma^{-1}(t)\underline{e}_\sigma)$, $\nu \in \{0,+,-\}$.

Let $\widehat{K}(t_+, t_-) = \mathrm{Hull}(\widehat{\eta}_+([0,t_+]) \cup \widehat{\eta}_-([0,t_-]))$, $(t_+, t_-) \in \mathbb{R}_+^2$. Then from $\eta_\sigma(\tau_\sigma + \cdot) = f_{K(\underline{\tau})} \circ \widehat{\eta}_\sigma$, $\sigma \in \{+,-\}$, we get $\widehat{K}(\underline{t}) = K(\underline{\tau} + \underline{t})/K(\underline{\tau})$, $\underline{t} \in \mathbb{R}_+^2$. By (2.1), for any $\sigma \in \{+,-\}$,

$$\widehat{K}(t_{-\sigma}\underline{e}_{-\sigma} + t\underline{e}_\sigma)/\widehat{K}(t_{-\sigma}\underline{e}_{-\sigma}) = K(\underline{\tau} + t_{-\sigma}\underline{e}_{-\sigma} + t\underline{e}_\sigma)/K(\underline{\tau} + t_{-\sigma}\underline{e}_{-\sigma}), \quad t, t_{-\sigma} \ge 0.$$

Applying Lemma 4.3 to the stopping time $\underline{\tau} + t_{-\sigma}\underline{e}_{-\sigma}$, we find that a.s. for any $t_{-\sigma} \in \mathbb{Q}_+$, $\widehat{K}(t_{-\sigma}\underline{e}_{-\sigma} + t\underline{e}_\sigma)/\widehat{K}(t_{-\sigma}\underline{e}_{-\sigma})$, $t \ge 0$, are generated by a chordal Loewner curve with some speed, which intersects $\mathbb{R}$ at a Lebesgue measure zero set. So $(\widehat{\eta}_+, \widehat{\eta}_-)$ is a.s. a commuting pair of chordal Loewner curves with some speeds.

Now $(\widetilde{\eta}_+, \widetilde{\eta}_-)$ is the normalization of $(\widehat{\eta}_+, \widehat{\eta}_-)$. We need to show that the conditional law of $(\widetilde{\eta}_+, \widetilde{\eta}_-)$ given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is that of a commuting pair of chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curves started from $(W_+, W_-; V_0, V_+, V_-)|_{\underline{\tau}}$. Let $\widetilde{K}_\sigma(t) = \mathrm{Hull}(\eta_\sigma([0,t]))$, $t \ge 0$, $\sigma \in \{+,-\}$, and $\widetilde{K}(t_+, t_-) = \mathrm{Hull}(K_+(t_+) \cup K_-(t_-))$, $(t_+, t_-) \in \mathbb{R}_+^2$. For $\sigma \in \{+,-\}$ and $t \ge 0$, let $\widetilde{F}^\sigma_t$ denote the σ-algebra generated by $\mathcal{F}^{(+)}_{\underline{\tau}}$ and $\widetilde{\eta}_\sigma(s)$, $s\le t$. It suffices to show that, for any $\sigma\in\{+,-\}$ and $t_{-\sigma}\ge 0$, $\widetilde{K}(t_{-\sigma}\underline{e}_{-\sigma}+t\underline{e}_\sigma)/\widetilde{K}(t_{-\sigma}\underline{e}_{-\sigma})$, $t\ge 0$, are a.s. generated by a chordal Loewner curves with some speed, whose normalization conditionally on $\widetilde{\mathcal{F}}^{-\sigma}_{t_{-\sigma}}$ has the law of a chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma})$ with force points located at $W_{-\sigma}$ and V_ν, $\nu\in\{0,+,-\}$, all valued at $\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}$.

It is easy to see that, for any $\underline{t}\in\mathbb{R}_+^2$, $\underline{\tau}+h_\oplus^{-1}(\underline{t})$ is an extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time. To see this, note that, for any $\underline{a}=(a_+,a_-)\in\mathbb{R}_+^2$,

$$\{\underline{\tau}+h_\oplus^{-1}(\underline{t})\le\underline{a}\}=\{\underline{\tau}\le\underline{a}\}\cap\{\mathrm{m}(a_+,\tau_-)-\mathrm{m}(\underline{\tau})\ge t_+\}\cap\{\mathrm{m}(\tau_+,a_-)-\mathrm{m}(\underline{\tau})\ge t_-\}\in\mathcal{F}^{(+)}_{\underline{a}}.$$

Applying Lemma 4.3 to $\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}$, we find that the family of $\mathbb{H}$-hulls

$$\widetilde{K}(t_{-\sigma}\underline{e}_{-\sigma}+t\underline{e}_\sigma)/\widetilde{K}(t_{-\sigma}\underline{e}_{-\sigma})=K(\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}+h_\sigma^{-1}(t)\underline{e}_\sigma)/K(\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}),\quad t\ge 0,$$

are generated by a chordal Loewner curve with some speed, whose normalization conditionally on $\mathcal{F}^{(+)}_{\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}}$ is that of a chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curve in $\mathbb{H}$ started from $W_\sigma(\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma})$ with force points located at $W_{-\sigma}$ and V_ν, $\nu\in\{0,+,-\}$, all valued at $\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}$.

Note that the above marked points are $\widetilde{\mathcal{F}}^{-\sigma}_{t_{-\sigma}}$-measurable since they are determined by $W_\pm(\underline{\tau})$, $V_\nu(\underline{\tau})$, $\nu\in\{0,+,-\}$, and $\widetilde{\eta}_{-\sigma}(t)$, $0\le t\le t_{-\sigma}$. To end the proof, it suffices to show that $\widetilde{\mathcal{F}}^{-\sigma}_{t_{-\sigma}}\subset\mathcal{F}^{(+)}_{\underline{\tau}+h^{-1}_{-\sigma}(t_{-\sigma})\underline{e}_{-\sigma}}$. By symmetry, we only need to work on the case $\sigma=+$.

For $t\ge 0$, let $\widehat{\mathcal{F}}^-_t$ be the σ-algebra generated by $\mathcal{F}^{(+)}_{\underline{\tau}}$ and $\widehat{\eta}_-(s)$, $s\le t$. Then $h_-^{-1}(t)$ are $(\widehat{\mathcal{F}}^-_t)$-stopping times for all $t\ge 0$. Since $\widetilde{\eta}_\pm=\widehat{\eta}_\pm\circ h_\pm^{-1}$, we get $\widetilde{\mathcal{F}}^-_{t_-}\subset\widehat{\mathcal{F}}^-_{h_-^{-1}(t_-)}$. Now it suffices to show that $\widehat{\mathcal{F}}^-_{h_-^{-1}(t_-)}\subset\mathcal{F}^{(+)}_{(\tau_+,\tau_-+h_-^{-1}(t_-))}$. Since $\underline{\tau}\le\underline{\tau}+h_-^{-1}(t_-)\underline{e}_-$, we have $\mathcal{F}^{(+)}_{\underline{\tau}}\subset\mathcal{F}^{(+)}_{\underline{\tau}+h_-^{-1}(t_-)\underline{e}_-}$. Since $\eta_-(\tau_-+t)=f_{K(\underline{\tau})}\circ\widehat{\eta}_-$, by continuity we can recover $\widehat{\eta}_-(s)$, $0\le s\le t$, using $\eta_-(s)$, $\tau_-\le s\le\tau_-+t$, and $K(\underline{\tau})$. Thus, for any $s_-\ge 0$, $\widehat{\mathcal{F}}^-_{s_-}\subset\mathcal{F}^{(+)}_{(\tau_+,\tau_-+s_-)}$. Let $A\in\widehat{\mathcal{F}}^-_{h_-^{-1}(t_-)}$. Fix $\underline{a}=(a_+,a_-)\in\mathbb{R}_+^2$. Then

$$A\cap\{(\tau_+,\tau_-+h_-^{-1}(t_-))<\underline{a}\}=\bigcup_{p\in\mathbb{Q}_+\cap(0,a_-)}(A\cap\{h_-^{-1}(t_-)\le p\}\cap\{(\tau_+,\tau_-+p)\le\underline{a}\})\in\mathcal{F}^{(+)}_{\underline{a}}.$$

where we used the fact that $A\cap\{h_-^{-1}(t_-)\le p\}\in\widehat{\mathcal{F}}^-_p\subset\mathcal{F}^{(+)}_{(\tau_+,\tau_-+p)}$ because $A\in\widehat{\mathcal{F}}^-_{h_-^{-1}(t_-)}$. Since this holds for any $\underline{a}\in\mathbb{R}_+^2$, by Proposition 2.24, $A\in\mathcal{F}^{(+)}_{(\tau_+,\tau_-+h_-^{-1}(t_-))}$. So we get $\widehat{\mathcal{F}}^-_{h_-^{-1}(t_-)}\subset\mathcal{F}^{(+)}_{(\tau_+,\tau_-+h_-^{-1}(t_-))}$, as desired. □

4.2 Relation with the independent coupling

Let $\mathbb{P}^{\underline{\rho}}$ denote the joint law of the driving functions of a commuting pair of chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curves in $\mathbb{H}$ started from $(w_+,w_-;v_0,v_+,v_-)$. When we want to emphasize the dependence of

w_+, w_-, v_0, v_+, v_-, we write it as $\mathbb{P}^{(\rho_0,\rho_+,\rho_-)}_{(w_+,w_-;v_0,v_+,v_-)}$. If $\rho_0 = 0$, i.e., v_0 does not play the role of a force point, we then write the measure as $\mathbb{P}^{(\rho_+,\rho_-)}_{(w_+,w_-;v_+,v_-)}$ or $\mathbb{P}^{(\rho_+,\rho_-)}$. If $\rho_0 = \rho_- = 0$, we then write the measure as $\mathbb{P}^{(\rho_+)}_{(w_+,w_-;v_+)}$ or $\mathbb{P}^{(\rho_+)}$.

The $\mathbb{P}^{\underline{\rho}}$ is a probability measure on Σ^2, where $\Sigma := \bigcup_{0<T\le\infty} C([0,T),\mathbb{R})$ was defined in [23, Section 2]. A random element in Σ is a continuous stochastic process with random lifetime. The space Σ^2 is equipped with an $\mathbb{R}_+^2$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ defined by $\mathcal{F}_{(t_+,t_-)} = \mathcal{F}^+_{t_+} \vee \mathcal{F}^-_{t_-}$, where $(\mathcal{F}^+_t)_{t\ge0}$ and $(\mathcal{F}^-_t)_{t\ge0}$ are the filtrations generated by the first function and the second function, respectively. A probability measure on Σ^2 is understood as the joint law of two stochastic processes with random lifetimes.

Let $\mathbb{P}^{\underline{\rho}}_+$ and $\mathbb{P}^{\underline{\rho}}_-$ denote the marginal laws of $\mathbb{P}^{\underline{\rho}}$ on Σ. Then $\mathbb{P}^{\underline{\rho}}$ is different from the product measure $\mathbb{P}^{\underline{\rho}}_i := \mathbb{P}^{\underline{\rho}}_+ \times \mathbb{P}^{\underline{\rho}}_-$. We will derive some relation between $\mathbb{P}^{\underline{\rho}}$ and $\mathbb{P}^{\underline{\rho}}_i$. Suppose now that $(\widehat{w}_+, \widehat{w}_-)$ follows the law $\mathbb{P}^{\underline{\rho}}_i$ instead of $\mathbb{P}^{\underline{\rho}}$. Then (4.1) holds for two independent Brownian motions B_+ and B_-, and η_+ and η_- are independent. Let $\mathcal{D}^{\rm disj}$ be as defined in Section 3.3 for such (η_+,η_-). Then $(\eta_+,\eta_-;\mathcal{D}^{\rm disj})$ is a disjoint commuting pair of chordal Loewner curves. Since B_+ and B_- are independent, for any $\sigma \in \{+,-\}$ and any finite $(\mathcal{F}^{-\sigma}_t)$-stopping time $t_{-\sigma}$, B_σ is a Brownian motion w.r.t. the filtration $(\mathcal{F}^\sigma_t \vee \mathcal{F}^{-\sigma}_{t_{-\sigma}})_{t\ge0}$, and we may view (4.1) as an $(\mathcal{F}^\sigma_t \vee \mathcal{F}^{-\sigma}_{t_{-\sigma}})_{t\ge0}$-adapted SDE. We will repeatedly apply Itô's formula (cf. [15]) in this subsection, where $\sigma \in \{+,-\}$, the variable $t_{-\sigma}$ of all functions is a fixed finite $(\mathcal{F}^{-\sigma}_t)$-stopping time, and all SDE are $(\mathcal{F}^\sigma_{t_\sigma} \vee \mathcal{F}^{-\sigma}_{t_{-\sigma}})_{t_\sigma\ge0}$-adapted in t_σ.

By (3.25) we get the SDE for W_σ (in t_σ):

$$\partial_\sigma W_\sigma = W_{\sigma,1}\partial\widehat{w}_\sigma + \Big(\frac{\kappa}{2} - 3\Big) W_{\sigma,2}\partial t_\sigma. \tag{4.3}$$

We will use the boundary scaling exponent b and central charge c defined by $\mathrm{b} = \frac{6-\kappa}{2\kappa}$ and $\mathrm{c} = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$. By (3.26) we get the SDE for $W^{\rm b}_{\sigma,N}$:

$$\frac{\partial_\sigma W^{\rm b}_{\sigma,N}}{W^{\rm b}_{\sigma,N}} = \mathrm{b}\,\frac{W_{\sigma,2}}{W_{\sigma,1}}\partial\widehat{w}_\sigma + \frac{\mathrm{c}}{6} W_{\sigma,S}\partial t_\sigma. \tag{4.4}$$

Next, we derive the SDE for $\partial_\sigma E_{W_\sigma,Y}$ for $Y \in \{W_{-\sigma}, V_0, V_+, V_-\}$. Note that $E_{W_\sigma,Y}(t_+,t_-)$ can be expressed as a product of a function in $t_{-\sigma}$ and a function $f(\underline{t}, W_\sigma(t_\sigma \underline{e}_\sigma), Y(t_\sigma\underline{e}_\sigma))$, where

$$f(\underline{t}, w, y) := \begin{cases} (g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(w) - g_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(y))/(w-y), & w \ne y; \\ g'_{K_{-\sigma,t_\sigma}(t_{-\sigma})}(w), & w = y. \end{cases} \tag{4.5}$$

Using (3.18,4.3) and (3.22-3.23) we see that $E_{W_\sigma,Y}$ satisfies the SDE

$$\frac{\partial_\sigma E_{W_\sigma,Y}}{E_{W_\sigma,Y}} \overset{\rm ae}{=} \Big[\frac{W_{\sigma,1}}{W_\sigma - Y} - \frac{W_{\sigma,1}}{W_\sigma - Y}\Big|_0^{-\sigma}\Big] d\widehat{w}_\sigma + \Big[\frac{2W_{\sigma,1}^2}{(W_\sigma - Y)^2} - \frac{2W_{\sigma,1}^2}{(W_\sigma - Y)^2}\Big|_0^{-\sigma}\Big]\partial t_\sigma$$
$$- \frac{\kappa}{W_\sigma - Y}\Big|_0^{-\sigma} \cdot \Big[\frac{W_{\sigma,1}}{W_\sigma - Y} - \frac{W_{\sigma,1}}{W_\sigma - Y}\Big|_0^{-\sigma}\Big]\partial t_\sigma + \Big(\frac{\kappa}{2} - 3\Big)\frac{W_{\sigma,2}}{W_\sigma - Y}\partial t_\sigma. \tag{4.6}$$

Define a positive continuous function $M_{i\underline{\rho}\to\underline{\rho}}$ on $\mathcal{D}^{\rm disj}$ by

$$M_{i\underline{\rho}\to\underline{\rho}}=F^{-\frac{c}{6}}\cdot E_{W_+,W_-}^{\frac{2}{\kappa}}\cdot\prod_{\sigma\in\{+,-\}}W_{\sigma,N}^{\rm b}\cdot\prod_{\nu\in\{0,+,-\}}V_{\nu,N}^{\frac{\rho_\nu(\rho_\nu+4-\kappa)}{4\kappa}}\cdot$$

$$\cdot\prod_{\sigma\in\{+,-\}}\Big[\prod_{\nu\in\{0,+,-\}}E_{W_\sigma,V_\nu}^{\frac{\rho_\nu}{\kappa}}\Big]\cdot\prod_{\nu_1<\nu_2\in\{0,+,-\}}E_{V_{\nu_1},V_{\nu_2}}^{\frac{\rho_{\nu_1}\rho_{\nu_2}}{2\kappa}}. \tag{4.7}$$

Then $M_{i\underline{\rho}\to\underline{\rho}}(t_+,t_-)=1$ if $t_+\cdot t_-=0$. Combining (4.1,3.21,4.6,3.27,4.4,3.29,3.31) and using the facts that $\widehat{w}_\sigma=W_\sigma|_0^{-\sigma}$, $\widehat{w}^\sigma_{-\sigma}=W_{-\sigma}|_0^{-\sigma}$ and $\widehat{v}^\sigma_\nu=V_\nu|_0^{-\sigma}$, we get the SDE for $M_{i\underline{\rho}\to\underline{\rho}}$ in t_σ when $t_{-\sigma}$ is a fixed $(\mathcal{F}^{-\sigma}_t)$-stopping time:

$$\frac{\partial_\sigma M_{i\underline{\rho}\to\underline{\rho}}}{M_{i\underline{\rho}\to\underline{\rho}}}={\rm b}\,\frac{W_{\sigma,2}}{W_{\sigma,1}}\partial B_\sigma-\Big[\frac{2}{\widehat{w}_\sigma-\widehat{w}^\sigma_{-\sigma}}+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu}{\widehat{w}_\sigma-\widehat{v}^\sigma_\nu}\Big]\frac{\partial B_\sigma}{\sqrt{\kappa}}+$$

$$+\Big[\frac{2W_{\sigma,1}}{W_\sigma-W_{-\sigma}}+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu W_{\sigma,1}}{W_\sigma-V_\nu}\Big]\frac{\partial B_\sigma}{\sqrt{\kappa}}. \tag{4.8}$$

This means that $M_{i\underline{\rho}\to\underline{\rho}}|^{-\sigma}_{t_{-\sigma}}$ is a local martingale in t_σ.

For $\sigma\in\{+,-\}$, let Ξ_σ denote the space of simple crosscuts of $\mathbb{H}$ that separate w_σ from $w_{-\sigma}$ and ∞. Here we do not require that the crosscuts separate w_σ from v_σ or v_0. For $\sigma\in\{+,-\}$ and $\xi_\sigma\in\Xi_\sigma$, let $\tau^\sigma_{\xi_\sigma}$ be the first time that η_σ hits the closure of ξ_σ; or the lifetime of η if such time does not exist. We see that $\tau^\sigma_{\xi_\sigma}\le{\rm hcap}_2({\rm Hull}(\xi_j))<\infty$. Let $\Xi=\{(\xi_+,\xi_-)\in\Xi_+\times\Xi_-,{\rm dist}(\xi_+,\xi_-)>0\}$. For $\underline{\xi}=(\xi_+,\xi_-)\in\Xi$, let $\tau_{\underline{\xi}}=(\tau^+_{\xi_+},\tau^-_{\xi_-})$. We may choose a countable set $\Xi^*\subset\Xi$ such that for every $\underline{\xi}=(\xi_+,\xi_-)\in\Xi$ there is $(\xi^*_+,\xi^*_-)\in\Xi^*$ such that ξ_σ is enclosed by ξ^*_σ, $\sigma\in\{+,-\}$.

Lemma 4.5. *For any $\underline{\xi}\in\Xi$, $|\log M_{i\underline{\rho}\to\underline{\rho}}|$ is uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$ by a constant depending only on $\kappa,\underline{\rho},w_+,w_-,v_0,v_+,v_-$ and $\underline{\xi}$.*

Proof. Fix $\underline{\xi}=(\xi_+,\xi_-)\in\Xi$. Let $K_{\xi_\sigma}={\rm Hull}(\xi_\sigma)$, $\sigma\in\{+,-\}$ and $K_{\underline{\xi}}=K_{\xi_+}\cup K(\xi_-)$. Then either $v_0\not\in\overline{K_{\xi_+}}$ or $v_0\not\in\overline{K_{\xi_-}}$. By symmetry, we assume that $v_0\not\in\overline{K_{\xi_+}}$. Pick $v_0^1<v_0^2\in(v_0,w_+)\setminus\overline{K_{\underline{\xi}}}$, and let V_0^j be the force point function started from v_0^j, $j=1,2$. By (3.13), $V_+\ge W_+\ge V_0^2>V_0^1\ge V_0\ge W_-\ge V_-$ on $[\underline{0},\tau_{\underline{\xi}}]$. Throughout the proof, a constant is a positive number that depends only on $w_+,w_-,v_0,v_+,v_-,\underline{\xi},v_0^1,v_0^2$, and a function defined on $[\underline{0},\tau_{\underline{\xi}}]$ is said to be uniformly bounded if its absolute value on $[\underline{0},\tau_{\underline{\xi}}]$ is bounded above by a constant. From the definition of $M_{i\underline{\rho}\to\underline{\rho}}$, it suffices to prove that $|\log F|$, $|\log E_{Y_1,Y_2}|$, $Y_1\ne Y_2\in\{W_+,W_-,V_0,V_+,V_-\}$, $|\log W_{\sigma,N}|$, $\sigma\in\{+,-\}$, and $|\log V_{\nu,N}|$, $\nu\in\{0,+,-\}$, are all uniformly bounded. By Proposition 2.2, $W_{+,1},W_{-,1}$ are uniformly bounded by 1.

For $\sigma\in\{+,-\}$, the function $(t_+,t_-)\mapsto t_\sigma$ is bounded on $[\underline{0},\tau_{\underline{\xi}}]$ by ${\rm hcap}_2(K_{\underline{\xi}})$. For any $\underline{t}\in[\underline{0},\tau_{\underline{\xi}}]$, since $g_{K_{\underline{\xi}}}=g_{K_{\underline{\xi}}/K(\underline{t})}\circ g_{K(\underline{t})}$, by Proposition 2.2 we get $0<g'_{K_{\underline{\xi}}}\le g'_{K(\underline{t})}\le1$ on $[v_0^1,v_0^2]$.

Since $[v_0^1, v_0^2]$ is a compact subset of $\mathbb{C}\setminus\overline{K_{\underline{\xi}}}$, $g'_{K_{\underline{\xi}}}$ on $[v_0^1, v_0^2]$ is bounded from below by a constant. So $|\log(g'_{K(\underline{t})})|$ is uniformly bounded on $[v_0^1, v_0^2]$. Since $V_0^j(\underline{t}) = g_{K(\underline{t})}(v_0^j)$, $j = 1, 2$, we see that $\frac{1}{V_0^2 - V_0^1}$ is uniformly bounded, which then implies that $\frac{1}{|W_\sigma - W_{-\sigma}|}$ and $\frac{1}{|W_\sigma - V_{-\sigma}|}$ are uniformly bounded, $\sigma \in \{+, -\}$. From (3.20) we see that $|\log F|$ is uniformly bounded. From (3.27,3.29) and the fact that $W_{-\sigma,N}|_0^\sigma = V_{-\sigma,N}|^\sigma = 1$, we see that $|\log W_{-\sigma,N}|$ and $|\log V_{-\sigma,N}|$, $\sigma \in \{+, -\}$, are uniformly bounded. We also know that $\frac{1}{|W_+ - V_0|} \le \frac{1}{|V_0^2 - V_0^1|}$ is uniformly bounded. From (3.29) with $\sigma = +$ and the fact that $V_{0,N}|_0^+ \equiv 1$ we find that $|\log V_{0,N}|$ is uniformly bounded.

Now we estimate $|\log E_{Y_1,Y_2}|$. From (3.14), for any $Y_1, Y_2 \in \{W_+, W_-, V_0, V_+, V_-\}$, $|Y_1 - Y_2| \le |V_+ - V_-|$ is uniformly bounded. If $Y_1 \in \{W_+, V_+\}$ and $Y_2 \in \{W_-, V_-\}$, then $\frac{1}{|Y_1 - Y_2|} \le \frac{1}{|V_0^1 - V_0^2|}$ is uniformly bounded. From (3.30) we see that $|\log E_{Y_1,Y_2}|$ is uniformly bounded. If $Y_1, Y_2 \in \{W_{-\sigma}, V_{-\sigma}\}$ for some $\sigma \in \{+, -\}$, then $\frac{1}{|Y_j - W_\sigma|}$, $j = 1, 2$, are uniformly bounded, and then the uniformly boundedness of $|\log E_{Y_1,Y_2}|$ follows from (3.31) and the fact that $E_{Y_1,Y_2}|_0^\sigma \equiv 1$. Finally, we consider the case that $Y_1 = V_0$. If $Y_2 \in \{W_+, V_+\}$, then $\frac{1}{|Y_2 - Y_1|} \le \frac{1}{|V_0^2 - V_0^1|}$, which is uniformly bounded. We can again use (3.30) to get the uniformly boundedness of $|\log E_{Y_1,Y_2}|$. If $Y_2 \in \{W_-, V_-\}$, then $\frac{1}{|Y_j - W_+|}$, $j = 1, 2$, are uniformly bounded. The uniformly boundedness of $|\log E_{Y_1,Y_2}|$ then follows from (3.31) with $\sigma = +$ and the fact that $E_{Y_1,Y_2}|_0^+ \equiv 1$. □

Corollary 4.6. *For any $\underline{\xi} \in \Xi$, $(M_{i\underline{\rho}\to\underline{\rho}}(\underline{t} \wedge \tau_{\underline{\xi}}))_{\underline{t}\in\mathbb{R}_+^2}$ is an $(\mathcal{F}_{\underline{t}})$-$M_{i\underline{\rho}\to\underline{\rho}}(\tau_{\underline{\xi}})$-Doob martingale w.r.t. $\mathbb{P}_i^{\underline{\rho}}$.*

Proof. This follows from (4.8), Lemma 4.5, and the same argument as in the proof of Corollary 3.2 of [22]. □

Lemma 4.7. *For any $\underline{\xi} = (\xi_+, \xi_-) \in \Xi$, $\mathbb{P}^{\underline{\rho}}$ is absolutely continuous w.r.t. $\mathbb{P}_i^{\underline{\rho}}$ on $\mathcal{F}_{\tau_{\underline{\xi}}}$, and the RN derivative is $M_{i\underline{\rho}\to\underline{\rho}}(\tau_{\underline{\xi}})$.*

Proof. Let $\underline{\xi} = (\xi_+, \xi_-) \in \Xi$. The above corollary implies that $\mathbb{E}_i^{\underline{\rho}}[M_{i\underline{\rho}\to\underline{\rho}}(\tau_{\underline{\xi}})] = M_{i\underline{\rho}\to\underline{\rho}}(\underline{0}) = 1$. So we may define a probability measure $\mathbb{P}_{\underline{\xi}}^{\underline{\rho}}$ by $d\mathbb{P}_{\underline{\xi}}^{\underline{\rho}} = M_{i\underline{\rho}\to\underline{\rho}}(\tau_{\underline{\xi}})d\mathbb{P}_i^{\underline{\rho}}$.

Since $M_{i\underline{\rho}\to\underline{\rho}}(t_+, t_-) = 1$ when $t_+t_- = 0$, from the above corollary we know that the marginal laws of $\mathbb{P}_{\underline{\xi}}^{\underline{\rho}}$ agree with that of $\mathbb{P}_i^{\underline{\rho}}$, which are $\mathbb{P}_+^{\underline{\rho}}$ and $\mathbb{P}_-^{\underline{\rho}}$. Suppose $(\widehat{w}_+, \widehat{w}_-)$ follows the law $\mathbb{P}_{\underline{\xi}}^{\underline{\rho}}$. Then $\widehat{w}_-$ follows the law $\mathbb{P}_-^{\underline{\rho}}$. Now we write $\tau_\pm$ for $\tau_{\xi_\pm}^\pm$, and $\underline{\tau}$ for $\tau_{\underline{\xi}}$. From Lemma 2.31 and Corollary 4.6, $\frac{d\mathbb{P}_{\underline{\xi}}^{\underline{\rho}}|\mathcal{F}_{(t_+,\tau_-)}}{d\mathbb{P}_i^{\underline{\rho}}|\mathcal{F}_{(t_+,\tau_-)}} = M_{i\underline{\rho}\to\underline{\rho}}(t_+ \wedge \tau_+, \tau_-)$, $0 \le t_+ < \infty$. From Girsanov Theorem and (4.8), we see that, under $\mathbb{P}_{\underline{\xi}}^{\underline{\rho}}$, $\widehat{w}_+$ satisfies the following SDE up to τ_+:

$$d\widehat{w}_+ = \sqrt{\kappa}dB_+^{\tau_-} + \kappa\,\mathrm{b}\,\frac{W_{+,2}}{W_{+,1}}\Big|_{\tau_-}^{-} dt_+ + \frac{2W_{+,1}}{W_+ - W_-}\Big|_{\tau_-}^{-} dt_+ + \sum_{\nu\in\{0,+,-\}} \frac{\rho_\nu W_{+,1}}{W_+ - V_\nu}\Big|_{\tau_-}^{-} dt_+,$$

where $B_+^{\tau_-}$ is a standard $(\mathcal{F}_{(t_+,\tau_-)})_{t_+\ge 0}$-Brownian motion under $\mathbb{P}^{\rho}_{\underline{\xi}}$. Using Lemma 3.13 and (3.25) we find that $W_+(\cdot,\tau_-)$ under $\mathbb{P}^{\rho}_{\underline{\xi}}$ satisfies the following SDE up to τ_+:

$$dW_+|_{\tau_-}^- \overset{\text{ae}}{=} \sqrt{\kappa}W_{+,1}|_{\tau_-}^- dB_+^{\tau_-} + \frac{2W_{+,1}^2}{W_+ - W_-}\Big|_{\tau_-}^- dt_+ + \sum_{\nu\in\{0,+,-\}} \frac{\rho_+ W_{+,1}^2}{W_+ - V_\nu}\Big|_{\tau_-}^- dt_+. \tag{4.9}$$

There is a similar SDE for $W_-(\tau_+,\cdot)$.

Note that the SDE (4.9) agrees with the SDE for $W_+(\cdot,\tau_-)$ if (η_+,η_-) is a commuting pair of chordal $\text{SLE}_\kappa(2,\underline{\rho})$ curves started from $(w_+,w_-;v_0,v_+,v_-)$. The same is true if τ_- is replaced by $t_-\wedge\tau_-$ for any deterministic $t_-\ge 0$. Thus, $\mathbb{P}^{\rho}_{\underline{\xi}}$ agrees with $\mathbb{P}^{\underline{\rho}}$ on $\mathcal{F}_{\tau_{\underline{\xi}}}$, which implies the conclusion of the lemma. □

Corollary 4.8. *If $\underline{T}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time, and is bounded above by $\tau_{\underline{\xi}}$ for some $\underline{\xi}\in\Xi$. Then $\mathbb{P}^{\underline{\rho}}|\mathcal{F}_{\underline{T}}$ is absolutely continuous w.r.t. $\mathbb{P}^{\rho}_{i}|\mathcal{F}_{\underline{T}}$, and the RN derivative is $M_{i\underline{\rho}\to\underline{\rho}}(\underline{T})$.*

Proof. This follows from Lemma 4.7, Proposition 2.31, and Corollary 4.6. □

4.3 Diffusion processes along a time curve

Now assume that $v_+ - v_0 = v_0 - v_-$. Let $\underline{u} = (u_+,u_-) : [0,T^u) \to \mathbb{R}_+^2$ be as in Section 3.4. By Lemma 3.22, a.s. $T^u = \infty$. Recall that for a function X on $\mathbb{R}_+^2$, we define $X^u = X\circ\underline{u}$. By Proposition 3.24, $\underline{u}(t)$ is an $(\mathcal{F}_{\underline{t}})$-stopping time for each $t\ge 0$. We then get an $\mathbb{R}_+$-indexed filtration $\mathcal{F}_t^u := \mathcal{F}_{\underline{u}(t)}$, $t\ge 0$, from Proposition 2.26. For $\underline{\xi} = (\xi_+,\xi_-)\in\Xi$, let $\tau^u_{\underline{\xi}}$ denote the first $t\ge 0$ such that $u_1(t) = \tau^1_{\xi_1}$ or $u_2(t) = \tau^2_{\xi_2}$, whichever comes first. Note that such time exists and is finite because $(\tau^1_{\xi_1},\tau^2_{\xi_2})\in\mathcal{D}$. The following proposition has the same form as [22, Lemma 4.2], whose proof can also be used here.

Proposition 4.9. *For $\underline{\xi}\in\Xi$, $\underline{u}(\tau^u_{\underline{\xi}})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time, $\tau^u_{\underline{\xi}}$ is an $(\mathcal{F}^u_t)_{t\ge 0}$-stopping time, and for any $t\ge 0$, $\underline{u}(t\wedge\tau^u_{\underline{\xi}})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time.*

First assume that $(\widehat{w}_+,\widehat{w}_-)$ follows the law $\mathbb{P}^{\rho}_{i}$. Let $\eta_\pm$ be the chordal Loewner curve driven by $\widehat{w}_\pm$. Let $\mathcal{D}^{\text{disj}}$ be as before. Let $\widehat{w}^{\pm}_{\mp}(t)$ and $\widehat{v}^{\pm}_{\nu}(t)$, $\nu\in\{0,+,-\}$ be the force point functions for $\eta_\pm$ started from $w_\mp$ and v_ν, $\nu\in\{0,+,-\}$, respectively. Define $\widehat{B}_\sigma$, $\sigma\in\{+,-\}$, by

$$\sqrt{\kappa}\widehat{B}_\sigma(t) = \widehat{w}_\sigma(t) - w_\sigma - \int_0^t \frac{2ds}{\widehat{w}_\sigma(s) - \widehat{w}^\sigma_{-\sigma}(s)} - \sum_{\nu\in\{0,+,-\}}\int_0^t \frac{\rho_\nu ds}{\widehat{w}_\sigma(s) - \widehat{v}^\sigma_\nu(s)}. \tag{4.10}$$

Then $\widehat{B}_+$ and $\widehat{B}_-$ are independent standard Brownian motions. So we get five $(\mathcal{F}_{\underline{t}})$-martingales on $\mathcal{D}^{\text{disj}}$: $\widehat{B}_+(t_+)$, $\widehat{B}_-(t_-)$, $\widehat{B}_+(t_+)^2 - t_+$, $\widehat{B}_-(t_-)^2 - t_-$, and $\widehat{B}_+(t_+)\widehat{B}_-(t_-)$. Fix $\underline{\xi}\in\Xi$. Using Propositions 2.31 and 3.24 and the facts that $u_\pm$ is uniformly bounded above on $[0,\tau_{\underline{\xi}}]$, we conclude that $\widehat{B}^u_\sigma(t\wedge\tau^u_{\underline{\xi}})$, $\widehat{B}^u_\sigma(t\wedge\tau^u_{\underline{\xi}})^2 - u_\sigma(t\wedge\tau^u_{\underline{\xi}})$, $\sigma\in\{+,-\}$, and $\widehat{B}^u_+(t\wedge\tau^u_{\underline{\xi}})\widehat{B}^u_-(t\wedge\tau^u_{\underline{\xi}})$ are all $(\mathcal{F}^u_t)$-martingales under $\mathbb{P}^{\underline{\rho}}_i$. Thus, the quadratic variation and covariation of $\widehat{B}^u_+$ and $\widehat{B}^u_-$ satisfy

$$d\langle \widehat{B}^u_+\rangle_t \overset{\text{ae}}{=} u_+'(t)dt,\quad d\langle \widehat{B}^u_-\rangle_t \overset{\text{ae}}{=} u_-'(t)dt,\quad d\langle \widehat{B}^u_+,\widehat{B}^u_-\rangle_t=0 \tag{4.11}$$

up to $\tau^u_{\underline{\xi}}$. From Lemmas 4.6 and 2.31 we know that $M^u_{i\underline{\rho}\to\underline{\rho}}(t\wedge\tau^u_{\underline{\xi}})$, $t\ge 0$ is an $(\mathcal{F}^u_t)_{t\ge0}$-martingale. Let T^u_{disj} denote the first t such that $\underline{u}(t)\not\in\mathcal{D}^{\text{disj}}$. Since $T^u_{\text{disj}}=\sup_{\underline{\xi}\in\Xi}\tau^u_{\underline{\xi}}=\sup_{\underline{\xi}\in\Xi^*}\tau^u_{\underline{\xi}}$, and Ξ^* is countable, we see that, T^u_{disj} is an $(\mathcal{F}^u_t)_{t\ge0}$-stopping time. We now compute the SDE for $M^u_{i\underline{\rho}\to\underline{\rho}}(t)$ up to T^u_{disj} in terms of $\widehat{B}^u_+$ and $\widehat{B}^u_-$. Using (4.7) we may express $M^u_{i\underline{\rho}\to\underline{\rho}}$ as a product of several factors. Among these factors, $E^u_{W_+,W_-}$, $(W^u_{\sigma,N})^{\text{b}}$, $(E^u_{W_\sigma,V_\nu})^{\rho_\nu/\kappa}$, $\sigma\in\{+,-\}$, $\nu\in\{0,+,-\}$, contribute the martingale part; and other factors are differentiable in t. For $\sigma\in\{+,-\}$, using (4.3,3.25,3.26) we get the $(\mathcal{F}^u_t)$-adapted SDEs:

$$dW^u_\sigma=W^u_{\sigma,1}d\widehat{w}^u_\sigma+\Big(\frac{\kappa}{2}-3\Big)W_{\sigma,2}u_\sigma'dt+\frac{2(W^u_{-\sigma,1})^2}{W^u_\sigma-W^u_{-\sigma}}\,u_{-\sigma}'dt, \tag{4.12}$$

$$\frac{dW^u_{\sigma,1}}{W^u_{\sigma,1}}=\frac{W^u_{\sigma,2}}{W^u_{\sigma,1}}\sqrt{\kappa}d\widehat{B}^u_\sigma+\text{drift terms}.$$

Since $W^u_{\sigma,N}=\frac{W^u_{\sigma,1}}{(W_{\sigma,1}|^\sigma_0)^u}$, and $(W_{\sigma,1}|^\sigma_0)^u(t)=W_{\sigma,1}(u_{-\sigma}(t)\underline{e}_{-\sigma})$ is differentiable in t, from the last displayed formula, we get the SDE for $W^{\text{b}}_{\sigma,N}$:

$$\frac{d(W^u_{\sigma,N})^{\text{b}}}{(W^u_{\sigma,N})^{\text{b}}}\overset{\text{ae}}{=}\text{b}\,\frac{W^u_{\sigma,2}}{W^u_{\sigma,1}}\sqrt{\kappa}d\widehat{B}^u_\sigma+\text{drift terms}.$$

For the SDE for $(E^u_{W_+,W_-})^{\frac{2}{\kappa}}$, note that when $X=W_+$ and $Y=W_-$, the numerators and denominators in (3.30) never vanish. So using (4.12) we get

$$\frac{d(E^u_{W_+,W_-})^{\frac{2}{\kappa}}}{(E^u_{W_+,W_-})^{\frac{2}{\kappa}}}\overset{\text{ae}}{=}\frac{2}{\kappa}\sum_{\sigma\in\{+,-\}}\Big[\frac{W^u_{\sigma,1}}{W^u_\sigma-W^u_{-\sigma}}-\frac{1}{\widehat{w}^u_\sigma-(\widehat{w}^\sigma_{-\sigma})^u}\Big]\sqrt{\kappa}d\widehat{B}^u_\sigma+\text{drift terms}.$$

We may express $E^u_{W_\sigma,V_\nu}(t)$ as a product of a function in $u_{-\sigma}(t)$, which is differentiable, and a function of the form $f(\underline{u}(t),\widehat{w}^u_\sigma(t),(\widehat{v}^\sigma_\nu)^u(t))$, where $f(\cdot,\cdot,\cdot)$ is given by (4.5). Using (4.12) we get the SDE for $(E^u_{W_\sigma,V_\nu})^{\frac{\rho_\nu}{\kappa}}$:

$$\frac{d(E^u_{W_\sigma,V_\nu})^{\frac{\rho_\nu}{\kappa}}}{(E^u_{W_\sigma,V_\nu})^{\frac{\rho_\nu}{\kappa}}}\overset{\text{ae}}{=}\frac{\rho_\nu}{\kappa}\Big[\frac{W^u_{\sigma,1}}{W^u_\sigma-V^u_\nu}-\frac{1}{\widehat{w}^u_\sigma-(\widehat{v}^\sigma_\nu)^u}\Big]\sqrt{\kappa}d\widehat{B}^u_\sigma+\text{drift terms}.$$

Here if $(\widehat{v}^\sigma_\nu)^u(t)=\widehat{w}^u_\sigma(t)^\pm$, we understand the function inside the square brackets as

$$\lim_{v\to\widehat{w}_\sigma(u_\sigma(t))}\frac{g'_{K_{-\sigma,u_\sigma(t)}(u_{-\sigma}(t))}(\widehat{w}_\sigma(u_\sigma(t)))}{g_{K_{-\sigma,u_\sigma(t)}(u_{-\sigma}(t))}(\widehat{w}_\sigma(u_\sigma(t)))-g_{K_{-\sigma,u_\sigma(t)}(u_{-\sigma}(t))}(v)}-\frac{1}{\widehat{w}_\sigma(u_\sigma(t))-v}=\frac{1}{2}\frac{W^u_{\sigma,2}(t)}{W^u_{\sigma,1}(t)}.$$

Combining the last three displayed formulas and using the fact that $M^u_{i\underline{\rho}\to\underline{\rho}}$ and $\widehat{B}^u_\pm$ are all $(\mathcal{F}^u_t)$-local martingales under $\mathbb{P}^{\underline{\rho}}_i$, we get

$$\frac{dM^u_{i\underline{\rho}\to\underline{\rho}}}{M^u_{i\underline{\rho}\to\underline{\rho}}} \overset{\text{ae}}{=} \sum_{\sigma\in\{+,-\}}\Big[\kappa\,\mathrm{b}\,\frac{W^u_{\sigma,2}}{W^u_{\sigma,1}}+2\Big[\frac{W^u_{\sigma,1}}{W^u_\sigma-W^u_{-\sigma}}-\frac{1}{\widehat{w}^u_\sigma-(\widehat{w}^\sigma_{-\sigma})^u}\Big]+ \\ +\sum_{\nu\in\{0,+,-\}}\rho_\nu\Big[\frac{W^u_{\sigma,1}}{W^u_\sigma-V^u_\nu}-\frac{1}{\widehat{w}^u_\sigma-(\widehat{v}^\sigma_\nu)^u}\Big]\Big]\frac{d\widehat{B}^u_\sigma}{\sqrt{\kappa}}, \tag{4.13}$$

where if $(\widehat{v}^\sigma_\nu)^u(t)=\widehat{w}^u_\sigma(t)^\pm$, the function inside the square brackets is understood as $\frac{1}{2}\frac{W^u_{\sigma,2}(t)}{W^u_{\sigma,1}(t)}$. From Corollary 4.8 and Proposition 4.9 we know that, for any $\underline{\xi}\in\Xi$ and $t\ge 0$,

$$\frac{d\mathbb{P}^{\underline{\rho}}|\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}}{d\mathbb{P}^{\underline{\rho}}_i|\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}}=M^u_{i\underline{\rho}\to\underline{\rho}}(t\wedge\tau^u_{\underline{\xi}}). \tag{4.14}$$

We will use a Girsanov argument to derive the SDEs for $\widehat{w}^u_+$ and $\widehat{w}^u_-$ up to T^u_{disj} under $\mathbb{P}^{\underline{\rho}}$.

For $\sigma\in\{+,-\}$, define a process $\widetilde{B}^u_\sigma(t)$ such that $\widetilde{B}^u(t)=0$ and

$$d\widetilde{B}^u_\sigma = d\widehat{B}^u_\sigma - \Big[\kappa\,\mathrm{b}\,\frac{W^u_{\sigma,2}}{W^u_{\sigma,1}}+\Big[\frac{2W^u_{\sigma,1}}{W^u_\sigma-W^u_{-\sigma}}-\frac{2}{\widehat{w}^u_\sigma-(\widehat{w}^\sigma_{-\sigma})^u}\Big] \\ +\sum_{\nu\in\{0,+,-\}}\Big[\frac{\rho_\nu W^u_{\sigma,1}}{W^u_\sigma-V^u_\nu}-\frac{\rho_\nu}{\widehat{w}^u_\sigma-(\widehat{v}^\sigma_\nu)^u}\Big]\Big]\frac{u'_\sigma(t)}{\sqrt{\kappa}}\,dt. \tag{4.15}$$

Lemma 4.10. *For any $\sigma\in\{+,-\}$ and $\underline{\xi}\in\Xi$, $|\widetilde{B}^u_\sigma|$ is bounded on $[0,\tau^u_{\underline{\xi}}]$ by a constant depending only on $\kappa,\underline{\rho},w_+,w_-,v_0,v_+,v_-$ and $\underline{\xi}$.*

Proof. Throughout the proof, a positive number that depends only on $\kappa,\underline{\rho},w_+,w_-,v_0,v_+,v_-$ and $\underline{\xi}$ is called a constant. It is clear that $\widehat{B}^u_+(t)=U(u_+(t),0)-U(0,0)$ and $\widehat{B}^u_-(t)=U(0,u_-(t))-U(0,0)$, where $U:=W_++W_-+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu}{2}V_\nu$. By Proposition 2.3, V_+ and V_- are bounded in absolute value by a constant on $[\underline{0},\tau_{\underline{\xi}}]$, and so are W_+,V_0,W_-,U because $V_+\ge W_+\ge V_0\ge W_-\ge V_-$. Thus, $\widehat{B}^u_\sigma$, $\sigma\in\{+,-\}$, are bounded in absolute value by a constant on $[0,\tau^u_{\underline{\xi}}]$. By (3.14) and that $V^u_+(t)-V^u_-(t)=e^{2t}(v_+-v_-)$ for $0\le t<T^u$, we know that $e^{2\tau^u_{\underline{\xi}}}\le 4\,\mathrm{diam}(\xi_+\cup\xi_-\cup[v_-,v_+])/|v_+-v_-|$. This means that $\tau^u_{\underline{\xi}}$ is bounded above by a constant. Since $\underline{u}([0,\tau^u_{\underline{\xi}}])\subset[\underline{0},\tau_{\underline{\xi}}]$, it remains to show that, for $\sigma\in\{+,-\}$,

$$\frac{W_{\sigma,2}}{W_{\sigma,1}},\quad \frac{W_{\sigma,1}}{W_\sigma-W_{-\sigma}}-\frac{1}{\widehat{w}_\sigma-\widehat{w}^\sigma_{-\sigma}},\quad \frac{W_{\sigma,1}}{W_\sigma-V_\nu}-\frac{1}{\widehat{w}_\sigma-\widehat{v}^\sigma_\nu},\quad \nu\in\{0,+,-\},$$

are all bounded in absolute value on $[\underline{0},\tau_{\underline{\xi}}]$ by a constant.

Because $\frac{1}{\widehat w_\sigma-\widehat w^\sigma_{-\sigma}}=\frac{W_{\sigma,1}}{W_\sigma-W_{-\sigma}}\Big|_0^{-\sigma}$, the boundedness of $\frac{W_{\sigma,1}}{W_\sigma-W_{-\sigma}}-\frac{1}{\widehat w_\sigma-\widehat w^\sigma_{-\sigma}}$ on $[\underline{0},\tau_{\underline{\xi}}]$ simply follows from the boundedness of $\frac{W_{\sigma,1}}{W_\sigma-W_{-\sigma}}$, which in turn follows from $0\le W_{\sigma,1}\le 1$ and that $|W_\sigma-W_{-\sigma}|$ is bounded from below on $[\underline{0},\tau_{\underline{\xi}}]$ by a positive constant, where the latter bound was given in the proof of Lemma 4.5.

For the boundedness of $\frac{W_{\sigma,2}}{W_{\sigma,1}}$ on $[\underline{0},\tau_{\underline{\xi}}]$, we assume $\sigma=+$ by symmetry. Since $W_{+,j}(t_+,t_-)=g^{(j)}_{K_{-,t_+}(t_-)}(\widehat w_+(t_+))$, $j=1,2$, and $K_{-,t_+}(\cdot)$ are chordal Loewner hulls driven by $W_-(t_+,\cdot)$ with speed $W_{-,1}(t_+,\cdot)^2$, by differentiating $\frac{g''_{K_{-,t_+}(t_-)}(\widehat w_+(t_+))}{g'_{K_{-,t_+}(t_-)}(\widehat w_+(t_+))}$ w.r.t. t_-, we get

$$\frac{W_{+,2}(t_+,t_-)}{W_{+,1}(t_+,t_-)}=\int_0^{t_-}\frac{4W_{-,1}^2W_{+,1}}{(W_+-W_-)^3}\Big|_{(t_+,s_-)}ds.$$

From the facts that $0\le W_{+,1},W_{-,1}\le 1$ and that $|W_+-W_-|$ is bounded from below by a constant on $[\underline{0},\tau_{\underline{\xi}}]$, we see that the integrand in the above displayed is bounded in absolute value by a constant, from which follows the boundedness of $\frac{W_{+,2}}{W_{+,1}}$.

For the boundedness of $\frac{W_{\sigma,1}}{W_\sigma-V_\nu}-\frac{1}{\widehat w_\sigma-\widehat v^\sigma_\nu}$ on $[\underline{0},\tau_{\underline{\xi}}]$ with $\sigma=+$, we note that $W_{+,1}(t_+,t_-)=g'_{K_{-,t_+}(t_-)}(\widehat w_+(t_+))$, $W_+(t_+,t_-)=g_{K_{-,t_+}(t_-)}(\widehat w_+(t_+))$, and $V_\nu(t_+,t_-)=g^{\eta_{-,t_+}(0)}_{K_{-,t_+}(t_-)}(\widehat v^+_\nu(t_+))$. By differentiating w.r.t. t_-, we get

$$\frac{W_{+,1}(t_+,t_-)}{W_+(t_+,t_-)-V_\nu(t_+,t_-)}-\frac{1}{\widehat w_+(t_+)-\widehat v^+_\nu(t_+)}=\int_0^{t_-}\frac{2W_{-,1}^2W_{+,1}}{(W_+-W_-)^2(V_\nu-W_-)}\Big|_{(t_+,s_-)}ds.$$

Since $0\le W_{+,1}\le 1$, $|W_+-W_-|$ is bounded from below by a constant on $[\underline{0},\tau_{\underline{\xi}}]$, and $V_\nu-W_-$ does not change sign (but could be 0), it suffices to show that $\big|\int_0^{t_-}\frac{2W_{-,1}^2}{V_\nu-W_-}|_{(t_+,s_-)}ds\big|$ is bounded by a constant on $[\underline{0},\tau_{\underline{\xi}}]$. This holds because the integral equals $V_\nu(t_+,t_-)-V_\nu(t_+,0)$, and $|V_\nu|$ is bounded by a constant on $[\underline{0},\tau_{\underline{\xi}}]$. The boundedness in the case $\sigma=-$ holds symmetrically. □

Lemma 4.11. *Under $\mathbb{P}^{\underline{\rho}}$, there is a stopped planar Brownian motion $\underline{B}(t)=(B_+(t),B_-(t))$, $0\le t<T^u_{\rm disj}$, such that, for $\sigma\in\{+,-\}$, $\widehat w^u_\sigma$ satisfies the SDE*

$$d\widehat w^u_\sigma\overset{\rm ae}{=}\sqrt{\kappa u'_\sigma}dB_\sigma+\Big[\kappa\,\mathrm{b}\frac{W^u_{\sigma,2}}{W^u_{\sigma,1}}+\frac{2W^u_{\sigma,1}}{W^u_\sigma-W^u_{-\sigma}}+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu W^u_{\sigma,1}}{W^u_\sigma-V^u_\nu}\Big]u'_\sigma dt,\quad 0\le t<T^u_{\rm disj}.$$

Here by saying that $(B_+(t),B_-(t))$, $0\le t<T^u_{\rm disj}$, is a stopped planar Brownian motion, we mean that $B_+(t)$ and $B_-(t)$, $0\le t<T^u_{\rm disj}$, are local martingales with $d\langle B_\sigma\rangle_t=t$, $\sigma\in\{+,-\}$, $d\langle B_+,B_-\rangle_t=0$, $0\le t<T^u_{\rm disj}$.

Proof. For $\sigma\in\{+,-\}$, define $\widetilde B^u_\sigma$ using (4.15). By (4.13), $\widetilde B^u_\sigma(t)M^u_{i\underline{\rho}\to\underline{\rho}}(t)$, $0\le t<T^u_{\rm disj}$, is an $(\mathcal{F}^u_t)$-local martingale under $\mathbb{P}^{\underline{\rho}}_i$. By Lemmas 4.5 and 4.10, for any $\underline{\xi}\in\Xi$, $\widetilde B^u_\sigma(t\wedge\tau^u_{\underline{\xi}})M^u_{i\underline{\rho}\to\underline{\rho}}(t\wedge\tau^u_{\underline{\xi}})$,

$t\ge 0$, is an $(\mathcal{F}^u_t)$-martingale under $\mathbb{P}^{\underline{\rho}}_i$. Since this process is $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})$-adapted, and $\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}\subset\mathcal{F}_{\underline{u}(t)}=\mathcal{F}^u_t$, it is also an $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})$-martingale. From (4.14) we see that $(\widetilde B^u_\sigma(t\wedge\tau^u_{\underline{\xi}}))_{t\ge 0}$, is an $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})_{t\ge0}$-martingale under $\mathbb{P}^{\underline{\rho}}$. A standard argument shows that $(\widetilde B^u_\sigma(t\wedge\tau^u_{\underline{\xi}}))_{t\ge0}$ is an $(\mathcal{F}^u_t=\mathcal{F}_{\underline{u}(t)})_{t\ge0}$-martingale under $\mathbb{P}^{\underline{\rho}}$. Since $T^u_{\rm disj}=\sup_{\underline{\xi}\in\Xi^*}\tau^u_{\underline{\xi}}$, we see that, for $\sigma\in\{+,-\}$, $\widetilde B^u_\sigma(t)$, $0\le t<T^u_{\rm disj}$, is an $(\mathcal{F}^u_t)$-local martingale under $\mathbb{P}^{\underline{\rho}}$.

From (4.11) we know that, under $\mathbb{P}^{\underline{\rho}}_i$,

$$\langle \widetilde B^u_\sigma(\cdot\wedge\tau^u_{\underline{\xi}})\rangle_t=u_\sigma(t\wedge\tau^u_{\underline{\xi}}),\quad \sigma\in\{+,-\};\quad \langle \widetilde B^u_+(\cdot\wedge\tau^u_{\underline{\xi}}),\widetilde B^u_-(\cdot\wedge\tau^u_{\underline{\xi}})\rangle_t=00 \tag{4.16}$$

Since $\mathbb{P}^{\underline{\rho}}\ll\mathbb{P}^{\underline{\rho}}_i$ on $\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}$ for any $t\ge0$, we also have (4.16) under $\mathbb{P}^{\underline{\rho}}$. Since $T^u_{\rm disj}=\sup_{\underline{\xi}\in\Xi^*}\tau^u_{\underline{\xi}}$, we conclude that, under $\mathbb{P}^{\underline{\rho}}$,

$$\langle \widetilde B^u_\sigma\rangle_t=u_\sigma(t),\quad\sigma\in\{+,-\};\quad\langle\widetilde B^u_+,\widetilde B^u_-\rangle_t\equiv0,\quad 0\le t<T^u_{\rm disj}.$$

Since $\widetilde B^u_\sigma(t)$, $0\le t<T^u_{\rm disj}$, $\sigma\in\{+,-\}$, are $(\mathcal{F}^u_t)$-local martingales under $\mathbb{P}^{\underline{\rho}}$, we get the stopped planar Brownian motion $(B_+(t),B_-(t))$, $0\le t<T^u_{\rm disj}$, such that $d\widetilde B^u_\sigma(t)=\sqrt{u'_\sigma(t)}dB_\sigma(t)$. Using (4.10) and (4.15) we then complete the proof. □

From now on, we work under the probability measure $\mathbb{P}^{\underline{\rho}}$. Combining Lemma 4.11 with (4.12) and (3.18), we get an SDE for $W^u_\sigma-V^u_0$ up to $T^u_{\rm disj}$:

$$\begin{aligned}d(W^u_\sigma-V^u_0)\overset{\rm ae}{=}{}&W^u_{\sigma,1}\sqrt{\kappa u'_\sigma}dB^u_\sigma+\sum_{\nu\in\{0,+,-\}}\frac{\rho_\nu(W^u_{\sigma,1})^2u'_\sigma}{W^u_\sigma-V^u_\nu}\,dt+\frac{2(W^u_{\sigma,1})^2u'_\sigma}{W^u_\sigma-W^u_{-\sigma}}\,dt\\&+\frac{2(W^u_{-\sigma,1})^2u'_{-\sigma}}{W^u_\sigma-W^u_{-\sigma}}\,dt+\frac{2(W^u_{\sigma,1})^2u'_\sigma}{W^u_\sigma-V^u_0}\,dt+\frac{2(W^u_{-\sigma,1})^2u'_{-\sigma}}{W^u_{-\sigma}-V^u_0}\,dt.\end{aligned}$$

Recall that $R_\sigma=\frac{W^u_\sigma-V^u_0}{V^u_\sigma-V^u_0}\in[0,1]$, $\sigma\in\{+,-\}$, and $\underline{R}=(R_+,R_-)$. Combining the above SDE with (3.33), we find that R_σ, $\sigma\in\{+,-\}$, satisfies the following SDE up to $T^u_{\rm disj}$:

$$dR_\sigma\overset{\rm ae}{=}\sigma\sqrt{\frac{\kappa R_\sigma(1-R_\sigma^2)}{R_++R_-}}dB_\sigma+\frac{(2+\rho_0)-(\rho_\sigma-\rho_{-\sigma})R_\sigma-(\rho_++\rho_-+\rho_0+6)R_\sigma^2}{R_++R_-}\,dt. \tag{4.17}$$

We will later show in Theorem 4.13 that (4.17) holds throughout $\mathbb{R}_+$.

Let $X=R_+-R_-$ and $Y=1-R_+R_-$. From (4.17) we know that X and Y satisfy the following SDEs up to $T^u_{\rm disj}$:

$$dX=dM_X-[(\rho_++\rho_-+\rho_0+6)X+(\rho_+-\rho_-)]dt, \tag{4.18}$$

$$dY=dM_Y-[(\rho_++\rho_-+\rho_0+6)Y-(\rho_++\rho_-+4)]dt, \tag{4.19}$$

where M_X and M_Y are local martingales whose quadratic variation and covariation satisfy the following equations up to $T^u_{\rm disj}$:

$$d\langle X,X\rangle=\kappa(Y-X^2)dt,\quad d\langle X,Y\rangle=\kappa(X-XY)dt,\quad d\langle Y,Y\rangle=\kappa(Y-Y^2)dt. \tag{4.20}$$

Let Δ denote the triangle domain $\{(x,y):|x|<y<1\}$. Then $(X,Y)\in\overline{\Delta}$ because $Y\le 1$ and $Y\pm X=(1\pm R_+)(1\mp R_-)\ge 0$ as $R_+,R_-\in[0,1]$.

Lemma 4.12. *If R_+ and R_- satisfy (4.17) for a stopped planar Brownian motion (B_+,B_-) up to some stopping time τ, then a.s. $\lim_{t\uparrow\tau}\underline{R}(t)\ne\underline{0}$.*

Proof. We know that $X:=R_+-R_-$ and $Y:=1-R_+R_-$ satisfy (4.18,4.19,4.20) up to τ, and as $t\uparrow\tau$, $\underline{R}(t)\to\underline{0}$ iff $(X(t),Y(t))\to(0,1)$. From (4.19,4.20) there is a stopped Brownian motion $B_Y(t)$, $0\le t<\tau$, such that Y satisfies the following SDE:

$$dY=\sqrt{\kappa Y(1-Y)}dB_Y-[(\rho_++\rho_-+\rho_0+6)Y-(\rho_++\rho_-+4)]dt,\quad 0\le t<\tau.$$

Define $R_0(t)=X(t)/Y(t)$ whenever $Y(t)\ne 0$. It suffices to show that $(R_0(t),Y(t))$ does not tend to $(0,1)$ as $t\uparrow\tau$. Assume $Y(0)\ne 0$. Let T be τ or the first time that $Y(t)=0$, whichever comes first. From (4.20) we know that R_0 satisfies $d\langle R_0\rangle_t=(1-R_0^2)/Y dt$ and $d\langle R_0,Y\rangle_t=0$. Combining this with (4.18,4.19), we see that there exists B_{R_0} such that $(B_{R_0}(t),B_Y(t))$, $0\le t<T$, is a stopped planar Brownian motion, and R_0 satisfies the following SDE:

$$dR_0=\sqrt{\frac{\kappa(1-R_0^2)}{Y}}dB_{R_0}-\frac{(\rho_++\rho_-+4)R_0+(\rho_+-\rho_-)}{Y}\,dt,\quad 0\le t<T.$$

Let $v(t)=\int_0^t\kappa/Y(s)ds$, $0\le t<T$, and $\widetilde{T}=\sup v([0,T))$. Let $\widetilde{R}_0(t)=R_0(v^{-1}(t))$ and $\widetilde{Y}(t)=Y(v^{-1}(t))$, $0\le t<\widetilde{T}$. Then there is a stopped planar Brownian motion $(\widetilde{B}_{R_0}(t),\widetilde{B}_Y(t))$, $0\le t<\widetilde{T}$, such that $\widetilde{R}_0$ and $\widetilde{Y}$ satisfy the following SDEs on $[0,\widetilde{T})$:

$$d\widetilde{R}_0=\sqrt{1-\widetilde{R}_0^2}d\widetilde{B}_{R_0}-(a_{R_0}\widetilde{R}+b_{R_0})dt, \tag{4.21}$$

$$d\widetilde{Y}=\widetilde{Y}\sqrt{1-\widetilde{Y}}d\widetilde{B}_Y-\widetilde{Y}(a_Y(\widetilde{Y}-1)+b_Y)dt, \tag{4.22}$$

where $a_Y=(\rho_++\rho_-+\rho_0+6)/\kappa$, $b_Y=(\rho_0+2)/\kappa$, $a_{R_0}=a_Y-b_Y$, $b_{R_0}=(\rho_+-\rho_-)/\kappa$.

Let $\Theta=\arcsin(\widetilde{R}_0)$ and $\Phi=\log(\frac{1+\sqrt{1-\widetilde{Y}}}{1-\sqrt{1-\widetilde{Y}}})$. Then $\Theta\in[-\pi/2,\pi/2]$ and $\Phi\in\mathbb{R}_+$. Using (4.22,4.21) we find that Θ and Φ satisfy the following SDEs on $[0,\widetilde{T})$:

$$d\Theta=d\widetilde{B}_{R_0}-(a_{R_0}-\frac{1}{2})\tan\Theta dt-b_{R_0}\sec\Theta dt;$$

$$d\Phi=-d\widetilde{B}_Y+\Big(b_Y-\frac{1}{4}\Big)\coth\Big(\frac{\Phi}{2}\Big)dt+\Big(\frac{3}{4}-a_Y\Big)\tanh\Big(\frac{\Phi}{2}\Big)dt.$$

Moreover, $\lim_{t\uparrow T}(R_0(t), Y(t)) = (0,1)$ is equivalent to $\lim_{t\uparrow \widetilde T}(\Theta(t),\Phi(t)) = (0,0)$. As $\Theta(t)\to 0$, Θ behaves like a standard Brownian motion; while as $\Phi(t)\to 0$, Φ behaves like a Bessel process of dimension δ such that $\frac{\delta-1}{2} = 2(b_Y - \frac14)$. Since Θ and Φ are independent, as $(\Theta,\Phi)\to(0,0)$, $\sqrt{\Theta^2+\Phi^2}$ behaves like a Bessel process of dimension $\delta+1 = 4b_Y+1 = \frac{4}{\kappa}(\rho_0+2)+1$. Since $\rho_0 \ge \frac{\kappa}{4}-2$, we get $\delta+1\ge 2$. Thus, a.s. $\lim_{t\uparrow\widetilde T}\sqrt{\Theta(t)^2+\Phi(t)^2}\ne 0$, which implies that $\lim_{t\uparrow\widetilde T}(\Theta(t),\Phi(t))\ne(0,0)$. The above argument can be made rigorous using Girsanov Theorem on a sequence of stopping times. So on the event $\{T=\tau\}\supset\{Y(t)\ne 0 \text{ on } [0,\tau)\}$, a.s. $\lim_{t\uparrow T}(R_0(t),Y(t))\ne(0,1)$. From the Markov property of (X,Y), we see that, for any $q\in\mathbb{Q}_+$, on the event $\{q<\tau\}\cap\{Y(t)\ne 0 \text{ on } [q,\tau)\}$, a.s. $\lim_{t\uparrow T}(R_0(t),Y(t))\ne(0,1)$. Since $\{\lim_{t\uparrow T} Y(t) = 1\}\subset\bigcup_{q\in\mathbb{Q}_+}\{q<\tau\}\cap\{Y(t)\ne 0\text{ on }[q,\tau)\}$, we get a.s. $\lim_{t\uparrow T}(R_0(t),Y(t))\ne(0,1)$, which implies that $\lim_{t\uparrow\tau}\underline{R}(t)\ne\underline{0}$. □

Theorem 4.13. *Under $\mathbb{P}^{\underline{\rho}}$, R_+ and R_- satisfy (4.17) throughout $\mathbb{R}_+$ for a pair of independent Brownian motions B_+ and B_-.*

Proof. We already know that R_+ and R_- satisfy (4.17) for a stopped planar Brownian motion (B_+,B_-) up to $T^u_{\rm disj}$, the first t such that $\eta_+([0,u_+(t)])$ intersects $\eta_-([0,u_-(t)])$, If $\rho_0\ge\frac{\kappa}{2}-2$, a.s. $T^u_{\rm disj}=\infty$, and so (4.17) holds throughout $\mathbb{R}_+$, and B_+ and B_- are independent Brownian motions. For the rest of the proof, assume that $\rho_0<\frac{\kappa}{2}-2$. Then a.s. $T^u_{\rm disj}<\infty$. Set $n=0$. Let $w^n_+=w_+$, $w^n_-=w_-$, $v^n_\nu=v_\nu$, $\nu\in\{0,+,-\}$, $\eta^n_+=\eta_+$, and $\eta^n_-=\eta_-$.

Let m^n denote the capacity function for (η^n_+,η^n_-), let W^n_+ and W^n_- be the driving functions, and let V^n_ν, $\nu\in\{0,+,-\}$, be the force point functions started from v^n_0,v^n_+,v^n_-, respectively. Let $\mathcal{F}^n_{(t_+,t_-)}$ be the σ-algebra generated by $\eta^n_+|_{[0,t_+]}$ and $\eta^n_-|_{[0,t_-]}$, $(t_+,t_-)\in\mathbb{R}^2_+$. Since $v^n_+\ge w^n_+\ge v^n_0\ge w^n_-\ge v^n_-$, and $v^n_+-v^n_0=v^n_0-v^n_-$, we have the time curve $\underline{u}^n=(u^n_+,u^n_-):\mathbb{R}_+\to\mathbb{R}^2_+$ such that $V^n_\sigma(\underline{u}(t))-V^n_0(\underline{u}(t))=e^{2t}(v^n_\sigma-v^n_0)$, $t\ge0$, $\sigma\in\{+,-\}$. For each $t\ge 0$, $\underline{u}^n(t)$ is an $(\mathcal{F}^n_{\underline{t}})_{\underline{t}\in\mathbb{R}^2_+}$-stopping time. Define $\mathcal{F}^{n,u}_t=\mathcal{F}^n_{\underline{u}(t)}$, $t\ge0$. Let $R^n_\sigma(t)=\frac{W^n_\sigma(\underline{u}^n(t))-V^n_0(\underline{u}(t))}{V^n_\sigma(\underline{u}^n(t))-V^n_0(\underline{u}(t))}$, $t\ge0$, $\sigma\in\{+,-\}$. Then there is a stopped $(\mathcal{F}^{n,u}_t)_{t\ge0}$-planar Brownian motions $(B^n_+(t),B^n_-(t))$, $0\le t<\tau^n$, where τ^n is the first t such that $\eta^n_+([0,u^n_+(t)])$ intersects $\eta^n_-([0,u^n_-(t)])$, which is a finite $(\mathcal{F}^{n,u}_t)$-stopping time, such that R^n_+ and R^n_- satisfy the $(\mathcal{F}^{n,u}_t)$-adapted SDE (4.17) up to τ^n. Then $\underline{\tau}^n:=\underline{u}^n(\tau^n)$ is an $(\mathcal{F}^n_{\underline{t}})$-stopping time. From Lemma 4.12 we have a.s. $(R^n_+(\tau^n),R^n_-(\tau^n))\ne(0,0)$, which implies that $W^n_+(\underline{\tau}^n)\ne W^n_-(\underline{\tau}^n)$.

Set $w^{n+1}_\sigma=W^n_\sigma(\underline{\tau}^n)$, $\sigma\in\{+,-\}$; $v^{n+1}_\nu=V^n_\nu(\underline{\tau}^n)$ if $V^n_\nu(\underline{\tau}^n)\not\in\{w^{n+1}_+,w^{n+1}_-\}$, $\nu\in\{0,+,-\}$; $v^{n+1}_\sigma=W^n_\sigma(\underline{\tau}^n)^\sigma$ if $V^n_\sigma(\underline{\tau}^n)=W^n_\sigma(\underline{\tau}^n)$, $\sigma\in\{+,-\}$; and $v^{n+1}_0=W^n_\sigma(\underline{\tau}^n)^{-\sigma}$ if $V^n_0(\underline{\tau}^n)=W^n_\sigma(\underline{\tau}^n)$, $\sigma\in\{+,-\}$. By Lemma 4.4, there a.s. exists a commuting pair of chordal Loewner curves $(\widehat\eta^{n+1}_+,\widehat\eta^{n+1}_-)$ with some speeds, which up to a conformal map agrees with the part of (η^n_+,η^n_-) after $\underline{\tau}^n$. Moreover, if one defines $h^n_\sigma(t)=\mathrm{m}^n(\underline{\tau}^n+t\underline{e}_\sigma)-\mathrm{m}^n(\underline{\tau}^n)$, $t\ge n$, and let $\eta^{n+1}_\sigma=\widehat\eta^{n+1}_\sigma\circ(h^n_\sigma)^{-1}$, $\sigma\in\{+,-\}$, then $(\eta^{n+1}_+,\eta^{n+1}_-)$ is the normalization of $(\widehat\eta^{n+1}_+,\widehat\eta^{n+1}_-)$, and its conditional law given $\mathcal{F}^n_{\underline{\tau}^n}$ is that of a commuting pair of chordal $\mathrm{SLE}_\kappa(2,\underline{\rho})$ curves in $\mathbb{H}$ started from $(w^{n+1}_+,w^{n+1}_-;v^{n+1}_0,v^{n+1}_+,v^{n+1}_-)$.

Since $v^{n+1}_+\ge w^{n+1}_+\ge v^{n+1}_0\ge w^{n+1}_-\ge v^{n+1}_-$ and $v^{n+1}_+-v^{n+1}_0=v^{n+1}_0-v^{n+1}_-$, the argument in the previous two paragraphs also work with $n+1$ in place of n, except that now $\mathcal{F}^{n+1}_{(t_+,t_-)}$ is the σ-algebra generated by $\mathcal{F}^n_{\underline{\tau}^n}$, $\eta^{n+1}_+|_{[0,t_+]}$ and $\eta^{n+1}_-|_{[0,t_-]}$, $(t_+,t_-)\in\mathbb{R}_+^2$. So we may iterate the above procedure with $n=0,1,2,3$, and etc.

Fix any $n\in\mathbb{N}\cup\{0\}$. By Lemma 3.18 and that $\widehat\eta^{n+1}_\sigma=\eta^n_\sigma\circ h^n_\sigma$, $\sigma\in\{+.-\}$, we see that, if $X\in\{W_+,W_-,V_0,V_+,V_-\}$, then $X^n(\underline{\tau}^n+\cdot)=X^{n+1}\circ h^n_\oplus$, where $h^n_\oplus:=h^n_+\oplus h^n_-$. Let $\widetilde{\underline{u}}^{n+1}(t)=h^n_\oplus(\underline{u}^n(\tau^n+t)-\underline{u}^n(\tau^n))$, $t\ge 0$. Then for $\nu\in\{0,+,-\}$, $V^{n+1}_\nu\circ\widetilde{\underline{u}}^{n+1}(t)=V^n_\nu(\underline{u}^n(\tau^n+t))$, $t\ge0$. By the definition of $\underline{u}^n$, we have for $\nu\in\{+,-\}$,

$$V^{n+1}_\nu\circ\widetilde{\underline{u}}^{n+1}(t)-V^{n+1}_0\circ\widetilde{\underline{u}}^{n+1}(t)=e^{2t}(V^n_\nu\circ\widetilde{\underline{u}}^{n+1}(0)-V^0_\nu\circ\widetilde{\underline{u}}^{n+1}(0)).$$

Since $\widetilde{\underline{u}}^{n+1}(0)=\underline{0}$, $\widetilde{\underline{u}}^{n+1}$ satisfies the same property as $\underline{u}^{n+1}$. By the uniqueness of the time curve, we have $\underline{u}^{n+1}=\widetilde{\underline{u}}^{n+1}=h^n_\oplus(\underline{u}^n(\tau^n+\cdot)-\underline{u}^n(\tau^n))$, which implies that, for $X\in\{W_+,W_-,V_0,V_+,V_-\}$, $X^{n+1}\circ\underline{u}^{n+1}=X^n(\underline{\tau}^n+(h^n_\oplus)^{-1}\circ\underline{u}^{n+1}(\cdot))=X^n\circ\underline{u}^n(\tau^n+\cdot)$. Thus, $R^{n+1}_\sigma=R^n_\sigma(\tau^n+\cdot)$, $\sigma\in\{+,-\}$. Since this holds for any $n\ge0$, and $R^0_\sigma=R_\sigma$, we get $R^n_\sigma=R_\sigma(\mu^{n-1}+\cdot)$, $\sigma\in\{+,-\}$, where $\mu^n=\sum_{k=0}^n\tau^k$, $n\ge0$.

Since B^{n+1}_+ and B^{n+1}_- are independent $(\mathcal{F}^{u,n+1}_t)_{t\ge0}$-Brownian motions, and $\mathcal{F}^{u,n+1}_0=\mathcal{F}^{u,n}_{\tau^n}$, we see that (B^{n+1}_+,B^{n+1}_-) is a planar Brownian motion independent of $\mathcal{F}^{u,n}_{\tau^n}$. Since $\mathcal{F}^{u,n}_{\tau^n}$ contains $\mathcal{F}^{u,k}_{\tau^k}$ for each $k\le n$, and $(B^k_+(t),B^k_-(t))$ is $(\mathcal{F}^{u,k}_t)_{t\ge0}$-adapted, we then conclude that (B^{n+1}_+,B^{n+1}_-) is independent of $(B^k_+(t),B^k_-(t))$, $0\le t<\tau^k$, $0\le k\le n$. Thus, $(B^k_+(t),B^k_-(t))$, $0\le t<\tau^k$, $k\ge0$, form an i.i.d. sequence of stopped planar Brownian motions.

Let $\mu_\infty=\lim\mu_n=\sum_{n=0}^\infty\tau^n$. Since τ^n, $n\ge0$, are i.i.d. positive random variables, we have a.s. $\mu^\infty=\infty$. We now define B_+ and B_- on $\mathbb{R}_+$ such that for $\sigma\in\{+,-\}$,

$$B_\sigma(t)=\sum_{j=0}^{n-1}B^j_\sigma(\tau^j)+B^n_\sigma(t-\mu_{n-1}),\quad\text{if }\mu_{n-1}\le t\le\mu_n,\quad n\ge0.$$

Then B_+ and B_- are independent Brownian motions. Since $R^n_\pm$ and $B^n_\pm$ satisfy (4.17) up to τ^n, we find that $R_\pm$ and $B_\pm$ satisfies (4.17) on $[0,\infty)$, and the proof is done. $\square$

Remark 4.14. The assumption $\rho_0\ge\frac{\kappa}{4}-2$ is used in the proof of Lemma 4.12, which is used twice in the proof of Theorem 4.13, and will also be used later in the proof of Lemma 5.15.

To emphasize the dependence of w_+,w_-,v_0,v_+,v_-, we write $\mathbb{P}^{\underline{\rho}}$ as $\mathbb{P}^{(\rho_0,\rho_+,\rho_-)}_{(w_+,w_-;v_0,v_+,v_-)}$. If $\rho_0=0$, i.e., v_0 does not play the role of a force point, we write the measure as $\mathbb{P}^{(\rho_+,\rho_-)}_{(w_+,w_-;v_+,v_-)}$.

4.4 Transition density

Suppose $R_+(t)$ and $R_-(t)$, $t\ge0$, satisfy the SDE (4.17) on $\mathbb{R}_+$. In this subsection, we are going to use orthogonal polynomials to derive the transition density of $\underline{R}(t)=(R_+(t),R_-(t))$, $t\ge0$, against the Lebesgue measure restricted to $[0,1]^2$. A similar approach was first used in [24, Appendix B] to calculate the transition density of radial Bessel processes, where one-variable orthogonal polynomials was used. Two-variable orthogonal polynomials was used in [22, Section 5] to calculate the transition density of a two-dimensional diffusion process. Here we will use another family of two-variable orthogonal polynomials to calculate the transition density of the ($\underline{R}$) here. In addition, we are going to derive the invariant density of ($\underline{R}$), and estimate the convergence of the transition density to the invariant density.

Recall that $X := R_+ - R_-$ and $Y := 1 - R_+R_-$ satisfy (4.18,4.19,4.20) throughout $\mathbb{R}_+$, and (X,Y) a.s. stays in $\overline{\Delta} \setminus \{(0,1)\}$. We will first find the transition density of $((X(t),Y(t)))$. Assume that the transition density $p(t,(x,y),(x^*,y^*))$ exists, and is smooth in (x,y), then it should be a solution to the PDE

$$-\partial_t p + \mathcal{L}p = 0, \tag{4.23}$$

where $\mathcal{L}$ is the second order differential operator defined by

$$\mathcal{L} = \frac{\kappa}{2}(y - x^2)\partial_x^2 + \kappa x(1-y)\partial_x\partial_y + \frac{\kappa}{2}y(1-y)\partial_y^2$$

$$-[(\rho_+ + \rho_- + \rho_0 + 6)x + (\rho_+ - \rho_-)]\partial_x - [(\rho_+ + \rho_- + \rho_0 + 6)y - (\rho_+ + \rho_- + 4)]\partial_y.$$

We perform a change of coordinate $(x,y) \mapsto (r,h)$ by $x = rh$ and $y = h$ (for $y \neq 0$). Direct calculation shows that

$$\partial_r = h\partial_x, \quad \partial_h = r\partial_x + \partial_y, \quad \partial_r^2 = h^2\partial_x^2, \quad \partial_h^2 = r^2\partial_x^2 + 2r\partial_x\partial_y + \partial_y^2, \quad \partial_r\partial_h = rh\partial_x^2 + h\partial_x\partial_y.$$

Let

$$\alpha_0 = \frac{2}{\kappa}(\rho_0 + 2) - 1, \quad \alpha_\pm = \frac{2}{\kappa}(\rho_\pm + 2) - 1, \quad \beta = \alpha_+ + \alpha_- + 1;$$

$$\lambda_n = -n(n + \alpha_0 + \beta + 1), \quad \lambda_n^{(r)} = -n(n+\beta), \quad n \geq 0.$$

Define two differential operators for the coordinate (r,h) by

$$\mathcal{L}^{(r)} = (1 - r^2)\partial_r^2 - [(\alpha_+ + \alpha_- + 2)r + (\alpha_+ - \alpha_-)]\partial_r;$$

$$\mathcal{L}^{(h)} = h(1-h)\partial_h^2 - [(\alpha_0 + \beta + 2)h - (\beta + 1)]\partial_h.$$

Direct calculation shows that, when $y \neq 0$, $\mathcal{L} = \frac{\kappa}{2}[\mathcal{L}^{(h)} + \frac{1}{h}\mathcal{L}^{(r)}]$, and

$$[\mathcal{L}^{(h)} + \frac{1}{h}\lambda_n^{(r)}]h^n = h^n[\mathcal{L}^{(h)} - 2n(h-1)\partial_h + \lambda_n],$$

where each h^n in the formula is understood as a multiplication operator. From (2.5) we know that Jacobi polynomials $P_n^{(\alpha_+,\alpha_-)}(r)$, $n \geq 0$, satisfy that

$$\mathcal{L}^{(r)}P_n^{(\alpha_+,\alpha_-)}(r) = \lambda_n^{(r)}P_n^{(\alpha_+,\alpha_-)}(r), \quad n = 0,1,2,\dots;$$

and the functions $P_m^{(\alpha_0,\beta+2n)}(2h-1)$, $m \geq 0$, satisfy that

$$(\mathcal{L}^{(h)} - 2n(h-1)\partial_h + \lambda_n)P_m^{(\alpha_0,\beta+2n)}(2h-1) = \lambda_{m+n}P_m^{(\alpha_0,\beta+2n)}(2h-1), \quad m = 0,1,2,\cdots.$$

For $n \geq 0$, define a two-variable polynomial $Q_n^{(\alpha_+,\alpha_-)}(x,y)$ such that

$$Q_n^{(\alpha_+,\alpha_-)}(x,y) = y^n P_n^{(\alpha_+,\alpha_-)}(x/y), \quad \text{if } y \neq 0.$$

Such $Q_n^{(\alpha_+,\alpha_-)}(x,y)$ is homogeneous of degree n with nonzero coefficient for x^n. For every pair of integers $n,m\ge 0$, define a two-variable polynomial $v_{n,m}(x,y)$ of degree $n+m$ by

$$v_{n,m}(x,y)=P_m^{(\alpha_0,\beta+2n)}(2y-1)Q_n^{(\alpha_+,\alpha_-)}(x,y).$$

Then $v_{n,m}$ is also a polynomial in r,h with the expression:

$$v_{n,m}=h^nP_m^{(\alpha_0,\beta+2n)}(2h-1)P_n^{(\alpha_+,\alpha_-)}(r). \tag{4.24}$$

From the above displayed formulas, we find that, on $\mathbb{R}^2\setminus\{y\neq 0\}$,

$$\frac{2}{\kappa}\mathcal{L}v_{n,m}=[\mathcal{L}^{(h)}+\frac{1}{h}\lambda_n^{(r)}](h^nP_m^{(\alpha_0,\beta+2n)}(2h-1)P_n^{(\alpha_+,\alpha_-)}(r))$$
$$=h^n[\mathcal{L}^{(h)}-2n(h-1)\partial_h+\lambda_n](P_m^{(\alpha_0,\beta+2n)}(2h-1)P_n^{(\alpha_+,\alpha_-)}(r))=\lambda_{n+m}v_{n,m}.$$

Since $v_{n,m}$ is a polynomial in x,y, by continuity the above equation holds throughout $\mathbb{R}^2$. Thus, for every $n,m\ge 0$, $v_{n,m}(x,y)e^{\frac{\kappa}{2}\lambda_{n+m}t}$ solves (4.23), and the same is true for any linear combination of such functions. From (4.24) we get an upper bound of $\|v_{n,m}\|_\infty:=\sup_{(x,y)\in\Delta}|v_{n,m}(x,y)|$:

$$\|v_{n,m}\|_\infty\le\|P_m^{(\alpha_0,\beta+2n)}\|_\infty\|P_n^{(\alpha_+,\alpha_-)}\|_\infty. \tag{4.25}$$

Since $P_n^{(\alpha_+,\alpha_-)}$, $n\ge 0$, are mutually orthogonal w.r.t. the weight function $\Psi^{(\alpha_+,\alpha_-)}$, and for any fixed $n\ge 0$, $P_m^{(\alpha_0,\beta+2n)}(2h-1)$, $m\ge 0$, are mutually orthogonal w.r.t. the weight function $\Psi^{(\alpha_0,\beta+2n)}(2h-1)=\mathbf{1}_{(0,1)}(h)2^{\alpha_0+\beta+2n}(1-h)^{\alpha_0}h^{\beta+2n}$, we conclude that $v_{n,m}(x,y)$, $n,m\in\mathbb{N}\cup\{0\}$, are mutually orthogonal w.r.t. the weight function

$$\Psi(x,y):=\mathbf{1}_\Delta(x,y)\frac{1}{y}\Big(1-\frac{x}{y}\Big)^{\alpha_+}\Big(1+\frac{x}{y}\Big)^{\alpha_-}(1-y)^{\alpha_0}y^\beta$$
$$=\mathbf{1}_\Delta(x,y)(y-x)^{\alpha_+}(y+x)^{\alpha_-}(1-y)^{\alpha_0}.$$

Moreover, we have

$$\|v_{n,m}\|_\Psi^2=2^{-(\alpha_0+\beta+2n+1)}\|P_m^{(\alpha_0,\beta+2n)}\|^2_{\Psi^{(\alpha_0,\beta+2n)}}\cdot\|P_n^{(\alpha_+,\alpha_-)}\|^2_{\Psi^{(\alpha_+,\alpha_-)}}. \tag{4.26}$$

Let $f(x,y)$ be a polynomial in two variables. Then f can be expressed by a linear combination $f(x,y)=\sum_{n=0}^\infty\sum_{m=0}^\infty a_{n,m}v_{n,m}(x,y)$ (note that every polynomial in x,y of degree less than k can be expressed as a linear combination of $v_{n,m}$ with $n+m<k$), where $a_{n,m}:=\langle f,v_{(n,m)}\rangle_\Psi/\|v_{n,m}\|_\Psi^2$ are zero for all but finitely many (n,m). Define

$$f(t,(x,y))=\sum_{n=0}^\infty\sum_{m=0}^\infty a_{n,m}v_{n,m}(x,y)e^{\frac{\kappa}{2}\lambda_{n+m}t}=\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{\langle f,v_{n,m}\rangle_\Psi}{\|v_{n,m}\|_\Psi^2}\cdot v_{n,m}(x,y)e^{\frac{\kappa}{2}\lambda_{n+m}t}.$$

Then $f(t,(r,s))$ solves (4.23). Let $(X(t),Y(t))$ be a stochastic process in Δ, which solves (4.18,4.19,4.20) with initial value (x,y). Fix $t_0>0$ and define $M_t=f(t_0-t,(X(t),Y(t)))$, $0\le t\le t_0$. By Itô's formula, (M_t) is a bounded martingale, which implies that

$$\mathbb{E}[f(X(t_0),Y(t_0))]=\mathbb{E}[M_{t_0}]=M_0=f(t_0,(x,y))$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\iint_{\Delta} f(x^*,y^*)\Psi(x^*,y^*)\frac{v_{n,m}(x^*,y^*)v_{n,m}(x,y)}{\|v_{n,m}\|_\Psi^2}\cdot e^{\frac{\kappa}{2}\lambda_{n+m}t_0}dx^*dy^*. \tag{4.27}$$

For $t>0$, $(x,y)\in\overline{\Delta}$, and $(x^*,y^*)\in\Delta$, define

$$p(t,(x,y),(x^*,y^*)) = \mathbf{1}_\Delta(x^*,y^*)\Psi(x^*,y^*)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{\frac{\kappa}{2}\lambda_{n+m}t}\frac{v_{n,m}(x,y)v_{n,m}(x^*,y^*)}{\|v_{n,m}\|_\Psi^2}. \tag{4.28}$$

Let $p(x^*,y^*) = C_\Psi \mathbf{1}_\Delta(x^*,y^*)\Psi(x^*,y^*)$, where $C_\Psi = 1/\|v_{n,m}\|_\Psi^2$. Note that $\lambda_0=0$ and $v_{0,0}\equiv 1$ since $P_0^{\alpha_0,\beta} = P_0^{\alpha_+,\alpha_-}\equiv 1$. So $p(x^*,y^*)$ corresponds to the first term in the series.

Lemma 4.15. *For any $t_0>0$, the series in (4.28) (without the factor $\Psi(x^*,y^*)$) converges uniformly on $[t_0,\infty)\times\overline{\Delta}\times\Delta$, and there is $C_{t_0}\in(0,\infty)$ depending only on κ, $\underline{\rho}$, and t_0 such that for any $(x,y)\in\overline{\Delta}$ and $(x^*,y^*)\in\Delta$,*

$$|p(t,(x,y),(x^*,y^*)) - p(x^*,y^*)| \le C_{t_0}e^{-(\rho_++\rho_-+\rho_0+6)t}\Psi(x^*,y^*), \quad t\ge t_0. \tag{4.29}$$

Moreover, for any $t>0$ and $(x^*,y^*)\in\Delta$,

$$p(x^*,y^*) = \iint_\Delta p(x,y)p(t,(x,y),(x^*,y^*))dxdy. \tag{4.30}$$

Proof. The uniform convergence of the series in (4.28) follows from (4.29), which in turn follows from Stirling's formula, (4.25,4.26,2.4,2.7), and the facts that $0>\lambda_1 = -\frac{2}{\kappa}(\rho_++\rho_-+\rho_0+6)>\lambda_n$ for any $n>1$ and $\lambda_n \asymp -n^2$ for big n. Formula (4.30) follows from the orthogonality of $v_{n,m}$ w.r.t. $\langle\cdot,\cdot\rangle_\Psi$ and the uniform convergence of the series in (4.28). □

Lemma 4.16. *The process $((X(t),Y(t)))$ that satisfies (4.18,4.19,4.20) has a transition density: $p(t,(x,y),(x^*,y^*))$, and an invariant density: $p(x^*,y^*)$.*

Proof. Fix $(x,y)\in\overline{\Delta}\setminus\{(0,1)\}$. Let $(X(t),Y(t))$ be the process that satisfies (4.18,4.19,4.20) with initial value (x,y). It suffices to show that, for any continuous function f on $\overline{\Delta}$, we have

$$\mathbb{E}[f(X(t_0),Y(t_0))] = \iint_\Delta p_{t_0}((x,y),(x^*,y^*))f(x^*,y^*)dx^*dy^*. \tag{4.31}$$

By Stone-Weierstrass theorem, f can be approximated by a polynomial in two variables uniformly on Δ. Thus, it suffices to show that (4.31) holds whenever f is a polynomial in x,y, which follows immediately from (4.27). The statement on $p(x^*,y^*)$ follows from (4.30). □

Since $X=R_+-R_-$, $Y=1-R_+R_-$, and the Jacobian of the transformation is $-(R_++R_-)$, we arrive at the following result.

Corollary 4.17. *The process* $(\underline{R}(t))$ *has a transition density:*

$$p^R(t,\underline{r},\underline{r}^*) := \mathbf{1}_{(0,1)^2}(\underline{r}^*)\cdot p(t,(r_+-r_-,1-r_+r_-),(r_+^*-r_-^*,1-r_+^*r_-^*))\cdot(r_+^*+r_-^*),$$

and an invariant density: $p^R(\underline{r}^*) := \mathbf{1}_{(0,1)^2}(\underline{r}^*)\cdot p(r_+^*-r_-^*,1-r_+^*r_-^*)\cdot(r_+^*+r_-^*)$*; and for any* $t_0>0$*, there is* $C_{t_0}\in(0,\infty)$ *depending only on* κ*,* $\underline{\rho}$*, and* t_0 *such that for any* $\underline{r}\in[0,1]^2$ *and* $\underline{r}^*\in(0,1)^2$*,*

$$|p^R(t,\underline{r},\underline{r}^*)-p^R(\underline{r}^*)|\le C_{t_0}e^{-(\rho_++\rho_-+\rho_0+6)t}p^R(\underline{r}^*),\quad t\ge t_0.$$

5 Other Commuting Pair of SLE Curves

In this section, we study three commuting pairs of SLE_κ-type curves, and compare them with the commuting $\mathrm{SLE}_\kappa(\underline{\rho})$ curves in the previous section. It turns out that each of them is "locally" absolutely continuous w.r.t. a commuting pair of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curves for some suitable force values. So the results in the previous section can be applied here.

5.1 Two curves in a 2-SLE_κ

First, we consider 2-SLE_κ. Let $\kappa\in(0,8)$. Let $v_-<w_-<w_+<v_+\in\mathbb{R}$. Suppose that $(\widehat\eta_+,\widehat\eta_-)$ is a 2-SLE_κ in $\mathbb{H}$ with link pattern $(w_+\to v_+;w_-\to v_-)$. Then for $\sigma\in\{+,-\}$, $\widehat\eta_\sigma$ is an hSLE_κ curve in $\mathbb{H}$ from w_σ to v_σ with force points $w_{-\sigma}$ and $v_{-\sigma}$.

Stop $\widehat\eta_+$ and $\widehat\eta_-$ at the first time that they disconnect ∞ from any of its force points, and parametrize the stopped curves by $\mathbb{H}$-capacity. Then we get two chordal Loewner curves, which are denoted by η_+ and η_-. For $\sigma\in\{+,-\}$, η_σ is an hSLE_κ curve in $\mathbb{H}$ from w_σ to v_σ with force points $w_{-\sigma}$ and $v_{-\sigma}$, in the chordal coordinate. Let $\widehat w_\sigma(t)$, $0\le t<T_\pm$ (the lifetime), be the chordal Loewner driving function for $\eta_\pm$; let $K_\sigma(\cdot)$ be the chordal Loewner hulls driven by $\widehat w_\sigma$; and let $(\mathcal{F}^\sigma_t)_{t\ge0}$ be the filtration generated by η_σ. For $\sigma\in\{+,-\}$, if $\tau_{-\sigma}$ is a stopping time for $\eta_{-\sigma}$, then conditionally $\mathcal{F}^{-\sigma}_{\tau_{-\sigma}}$ and the event that $\tau_{-\sigma}<T_{-\sigma}$, the whole η_σ and the part of $\widehat\eta_{-\sigma}$ after $\eta(\tau_{-\sigma})$ together form a 2-SLE_κ in $\mathbb{H}\setminus K_{-\sigma}(\tau_{-\sigma})$ with link pattern $(w_\sigma\to v_\sigma;\eta_{-\sigma}(\tau_{-\sigma})\to v_{-\sigma})$. Thus, the conditional law of $\widehat\eta_\sigma$ is that of an hSLE_κ curve from w_σ to v_σ in $\mathbb{H}\setminus K_{-\sigma}(\tau_{-\sigma})$ with force points $\eta_{-\sigma}(\tau_{-\sigma})$ and $v_{-\sigma}$. This implies that there a.s. exists a chordal Loewner curve $\eta_{\sigma,\tau_{-\sigma}}$ with some speed such that $\eta_\sigma=f_{K_{-\sigma}(\tau_{-\sigma})}\circ\eta_{\sigma,\tau_{-\sigma}}$, and the conditional law of the normalization of $\eta_{\sigma,\tau_{-\sigma}}$ given $\mathcal{F}^{-\sigma}_{\tau_{-\sigma}}$ is that of an hSLE_κ curve in $\mathbb{H}$ from $g_{K_{-\sigma}(\tau_{-\sigma})}(w_\sigma)$ to $g_{K_{-\sigma}(\tau_{-\sigma})}(v_\sigma)$ with force points $\widehat w_{-\sigma}(\tau_{-\sigma})$ and $g_{K_{-\sigma}(\tau_{-\sigma})}(v_{-\sigma})$, in the chordal coordinate.

Thus, (η_+,η_-) a.s. satisfies the conditions in Definition 3.2 with $\mathcal{D}_1:=\mathcal{I}_+\times\mathcal{I}_-$, $I_\sigma=[0,T_\sigma)$ and $\mathcal{I}^*_\sigma=\mathcal{I}_\sigma\cap\mathbb{Q}$, $\sigma\in\{+,-\}$. So $(\eta_+,\eta_-;\mathcal{D}_1)$ is a.s. a commuting pair of chordal Loewner curves. We now adopt the functions from Section 3. Define a function M_1 on $\mathcal{D}_1$ by

$$M_1=\prod_{\sigma\in\{+,-\}}\Big(|W_\sigma-V_\sigma|^{\frac{8}{\kappa}-1}|W_\sigma-V_{-\sigma}|^{\frac{4}{\kappa}}\Big)\cdot F_{\kappa,2}\Big(\frac{(W_+-W_-)(V_+-V_-)}{(W_+-V_-)(V_+-W_-)}\Big)^{-1}. \tag{5.1}$$

Since $F_{\kappa,2}$ is continuous and positive on $[0,1]$, $|W_\sigma - V_\sigma|, |W_\sigma - V_{-\sigma}| \le |V_+ - V_-|$, and $\frac{8}{\kappa}-1, \frac{4}{\kappa} > 0$, we get an upper bound of M_1 as follows, where $C>0$ depends only on κ:

$$M_1 \le C|V_+ - V_-|^{2(\frac{12}{\kappa}-1)}. \tag{5.2}$$

Let $\mathcal{F}_{(t_+,t_-)} = \mathcal{F}^+_{t_+} \vee \mathcal{F}^-_{t_-}$ for $(t_+,t_-) \in \mathbb{R}_+^2$. We will prove that M_1 extends to continuously $\mathbb{R}_+^2$, and becomes $(\mathcal{F}_{\underline{t}})$-martingale, which acts as Radon-Nikodym derivatives between measures. We first need some deterministic properties of M_1.

For $\sigma \in \{+,-\}$ and $R > |v_+ - v_-|/2$, let τ^σ_R be the first time that $|\eta_\sigma(t) - (v_+ + v_-)/2| = R$ if such time exists; otherwise $\tau^\sigma_R = T_\sigma$. Let $\underline{\tau}_R = (\tau^+_R, \tau^-_R)$. Note that $\tau^+_R, \tau^-_R \le \mathrm{m}(\underline{\tau}_R) \le R^2/2$ because if $K \subset \{z \in \mathbb{H} : |z - (v_+ + v_-)/2| \le R\}$, then $\mathrm{hcap}_2(K) \le R^2/2$.

Lemma 5.1. *M_1 a.s. extends continuously to $\mathbb{R}_+^2$ with $M_1 \equiv 0$ on $\mathbb{R}_+^2 \setminus \mathcal{D}_1$.*

Proof. It suffices to show that for $\sigma \in \{+,-\}$, as $t_\sigma \uparrow T_\sigma$, $M_1 \to 0$ uniformly in $t_{-\sigma} \in [0, T_{-\sigma})$. By symmetry, we may assume that $\sigma = +$. For a fixed $t_- \in [0,T_-)$, as $t_+ \uparrow T_+$, $\eta_+(t_+)$ tends to either some point on $[v_+,\infty)$ or some point on $(-\infty, v_-)$. We know that $F_{\kappa,2}$ is continuous and positive on $[0,1]$. So the factor $F_{\kappa,2}\Big(\frac{(W_+-W_-)(V_+-V_-)}{(W_+-V_-)(V_+-W_-)}\Big)^{-1}$ is uniformly bounded on $\mathcal{D}_1$. Since the union of (the whole) η_+ and η_- is bounded, by (3.14) $|V_+ - V_-|$ is bounded on $\mathcal{D}_1$, which implies that $|W_\pm - V_\pm|$ and $|W_\pm - V_\mp|$ are also bounded on $\mathcal{D}_1$. Thus, it suffices to show that when η_+ terminates at $[v_+,\infty)$, $W_+ - V_+ \to 0$ as $t_+ \uparrow T_+$, uniformly in $[0,T_-)$; and when η_+ terminates at $(-\infty, v_-)$, $W_- - V_- \to 0$ as $t_+ \uparrow T_+$, uniformly in $[0,T_-)$.

For any $\underline{t} = (t_+,t_-) \in \mathcal{D}_1$, neither $\eta_+([0,t_+])$ nor $\eta_-([0,t_-])$ hit $(-\infty, v_-] \cup [v_+,\infty)$, which implies that $v_+, v_- \not\in \overline{K(\underline{t})}$ and $V_\pm(\underline{t}) = g_{K(\underline{t})}(v_\pm)$. Suppose that η_+ terminates at $x_0 \in [v_+,\infty)$. Since SLE_κ is not boundary-filling for $\kappa \in (0,8)$, we know that $\mathrm{dist}(x_0,\eta_-) > 0$. Let $r = \min\{|w_+ - v_+|, \mathrm{dist}(x_0,\eta_-)\} > 0$. Fix $\varepsilon \in (0,r)$. Since $x_0 = \lim_{t\uparrow T_+} \eta_+(t)$, there is $\delta > 0$ such that $|\eta_+(t) - x_0| < \varepsilon$ for $t \in (T_+ - \delta, T_+)$. Fix $t_+ \in (T_+ - \delta, T_+)$ and $t_- \in [0,T_-)$. Let J be the connected component of $\{|z - x_0| = \varepsilon\} \cap (\mathbb{H} \setminus K(\underline{t}))$ whose closure contains $x_0 + \varepsilon$. Then J disconnects v_+ and $\eta_+(t_+,T_+)$ from ∞ in $\mathbb{H} \setminus K(\underline{t})$. Thus, $g_{K(\underline{t})}(J)$ disconnects $V_+(\underline{t})$ and $W_+(\underline{t})$ from ∞. Since $\eta_+ \cup \eta_-$ is bounded, there is a (random) $R \in (0,\infty)$ such that $\eta_+ \cup \eta_- \subset \{|z - x_0| < R\}$. Let $\xi = \{|z - x_0| = 2R\} \cap \mathbb{H}$. By comparison principle, the extremal length ([1]) of the family of curves in $\mathbb{H}\setminus K(\underline{t})$ that separate J from ξ is bounded above by $\frac{\pi}{\log(R/\varepsilon)}$. By conformal invariance, the extremal length of the family of curves in $\mathbb{H}$ that separate $g_{K(\underline{t})}(J)$ from $g_{K(\underline{t})}(\xi)$ is also bounded above by $\frac{\pi}{\log(R/\varepsilon)}$. Now $g_{K(\underline{t})}(\xi)$ and $g_{K(\underline{t})}(J)$ are crosscuts of $\mathbb{H}$ such that the former encloses the latter. Let D denote the subdomain of $\mathbb{H}$ bounded by $g_{K(\underline{t})}(\xi)$. From Proposition 2.3 we know that $D \subset \{|z - x_0| \le 5R\}$. So the Euclidean area of D is less than $13\pi R^2$. By the definition of extremal length, there is a curve γ in D that separates $g_{K(\underline{t})}(J)$ from $g_{K(\underline{t})}(\xi)$ with Euclidean distance less than $2\sqrt{13\pi R^2 * \frac{\pi}{\log(R/\varepsilon)}} < 8\pi R * \log(R/\varepsilon)^{-1/2}$. Since $g_{K(\underline{t})}(J)$ disconnects $V_+(\underline{t})$ and $W_+(\underline{t})$ from ∞, γ also separates $V_+(\underline{t})$ and $W_+(\underline{t})$ from ∞. Thus, $|W_+(\underline{t}) - V_+(\underline{t})| < 8\pi R * \log(R/\varepsilon)^{-1/2}$ if $t_+ \in (T_+ - \delta, T_+)$ and $t_- \in [0,T_-)$. This proves the uniform convergence of $\lim_{t_+\uparrow T_+} |W_+ - V_+| = 0$ in $t_- \in [0,T_-)$ in the case that

$\lim_{t_+\uparrow T_+}\eta_+(t_+)\in[v_+,\infty)$. The proof of the uniform convergence of $\lim_{t_+\uparrow T_+}|W_+-V_-|=0$ in $t_-\in[0,T_-)$ in the case that $\lim_{t_+\uparrow T_+}\eta_+(t_+)\in(-\infty,v_-]$ is similar. $\square$

From now on, we understand M_1 as a continuous stochastic process defined on $\mathbb{R}_+^2$ with constant zero on $\mathbb{R}_+^2\setminus\mathcal{D}_1$.

Lemma 5.2. *Let $R>0$. Then $M_1(\underline{t}\wedge\underline{\tau}_R)$, $\underline{t}\in\mathbb{R}_+^2$, is an $M_1(\underline{\tau}_R)$-Doob martingale w.r.t. the filtration $(\mathcal{F}^+_{t_+\wedge\tau_R^+}\vee\mathcal{F}^-_{t_-\wedge\tau_R^-})_{(t_+,t_-)\in\mathbb{R}_+^2}$. Moreover, if the underlying probability measure is weighted by $M_1(\underline{\tau}_R)/M_1(\underline{0})$, then the new law of $(\widehat{w}_+,\widehat{w}_-)$ agrees with the $\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}$ on the σ-algebra $\mathcal{F}^+_{\tau_R^+}\vee\mathcal{F}^-_{\tau_R^-}$.*

Proof. Fix $t_-\ge 0$. Let $\widehat{\tau}^R_-=t_-\wedge\tau_R^-$, $u(t)=\mathrm{m}(t,\widehat{\tau}^R_-)-\mathrm{m}(0,\widehat{\tau}^R_-)$, and $\widetilde{\eta}_{+,\widehat{\tau}_R^-}=\eta_{+,\widehat{\tau}_R^-}\circ u^{-1}$. Then $\widetilde{\eta}_{+,\widehat{\tau}_R^-}$ is the normalization of $\eta_{+,\widehat{\tau}-_R}$, and the conditional law of $\widetilde{\eta}_{+,\widehat{\tau}_R^-}$ given $\mathcal{F}^-_{\widehat{\tau}_R^-}$ is that of an hSLE$_\kappa$ curve in $\mathbb{H}$ from $W_+(0,\widehat{\tau}_R^-)$ to $V_+(0,\widehat{\tau}_R^-)$ with force points $W_-(0,\widehat{\tau}_R^-)$ and $V_-(0,\widehat{\tau}_R^-)$, in the chordal coordinate. Moreover, the driving function for $\widetilde{\eta}_{+,\widehat{\tau}_R^-}$ is $W_+(u^{-1}(t),\widehat{\tau}_R^-)$, and by Lemmas 3.13 and 3.12, the force point functions started from $V_+(0,\widehat{\tau}_R^-)$, $W_-(0,\widehat{\tau}_R^-)$ and $V_-(0,\widehat{\tau}_R^-)$ are $V_+(u^{-1}(t),\widehat{\tau}_R^-)$, $W_-(u^{-1}(t),\widehat{\tau}_R^-)$ and $V_-(u^{-1}(t),\widehat{\tau}_R^-)$, respectively. Thus, $M_1(u^{-1}(t),\widehat{\tau}_R^-)$ agrees with the M given in Proposition 2.20 with $\rho=2$, $w_0=W_+(0,\widehat{\tau}_R^-)$, $w_\infty=V_+(0,\widehat{\tau}_R^-)$, $v_1=W_-(\cdot,\widehat{\tau}_R^-)$ and $v_2=V_-(\cdot,\widehat{\tau}_R^-)$.

For $t\ge 0$, let $\widetilde{\mathcal{F}}_t$ denote the σ-algebra generated by $\mathcal{F}^-_{\widehat{\tau}_R^-}$ and $\widetilde{\eta}_{+,\widehat{\tau}_R^-}(s)$, $0\le s\le t$. Let $\widetilde{T}_+$ denote the lifetime of $\widetilde{\eta}_{+,\widehat{\tau}_R^-}$. Then u maps $[0,T_+)$ onto $[0,\widetilde{T}_+)$. By Proposition 2.20, $M_1(u^{-1}(t),\widehat{\tau}_R^-)$, $0\le t<\widetilde{T}_+$, is a local martingale w.r.t. the filtration $(\widetilde{\mathcal{F}}_t)_{t\ge 0}$. By the definition of $\widetilde{\eta}_{+,\widehat{\tau}_R^-}$, for any $0\le t<T_+$, $\eta_+(t)=f_{K_-(\widehat{\tau}_R^-)}\circ\widetilde{\eta}_{+,\widehat{\tau}_R^-}(u(t))$. Extend u to $\mathbb{R}_+$ such that if $t\ge T_+$ then $u(t)=\widetilde{T}_+$. Then for every $t\ge 0$, $u(t)$ is an $(\widetilde{\mathcal{F}}_t)$-stopping time because for any $a\ge 0$, $u(t)>a$ if and only if $a<\widetilde{T}_+$ and $\mathrm{hcap}_2(\mathrm{Hull}(f_{K_-(\widehat{\tau}_R^-)}\circ\widetilde{\eta}_{+,\widehat{\tau}_R^-}([0,a])))<t$. So we get a filtration $(\widetilde{\mathcal{F}}_{u(t)})_{t\ge 0}$, and $M_1(t,\widehat{\tau}_R^-)$, $0\le t<T_+$, is an $(\widetilde{\mathcal{F}}_{u(t)})_{t\ge 0}$-local martingale.

From $\eta_+(t)=f_{K_-(\widehat{\tau}_R^-)}\circ\widetilde{\eta}_{+,\widehat{\tau}_R^-}(u(t))$, $0\le t<T_+$, we know that $\mathcal{F}^+_t\vee\mathcal{F}^-_{\widehat{\tau}_R^-}\subset\widetilde{\mathcal{F}}_{u(t)}$ for $t\ge 0$. Since τ_R^+ is an $(\mathcal{F}^+_t)_{t\ge 0}$-stopping time, it is also an $(\widetilde{\mathcal{F}}_{u(t)})_{t\ge 0}$-stopping time. Since $\widehat{\tau}_R^-\le\tau_R^-$, by the boundedness of M_1 on $[\underline{0},\underline{\tau}_R]$, $M_1(t\wedge\tau_R^+,\widehat{\tau}_R^-)$, $t\ge 0$, is a bounded $(\widetilde{\mathcal{F}}_{u(t)})_{t\ge 0}$-martingale. Since $\mathcal{F}^+_{t_+\wedge\tau_R^+}\vee\mathcal{F}^-_{\widehat{\tau}_R^-}\subset\widetilde{\mathcal{F}}_{u(t_+)}$ and $\widehat{\tau}_R^-=t_-\wedge\tau_R^-$, we conclude that $M_1(t_+\wedge\tau_R^+,t_-\wedge\tau_R^-)$, $t_+\ge 0$, is a bounded $(\mathcal{F}^+_{t_+\wedge\widehat{\tau}_R^+}\vee\mathcal{F}^-_{t_-\wedge\tau_R^-})_{t_+\ge 0}$-martingale. This holds for any $t_-\ge 0$. Symmetrically, for any $t_+\ge 0$, $M_1(t_+\wedge\tau_R^+,t_-\wedge\tau_R^-)$, $t_-\ge 0$, is a bounded $(\mathcal{F}^+_{t_+\wedge\tau_R^+}\vee\mathcal{F}^-_{t_-\wedge\tau_R^-})_{t_-\ge 0}$-martingale. Thus, $M_1(\underline{t}\wedge\underline{\tau}_R)$, $\underline{t}\in\mathbb{R}_+^2$, is a bounded $(\mathcal{F}^+_{t_+\wedge\tau_R^+}\vee\mathcal{F}^-_{t_-\wedge\tau_R^-})_{(t_+,t_-)\in\mathbb{R}_+^2}$-martingale. Since $M_1(\underline{t}\wedge\underline{\tau}_R)\to M_1(\underline{\tau}_R)$ as $t_+,t_-\to\infty$, $M_1(\underline{t}\wedge\underline{\tau}_R)$ is an $M_1(\underline{\tau}_R)$-Doob martingale.

By weighting the underlying probability measure by $M_1(\underline{\tau}_R)/M_1(\underline{0})$, we get another probability measure. To describe the joint law of $\widehat{w}_+$ and $\widehat{w}_-$ restricted to $\mathcal{F}_{\underline{\tau}_R}$ under the new probability measure, we study the new marginal law of η_- up to τ_R^- and the new conditional law of η_+ up to τ_R^+ given that part of η_-. We may do the weighting in two steps. First, weight the original measure by $N_1 := M_1(0,\tau_R^-)/M_1(0,0)$ to get a new measure $\mathbb{P}_1$; second, weight $\mathbb{P}_1$ by $N_2 := M_1(\tau_R^+,\tau_R^-)/M_1(0,\tau_R^-)$ to get $\mathbb{P}_2$. Since N_1 depends only on η_-, after the first step, the conditional law of η_+ given any part of η_- does not change. By Proposition 2.20, the η_- up to τ_R^- under $\mathbb{P}_1$ is a chordal $\mathrm{SLE}_\kappa(2,2,2)$ curve in $\mathbb{H}$ started from w_- with force points v_-, w_+, v_+, respectively, up to τ_R^-. Since $N_1 = 0$ when $\tau_R^- = T_-$, $\mathbb{P}_1$ is supported by $\{\tau_R^- < T_-\}$, on which $M_1(0,\tau_R^-) > 0$. So N_2 is $\mathbb{P}_1$-a.s. well defined. Since $\mathbb{E}[N_2|\mathcal{F}^-_{\tau_R^-}] = 1$, after the second step, the law of η_- up to τ_R^- does not change further. To describe the conditional law of η_+ up to $\tau_R^+ = \tau_R^+(\eta_+)$ given η_- up to τ_R^-, it suffices to consider the conditional law of η_{+,τ_R^-} up to $\tau_R^+(\eta_+)$ since we may recover η_+ using $\eta_+ = f_{K_-(\tau_R^-)} \circ \eta_{+,\tau_R^-}$. By Proposition 2.20 again, the conditional law of the normalization of η_{+,τ_R^-} up to $\tau_R^+(\eta_+)$ under $\mathbb{P}_2$ is that of a chordal $\mathrm{SLE}_\kappa(2,2,2)$ curve in $\mathbb{H}$ started from $W_+(0,\tau_R^-)$ with force points at $V_+(0,\tau_R^-)$, $W_-(0,\tau_R^-)$ and $V_-(0,\tau_R^-)$, respectively. Thus, under $\mathbb{P}_2$ the joint law of η_+ up to τ_R^+ and η_- up to τ_R^- agrees with that of a commuting pair of $\mathrm{SLE}_\kappa(2,2,2)$ curves started from $(w_+,w_-;v_+,v_-)$ respectively up to τ_R^+ and τ_R^-. This means that $\mathbb{P}_2 = \mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}$ on $\mathcal{F}^+_{t_+\wedge\tau_R^+} \vee \mathcal{F}^-_{t_-\wedge\tau_R^-}$, as desired. $\square$

We let $\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}$ denote the joint law of the driving functions $\widehat{w}_+$ and $\widehat{w}_-$ here. From the lemma, we find that, for any $\underline{t} = (t_+,t_-) \in \mathbb{R}_+^2$ and $R > 0$,

$$\frac{d\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}|(\mathcal{F}^+_{t_+\wedge\tau_R^+} \vee \mathcal{F}^-_{t_-\wedge\tau_R^-})}{d\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}|(\mathcal{F}^+_{t_+\wedge\tau_R^+} \vee \mathcal{F}^-_{t_-\wedge\tau_R^-})} = \frac{M_1(\underline{t}\wedge\underline{\tau}_R)}{M_1(\underline{0})}, \quad R > 0. \tag{5.3}$$

Theorem 5.3. *Under $\mathbb{P}_{(\underline{w}\to\underline{v})}$, $M_1(\underline{t})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingale; and for any extended $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{\tau}$,*

$$\frac{d\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}_{\underline{\tau}} \cap \{\underline{\tau}\in\mathbb{R}_+^2\}}{d\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}|\mathcal{F}_{\underline{\tau}} \cap \{\underline{\tau}\in\mathbb{R}_+^2\}} = \frac{M_1(\underline{\tau})}{M_1(\underline{0})}. \tag{5.4}$$

Proof. Since $\mathcal{F}^+_{t_+\wedge\tau_R^+} \vee \mathcal{F}^-_{t_-\wedge\tau_R^-}$ agrees with $\mathcal{F}^+_{t_+} \vee \mathcal{F}^-_{t_-} = \mathcal{F}_{\underline{t}}$ on $\{\underline{t} \le \underline{\tau}_R\}$, from (5.3) we get

$$\frac{d\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}|(\mathcal{F}_{\underline{t}} \cap \{\underline{t}\le\underline{\tau}_R\})}{d\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}|(\mathcal{F}_{\underline{t}} \cap \{\underline{t}\le\underline{\tau}_R\})} = \frac{M_1(\underline{t})}{M_1(\underline{0})}, \quad \forall \underline{t}\in\mathbb{R}_+^2, \quad R > 0,$$

which implies by sending $R\to\infty$ that

$$\frac{d\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}_{\underline{t}}}{d\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}|\mathcal{F}_{\underline{t}}} = \frac{M_1(\underline{t})}{M_1(\underline{0})}, \quad \forall \underline{t}\in\mathbb{R}_+^2. \tag{5.5}$$

From this we conclude that M_1 is an $(\mathcal{F}_{\underline{t}})$-martingale under $\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}$.

Let $\underline{\tau}$ be an extended $(\mathcal{F}_{\underline{t}})$-stopping time. Fix $A \in \mathcal{F}_{\underline{\tau}}$ such that $A \subset \{\underline{\tau} \in \mathbb{R}_+^2\}$. Let $\underline{t} \in \mathbb{R}_+^2$. Define the $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{\tau}^{\underline{t}}$ as in Proposition 2.28, which gives $A \cap \{\underline{\tau} \le \underline{t}\} \in \mathcal{F}_{\underline{\tau}^{\underline{t}}} \subset \mathcal{F}_{\underline{t}}$. Using (5.5) and applying Proposition 2.31 to the stopping times $\underline{t}, \underline{\tau}^{\underline{t}}$ and the martingale M_1, we get $\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}[A \cap \{\underline{\tau} \le \underline{t}\}] = \mathbb{E}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}[\mathbf{1}_{A\cap\{\underline{\tau}\le\underline{t}\}}\frac{M_1(\underline{\tau})}{M_1(\underline{0})}]$. Sending both coordinates of $\underline{t}$ to ∞, we get $\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}[A] = \mathbb{E}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}[\mathbf{1}_A\frac{M_1(\underline{\tau})}{M_1(\underline{0})}]$. So we get the desired (5.4). □

Corollary 5.4. *For any extended $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{\tau}$,*

$$\frac{d\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}|\mathcal{F}_{\underline{\tau}} \cap \{\underline{\tau} \in \mathcal{D}_1\}}{d\mathbb{P}^{(2,2)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}_{\underline{\tau}} \cap \{\underline{\tau} \in \mathcal{D}_1\}} = \frac{M_1(\underline{\tau})^{-1}}{M_1(\underline{0})^{-1}}.$$

Proof. This follows from Theorem 5.3 and the fact that $M_1 > 0$ on $\mathcal{D}_1$. □

For convenience, we write $\mathbb{P}_1$ for $\mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}$. Assume now that $v_0 := (v_+ + v_-)/2 \in [w_-, w_+]$. We understand v_0 as $w_\sigma^{-\sigma}$ if $(v_+ + v_-)/2 = w_\sigma$, $\sigma \in \{+,-\}$. Let V_0 be the force point function started from v_0. By Section 3.4, we may define the time curve $\underline{u} : [0, T^u) \to \mathcal{D}_1$ such that $V_\sigma(\underline{u}(t)) - V_0(\underline{u}(t)) = e^{2t}(v_\sigma - v_0)$, $0 \le t < T^u$, $\sigma \in \{+,-\}$, and $\underline{u}$ can not be extended beyond T^u with such property. We follow the notation there, for every X defined on $\mathcal{D}$, we use X^u to denote the function $X \circ \underline{u}$ defined on $[0, T^u)$. We also define the processes $R_\sigma = \frac{W^u_\sigma - V^u_0}{V^u_\sigma - V^u_0} \in [0,1]$, $\sigma \in \{+,-\}$, and $\underline{R} = (R_+, R_-)$. Since T_σ is an $(\mathcal{F}^\sigma_t)_{t\ge0}$-stopping time for $\sigma \in \{+,-\}$, $\mathcal{D}_1 = [0, T_+) \times [0, T_-)$ is an $(\mathcal{F}_{\underline{t}})$-stopping region. We now extend $\underline{u}$ to $\mathbb{R}_+$ such that if $s \ge T^u$ then $\underline{u}(s) = \lim_{t\uparrow T^u} \underline{u}(t)$. By Proposition 3.24, for any $t \ge 0$, $\underline{u}(t)$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-stopping time.

Let $I = v_+ - v_0 = v_0 - v_-$. Let $\alpha_1 = 2(\frac{12}{\kappa} - 1)$ and define

$$G_1(r_+, r_-) = \prod_{\sigma\in\{+,-\}} (1 - r_\sigma)^{\frac{8}{\kappa}-1}(1 + r_\sigma)^{\frac{4}{\kappa}} F_{\kappa,2}\Big(\frac{2(r_+ + r_-)}{(1 + r_+)(1 + r_-)}\Big)^{-1}. \tag{5.6}$$

Then $M^u_1(t) = (e^{2t}I)^{\alpha_1} G_1(\underline{R}(t))$ on $[0, T^u)$. So we obtain the following lemma.

We are going to derive the transition density of the process $(\underline{R}(t))_{0\le t<T^u}$ under $\mathbb{P}_1$. In fact, T^u is $\mathbb{P}_1$-a.s. finite, and by saying that $\widetilde{p}^R_1(t, \underline{r}, \underline{r}^*)$ is the transition density of $(\underline{R})$ under $\mathbb{P}_1$, we mean that, if $(\underline{R}(t))$ starts from $\underline{r}$, then for any bounded measurable function f on $(0,1)^2$, and any $t > 0$,

$$\mathbb{E}_1[\mathbf{1}_{\{T^u>t\}} f(\underline{R}(t))] = \int_{[0,1]^2} f(\underline{r}^*)\widetilde{p}^R_1(t, \underline{r}, \underline{r}^*)d\underline{r}^*.$$

Applying Corollary 5.4 to the $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{u}(t)$ for any deterministic $t \ge 0$, and using that $\underline{u}(t) \in \mathcal{D}_1$ iff $t < T^u$, we get

$$\frac{d\mathbb{P}_1|\mathcal{F}^u_t \cap \{T^u > t\}}{d\mathbb{P}^{(2,2)}|\mathcal{F}^u_t \cap \{T^u > t\}} = \frac{M^u_1(t)^{-1}}{M^u_1(0)^{-1}} = e^{-2\alpha_1 t}\frac{G_1(\underline{R}(0))}{G_1(\underline{R}(t))}, \quad t \ge 0.$$

Combining it with Corollary 4.17, we get the following transition density.

Lemma 5.5. *Let $p_1^R(t,\underline{r},\underline{r}^*)$ be the function $p^R(t,\underline{r},\underline{r}^*)$ given in Corollary 4.17 with $\rho_0 = 0$ and $\rho_+ = \rho_- = 2$. Then under $\mathbb{P}_1$, the transition density of $(\underline{R})$ is*

$$\widetilde{p}_1^R(t,\underline{r},\underline{r}^*) := e^{-2\alpha_1 t} p_1^R(t,\underline{r},\underline{r}^*) \cdot \frac{G_1(\underline{r})}{G_1(\underline{r}^*)}.$$

5.2 Opposite pair of iSLE$_\kappa(\rho)$ curves, the generic case

Second, we consider another pair of random curves. Let κ and ρ be as in Proposition 2.21, i.e., $\kappa \in (0,4]$ and $\rho > -2$ or $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 2$. Let $w_- < w_+ \in \mathbb{R}$. Let $v_- \in (-\infty, w_-) \cup \{w_-^-\}$ and $v_+ \in (w_+, \infty) \cup \{w_+^+\}$. Let $\widehat{\eta}_+$ be an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_+ to w_- with force points v_+ and v_-, and let $\widehat{\eta}_-$ be its reversal. Then $\widehat{\eta}_-$ is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_- to w_+ with force points v_- and v_+.

For $\sigma \in \{+,-\}$, stop $\widehat{\eta}_\sigma$ at the first time that it disconnects $w_{-\sigma}$ from ∞, and parametrize the stopped curve by $\mathbb{H}$-capacity. The chordal Loewner curve: $\eta_\sigma(t)$, $0 \leq t < T_\sigma$ (lifetime), is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_σ to $w_{-\sigma}$ with force points v_σ and $v_{-\sigma}$, in the chordal coordinate. Let $\widehat{w}_\sigma$ denote the driving function. We still let $K_\sigma(\cdot)$ and $(\mathcal{F}_t^\sigma)_{t\geq 0}$ denote the $\mathbb{H}$-hulls and the filtration generated by η_σ, $\sigma \in \{+,-\}$, and let $K(t_+,t_-) = \mathrm{Hull}(K_+(t_+) \cup K_-(t_-))$. From the DMP and reversibility of iSLE$_\kappa(\rho)$, we know that, for $\sigma \in \{+,-\}$, if $\tau_{-\sigma}$ is a stopping time for $\eta_{-\sigma}$, then conditionally on $\mathcal{F}_{\tau_{-\sigma}}^{-\sigma}$ and the event that $\tau_{-\sigma} < T_{-\sigma}$, the other curve $\widehat{\eta}_\sigma$ from its beginning up to the time that it hits $\eta(\tau_{-\sigma})$ is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H} \setminus K_{-\sigma}(\tau_{-\sigma})$ from w_σ to $\eta_{-\sigma}(\tau_{-\sigma})$ with force points being v_σ and another point, which is the point on $\{v_{-\sigma}\} \cup \overline{K_{-\sigma}(\tau_{-\sigma})} \cap \mathbb{R}$ that is closest to $(-\sigma)\cdot\infty$. Thus, a.s. there is a chordal Loewner curve $\eta_{\sigma,\tau_{-\sigma}}$ with some speed, such that the part of η_σ up to the time that it disconnects $\eta_{-\sigma}(\tau_{-\sigma})$ from ∞ equals the $f_{K_{-\sigma}(\tau_{-\sigma})}$-image of $\eta_{\sigma,\tau_{-\sigma}}$, and the conditional law of the normalization of $\eta_{\sigma,\tau_{-\sigma}}$ given $\mathcal{F}_{\tau_{-\sigma}}^{-\sigma}$ is that of an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from $g_{K_{-\sigma}(\tau_{-\sigma})}(w_\sigma)$ to $\widehat{w}_{-\sigma}(\tau_{-\sigma})$ with force points $g_{K_{-\sigma}(\tau_{-\sigma})}(v_\sigma)$ and $g_{K_{-\sigma}(\tau_{-\sigma})}^{w_{-\sigma}}(v_{-\sigma})$ (Definition 2.11), in the chordal coordinate.

Thus, a.s. η_+ and η_- satisfy the conditions in Definition 3.2 with $\mathcal{I}_\pm = [0,T_\pm)$, $\mathcal{I}_\pm^* = \mathcal{I}_\pm \cap \mathbb{Q}$, and

$$\mathcal{D}_2(\eta_+,\eta_-) := \{(t_+,t_-) \in \mathcal{I}_+ \times \mathcal{I}_- : \exists \underline{t}' = (t_+',t_-') \in \mathcal{I}_+ \times \mathcal{I}_- \text{ with } t_+' > t_+ \text{ and } t_-' > t_- \text{ such that } K(\cdot,\cdot) \text{ is strictly increasing in both variables on } [\underline{0},\underline{t}']\}, \tag{5.7}$$

which is an HC region. So $(\eta_+,\eta_-;\mathcal{D}_2(\eta_+,\eta_-))$ is a.s. a commuting pair of chordal Loewner curves. Let W_+ and W_- be the driving functions, and let V_+ and V_- be the force point functions started from v_+ and v_-, respectively. Let $(\mathcal{F}_{\underline{t}}^{(+)})_{\underline{t}\in\mathbb{R}_+^2}$ be the right-continuous augmentation of $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$. Then $\mathcal{D}_2(\eta_+,\eta_-)$ is an $(\mathcal{F}_{\underline{t}}^{(+)})$-stopping region.

We now write $\mathcal{D}_2(\eta_+,\eta_-)$ simply as $\mathcal{D}_2$. Define a non-negative function M_2 on $\mathcal{D}_2$ by

$$M_2 = |W_+ - W_-|^{\frac{8}{\kappa}-1} |V_+ - V_-|^{\frac{\rho(2\rho+4-\kappa)}{2\kappa}} \prod_{\sigma\in\{+,-\}} |W_\sigma - V_{-\sigma}|^{\frac{2\rho}{\kappa}} \cdot F_{\kappa,\rho}\Big(\frac{(V_+ - W_+)(W_- - V_-)}{(W_+ - V_-)(V_+ - W_-)}\Big)^{-1}. \tag{5.8}$$

It is well defined and continuous on $\mathcal{D}_2$ because $\frac{8}{\kappa}-1>0$, and $W_+-V_-, V_+-W_-, V_+-V_-$ are not zero, where the latter facts follow from that η_+ does not hit $(-\infty, v_-]$ before T_+ and that η_- does not hit $[v_+,\infty)$ before T_-. We first present some deterministic results on M_2.

Lemma 5.6. *There is a constant $C\in(0,\infty)$ depending only on κ and ρ such that*

$$M_2 \le C\Big(\frac{|W_+-W_-|}{|V_+-V_-|}\Big)^{(\frac{8}{\kappa}-1)\wedge\frac{2\rho-(\kappa-8)}{\kappa}} |V_+-V_-|^{\frac{(\rho+2)(2\rho-(\kappa-8))}{2\kappa}}. \tag{5.9}$$

In particular, since $(\frac{8}{\kappa}-1)\wedge\frac{2\rho-(\kappa-8)}{\kappa}>0$ and $|W_+-W_-|\le|V_+-V_-|$, we get a simpler upper bound: $M_2\le C|V_+-V_-|^{\frac{(\rho+2)(2\rho-(\kappa-8))}{2\kappa}}$; using that $|V_+-V_-|\ge|v_+-v_-|$, we get another upper bound: $M_2\le C'|W_+-W_-|^{(\frac{8}{\kappa}-1)\wedge\frac{2\rho-(\kappa-8)}{\kappa}}|V_+-V_-|^{\frac{(\rho+2)(2\rho-(\kappa-8))}{2\kappa}}$, where $C'\in(0,\infty)$ depends only on $\kappa,\rho,|v_+-v_-|$.

Proof. It suffices to prove (5.9). First, the factor $F_{\kappa,\rho}(\frac{(V_+-W_+)(W_--V_-)}{(W_+-V_-)(V_+-W_-)})^{-1}$ in (5.8) is bounded from below and above by positive constants depending only on κ and ρ because $F_{\kappa,\rho}$ is continuous and positive on $[0,1]$. Since $V_-\le W_-\le W_+\le V_+$, we have $(W_+-V_-)+(V_+-W_-)\ge V_+-V_-$. So one of W_+-V_- and V_+-W_- is at least $(V_+-V_-)/2$. By symmetry, we only need to consider the case that $V_+-W_-\ge(V_+-V_-)/2$. In that case, $|V_+-W_-|\asymp|V_+-V_-|$, and we have

$$\begin{aligned}M_2 &\asymp |W_+-W_-|^{\frac{8}{\kappa}-1}|V_+-V_-|^{\frac{\rho(2\rho+4-\kappa)}{2\kappa}+\frac{2\rho}{\kappa}}|W_+-V_-|^{\frac{2\rho}{\kappa}}\\ &= \Big(\frac{|W_+-W_-|}{|W_+-V_-|}\Big)^{\frac{8}{\kappa}-1}\Big(\frac{|W_+-V_-|}{|V_+-V_-|}\Big)^{\frac{2\rho-(\kappa-8)}{\kappa}}|V_+-V_-|^{\frac{(\rho+2)(2\rho-(\kappa-8))}{2\kappa}}\\ &\le \Big(\frac{|W_+-W_-|}{|V_+-V_-|}\Big)^{(\frac{8}{\kappa}-1)\wedge\frac{2\rho-(\kappa-8)}{\kappa}}|V_+-V_-|^{\frac{(\rho+2)(2\rho-(\kappa-8))}{2\kappa}},\end{aligned}$$

as desired, where in the last step we used that $\frac{|W_+-W_-|}{|W_+-V_-|},\frac{|W_+-V_-|}{|V_+-V_-|}\le 1$ and the inequality that for $0\le x,y\le 1$ and $a,b>0$, $x^ay^b\le(xy)^{a\wedge b}$. □

Lemma 5.7. *M_2 a.s. extends continuously to $\mathbb{R}_+^2$ with $M_2\equiv 0$ on $\mathbb{R}_+^2\setminus\mathcal{D}_2$.*

Proof. Since for $\sigma\in\{+,-\}$, η_σ a.s. extends continuously to $[0,T_\sigma]$, by Remark 3.9, W_+ and W_- a.s. extend continuously to $\overline{\mathcal{D}_2}$. From (3.14) we know that a.s. $|V_+-V_-|$ is bounded on $\mathcal{D}_2$. Thus, by Lemma 5.6 it suffices to show that (the continuations of) W_+ and W_- a.s. agree on $\partial\mathcal{D}_2\cap\mathbb{R}_+^2$. Define subsets of $\partial\mathcal{D}_2$:

$$A_+=\{(t_+,T_-^{\mathcal{D}_2}(t_+)):t_+\in\mathbb{Q}\cap(0,T_+)\},\quad A_-=\{(T_+^{\mathcal{D}_2}(t_-),t_-):t_-\in\mathbb{Q}\cap(0,T_-)\}.$$

Then $A_+\cup A_-$ is dense in $\partial\mathcal{D}_2\cap(0,\infty)^2$. Thus, it suffices to show that W_+ and W_- a.s. agree on $A_+\cup A_-$. By symmetry, we only need to show that W_+ and W_- a.s. agree on A_+. Since A_+ is countable, it suffices to show that, for any $s_+\in\mathbb{Q}_+$, on the event that $s_+<T_+$, a.s. $W_+(s_+,T_-^{\mathcal{D}_2}(s_+))=W_-(s_+,T_-^{\mathcal{D}_2}(s_+))$. Since $W_+\ge W_-$, if the equality does not hold, then

there exists $s_- \in \mathbb{Q}$ with $(s_+, s_-) \in \mathcal{D}_2$ such that $\inf_{t_- \in [s_-, T_-^{\mathcal{D}_2}(s_+))} W_+(s_+, t_-) - W_-(s_+, t_-) > 0$. Thus, it suffices to show that, for any $(s_+, s_-) \in \mathbb{Q}_+^2$, on the event that $(s_+, s_-) \in \mathcal{D}_2$, a.s. $\inf_{t_- \in [s_-, T_-^{\mathcal{D}_2}(s_+))}(W_+(s_+, t_-) - W_-(s_+, t_-)) = 0$.

Fix $(s_+, s_-) \in \mathbb{Q}_+^2$. We will show that the probability of the event E that $(s_+, s_-) \in \mathcal{D}_2$ and $\inf_{t_- \in [s_-, T_-^{\mathcal{D}_2}(s_+))}(W_+(s_+, t_-) - W_-(s_+, t_-)) > 0$ is zero. Suppose the event E happens. Since $(s_+, s_-) \in \mathcal{D}_2$, we may choose a (random) sequence $\delta_n \downarrow 0$ such that $\eta_+(s_+ + \delta_n) \in \mathbb{H} \setminus K(s_+, s_-)$ for all n. Let $z_n = g_{K(s_+, s_-)}(\eta_+(s_+ + \delta_n)) \in K(s_+ + \delta_n, s_-)/K(s_+, s_-)$, $n \in \mathbb{N}$, then $z_n \to W_+(s_+, s_-)$ by (3.6). Since $K_{-,s_+}(t_-)$, $0 \le t_- < T_-^{\mathcal{D}_2}(s_+)$, are chordal Loewner hulls driven by $W_-(s_+, \cdot)$ with speed $d\,\mathrm{m}(s_+, \cdot)$, by Proposition 2.6, $K_{-,s_+}(s_- + t)/K_{-,s_+}(s_-)$, $0 \le t < T_-^{\mathcal{D}_2}(s_+) - s_-$, are chordal Loewner hulls driven by $W_-(s_+, s_- + \cdot)$ with speed $d\,\mathrm{m}(s_+, s_- + \cdot)$. By Lemma 3.13, $W_+(s_+, t) = g_{K(s_+,t)/K_+(s_+)}^{W_-(s_+,0)}(\widehat{w}_+(s_+))$. By Proposition 2.12, $W_+(s_+, s_- + t) = g_{K_{-,s_+}(s_-+t)/K_{-,s_+}(s_-)}^{W_-(s_+,s_-)}(W_+(s_+, s_-))$, $0 \le t < T_-^{\mathcal{D}_2}(s_+) - s_-$. Since $W_+(s_+, t_-) > W_-(s_+, t_-)$ for $s_- \le t_- < T_-^{\mathcal{D}_2}(s_+)$, we find that $W_+(s_+, t_-)$ has positive distance from $K_{-,s_+}(t_-)/K_{-,s_+}(s_-)$ for all $t_- \in [s_-, T_-^{\mathcal{D}_2}(s_+))$. Moreover, from that $\lim_{t_- \uparrow T_-^{\mathcal{D}_2}(s_+)} W_+(s_+, t_-) - W_-(s_+, t_-) > 0$, we know that $W_+(s_+, s_-)$ has positive distance from the $\mathbb{H}$-hull generated by the union of $K_{-,s_+}(t_-)/K_{-,s_+}(s_-) = K(s_+, t_-)/K(s_+, s_-)$ over all $t_- \in [s_-, T_-^{\mathcal{D}_2}(s_+))$, which is $K(s_+, T_-^{\mathcal{D}_2}(s_+))/K(s_+, s_-)$. Since $z_n \to W_+(s_+, s_-)$, for n big enough, z_n is not contained in $K(s_+, T_-^{\mathcal{D}_2}(s_+))/K(s_+, s_-)$. Thus, for n big enough, $\eta_+(s_+ + \delta_n) = f_{K(s_+,s_-)}(z_n)$ is not contained in $K(s_+, T_-^{\mathcal{D}_2}(s_+)) \setminus K(s_+, s_-)$, which implies that $\eta_+(s_+ + \delta_n) \in \mathbb{H} \setminus K(s_+, T_-^{\mathcal{D}_2}(s_+))$ because $\eta_+(s_+ + \delta_n) \in \mathbb{H} \setminus K(s_+, s_-)$.

By the DMP and reversibility of $\mathrm{iSLE}_\kappa(\rho)$, conditionally first on $\eta_+([0, s_+])$ and then on $\eta_-([0, T_-^{\mathcal{D}_2}(s_+)])$, the part of η_+ after s_+ and the part of η_- after $T_-^{\mathcal{D}_2}(s_+)$ are two pieces of the same $\mathrm{iSLE}_\kappa(\rho)$ curve in the closure of one connected component of $\mathbb{H} \setminus (\eta_+([0, s_+]) \cup \eta_-([0, T_-^{\mathcal{D}_2}(s_+)]))$ (with opposite directions). Since $\eta_+(s_+ + \delta_n) \in \mathbb{H} \setminus K(s_+, T_-^{\mathcal{D}_2}(s_+))$ for n big enough, this connected component has to be $\mathbb{H} \setminus K(s_+, T_-^{\mathcal{D}_2}(s_+))$. So a.s. $K(\cdot, \cdot)$ is strictly increasing on $[0, s_+ + \delta] \times [0, T_-^{\mathcal{D}_2}(s_+) + \varepsilon]$ in both variables for some $\delta, \varepsilon > 0$, which contradicts that $(s_+, T_-^{\mathcal{D}_2}(s_+)) \notin \mathcal{D}_2$. Thus, the event E has probability zero, and the proof is done. $\square$

From now on, we understand M_2 as the continuous extension defined in Lemma 5.7. Let $\tau_R^\pm$ and $\underline{\tau}_R$, $R > 0$, be as in the last subsection.

Lemma 5.8. *For any $R > 0$, $(M_2(\underline{t} \wedge \underline{\tau}_R))_{\underline{t} \in \mathbb{R}_+^2}$ is an $M_2(\underline{\tau}_R)$-Doob martingale w.r.t. the filtration $(\mathcal{F}^+_{t_+ \wedge \tau_R^+} \vee \mathcal{F}^-_{t_- \wedge \tau_R^-})_{(t_+, t_-) \in \mathbb{R}_+^2}$. Moreover, if the underlying probability measure is weighted by $M_2(\underline{\tau}_R)/M_1(\underline{0})$, then the new law of $(\widehat{w}_+, \widehat{w}_-)$ agrees with the probability measure $\mathbb{P}^{(\rho,\rho)}_{(w_+, w_-; v_+, v_-)}$ on the σ-algebra $\mathcal{F}^+_{\tau_R^+} \vee \mathcal{F}^-_{\tau_R^-}$.*

Proof. We follow the argument in the proof of Lemma 5.2, where the key ingredient is Proposition 2.20, except that here we use Lemma 5.6 instead of (5.2). $\square$

We now use $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$ to denote the joint law of the $\widehat{w}_+$ and $\widehat{w}_-$ here.

Theorem 5.9. *Under $\mathbb{P}^\rho_{w\leftrightarrow w}$, $M_2(\underline{t})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingale; and for any extended $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{\tau}$,*

$$\frac{d\mathbb{P}^{(\rho,\rho)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathbb{R}_+^2\}}{d\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}|\mathcal{F}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathbb{R}_+^2\}}=\frac{M_2(\underline{\tau})}{M_2(\underline{0})}.$$

Proof. This is similar to the proof of Theorem 5.3 except that here we use Lemma 5.8. □

Corollary 5.10. *Let $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$ be the right-continuous augmentation of $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$. Then $M_2(\underline{t})$ is an $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$-martingale under $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$, and for any extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time $\underline{\tau}$,*

$$\frac{d\mathbb{P}^{(\rho,\rho)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}^{(+)}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathbb{R}_+^2\}}{d\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}|\mathcal{F}^{(+)}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathbb{R}_+^2\}}=\frac{M_2(\underline{\tau})}{M_2(\underline{0})}. \tag{5.10}$$

Proof. By Proposition 2.30, M_2 is an $(\mathcal{F}^{(+)}_{\underline{t}})$-martingale under $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$. Using Theorem 5.9 and Proposition 2.31, we easily get (5.10) in the case that $\underline{\tau}$ is a bounded $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time. The results extends to the general case by Proposition 2.28. □

Lemma 5.11. *For any extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time $\underline{\tau}$, $M_2(\underline{\tau})$ is $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$-a.s. positive on the event $\{\underline{\tau}\in\mathcal{D}_2\}$.*

Proof. Let $\underline{\tau}$ be an extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time. Then $\{\underline{\tau}\in\mathcal{D}_2\}\in\mathcal{F}^{(+)}_{\underline{\tau}}$ because for any $\underline{a}\in\mathbb{R}_+^2$,

$$\{\underline{\tau}\in\mathcal{D}_2\}\cap\{\underline{\tau}<\underline{a}\}=\bigcup_{\underline{t}<\underline{t}'\in[\underline{0},\underline{a})\cap\mathbb{Q}_+^2}(\{\underline{\tau}\le\underline{t}\}\cap\{K(\cdot,\cdot)\text{ is strictly increasing on }[\underline{0},\underline{t}']\})\in\mathcal{F}_{\underline{a}}.$$

Let $A=\{\underline{\tau}\in\mathcal{D}_2\}\cap\{M_2(\underline{\tau})=0\}\in\mathcal{F}^{(+)}_{\underline{\tau}}$. We are going to show that $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}[A]=0$. Since $M_2(\underline{\tau})=0$ on $A\in\mathcal{F}^{(+)}_{\underline{\tau}}\cap\mathbb{R}_+^2$, by Corollary 5.10, $\mathbb{P}^{(\rho,\rho)}_{(w_+,w_-;v_+,v_-)}[A]=0$. Applying Corollary 5.10 to $\underline{\tau}+\underline{t}$, where $\underline{t}\in\mathbb{Q}_+^2$, we find that $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$-a.s $M_2(\underline{\tau}+\underline{t})=0$ on A. Thus, on the event A, $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$-a.s. $M_2(\underline{\tau}+\underline{t})=0$ for any $\underline{t}\in\mathbb{Q}_+^2$, which implies by the continuity that $M_2\equiv 0$ on $\underline{\tau}+\mathbb{R}_+^2$, which further implies that $W_+\equiv W_-$ on $(\underline{\tau}+\mathbb{R}_+^2)\cap\mathcal{D}_2$, which in turn implies by Lemma 3.7 that $\eta_+(\tau_++t_+)=\eta_-(\tau_-+t_-)$ for any $\underline{t}=(t_+,t_-)\in\mathbb{R}_+^2$ such that $\underline{\tau}+\underline{t}\in\mathcal{D}_2$, and so $K(\cdot,\cdot)$ can not be strictly increasing on $[0,\underline{\tau}+\underline{t}]$ for any $\underline{t}>\underline{0}$, which then contradicts that $\underline{\tau}\in\mathcal{D}_2$. So we have $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}[A]=0$. □

Corollary 5.12. *For any extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time $\underline{\tau}$,*

$$\frac{d\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}|\mathcal{F}^{(+)}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathcal{D}_2\}}{d\mathbb{P}^{(\rho,\rho)}_{(w_+,w_-;v_+,v_-)}|\mathcal{F}^{(+)}_{\underline{\tau}}\cap\{\underline{\tau}\in\mathcal{D}_2\}}=\frac{M_2(\underline{\tau})^{-1}}{M_2(\underline{0})^{-1}}.$$

Proof. This follows from Corollary 5.10 and Lemma 5.11. □

The following lemma describes the DMP for $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$, which is similar to Lemma 4.4.

Theorem 5.13. *Suppose $(\widehat{w}_+,\widehat{w}_-)$ follows the law $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$. We write $\mathcal{D}_2$ for the $\mathcal{D}_2(\eta_+,\eta_-)$. Let $\underline{\tau}=(\tau_+,\tau_-)$ be an extended $(\mathcal{F}^{(+)}_{\underline{t}})_{\underline{t}\in\mathbb{R}^2_+}$-stopping time. Then on the event that $\underline{\tau}\in\mathcal{D}_2$, a.s. there is another random commuting pair of chordal Loewner curves $(\widehat{\eta}_+,\widehat{\eta}_-;\widehat{\mathcal{D}}_2)$ with some speeds, which agrees with the part of $(\eta_+,\eta_-;\mathcal{D}_2)$ after $\underline{\tau}$. Moreover, $\widehat{\mathcal{D}}_2=\mathcal{D}_2(\widehat{\eta}_+,\widehat{\eta}_-)$ as in (5.7), and the normalization of $(\widehat{\eta}_+,\widehat{\eta}_-;\widehat{\mathcal{D}}_2)$, denoted by $(\widetilde{\eta}_+,\widetilde{\eta}_-;\widetilde{\mathcal{D}}_2)$, satisfies the following properties. For $\sigma\in\{+,-\}$, let $\widetilde{\mathcal{F}}^{\sigma}_t$ be the σ-algebra generated by $\mathcal{F}^{(+)}_{\underline{\tau}}$ and $\widetilde{\eta}_\sigma(s)$, $s\le t$. Let $\widetilde{\mathcal{F}}_{(t_+,t_-)}=\widetilde{\mathcal{F}}^+_{t_+}\vee\widetilde{\mathcal{F}}^-_{t_-}$, and $(\widetilde{\mathcal{F}}^{(+)}_{\underline{t}})$ be the right-continuous augmentation of $(\widetilde{\mathcal{F}}_{\underline{t}})$. Then for any extended $(\widetilde{\mathcal{F}}^{(+)}_{\underline{t}})$-stopping time $\underline{\widetilde{S}}$, we have $\mathbb{P}[\underline{\widetilde{S}}\in\widetilde{\mathcal{D}}_2|\mathcal{F}^{(+)}_{\underline{\tau}},\underline{\tau}\in\mathcal{D}_2]=\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(W_+\leftrightarrow W_-;V_+,V_-)|_{\underline{\tau}}}[\underline{\widetilde{S}}\in\widetilde{\mathcal{D}}_2]$. Here if for some $\sigma\in\{+,-\}$, $V_\sigma(\underline{\tau})=W_\sigma(\underline{\tau})$, then $V_\sigma(\underline{\tau})$ is treated as $W_\sigma(\underline{\tau})^\sigma$.*

Remark 5.14. A stronger statement should be true: the conditional joint law of the driving functions for $\widetilde{\eta}_+$ and $\widetilde{\eta}_-$ given $\mathcal{F}^{(+)}_{\underline{\tau}}$ is $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(W_+\leftrightarrow W_-;V_+,V_-)|_{\underline{\tau}}}$. But the statement of the lemma is sufficient for our purpose.

Proof. Suppose that $\underline{\tau}\in\mathcal{D}_2$ happens. To prove the existence of $(\widehat{\eta}_+,\widehat{\eta}_-;\widehat{\mathcal{D}}_2)$, which agrees with the part of $(\eta_+,\eta_-;\mathcal{D}_2)$ after $\underline{\tau}$, by Lemma 3.17, it suffices to show that, for any $\sigma\in\{+,-\}$ and any $\underline{q}=(q_+,q_-)\in\mathbb{Q}^2_+$, on the event $\underline{\tau}+\underline{q}\in\mathcal{D}_2$, a.s. $K(\underline{\tau}+q_{-\sigma}\underline{e}_{-\sigma}+t\underline{e}_\sigma)/K(\underline{\tau}+q_{-\sigma}\underline{e}_{-\sigma})$, $0\le t\le q_\sigma$, are generated by a chordal Loewner curve with some speed, which intersects $\mathbb{R}$ at a Lebesgue measure zero set. This follows from Lemma 4.4 and Corollary 5.12 (applied to $\underline{\tau}+\underline{q}$). Let $\widehat{K}(\cdot,\cdot)$ be the hull function for $(\widehat{\eta}_+,\widehat{\eta}_-)$. Since $\eta_\sigma(\tau_\sigma+\cdot)=f_{K(\underline{\tau})}\circ\widehat{\eta}_\sigma$, $\sigma\in\{+,-\}$, we get $\widehat{K}=K(\underline{\tau}+\cdot)/K(\underline{\tau})$. So $\widehat{\mathcal{D}}_2=\{\underline{t}-\underline{\tau}:\underline{t}\in\mathcal{D}_2,\underline{t}\ge\underline{\tau}\}=\mathcal{D}_2(\widehat{\eta}_+,\widehat{\eta}_-)$.

Let $(\widetilde{\eta}_+,\widetilde{\eta}_-;\widetilde{\mathcal{D}}_2)$ be the normalization of $(\widehat{\eta}_+,\widehat{\eta}_-;\widehat{\mathcal{D}}_2)$. Let $h_\sigma(t)=\mathrm{m}(\underline{\tau}+t\underline{e}_\sigma)-\mathrm{m}(\underline{\tau})$, $t\ge0$, $\sigma\in\{+,-\}$, and $h_\oplus=h_+\oplus h_-$. Then $\widetilde{\eta}_\sigma=\widehat{\eta}_\sigma\circ h_\sigma^{-1}$, $\sigma\in\{+,-\}$, and $\widetilde{\mathcal{D}}_2=h_\oplus(\widehat{\mathcal{D}}_2)$. We add tilde to denote the functions from Section 3 and M_2 in (5.8) for $(\widetilde{\eta}_+,\widetilde{\eta}_-;\widetilde{\mathcal{D}}_2)$. By Lemma 3.17, for $X\in\{W_+,W_-,V_+,V_-\}$, $\widetilde{X}=X(\underline{\tau}+h_\oplus^{-1}(\cdot))$. So $\widetilde{M_2}=M_2(\underline{\tau}+h_\oplus^{-1}(\cdot))$.

The argument at the end of the proof of Lemma 4.4 works here to show that, for any $\underline{t}\in\mathbb{R}^2_+$, $\underline{\tau}+h_\oplus^{-1}(\underline{t})$ is an extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time, and $\widetilde{\mathcal{F}}_{\underline{t}}\subset\mathcal{F}^{(+)}_{\underline{\tau}+h_\oplus^{-1}(\underline{t})}$. Let $\underline{\widetilde{S}}$ be an extended $(\widetilde{\mathcal{F}}^{(+)}_{\underline{t}})$-stopping time. Let $\underline{S}=\underline{\tau}+h_\oplus^{-1}(\underline{\widetilde{S}})$. Then $\underline{S}$ is an extended $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time because for any $\underline{a}\in\mathbb{R}^2_+$,

$$\{\underline{S}<\underline{a}\}=\bigcup_{\underline{p}\in\mathbb{Q}^2_+}(\{\underline{\widetilde{S}}<\underline{p}\}\cap\{\underline{\tau}+h_\oplus^{-1}(\underline{p})<\underline{a}\})\in\mathcal{F}_{\underline{a}},$$

where we used that $\{\widetilde{\underline{S}} < \underline{p}\} \in \widetilde{\mathcal{F}}_{\underline{p}} \subset \mathcal{F}^{(+)}_{\underline{\tau}+h_\oplus^{-1}(\underline{p})}$. We now write $\mathbb{P}$ for $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$, $\widetilde{\mathbb{P}}$ for $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(W_+\leftrightarrow W_-;V_+,V_-)|_{\underline{\tau}}}$, $\mathbb{P}'$ for $\mathbb{P}^{(\rho,\rho)}_{(w_+,w_-;v_+,v_-)}$, and $\widetilde{\mathbb{P}}'$ for $\mathbb{P}^{(\rho,\rho)}_{(W_+\leftrightarrow W_-;V_+,V_-)|_{\underline{\tau}}}$. To prove that $\mathbb{P}[\widetilde{\underline{S}} \in \widetilde{\mathcal{D}}_2|\mathcal{F}^{(+)}_{\underline{\tau}}, \underline{\tau} \in \mathcal{D}_2] = \widetilde{\mathbb{P}}[\widetilde{\underline{S}} \in \widetilde{\mathcal{D}}_2]$, it suffices to show that, for any $A \in \mathcal{F}^{(+)}_{\underline{\tau}}$ with $A \subset \{\underline{\tau} \in \mathcal{D}_2\}$, we have

$$\mathbb{P}[A \cap \{\widetilde{\underline{S}} \in \widetilde{\mathcal{D}}_2\}] = \mathbb{E}[\mathbf{1}_A \widetilde{\mathbb{P}}[\widetilde{\underline{S}} \in \widetilde{\mathcal{D}}_2]]. \tag{5.11}$$

Note that $\widetilde{\underline{S}} \in \widetilde{\mathcal{D}}_2$ if and only if $\underline{S} \in \mathcal{D}_2$. By Corollary 5.12, the LHS of (5.11) equals

$$\mathbb{P}[A \cap \{\underline{S} \in \mathcal{D}_2\}] = \mathbb{E}'\Big[\mathbf{1}_{A\cap\{\underline{S}\in\mathbb{R}_+^2\}} \frac{M_2(\underline{S})}{M_2(\underline{0})}\Big].$$

Applying Corollary 5.12 twice (to $\mathbb{P}$ and $\widetilde{\mathbb{P}}$), we find that the RHS of (5.11) equals

$$\mathbb{E}'\Big[\mathbf{1}_A \frac{M_2(\underline{\tau})}{M_2(\underline{0})} \widetilde{\mathbb{E}}'\Big[\mathbf{1}_{\{\widetilde{\underline{S}}\in\mathbb{R}_+^2\}} \frac{\widetilde{M}_2(\widetilde{\underline{S}})}{\widetilde{M}_2(\underline{0})}\Big]\Big] = \mathbb{E}'\Big[\widetilde{\mathbb{E}}'\Big[\mathbf{1}_{A\cap\{\underline{S}\in\mathbb{R}_+^2\}} \frac{M_2(\underline{S})}{M_2(\underline{0})}\Big]\Big] = \mathbb{E}'\Big[\mathbf{1}_{A\cap\{\underline{S}\in\mathbb{R}_+^2\}} \frac{M_2(\underline{S})}{M_2(\underline{0})}\Big],$$

where in the first equality, we used $\widetilde{M}_2(\widetilde{\underline{S}}) = M_2(\underline{S})$ and $\widetilde{M}_2(\underline{0}) = M_2(\underline{\tau})$, and in the second equality we used Lemma 4.4. So we get (5.11), and the proof is done. □

For convenience, we write $\mathbb{P}_2$ for $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$. We now also assume that $v_0 := (v_+ + v_-)/2 \in [w_-, w_+]$, and let V_0 be the force point function started from v_0. We may define the time curve $\underline{u} : [0, T^u) \to \mathcal{D}_2$ and the processes $R_\sigma(t)$, $\sigma \in \{+,-\}$, and $\underline{R}(t)$ as in Section 3.4, and extend $\underline{u}$ to $\mathbb{R}_+$ such that $\underline{u}(s) = \lim_{t\uparrow T^u} \underline{u}(t)$ for $s \ge T^u$. Since $\mathcal{D}_2$ is an $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping region, by Proposition 3.24, for any $t \ge 0$, $\underline{u}(t)$ is an $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time.

Applying Corollary 5.12 to $\underline{u}(t)$ for any deterministic $t \ge 0$, we get

$$\frac{d\mathbb{P}_2|\mathcal{F}^{(+)}_{\underline{u}(t)} \cap \{t < \widetilde{T}^u\}}{d\mathbb{P}^{(\rho,\rho)}|\mathcal{F}^{(+)}_{\underline{u}(t)} \cap \{t < \widetilde{T}^u\}} = \frac{M_2^u(t)^{-1}}{M_2^u(0)^{-1}} = e^{-2\alpha_2 t} \frac{G_2(\underline{R}(0))}{G_2(\underline{R}(t))},$$

where $\alpha_2 = \frac{(\rho+2)(2\rho+8-\kappa)}{2\kappa}$ and

$$G_2(r_+, r_-) := 2^{\frac{\rho(2\rho+4-\kappa)}{2\kappa}} (r_+ + r_-)^{\frac{8}{\kappa}-1} \prod_{\sigma\in\{+,-\}} (1 + r_\sigma)^{\frac{2\rho}{\kappa}} \cdot F_{\kappa,\rho}\Big(\frac{(1 - r_+)(1 - r_-)}{(1 + r_+)(1 + r_-)}\Big)^{-1}. \tag{5.12}$$

So we obtain the following lemma.

Lemma 5.15. *Let $p_2^R(t, \underline{r}, \underline{r}^*)$ be the function $p^R(t, \underline{r}, \underline{r}^*)$ given in Corollary 4.17 with $\rho_0 = 0$ and $\rho_+ = \rho_- = \rho$. Then under $\mathbb{P}_2$, the transition density of $(\underline{R})$ is*

$$\widetilde{p}_2^R(t, \underline{r}, \underline{r}^*) := e^{-2\alpha_2 t} p_2^R(t, \underline{r}, \underline{r}^*) \cdot \frac{G_2(\underline{r})}{G_2(\underline{r}^*)}.$$

5.3 Opposite pair of iSLE$_\kappa(\rho)$ curves, a limit case

Third, we consider another pair of random curves. Let κ and ρ be as in the last subsection. Let $\widehat\eta_+$ be an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_+ to w_- with force points $v_+,-\infty$. So its reversal $\widehat\eta_-$ is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_- to w_+ with force points $-\infty,v_+$. Define the chordal Loewner curves $\eta_+(t_+)$, $0\le t_+<T_+$, and $\eta_-(t_-)$, $0\le t_-<T_-$, with driving functions $\widehat w_+$ and $\widehat w_-$, respectively, in the same way as in the previous subsection. Define $\mathcal{D}_3=\mathcal{D}_2(\eta_+,\eta_-)$ using (5.7). Then $(\eta_+,\eta_-;\mathcal{D}_3)$ is a.s. a commuting pair of chordal Loewner curves. Let W_+ and W_- be the driving functions, and let V_+ be the force point function started from v_+.

Define a non-negative function M_3 on $\mathcal{D}$ by

$$M_3=|W_+-W_-|^{\frac8\kappa-1}|V_+-W_-|^{\frac{2\rho}{\kappa}}\cdot F_{\kappa,\rho}\Big(\frac{V_+-W_+}{V_+-W_-}\Big)^{-1}.$$

Let V_- be the force point function started from w_-^-. Since $V_+\ge W_+\ge W_-\ge V_-$, there are $C>0$ depending on κ,ρ and C' depending on κ,ρ and $|v_+-w_-|$ such that

$$\begin{aligned}M_3&\le C\Big(\frac{W_+-W_-}{V_+-W_-}\Big)^{\frac8\kappa-1}\Big(\frac{V_+-W_-}{V_+-V_-}\Big)^{\frac2\kappa(\rho-(\frac\kappa2-4))}(V_+-V_-)^{\frac2\kappa(\rho-(\frac\kappa2-4))}\\&\le C'(W_+-W_-)^{(\frac8\kappa-1)\wedge\frac2\kappa(\rho-(\frac\kappa2-4))}(V_+-V_-)^{\frac2\kappa(\rho-(\frac\kappa2-4))}.\end{aligned}$$

Here we use the fact that $\frac8\kappa-1,\frac2\kappa(\rho-(\frac\kappa2-4))>0$, $V_+\ge v_+$, and $V_-\le w_-$. Then the exactly same proof of Lemma 5.7 can be used here to prove the following lemma.

Lemma 5.16. *M_3 a.s. extends continuously to $\mathbb{R}_+^2$ with $M_3\equiv0$ on $\mathbb{R}_+^2\setminus\mathcal{D}_3$.*

We now understand M_3 as the continuous extension defined on $\mathbb{R}_+^2$. Let $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+)}$ denote the joint law of $\widehat w_+$ and $\widehat w_-$. Then similar arguments as in the previous subsection give the following propositions.

Theorem 5.17. *Under $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+)}$, M_3 is an $(\mathcal{F}_{\underline t})_{\underline t\in\mathbb{R}_+^2}$-martingale; and for any extended $(\mathcal{F}^{(+)}_{\underline t})$-stopping time $\underline T$,*

$$\frac{d\mathbb{P}^{(\rho)}_{(w_+,w_-;v_+)}|\mathcal{F}^{(+)}_{\underline T}\cap\{\underline T\in\mathbb{R}_+^2\}}{d\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+)}|\mathcal{F}^{(+)}_{\underline T}\cap\{\underline T\in\mathbb{R}_+^2\}}=\frac{M_3(\underline T)}{M_3(\underline 0)}.$$

Corollary 5.18. *For any extended $(\mathcal{F}^{(+)}_{\underline t})$-stopping time $\underline T$, $M_3(\underline T)$ is $\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+)}$-a.s. positive on the event $\{\underline T\in\mathcal{D}_3\}$, and*

$$\frac{d\mathbb{P}^{\mathrm{iSLE}(\rho)}_{(w_+\leftrightarrow w_-;v_+)}|\mathcal{F}^{(+)}_{\underline T}\cap\{\underline T\in\mathcal{D}_3\}}{d\mathbb{P}^{(\rho)}_{(w_+,w_-;v_+)}|\mathcal{F}^{(+)}_{\underline T}\cap\{\underline T\in\mathcal{D}_3\}}=\frac{M_3(\underline T)^{-1}}{M_3(\underline 0)^{-1}}.$$

Theorem 5.19. *The statement in Theorem 5.13 holds with* $\mathbb{P}^{\rm iSLE(\rho)}_{(w_+\leftrightarrow w_-;v_+)}$ *and* $\mathbb{P}^{\rm iSLE(\rho)}_{(W_+\leftrightarrow W_-;V_+)|_{\underline{\tau}}}$ *in place of* $\mathbb{P}^{\rm iSLE(\rho)}_{(w_+\leftrightarrow w_-;v_+,v_-)}$ *and* $\mathbb{P}^{\rm iSLE(\rho)}_{(W_+\leftrightarrow W_-;V_+,V_-)|_{\underline{\tau}}}$*, respectively.*

In this subsection, we have marked points $v_+ > w_+ > w_-$. We introduce two more marked points v_0 and v_- by $v_0 = (w_+ + w_-)/2$ and $v_- = 2v_0 - v_+$. Let V_0 and V_- be the force point functions started from v_0 and v_-, respectively. For convenience, we write $\mathbb{P}_3$ for $\mathbb{P}^{\rm iSLE(\rho)}_{(w_+\leftrightarrow w_-;v_+)}$. Under $\mathbb{P}_3$, we may define the time curve $\underline{u} : [0, T^u) \to \mathcal{D}_2$ and the processes $R_\sigma(t)$, $\sigma \in \{+,-\}$, and $\underline{R}(t)$ as in Section 3.4. Then for each $t \ge 0$, (the extended) $\underline{u}(t)$ is an $(\mathcal{F}^{(+)}_{\underline{t}})$-stopping time.

Applying Corollary 5.18 to $\underline{u}(t)$ for any deterministic $t \ge 0$, we get

$$\frac{d\mathbb{P}_3|\mathcal{F}^{(+)}_{\underline{u}(t)} \cap \{t < T^u\}}{d\mathbb{P}^{(\rho)}|\mathcal{F}^{(+)}_{\underline{u}(t)} \cap \{t < T^u\}} = \frac{M^u_3(t)^{-1}}{M^u_3(0)^{-1}} = e^{-2\alpha_3 t}\frac{G_3(\underline{R}(0))}{G_3(\underline{R}(t))},$$

where $\alpha_3 = \frac{2\rho+8-\kappa}{\kappa}$ and

$$G_3(r_+, r_-) := (r_+ + r_-)^{\frac{8}{\kappa}-1}(1 + r_-)^{\frac{2\rho}{\kappa}} \cdot F_{\kappa,\rho}\Big(\frac{1 - r_+}{1 + r_-}\Big)^{-1}. \tag{5.13}$$

Using an argument similar to the proof of Lemma 5.15, we get the following lemma.

Lemma 5.20. *Let* $p^R_3(t, \underline{r}, \underline{r}^*)$ *be the function given in Corollary 4.17 with* $\rho_0 = \rho_- = 0$ *and* $\rho_+ = \rho$*. Then under* $\mathbb{P}_3$*, the transition density of* $(\underline{R})$ *is*

$$\widetilde{p}^R_3(t, \underline{r}, \underline{r}^*) := e^{-2\alpha_3 t} p^R_3(t, \underline{r}, \underline{r}^*) \cdot \frac{G_3(\underline{r})}{G_3(\underline{r}^*)}.$$

Using Lemmas 5.5, 5.15, and 5.20, we can obtain a quasi-invariant density of $\underline{R}$ under either $\mathbb{P}_1$, $\mathbb{P}_2$, or $\mathbb{P}_3$ as follows. For $j = 1, 2, 3$, let $p^R_j(\underline{r}^*)$ be the invariant density of $\underline{R}$ under $\mathbb{P}^{(2,2)}$, $\mathbb{P}^{(\rho,\rho)}$, and $\mathbb{P}^{(\rho)}$, respectively, given by Corollary 4.17, and let G_1, G_2, G_3 be given by (5.6,5.12,5.13), respectively. Define

$$\mathcal{Z}_j = \int_{(0,1)^2} \frac{p^R_j(\underline{r}^*)}{G_j(\underline{r}^*)} d\underline{r}^*, \quad \widetilde{p}^R_j = \frac{1}{\mathcal{Z}_j}\frac{p^R_j}{G_j}, \quad j = 1, 2, 3. \tag{5.14}$$

It is straightforward to check that $\mathcal{Z}_j \in (0, \infty)$, $j = 1, 2, 3$.

Lemma 5.21. *Let* $\alpha_1, \alpha_2, \alpha_3$ *be given by Lemmas 5.5, 5.15, and 5.20, respectively.*

(i) For any $j \in \{1, 2, 3\}$*,* $t > 0$ *and* $\underline{r}^* \in [0, 1]^2$*,*

$$\int_{[0,1]^2} \widetilde{p}^R_j(\underline{r})\widetilde{p}^R_j(t, \underline{r}, \underline{r}^*) d\underline{r} = e^{-2\alpha_j t}\widetilde{p}^R_j(\underline{r}^*).$$

This means, under the law $\mathbb{P}_j$*, if the process* $(\underline{R})$ *starts from a random point in* $(0, 1)^2$ *with density* $\widetilde{p}^R_j$*, then for any deterministic* $t \ge 0$*, the density of (the survived)* $\underline{R}(t)$ *is* $e^{-2\alpha_j t}\widetilde{p}^R_j$*. So we call* $\widetilde{p}^R_j$ *a quasi-invariant distribution for* $(\underline{R})$ *under* $\mathbb{P}_j$*.*

(ii) Let $\beta_1 = 10$, $\beta_2 = 2\rho + 6$, and $\beta_3 = \rho + 6$. For $j \in \{1, 2, 3\}$, under the law $\mathbb{P}_j$, for any $\underline{r} \in (0,1)^2$, if $\underline{R}$ starts from $\underline{r}$, then

$$\mathbb{P}_j[T^u > t] = \mathcal{Z}_j G_j(\underline{r}) e^{-2\alpha_j t}(1 + O(e^{-\beta_j t})); \tag{5.15}$$

$$\widetilde{p}_j^R(t, \underline{r}, \underline{r}^*) = \mathbb{P}_j[T^u > t]\widetilde{p}_j^R(\underline{r}^*)(1 + O(e^{-\beta_j t})). \tag{5.16}$$

Here we emphasize that the implicit constants in the O symbols do not depend on $\underline{r}$.

Proof. Part (i) follows easily from (4.30). For part (ii), suppose $\underline{R}$ starts from $\underline{r}$. Using Corollary 4.17, Lemmas 5.5, 5.15, and 5.20, and formulas (5.14), we get

$$\mathbb{P}_j[T^u > t] = \int_{(0,1)^2} \widetilde{p}_j^R(t, \underline{r}, \underline{r}^*) d\underline{r}^* = \int_{(0,1)^2} e^{-2\alpha_j t} p_j^R(t, \underline{r}, \underline{r}^*) \frac{G_j(\underline{r})}{G_j(\underline{r}^*)} d\underline{r}^*$$

$$= \int_{(0,1)^2} e^{-2\alpha_j t} p_j^R(\underline{r}^*)(1 + O(e^{-\beta_j t})) \frac{G_j(\underline{r})}{G_j(\underline{r}^*)} d\underline{r}^* = \mathcal{Z}_j G_j(\underline{r}) e^{-2\alpha_j t}(1 + O(e^{-\beta_j t})),$$

which is (5.15). We also have

$$\widetilde{p}_j^R(t, \underline{r}, \underline{r}^*) = e^{-2\alpha_j t} p_j^R(t, \underline{r}, \underline{r}^*) \frac{G_j(\underline{r})}{G_j(\underline{r}^*)}$$

$$= e^{-2\alpha_j t} p_j^R(\underline{r}^*)(1 + O(e^{-\beta_j t})) \frac{G_j(\underline{r})}{G_j(\underline{r}^*)} = e^{-2\alpha_j t} \mathcal{Z}_j \widetilde{p}_j^R(\underline{r}^*)(1 + O(e^{-\beta_j t})) G_j(\underline{r}),$$

which together with (5.15) implies (5.16). □

6 Boundary Green's Functions

We are going to prove the main theorems and some other important theorems in this section.

Lemma 6.1. *Let U_1 and U_2 be two simply connected subdomains of the Riemann sphere $\widehat{\mathbb{C}}$, both of which contain ∞ and do not contain 0. Suppose f maps U_1 conformally onto U_2 and fixes ∞. Suppose for $j = 1, 2$, f_j maps $\mathbb{D}^* := \widehat{\mathbb{C}} \setminus \{|z| \le 1\}$ conformally onto U_j and fixes ∞, such that $f_2 = f \circ f_1$. Let $a_j = \lim_{z \to \infty} |f_j(z)|/|z| > 0$, $j = 1, 2$, and $a = a_2/a_1$. If $R > 4a_1$, then $\{|z| > R\} \subset U_1$, and $\{|z| > aR + 4a_2\} \subset f(\{|z| > R\}) \subset \{|z| > aR - 4a_2\}$.*

Proof. That $\{|z| \ge R\} \subset U_1$ when $R > 4a_1$ follows from Koebe's 1/4 theorem applied to $J \circ f_1 \circ J$, where $J(z) := 1/z$. Define $g_j = f_j/a_j$, $j = 1, 2$. Fix $z_1 \in U_1$. Let $z_2 = f(z_1) \in U_2$, $w_0 = f_1^{-1}(z_1) = f_2^{-1}(z_2) \in \mathbb{D}^*$, and $w_j = g_j(w_0) = z_j/a_j$, $j = 1, 2$. Let $R_j = |z_j|$, $j = 1, 2$, and $r_j = |w_j|$, $j = 0, 1, 2$. Then $R_j = a_j r_j$, $j = 1, 2$. Applying Koebe's distortion theorem to $J \circ g_j \circ J$, we find that $r_0 + \frac{1}{r_0} - 2 \le r_j \le r_0 + \frac{1}{r_0} + 2$, $j = 1, 2$, which implies that $|R_1/a_1 - R_2/a_2| = |r_1 - r_2| \le 4$. Thus, $aR_1 - 4a_2 \le R_2 \le aR_1 + 4a_2$, which implies that f maps $\{|z| > R\}$ into $\{|z| > aR - 4a_2\}$, and $f(\{|z| = R\}) \subset \{|z| \le aR + 4a_2\}$. The latter inclusion implies that $f(\{|z| > R\}) \supset \{|z| > aR + 4a_2\}$ because $f(\infty) = \infty$. □

Theorem 6.2. *Let $\kappa\in(0,8)$. Let $v_-<w_-<w_+<v_+\in\mathbb{R}$ be such that $0\in[v_-,v_+]$. Let $(\widehat\eta_+,\widehat\eta_-)$ be a 2-SLE$_\kappa$ in $\mathbb{H}$ with link pattern $(w_+\to v_+;w_-\to v_-)$. Let $\alpha_1=2(\frac{12}{\kappa}-1)$, $\beta_1=10$, and $G_1(\underline{w};\underline{v})$ be as in (1.2). Then there is a constant $C>0$ depending only on κ such that, as $L\to\infty$,*

$$\mathbb{P}[\widehat\eta_\sigma\cap\{|z|>L\}\ne\emptyset,\sigma\in\{+,-\}]=CL^{-\alpha_1}G_1(\underline{w};\underline{v})\Big(1+O\Big(\frac{|v_+-v_-|}{L}\Big)^{\frac{\beta_1}{\beta_1+2}}\Big),\tag{6.1}$$

where the implicit constant depends only on κ.

Proof. Let $p(\underline{w};\underline{v};L)$ denote the LHS of (6.1). Construct the random commuting pair of chordal Loewner curves $(\eta_1,\eta_2;\mathcal{D})$ from $\widehat\eta_1$ and $\widehat\eta_2$ as in Section 5.1, where $\mathcal{D}=[0,T_+)\times[0,T_-)$, and T_σ is the lifetime of η_σ, $\sigma\in\{+,-\}$. We adopt the symbols from Sections 3.1. Note that, when $L>|v_+|\vee|v_-|$, $\widehat\eta_+$ and $\widehat\eta_-$ both intersect $\{|z|>L\}$ if and only if η_+ and η_- both intersect $\{|z|>L\}$. In fact, for any $\sigma\in\{+,-\}$, η_σ either disconnects v_j from ∞, or disconnects v_{-j} from ∞. If η_σ does not intersect $\{|z|>L\}$, then in the former case, $\widehat\eta_\sigma$ grows in a bounded connected component of $\mathbb{H}\setminus\eta_\sigma$ after the end of η_σ, and so can not hit $\{|z|>L\}$; and in the latter case $\eta_{-\sigma}$ grows in a bounded connected component of $\mathbb{H}\setminus\eta_\sigma$, and can not hit $\{|z|>L\}$. We first consider a very special case: $v_+=1$, $v_-=-1$, $w_+=r_+\in[0,1)$, and $w_-=-r_-\in(-1,0]$. Let $v_0=0$. Let V_ν be the force point function started from v_ν, $\nu\in\{0,+,-\}$. Since $|v_+-v_0|=|v_0-v_-|$, we may define a time curve $\underline{u}:[0,T^u)\to\mathcal{D}$ as in Section 3.4 and adopt the symbols from there. Define $p(\underline{r};L)=p(r_+,-r_-;1,-1;L)$.

Suppose $L>2e^6$, and so $\frac12\log(L/2)>3$. Let $t_0\in[3,\frac12\log(L/2))$. If both η_+ and η_- intersect $\{|z|>L\}$, then there is some $t'\in[0,T^u)$ such that either $\eta_+(u_+([0,t']))$ or $\eta_-(u_-([0,t']))$ intersects $\{|z|>L\}$, which by (3.32) implies that $L\le 2e^{2t'}$, and so $T^u>t'\ge\log(L/2)/2>t_0$. Thus, $\{\eta_\sigma\cap\{|z|>L\}\ne\emptyset,\sigma\in\{+,-\}\}\subset\{T^u>t_0\}$. By (3.32) again, $\operatorname{rad}_0(\eta_\sigma([0,u_\sigma(t_0)]))\le 2e^{2t_0}<L$. So $\eta_\sigma([0,u_\sigma(t_0)])$, $\sigma\in\{+,-\}$, do not intersect $\{|z|>L\}$.

Let $\widehat g^u_{t_0}(z)=(g_{K(\underline{u}(t_0))}(z)-V^u_0(t_0))/e^{2t_0}$. Then $\widehat g^u_{t_0}$ maps $\mathbb{C}\setminus(K(\underline{u}(t_0))^{\rm doub}\cup[v_-,v_+])$ conformally onto $\mathbb{C}\setminus[-1,1]$, and fixes ∞ with $\widehat g^u_{t_0}(z)/z\to e^{-2t_0}$ as $z\to\infty$. From $V^u_-\le v_-<0$, $V^u_+\ge v_+>0$, and $V^u_0=(V^u_++V^u_-)/2$, we get $|V^u_0(t_0)|\le|V^u_+(t_0)-V^u_-(t_0)|/2=e^{2t_0}$. Applying Lemma 6.1 to $f=\widehat g^u_{t_0}$ and $f_2(z)=(z+1/z)/2$ ($a_1=e^{2t_0}/2$ and $a_2=1/2$) and using that $L>2e^{2t_0}$, we get $\{|z|>L\}\subset\mathbb{C}\setminus(K(\underline{u}(t_0))^{\rm doub}\cup[v_-,v_+])$ and

$$\{|z|>L/e^{2t_0}-2\}\supset\widehat g^u_{t_0}(\{|z|>L\})\supset\{|z|>L/e^{2t_0}+2\}.\tag{6.2}$$

Note that both η_+ and η_- intersect $\{|z|>L\}$ if and only if $T^u>t_0$ and the $\widehat g^u_{t_0}$-image of the parts of η_σ after $u_\sigma(t_0)$, $\sigma\in\{+,-\}$, both intersect the $\widehat g^u_{t_0}$-image of $\{|z|>L\}$. From Proposition 2.32, conditionally on $\mathcal{F}^{(+)}_{\underline{u}(t_0)}$ and the event that $T^u>t_0$, the $\widehat g^u_{t_0}$-image of the parts of $\widehat\eta_\sigma$ after $\eta_\sigma(u_\sigma(t_0))$, $\sigma\in\{+,-\}$, form a 2-SLE$_\kappa$ in $\mathbb{H}$, with link pattern $(W^u_\sigma(t_0)-V^u_0(t_0))/e^{2t_0}=\sigma R_\sigma(t_0)\to(V^u_\sigma(t_0)-V^u_0(t_0))/e^{2t_0}=\sigma 1$, $\sigma\{+,-\}$. From (6.2) we get

$$p(\underline{R}(t_0);\frac{L}{e^{2t_0}}+2)\le\mathbb{P}[\eta_\sigma\cap\{|z|>L\}\ne\emptyset,\sigma\in\{+,-\}|\mathcal{F}^{(+)}_{\underline{u}(t_0)},T^u>t_0]\le p(\underline{R}(t_0);\frac{L}{e^{2t_0}}-2)].\tag{6.3}$$

We use the approach of [6] to prove the convergence of $\lim_{L\to\infty} L^{\alpha_1} p(\underline{r}, L)$. We first estimate $p(L) := \int_{[0,1]^2} p(\underline{r}; L)\widetilde{p}_1^R(\underline{r})d\underline{r}$, where $\widetilde{p}_1^R$ is the quasi-invariant density for the process $(\underline{R})$ under $\mathbb{P}_1 = \mathbb{P}^{2-\mathrm{SLE}}_{(w_+\to v_+;w_-\to v_-)}$ given in Lemma 5.21. This is the probability that the two curves in a 2-SLE$_\kappa$ in $\mathbb{H}$ with link pattern $(r_+ \to 1; -r_- \to -1)$ both hit $\{|z| > L\}$, where (r_+, r_-) is a random point that follows the density $\widetilde{p}_1^R$. From Lemma 5.21 we know that, for the deterministic time t_0, $\mathbb{P}[T^u > t_0] = e^{-\alpha_1 t_0}$ and the law of $(\underline{R}(t_0))$ conditionally on the event $\{T^u > t_0\}$ still has density $\widetilde{p}_1^R$. Thus, the conditional joint law of the $\widehat{g}^u_{t_0}$-images of the parts of $\widehat{\eta}_\sigma$ after $\eta_\sigma(u_\sigma(t_0))$, $\sigma \in \{+,-\}$ given $\mathcal{F}^u_{t_0}$ and the event that $T^u > t_0$ agrees with that of $(\widehat{\eta}_+, \widehat{\eta}_-)$. From (6.3) and that $\{\eta_\sigma \cap \{|z| > L\} \neq \emptyset, \sigma \in \{+,-\}\} \subset \{T^u > t_0\}$ we get

$$e^{-2\alpha_1 t_0} p(L/e^{2t_0} - 2) \geq p(L) \geq e^{-2\alpha_1 t_0} p(L/e^{2t_0} + 2), \quad \text{if } L > 2e^{2t_0}.$$

Let $q(L) = L^{\alpha_1} p(L)$. Then (if $t_0 \geq 3$ and $L > 2e^{2t_0}$)

$$(1 - 2e^{2t_0}/L)^{-\alpha_1} q(L/e^{2t_0} - 2) \geq q(L) \geq (1 + 2e^{2t_0}/L)^{-\alpha_1} q(L/e^{2t_0} + 2). \tag{6.4}$$

Suppose $L_0 > 4$ and $L \geq e^6(L_0 + 2)$. Let $t_1 = \log(L/(L_0 + 2))/2$ and $t_2 = \log(L/(L_0 - 2))/2$. Then $L/e^{2t_1} - 2 = L/e^{2t_2} + 2 = L_0$, $t_2 \geq t_1 \geq 3$ and $L = (L_0 - 2)e^{2t_2} > 2e^{2t_2} \geq 2e^{2t_1}$. From (6.4) (applied here with t_1 and t_2 in place of t_0 on the LHS and RHS, respectively) we get

$$(1 + 2/L_0)^{\alpha_1} q(L_0) \geq q(L) \geq (1 - 2/L_0)^{\alpha_1} q(L_0), \quad \text{if } L \geq e^6(L_0 + 2) > 6e^6. \tag{6.5}$$

From (3.32) we know that $T^u > t_0$ implies that both η_+ and η_- intersect $\{|z| > e^{2t_0}/64\}$. Since $\mathbb{P}[T^u > t_0] = e^{-2\alpha_1 t_0} > 0$ for all $t_0 \geq 0$, we see that p is positive on $[0, \infty)$, and so is q. From (6.5) we see that $\lim_{L\to\infty} q(L)$ converges to a point in $(0, \infty)$. Denote it by $q(\infty)$. By fixing $L_0 \geq 4$ and sending $L \to \infty$ in (6.5), we get

$$q(\infty) L_0^{-\alpha_1} (1 + 2/L_0)^{-\alpha_1} \leq p(L_0) \leq q(\infty) L_0^{-\alpha_1} (1 - 2/L_0)^{-\alpha_1}, \quad \text{if } L_0 > 4. \tag{6.6}$$

Now we estimate $p(\underline{r}; L)$ for a fixed deterministic $\underline{r} \in [0,1)^2 \setminus \{(0,0)\}$. The process $(\underline{R})$ starts from $\underline{r}$ and has transition density given by Lemma 5.5. Fix $L > 2e^6$ and choose $t_0 \in [3, \log(L/2)/2)$. Then both η_+ and η_- intersect $\{|z| > L\}$ implies that $T^u > t_0$. From Lemma 5.21 we know that $\mathbb{P}_1[T^u > t_0] = \mathcal{Z}_1 G_1(\underline{r}) e^{-2\alpha_1 t_0}(1 + O(e^{-\beta_1 t_0}))$ and the law of $\underline{R}(t_0)$ conditional on $\{T^u > t_0\}$ has a density on $(0,1)^2$, which equals $\widetilde{p}_1^R \cdot (1 + O(e^{-\beta_1 t_0}))$. Using (6.3,6.6) we get

$$p(\underline{r}; L) = \mathcal{Z}_1 q(\infty) G_1(\underline{r}) e^{-2\alpha_1 t_0} (L/e^{2t_0})^{-\alpha_1} (1 + O(e^{-\beta_1 t_0}))(1 + O(e^{2t_0}/L)).$$

For $L > e^{36}$, by choosing t_0 such that $e^{2t_0} = L^{2/(2+\beta_1)}$ and letting $C_0 = \mathcal{Z} q(\infty)$, we get

$$p(\underline{r}; L) = C_0 G_1(\underline{r}) L^{-\alpha_1} (1 + O(L^{-\beta_1/(2+\beta_1)})).$$

Since $G_1(r_+, r_-) = G_1(r_+, -r_-; 1, -1)$, we proved (6.1) for $v_\pm = \pm 1$, $w_+ \in [0,1)$, and $w_- \in (-1, 0]$. Since $G_1(aw_+ + b, aw_- + b; av_+ + b, av_- + b) = a^{-\alpha_1} G_1(w_+, w_-; v_+, v_-)$ for any $a > 0$ and $b \in \mathbb{R}$, by a translation and a dilation, we get (6.1) in the case that $(v_+ + v_-)/2 \in [w_-, w_+]$. Here the assumption that $0 \in [v_+, v_-]$ is used to control the amount of translation.

Finally, we consider all other cases, i.e., $(v_+ + v_-)/2 \notin [w_-, w_+]$. By symmetry, we may assume that $(v_+ + v_-)/2 < w_-$. Let $v_0 = (w_+ + w_-)/2$. Then $v_+ > w_+ > v_0 > w_- > v_-$, but $v_+ - v_0 < v_0 - v_-$. We still let V_ν be the force point functions started from v_ν, $\nu \in \{0, +, -\}$. By (3.18) V^ν satisfies the PDE $\partial_+ V_\nu \overset{\text{ae}}{=} \frac{2W_{+,1}^2}{V_\nu - W_+}$ on $\mathcal{D}_1^{\text{disj}}$ as defined in Section 3.3. Thus, on $\mathcal{D}_1^{\text{disj}}$, for any $\nu_1 \neq \nu_2 \in \{+, -, 0\}$, $\partial_+ \log|V_{\nu_1} - V_{\nu_2}| \overset{\text{ae}}{=} \frac{-2}{(V_{\nu_2} - W_+)(V_{\nu_1} - W_+)}$, which implies that

$$\frac{\partial_+(\frac{V_+ - V_0}{V_0 - V_-})}{\partial_+ \log|V_+ - V_-|} = \frac{V_+ - V_0}{W_+ - V_0} \cdot \frac{V_+ - V_-}{V_0 - V_-} > 1. \tag{6.7}$$

Fixing $t_- = 0$. The displayed formula means that $\frac{V_+(t,0) - V_0(t,0)}{V_0(t,0) - V_-(t,0)}$ is increasing with a rate faster than $\log|V_+(t,0) - V_-(t,0)|$. From the assumption, $\frac{V_+(0,0) - V_0(0,0)}{V_0(0,0) - V_-(0,0)} = \frac{v_+ - v_0}{v_0 - v_-} \in (0,1)$. Let τ be the first t such that $\frac{V_+(t,0) - V_0(t,0)}{V_0(t,0) - V_-(t,0)} = 1$; if such time does not exist, then set $\tau = T_+$. Then τ is an $(\mathcal{F}_{t_+}^+)_{t_+ \geq 0}$-stopping time, and from (6.7) we know that, for any $0 \leq t < \tau$, $|V_+(t,0) - V_-(t,0)| < e|v_+ - v_-|$, which implies by (3.14) that $\operatorname{diam}(\eta_+([0,t])) < e|v_+ - v_-|$. From (5.1) we know that $M_1 = G_1(\underline{W}; \underline{V})$. Here and below, we write $\underline{W}$ and $\underline{V}$ for (W_+, W_-) and (V_+, V_-), respectively. From Lemma 5.2 we know that for any $L \in (0, \infty)$, $(M_1(t \wedge \tau_L^+, 0))_{t \geq 0}$ is a Doob-martingale, where $M_1(t,0) = 0$ if $t \geq T_+$. Taking $L = (e+1)|v_+ - v_-|$, we find that $\tau_L^+ > \tau$. In fact, if $\eta_+([0,t])$ intersects $\{|z| > L\}$, then $\operatorname{diam}(\eta_+([0,t])) > L - |w_+| > L - |v_+ - v_-| \geq e|v_+ - v_-|$, which then implies that $|V_+(t,0) - V_-(t,0)| > e|v_+ - v_-|$ by (3.14), and so $t > \tau$ because $\operatorname{diam}(\eta_+([0,\eta])) \leq e|v_+ - v_-|$. So by Proposition 2.31,

$$\mathbb{E}[\mathbf{1}_{\{\tau < T_+\}} G_1(\underline{W}; \underline{V})|_{(\tau,0)}] = \mathbb{E}[M_1(\tau, 0)] = M_1(0,0) = G_1(\underline{w}; \underline{v}). \tag{6.8}$$

Using the same argument as in the proof of (6.3) with $(\tau, 0)$ in place of $\underline{u}(t_0)$ and $g_{K(\tau,0)}$ in place of $\widehat{g}_{t_0}^u$, we get

$$p((\underline{W}; \underline{V})|_{(\tau,0)}; L_+) \leq \mathbb{P}[\eta_\sigma \cap \{|z| = L\} \neq \emptyset, \sigma \in \{+, -\} | \mathcal{F}_\tau^+, \tau < T_+] \leq p((\underline{W}; \underline{V})|_{(\tau,0)}; L_-), \tag{6.9}$$

where $L_\mu = L + \mu \cdot |V_+(\tau, 0) - V_-(\tau, 0)|$, $\mu \in \{+, -\}$.

If $\tau < T_+$, from the definition of τ we know that $V_0(\tau, 0) = (V_+(\tau, 0) + V_-(\tau, 0))/2$. Since $W_+ \geq V_0 \geq W_-$, we have $(V_+(\tau, 0) + V_-(\tau, 0))/2 \in [W_-(\tau, 0), W_+(\tau, 0)]$. Also note that $V_-(\tau, 0) \leq v_- \leq 0$ and $V_+(\tau, 0) \geq v_+ \geq 0$. So we may apply the result in the particular case to get

$$\begin{aligned} p((\underline{W}; \underline{V})|_{(\tau,0)}; L_\pm) =& C_0 G_1(\underline{W}; \underline{V})|_{(\tau,0)} \cdot L_\pm^{-\alpha_1} \Big(1 + O\Big(\Big(\frac{|V_+(\tau,0) - V_-(\tau,0)|}{L_\pm}\Big)^{\beta_1/(2+\beta_1)}\Big)\Big) \\ =& C_0 G_1(\underline{W}; \underline{V})|_{(\tau,0)} \cdot L^{-\alpha_1} \Big(1 + O\Big(\Big(\frac{|v_+ - v_-|}{L}\Big)^{\beta_1/(2+\beta_1)}\Big)\Big). \end{aligned} \tag{6.10}$$

Here in the last step we used $|V_+(\tau, 0) - V_-(\tau, 0)| \leq e|v_+ - v_-|$ and $L_\pm = L(1 + O(|v_+ - v_-|/L))$. Plugging (6.10) into (6.9), taking expectation on both sides of (6.9), and using the fact that

$\tau < T_+$ when $\eta_+ \cap \{|z| = L\} \neq \emptyset$, we get

$$p(\underline{w}; \underline{v}; L) = C_0 \mathbb{E}[\mathbf{1}_{\{\tau < T_+\}} G_1(\underline{W}; \underline{V})|_{(\tau,0)}] \cdot L^{-\alpha_1}(1 + O((|v_+ - v_-|/L)^{\beta_1/(2+\beta_1)}))$$
$$= C_0 G_1(\underline{w}; \underline{v}) \cdot L^{-\alpha_1}(1 + O((|v_+ - v_-|/L)^{\beta_1/(2+\beta_1)})),$$

where in the last step we used (6.8). The proof is now complete. □

Theorem 6.3. *Let $\kappa \in (4,8)$. Then Theorem 6.2 holds with the same α_1, β_1, G_1 but a different positive constant C under either of the following two modifications:*

(i) the set $\{|z| > L\}$ is replaced by (L, ∞), $(-\infty, -L)$, or $(L, \infty) \cup (-\infty, -L)$;

(ii) the event that $\eta_\sigma \cap \{|z| > L\} \neq \emptyset$, $\sigma \in \{+,-\}$, is replaced by $\eta_+ \cap \eta_- \cap \{|z| > L\} \neq \emptyset$.

Proof. The same argument in the proof of Theorem 6.2 works here, where the assumption that $\kappa \in (4,8)$ is used to guarantee that the probability of the event is positive for all $L > 0$. □

Theorem 6.4. *Let $\kappa \in (0,4]$ and $\rho > -2$, or $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 2$. Let $w_- < w_+ \in \mathbb{R}$, $v_+ \in \{w_+^+\} \cup (w_+, \infty)$ and $v_- \in \{w_-^-\} \cup (-\infty, w_-)$ be such that $0 \in [v_-, v_+]$. Let $\widehat{\eta}$ be an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_+ to w_- with force points at v_+ and v_-. Let $\alpha_2 = \frac{\rho+2}{\kappa}(\rho - (\frac{\kappa}{2} - 4))$, $\beta_2 = 2\rho + 6$, and*

$$G_2(\underline{w}; \underline{v}) = |w_+ - w_-|^{\frac{8}{\kappa}-1} |v_+ - v_-|^{\frac{\rho(2\rho+4-\kappa)}{2\kappa}} \prod_{\sigma \in \{+,-\}} |w_\sigma - v_{-\sigma}|^{\frac{2\rho}{\kappa}} F_{\kappa,\rho}\Big(\frac{(v_+ - w_+)(w_- - v_-)}{(w_+ - v_-)(v_+ - w_-)}\Big)^{-1}.$$

Then there is a constant $C > 0$ depending only on κ, ρ such that, as $L \to \infty$,

$$\mathbb{P}[\widehat{\eta} \cap \{|z| > L\} \neq \emptyset] = C L^{-\alpha_2} G_2(\underline{w}; \underline{v}) \Big(1 + O\Big(\frac{|v_+ - v_-|}{L}\Big)^{\frac{\beta_2}{\beta_2+2}}\Big),$$

where the implicit constant depends only on κ, ρ. Moreover, if $\kappa \in (0,4]$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$, then the above statement holds (with a different positive constant C) if the set $\{|z| > L\}$ is replaced by (L, ∞), $(-\infty, -L)$, or $(L, \infty) \cup (-\infty, -L)$.

Proof. Let $p(\underline{w}; \underline{v}; L)$ denote the probability that $\widehat{\eta}$ intersects $\{|z| > L\}$, and let $p(\underline{r}; L) = p(r_+, -r_-; 1, -1; L)$ for $\underline{r} = (r_+, r_-) \in [0,1]^2 \setminus \{(0,0)\}$. Let $\widehat{\eta}_+ = \widehat{\eta}$ and $\widehat{\eta}_-$ be the time-reversal of $\widehat{\eta}$. Construct the random commuting pair of chordal Loewner curves $(\eta_+, \eta_-; \mathcal{D}_2)$ from $\widehat{\eta}_+$ and $\widehat{\eta}_-$ as in Section 5.2, where $\mathcal{D}_2$ is defined by (5.7). Then for $L > \max\{|v_+|, |v_-|\}$, $\widehat{\eta} \cap \{|z| > L\} \neq \emptyset$ if and only if $\eta_\sigma \cap \{|z| > L\} \neq \emptyset$, $\sigma \in \{+,-\}$.

The rest of the proof follows the same line as that of Theorem 6.2 except that we now apply Lemma 5.21 with $j = 2$ and use Lemma 5.8 and Theorem 5.13 in place of Lemma 5.2 and Proposition 2.32, respectively. More specifically, to obtain the counterpart of (6.3), we apply Theorem 5.13 to $\underline{\tau} = \underline{u}(t_0)$ and $\underline{\widetilde{S}} = \underline{\widetilde{S}}^\mu = (\widetilde{S}_+^\mu, \widetilde{S}_-^\mu)$, $\mu \in \{+,-\}$, where

$$\widetilde{S}_\sigma^\mu := \inf\{t : |\widetilde{\eta}_\sigma(t) - V_0^u(t)| > L + \mu \cdot 2e^{t_0}\}, \quad \sigma \in \{+,-\}.$$

By convention, if $\widetilde{S}_+^\mu$ or $\widetilde{S}_-^\mu$ is not well defined, then we set $\underline{\widetilde{S}}^\mu = \underline{\infty}$. To obtain the counterpart of (6.9), we apply Theorem 5.13 to $\underline{\tau} = (\tau, 0)$ and $\underline{\widetilde{S}} = \underline{\widetilde{S}}^\mu = (\widetilde{S}_+^\mu, \widetilde{S}_-^\mu)$, $\mu \in \{+,-\}$, where

$$\widetilde{S}_\sigma^\mu := \inf\{t : |\widetilde{\eta}_\sigma(t)| > L + \mu \cdot |V_+(\tau,0) - V_-(\tau,0)|\}, \quad \sigma \in \{+,-\}.$$

In either case $\underline{\widetilde{S}}^\mu$ is a stopping time w.r.t. the right-continuous augmentation of the filtration $(\widetilde{\mathcal{F}}_{\underline{t}})_{\underline{t}\in\mathbb{R}_+^2}$, where $\widetilde{F}_{(t_+,t_-)}$ is generated by $\widetilde{\eta}_+|_{[0,t_+]}$, $\widetilde{\eta}_-|_{[0,t_-]}$, and $\mathcal{F}_{\underline{\tau}}^{(+)}$. We also use the fact that $G_2(r_+, r_-) = G_2(r_+, -r_-; 1, -1)$.

Finally, the statement about the case $\kappa \in (0,4]$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$ follows from the same argument as above, where the conditions on κ and ρ guarantees that the probability that $\widehat{\eta}$ intersects (L,∞) or $(-\infty,-L)$ is positive for any $L > 0$. □

Corollary 6.5. *Let $\kappa \in (0,8)$. Let $v_- < w_- < w_+ < v_+ \in \mathbb{R}$ be such that $0 \in [v_-, v_+]$. Let $(\widehat{\eta}_w, \widehat{\eta}_v)$ be a 2-SLE$_\kappa$ in $\mathbb{H}$ with link pattern $(w_+ \leftrightarrow w_-; v_+ \leftrightarrow v_-)$. Let $\alpha_2 = 2(\frac{12}{\kappa} - 1)$, $\beta_2 = 10$, and $G_2(\underline{w}; \underline{v})$ be as in (1.3). Then there is a constant $C > 0$ depending only on κ such that, as $L \to \infty$,*

$$\mathbb{P}[\widehat{\eta}_u \cap \{|z| > L\} \neq \emptyset, u \in \{w,v\}] = CL^{-\alpha_2} G_2(\underline{w}; \underline{v}) \Big(1 + O\Big(\frac{|v_+ - v_-|}{L}\Big)^{\frac{\beta_2}{\beta_2+2}}\Big),$$

where the implicit constant depends only on κ.

Proof. This follows from Theorem 6.4 and the facts that $\widehat{\eta}_w$ is an hSLE$_\kappa$, i.e., iSLE$_\kappa(2)$ curve in $\mathbb{H}$ from w_+ to w_- with force points at v_+, v_-, and that when $L > \max\{|v_+|, |v_-|\}$, $\widehat{\eta}_w \cap \{|z| > L\} \neq \emptyset$ implies that $\widehat{\eta}_v \cap \{|z| > L\} \neq \emptyset$ as well. □

Theorem 6.6. *Let $\kappa \in (0,4]$ and $\rho > -2$, or $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 2$. Let $w_- < w_+ \in \mathbb{R}$ and $v_+ \in \{w_+^-\} \cup (w_+, \infty)$ be such that $0 \in [w_-, v_+]$. Let $\widehat{\eta}$ be an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from w_+ to w_- with force points at v_+ and ∞. Let $\alpha_3 = \frac{2}{\kappa}(\rho - (\frac{\kappa}{2} - 4))$, $\beta_3 = \rho + 6$, and*

$$G_3(\underline{w}; v_+) = |w_+ - w_-|^{\frac{8}{\kappa}-1} |v_+ - w_-|^{\frac{2\rho}{\kappa}} F_{\kappa,\rho}\Big(\frac{v_+ - w_+}{v_+ - w_-}\Big)^{-1}.$$

Then there is a constant $C > 0$ depending only on κ, ρ such that, as $L \to \infty$,

$$\mathbb{P}[\widehat{\eta} \cap \{|z| > L\} \neq \emptyset] = CL^{-\alpha_3} G_3(\underline{w}; v_+) \Big(1 + O\Big(\frac{|w_+ - v_-|}{L}\Big)^{\frac{\beta_3}{\beta_3+2}}\Big),$$

where the implicit constant depends only on κ, ρ. Moreover, if $\kappa \in (0,4]$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$, then the statement holds (with a different positive constant C) if the set $\{|z| > L\}$ is replaced by (L,∞) or $(L,\infty) \cup (-\infty,-L)$; if $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 2$, then the statement holds if $\{|z| > L\}$ is replaced by $(-\infty,-L)$ or $(L,\infty) \cup (-\infty,-L)$.

Proof. The proof follows the same line as that of Theorems 6.4 and 6.2 except that we now introduce $v_0 := (w_+ + w_-)/2$ and $v_- := 2v_0 - v_+$ as in Section 5.3. Then we can define the time curve $\underline{u}$ as in Section 3.4 without an additional assumption. We now apply Lemma 5.21 with $j = 3$ and use Theorem 5.13 in place of Proposition 2.32. Note that the $G_3(r_+, r_-)$ in (5.13) agrees with the $G_3(r_+, -r_-; 1, -1)$ here. The last sentence follows from the same argument and the fact that the events are positive for any $L > 0$ in each case. □

Corollary 6.7. *Let $\kappa \in (0, 8)$. Let $w_- < w_+ < v_+ \in \mathbb{R}$ be such that $0 \in [w_-, v_+]$. Let $(\widehat{\eta}_w, \widehat{\eta}_v)$ be a 2-SLE$_\kappa$ in $\mathbb{H}$ with link pattern $(w_+ \leftrightarrow w_-; v_+ \leftrightarrow \infty)$. Let $\alpha_3 = \frac{12}{\kappa} - 1$, $\beta_3 = 8$, and $G_3(\underline{w}; v_+)$ be as in (1.4). Then there is a constant $C > 0$ depending only on κ such that, as $L \to \infty$,*

$$\mathbb{P}[\widehat{\eta}_u \cap \{|z| > L\} \neq \emptyset, u \in \{w, v\}] = CL^{-\alpha_3} G_3(\underline{w}; v_+)\Big(1 + O\Big(\frac{|w_+ - v_-|}{R}\Big)^{\frac{\beta_3}{\beta_3+2}}\Big),$$

where the implicit constant depends only on κ.

Proof. This follows from Theorem 6.6 and the facts that $\widehat{\eta}_w$ is an hSLE$_\kappa$, i.e., iSLE$_\kappa(2)$ curve in $\mathbb{H}$ from w_+ to w_- with force points at v_+, ∞, and that $\widehat{\eta}_v \cap \{|z| > L\} \neq \emptyset$ for any $L > 0$. □

Proof of Theorem 1.1. This follows from Theorem 6.2, Corollary 6.5, and Corollary 6.7. □

Proof of Theorem 1.2. By symmetry, we may assume that $z_0 = 0$ and $w > v \geq 0$. Let $J(z) = -1/z$, which is a Möbius automorphism of $\mathbb{H}$, and swaps 0 and ∞. Now $J(\eta)$ is an SLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from $J(w)$ to 0 with the force point at $J(v)$, its reversal is an iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from 0 to $J(w)$ with force points at 0^+ and $J(v)$. Note that $\text{dist}(\eta, 0) < r$ iff $J(\eta) \cap \{|z| > 1/r\} \neq \emptyset$. So (i) follows from Theorem 6.4 by setting $w_+ = 0$, $w_- = -\frac{1}{w}$, $v_+ = 0^+$ and $v_- = -\frac{1}{v}$; and (ii) follows from Theorem 6.6 by setting $w_+ = 0$, $w_- = -\frac{1}{w}$, and $v_+ = 0^+$. □

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