Backward SLE and the symmetry of the welding

Steffen Rohde*[1] and Dapeng Zhan†[2]

[1] *University of Washington*
[2] *Michigan State University*

November 1, 2013

Abstract

The backward chordal Schramm-Loewner Evolution naturally defines a conformal welding homeomorphism of the real line. We show that this homeomorphism is invariant under the automorphism $x \mapsto -1/x$, and conclude that the associated solution to the welding problem (which is a natural renormalized limit of the finite time Loewner traces) is reversible. The proofs rely on an analysis of the action of analytic circle diffeomorphisms on the space of hulls, and on the coupling techniques of the second author.

Contents

*Research partially supported by NSF grant DMS-1068105.
†Research partially supported by NSF grants DMS-0963733, DMS-1056840, and Sloan fellowship.

1 Introduction

1.1 Introduction and results

The Schramm-Loewner Evolution SLE$_\kappa$, first introduced in [14], is a stochastic process of random conformal maps that has received a lot of attention over the last decade. We refer to the introductory text [6] for basic facts and definitions. In this paper we are largely concerned with chordal SLE$_\kappa$, which can be viewed as a family of random curves γ that join 0 and ∞ in the closure of the upper half plane $\overline{\mathbb{H}}$. A fundamental property of chordal SLE is *reversibility*: The law of γ is invariant under the automorphism $z \mapsto -1/z$ of $\mathbb{H}$, modulo time parametrization. This has first been proved by the second author in [18] for $\kappa \le 4$, and recently by Miller and Sheffield for $4 < \kappa \le 8$ in [9]. It is known to be false for $\kappa > 8$ ([13],[19]).

In the early years of SLE, Oded Schramm, Wendelin Werner and the first author made an attempt to prove reversibility along the following lines: The "backward" flow

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \sqrt{\kappa} B_t}, \quad f_0(z) = z, \quad 0 \le t \le T,$$

generates curves $\beta_T = \beta[0,T]$ whose law is that of the chordal SLE trace $\gamma[0,T]$ (up to translation by $\sqrt{\kappa}B_T$) . When $\kappa \le 4$, these curves are simple, and each point of β (with the exception of the endpoints) corresponds to two points on the real line under the conformal map f_t. The *conformal welding homeomorphism* ϕ of β_T is the auto-homeomorphism of the interval $f_T^{-1}(\beta_T)$ that interchanges these two points. In other words, it is the rule that describes which points on the real line are to be identified (laminated) in order to form the curve β_T. It is known [13] that, for $\kappa < 4$, the welding almost surely uniquely determines the curve. The welding homeomorphism can be obtained by restricting the backward flow to the real line: Two points $x \neq y$ on the real line are to be welded if and only if their swallowing times coincide, $\phi(x) = y$ if and only if $\tau_x = \tau_y$, see Section 3.5. An idea to prove reversibility was to prove the invariance of ϕ under $x \mapsto -1/x$, and to relate this to reversibility of a suitable limit of the curves β_T. But the attempts to prove invariance of ϕ failed, and this program was never completed successfully.

In this paper, we use the coupling techniques of the second author, introduced in [18] for his proof of reversibility of (forward) SLE traces. We use it to prove the invariance of the welding:

Theorem 1.1. *Let $\kappa \in (0, 4]$, and ϕ be a backward chordal SLE$_\kappa$ welding. Let $h(z) = -1/z$. Then $h \circ \phi \circ h$ has the same distribution as ϕ.*

As a consequence, in the range $\kappa \in (0, 4)$ where the SLE trace is conformally removable, we obtain the reversibility of suitably normalized limits of the β_T (see Section 6 for details):

Theorem 1.2. *Let $\kappa \in (0, 4)$, and β be a normalized global backward chordal SLE$_\kappa$ trace. Let $h(z) = -1/z$. Then $h(\beta \setminus \{0\})$ has the same distribution as $\beta \setminus \{0\}$ as random sets.*

In the important paper [16], Sheffield obtains a representation of the SLE welding in terms of a quantum gravity boundary length measure, and also relates it to a simple Jordan arc, which differs from our β only through normalization. A similar random welding homeomorphism is constructed in [2], where the main point is the very difficult existence of a curve solving the welding problem. Our approach to the welding is different: In order to prove Theorem 1.1, in Section 2 we develop a framework to study the effect of analytic perturbations of weldings on the corresponding hulls. We show in Section 4 that a Möbius image of a backward chordal SLE$_\kappa$ process is a backward radial SLE$(\kappa, -\kappa-6)$ process, and the welding is preserved under this conformal transformation. In Section 5 we apply the coupling technique to show that backward radial SLE$(\kappa, -\kappa - 6)$ started from an ordered pair of points (a, b) commutes with backward radial SLE$(\kappa, -\kappa-6)$ started from (b, a), and use this in Section 6 to prove Theorem 1.1.

In a subsequent paper [23] of the second author, Theorem 1.1 is used to study the ergodic properties of a forward SLE$_\kappa$ trace near the tip at a fixed capacity time.

1.2 Notation

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $I_{\mathbb{R}}(z) = \overline{z}$ and $I_{\mathbb{T}}(z) = 1/\overline{z}$ be the reflections about $\mathbb{R}$ and $\mathbb{T}$, respectively. Let e^i denote the map $z \mapsto e^{iz}$. Let $\cot_2(z) = \cot(z/2)$ and $\sin_2(z) = \sin(z/2)$. For a real interval J, let $C(J)$ denote the space of real valued continuous functions on J. An increasing or decreasing function in this paper is assumed to be strictly monotonic. We use $B(t)$ to denote a standard real Brownian motion. By $f : D \overset{\text{Conf}}{\twoheadrightarrow} E$ we mean that f maps D conformally onto E. By $f_n \xrightarrow{\text{l.u.}} f$ in U we mean that f_n converges to f uniformly on every compact subset of U. We will frequently use the notation $D_n \xrightarrow{\text{Cara}} D$ as in Definition A.1.

The outline of this paper is the following. In Section 2, we derive some fundamental results in Complex Analysis, which are interesting on their own. In Section 3, we review the properties of forward Loewner processes, and derive some properties of backward Loewner processes. In Section 4, we discuss how are backward Loewner processes transformed by conformal maps. In Section 5, we present and prove certain commutation relations between backward SLE$(\kappa; \vec{\rho})$ processes. In the last section, we prove the reversibility of backward chordal SLE_κ processes for $\kappa \in (0,4]$ and propose questions in other cases. In the appendix, we discuss some results on the topology of domains and hulls.

2 Extension of Conformal Maps

2.1 Interior hulls in $\mathbb{C}$

An interior hull (in $\mathbb{C}$) is a nonempty compact connected set $K \subset \mathbb{C}$ such that $\mathbb{C} \setminus K$ is also connected. For every interior hull K in $\mathbb{C}$, there are a unique $r \ge 0$ and a unique $\phi_K : \widehat{\mathbb{C}} \setminus K \overset{\mathrm{Conf}}{\twoheadrightarrow} \widehat{\mathbb{C}} \setminus r\overline{\mathbb{D}}$ such that $\phi_K(\infty) = \infty$ and $\phi_K'(\infty) := \lim_{z\to\infty} z/\phi_K(z) = 1$. We call $\mathrm{rad}(K) := r$ the radius of K and $\mathrm{cap}(K) := \ln(r)$ the capacity of K. The radius is 0 iff K contains only one point. In general, we have $\mathrm{rad}(K) \le \mathrm{diam}(K) \le 4\,\mathrm{rad}(K)$. We call K nondegenerate if it contains more than one point. For such K, there is a unique $\varphi_K : \widehat{\mathbb{C}} \setminus K \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{D}^*$ such that $\varphi_K(\infty) = \infty$ and $\varphi_K'(\infty) > 0$. In fact, $\varphi_K = \phi_K/\,\mathrm{rad}(K)$. Let $\psi_K = \varphi_K^{-1}$ for such K.

For any Jordan curve J in $\mathbb{C}$, let D_J denote the Jordan domain bounded by J, and let $D_J^* = \widehat{\mathbb{C}} \setminus (D_J \cup J)$. Suppose $f_J : \mathbb{D} \overset{\mathrm{Conf}}{\twoheadrightarrow} D_J$ and $f_J^* = \psi_{\overline{D_J}} : \mathbb{D}^* \overset{\mathrm{Conf}}{\twoheadrightarrow} D_J^*$. Then both f_J and f_J^* extend continuously to a homeomorphism from $\mathbb{T}$ onto J. Let $h = (f_J^*)^{-1} \circ f_J$. Then h is an orientation-preserving automorphism of $\mathbb{T}$. We call such h a conformal welding. Not every homeomorphism of $\mathbb{T}$ is a conformal welding, but it is well-known (and an easy consequence of the uniformization theorem) that every analytic automorphism is a conformal welding, and that the associated Jordan curve is analytic. See [8] for the quasiconformal theory of conformal welding, and [3] for deep generalizations and further references.

Lemma 2.1. *Let β be an analytic Jordan curve. Let $\Omega \subset \mathbb{C}$ be a neighborhood of $\mathbb{T}$. Suppose W is a conformal map defined in Ω, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Let $\Omega^\beta = \beta \cup D_\beta \cup \psi_{\overline{D_\beta}}(\Omega \cap \mathbb{D}^*)$. Then there is a conformal map V defined in Ω^β such that $V \circ \psi_{\overline{D_\beta}} = \psi_{\overline{D_{V(\beta)}}} \circ W$ in $\Omega \cap \mathbb{D}^*$.*

Proof. Fix a conformal map $f_\beta : \mathbb{D} \overset{\mathrm{Conf}}{\twoheadrightarrow} D_\beta$ and let $h_\beta = \varphi_{\overline{D_\beta}} \circ f_\beta$ be the associated conformal welding homeomorphism. Define $h = W \circ h_\beta$. Since β is analytic, h is analytic and there is an analytic Jordan curve γ and a conformal map $f_\gamma : \mathbb{D} \overset{\mathrm{Conf}}{\twoheadrightarrow} D_\gamma$ such that $h = h_\gamma = \varphi_{\overline{D_\gamma}} \circ f_\gamma$. Define $V = f_\gamma \circ f_\beta^{-1}$ on D_β. Since β and γ are analytic curves, V extends conformally to a neighborhood of β with $V(\beta) = \gamma$. On β, this extension (still denoted V) satisfies $V = (\psi_{\overline{D_\gamma}} \circ h_\gamma) \circ (h_\beta^{-1} \circ \psi_{\overline{D_\beta}}^{-1}) = \psi_{\overline{D_\gamma}} \circ W \circ \psi_{\overline{D_\beta}}^{-1}$. Therefore V extends conformally to all of Ω^β and satisfies the desired property. □

Theorem 2.2. *Let H be a nondegenerate interior hull. Let $\Omega \subset \mathbb{C}$ be a neighborhood of $\mathbb{T}$. Suppose W is a conformal map defined in Ω, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Let $\Omega^H = H \cup \psi_H(\Omega \cap \mathbb{D}^*)$. Then there is a conformal map V defined in Ω^H such that $V \circ \psi_H = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. If another conformal map $\widetilde{V}$ satisfies the properties of V, then $\widetilde{V} = aV + b$ for some $a > 0$ and $b \in \mathbb{C}$.*

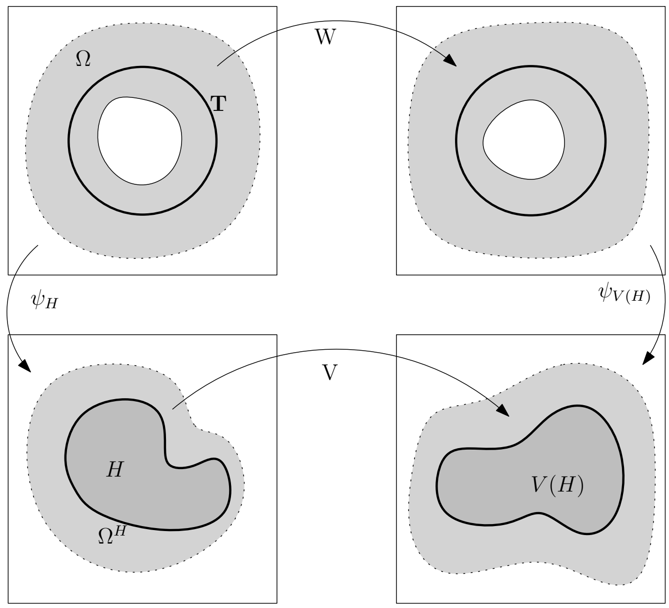

Figure 1: The situation of Theorem 2.2 and Lemma 2.1. Given H and W, V can be constructed to be analytic in H. In Lemma 2.1, the boundary of H is assumed to be an analytic Jordan curve, while in Theorem 2.2, no regularity assumption is made.

Proof. First, define a sequence of analytic Jordan curves (β_n) by

$$\beta_n = \psi_H(\{e^{\frac{1}{n}+i\theta} : 0 \le \theta \le 2\pi\}), \quad n \in \mathbb{N}.$$

Then $\beta_n \cup D_{\beta_n} \to H$ in $d_{\mathcal{H}}$ (see Appendix B). From Lemma 2.1, for each $n \in \mathbb{N}$, there is a conformal map V_n defined in $\Omega^{\beta_n} := \beta_n \cup D_{\beta_n} \cup \psi_{\beta_n}(\Omega \cap \mathbb{D}^*)$ such that $V_n \circ \psi_{\beta_n} = \psi_{V_n(\beta_n)} \circ W$ in $\Omega \cap \mathbb{D}^*$. Note that for any $a_n > 0$ and $b_n \in \mathbb{C}$, $a_n V_n + b_n$ satisfies the same property as V_n. Thus, we may assume that $0 \in V_n(\beta_n) \subset \overline{\mathbb{D}}$ and $V_n(\beta_n) \cap \mathbb{T} \neq \emptyset$. Let $\gamma_n = V_n(\beta_n)$, $n \in \mathbb{N}$. Then each γ_n is an interior hull contained in the interior hull $\overline{\mathbb{D}}$, and $\text{diam}(\gamma_n) \ge 1$. So $\text{rad}(\gamma_n) \ge 1/4$. From Corollary B.2, (γ_n) contains a subsequence which converges to some interior hull K contained in $\overline{\mathbb{D}}$ with radius at least $1/4$. So K is nondegenerate. By passing to a subsequence, we may assume that $\gamma_n \to K$. From $\beta_n \to H$ and $\gamma_n \to K$ we get $\psi_{\beta_n} \xrightarrow{\text{l.u.}} \psi_H$ in $\Omega \cap \mathbb{D}^*$ and $\psi_{\gamma_n} \xrightarrow{\text{l.u.}} \psi_K$ in $W(\Omega \cap \mathbb{D}^*)$. Thus, $\psi_{\beta_n}(\Omega \cap \mathbb{D}^*) \xrightarrow{\text{Cara}} \psi_H(\Omega \cap \mathbb{D}^*)$ by Lemma A.2.

Since $V_n \circ \psi_{\beta_n} = \psi_{\gamma_n} \circ W$ in $\Omega \cap \mathbb{D}^*$, we find that $V_n = \psi_{\gamma_n} \circ W \circ \psi_{\beta_n}^{-1}$ in $\psi_{\beta_n}(\Omega \cap \mathbb{D}^*)$. Let $V = \psi_K \circ W \circ \psi_H^{-1}$ in $\psi_H(\Omega \cap \mathbb{D}^*)$. Then $V_n \xrightarrow{\text{l.u.}} V$ in $\psi_H(\Omega \cap \mathbb{D}^*)$. We may find $r > 1$ such that for any $s \in (1, r]$, $s\mathbb{T} \subset \Omega \cap \mathbb{D}^*$. Then $\psi_H(r\mathbb{T})$ is a Jordan curve in $\psi_H(\Omega \cap \mathbb{D}^*)$ surrounding H, and the Jordan domain bounded by $\psi_H(r\mathbb{T})$ is contained in $\Omega^H = H \cup \psi_H(\Omega \cap \mathbb{D}^*)$. Since $\psi_H(r\mathbb{T})$ is a compact subset of $\psi_H(\Omega \cap \mathbb{D}^*)$, we have $V_n \to V$ uniformly on $\psi_H(r\mathbb{T})$. It is easy to see that $\Omega^{\beta_n} \xrightarrow{\text{Cara}} \Omega^H$. For n big enough, $\psi_H(r\mathbb{T})$ together with its interior is contained in Ω^{β_n}. From the maximum principle, V_n converges uniformly in the interior of $\psi_H(r\mathbb{T})$ to a conformal map which extends V. We still use V to denote the extended conformal map. Then V is a conformal map defined in Ω^H, and $V_n \stackrel{\text{l.u.}}{\longrightarrow} V$ in Ω^H. Letting $n \to \infty$ in the equality $V_n \circ \psi_{\beta_n} = \psi_{\gamma_n} \circ W$ in $\Omega \cap \mathbb{D}^*$, we conclude that $V \circ \psi_H = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. So the existence part is proved.

If $\widetilde{V} = aV + b$ for some $a > 0$ and $b \in \mathbb{C}$, then $\psi_{\widetilde{V}(H)} = a\psi_{V(H)} + b$, which implies $\widetilde{V} \circ \psi_H = \psi_{\widetilde{V}(H)} \circ W$. Finally, suppose $\widetilde{V}$ satisfies the properties of V. Then $\widetilde{V} \circ V^{-1}$ is a conformal map in $V(\Omega^H)$. Since $V \circ \psi_H = \psi_{V(H)} \circ W$ and $\widetilde{V} \circ \psi_H = \psi_{\widetilde{V}(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$, we find that $\widetilde{V} \circ V^{-1} = \psi_{\widetilde{V}(H)} \circ \psi_{V(H)}^{-1}$ in $\psi_{V(H)}(W(\Omega \cap \mathbb{D}^*)) = V(\Omega^H) \setminus V(H)$. Note that $\psi_{\widetilde{V}(H)} \circ \psi_{V(H)}^{-1}$ is a conformal map defined in $\widehat{\mathbb{C}} \setminus V(H)$. Since $V(\Omega^H) \cup (\widehat{\mathbb{C}} \setminus V(H)) = \widehat{\mathbb{C}}$, we may define an analytic function h in $\mathbb{C}$ such that $h = \widetilde{V} \circ V^{-1}$ in $V(\Omega^H)$ and $h = \psi_{\widetilde{V}(H)} \circ \psi_{V(H)}^{-1}$ in $\mathbb{C} \setminus V(H)$. From the properties of $\psi_{\widetilde{V}(H)}$ and $\psi_{V(H)}$, we have $h(\infty) = \infty$ and $h'(\infty) > 0$. Thus, $h(z) = az + b$ for some $a > 0$ and $b \in \mathbb{C}$, which implies that $\widetilde{V} = aV + b$. □

Now we obtain a new proof of the following well-known result about conformal welding.

Corollary 2.3. *Let W be conformal in a neighborhood of $\mathbb{T}$, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. If h is a conformal welding, then $W \circ h$ and $h \circ W$ are also conformal weldings.*

Proof. Apply Theorem 2.2 to $H = \overline{D_J}$, where J is the Jordan curve for the conformal welding h. We find a conformal map V defined in $\Omega^H = D_J \cup f_J^*(\Omega \cap \mathbb{D}^*)$ such that $V \circ f_J^* = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. Let $J' = V(J)$. Then J' is also a Jordan curve, $V(H) = \overline{D_{J'}}$, and $\psi_{V(H)} = f_{J'}^*$. Let $f_{J'} = V \circ f_J$. Then $f_{J'} : \mathbb{D} \stackrel{\text{Conf}}{\twoheadrightarrow} D_J$. Thus,

$$W \circ h = W \circ (f_J^*)^{-1} \circ f_J = \psi_{H(H)}^{-1} \circ V \circ f_J = (f_{J'}^*)^{-1} \circ f_{J'},$$

which implies that $W \circ h$ is a conformal welding. As for $h \circ W$, note that $(h \circ W)^{-1} = W^{-1} \circ h^{-1}$ and that h is a conformal welding if and only if h^{-1} is a conformal welding. □

2.2 Hulls in the upper half plane

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. A subset K of $\mathbb{H}$ is called an $\mathbb{H}$-hull if it is bounded, relatively closed in $\mathbb{H}$, and $\mathbb{H} \setminus K$ is simply connected. For every $\mathbb{H}$-hull K, there is are a unique $c \ge 0$ and a unique $g_K : \mathbb{H} \setminus K \stackrel{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$ such that $g_K(z) = z + \frac{c}{z} + O(\frac{1}{z^2})$ as $z \to \infty$. The number c is called the $\mathbb{H}$-capacity of K, and is denoted by $\operatorname{hcap}(K)$. Let $f_K = g_K^{-1}$. The empty set is an $\mathbb{H}$-hull with $\operatorname{hcap}(\emptyset) = 0$ and $g_\emptyset = f_\emptyset = \operatorname{id}_{\mathbb{H}}$.

Definition 2.4. *Let K_1 and K_2 be $\mathbb{H}$-hulls.*

1. *If $K_1 \subset K_2$, define $K_2/K_1 = g_{K_1}(K_2 \setminus K_1)$. We call K_2/K_1 a quotient hull of K_2, and write $K_2/K_1 \prec K_2$.*
2. *The product $K_1 \cdot K_2$ is defined to be $K_1 \cup f_{K_1}(K_2)$.*

The following facts are easy to check.

1. K_2/K_1 and $K_1 \cdot K_2$ in the definition are also $\mathbb{H}$-hulls.
2. For any two $\mathbb{H}$-hulls K_1 and K_2, $K_1 \subset K_1 \cdot K_2$ and $K_2 = (K_1 \cdot K_2)/K_1 \prec K_1 \cdot K_2$. If $K_1 \subset K_2$, then $K_1 \cdot (K_2/K_1) = K_2$.

3. The space of $\mathbb{H}$-hulls with the product "$\cdot$" is a semigroup with identity element $\emptyset$, and "$\prec$" is a transitive relation of this space.
4. $f_{K_1 \cdot K_2} = f_{K_1} \circ f_{K_2}$ in $\mathbb{H}$; $g_{K_1 \cdot K_2} = g_{K_2} \circ g_{K_1}$ in $\mathbb{H} \setminus (K_1 \cdot K_2)$.
5. $\text{hcap}(K_1 \cdot K_2) = \text{hcap}(K_1) + \text{hcap}(K_2)$. If $K_1 \subset K_2$ or $K_1 \prec K_2$, then $\text{hcap}(K_1) \le \text{hcap}(K_2)$.

From $f_{K_1 \cdot K_2} = f_{K_1} \circ f_{K_2}$ in $\mathbb{H}$ we can conclude that $f_{K_1} = f_{K_1 \cdot K_2} \circ g_{K_2}$ in $\mathbb{H} \setminus K_2$. So f_{K_1} is an analytic extension of $f_{K_1 \cdot K_2} \circ g_{K_2}$, which means that K_1 is uniquely determined by $K_1 \cdot K_2$ and K_2. So the following definition makes sense.

Definition 2.5. *Let K_1 and K_2 be $\mathbb{H}$-hulls such that $K_1 \prec K_2$. We use $K_2 : K_1$ to denote the unique $\mathbb{H}$-hull $K \subset K_2$ such that $K_2/K = K_1$.*

For an $\mathbb{H}$-hull K, the base of K is the set $B_K = \overline{K} \cap \mathbb{R}$. Let the double of K be defined by $\widehat{K} = K \cup I_{\mathbb{R}}(K) \cup B_K$, where $I_{\mathbb{R}}(z) := \overline{z}$. Then g_K extends to a conformal map (still denoted by g_K) in $\widehat{\mathbb{C}} \setminus \widehat{K}$, which satisfies $g_K(\infty) = \infty$, $g_K'(\infty) = 1$, and $g_K \circ I_{\mathbb{R}} = I_{\mathbb{R}} \circ g_K$. Moreover, $g_K(\widehat{\mathbb{C}} \setminus \widehat{K}) = \widehat{\mathbb{C}} \setminus S_K$ for some compact $S_K \subset \mathbb{R}$, which is called the support of K. So f_K extends to a conformal map from $\widehat{\mathbb{C}} \setminus S_K$ onto $\widehat{\mathbb{C}} \setminus \widehat{K}$.

Lemma 2.6. *f_K can not be extended analytically beyond $\widehat{\mathbb{C}} \setminus S_K$.*

Proof. Suppose f_K can be extended analytically near $x_0 \in \mathbb{R}$, then the image of f_K contains a neighborhood of $f_K(x_0) \in \mathbb{R}$. So $f_K(\mathbb{H}) = \mathbb{H} \setminus K$ contains a neighborhood of $f_K(x_0)$ in $\mathbb{H}$. This then implies that $f_K(x_0) \in \mathbb{R} \setminus B_K$. Thus, there is $y_0 \in \mathbb{R} \setminus S_K$ such that $f_K(y_0) = f_K(x_0)$. Since f_K is conformal in $\mathbb{H}$, we must have $x_0 = y_0 \in \mathbb{R} \setminus S_K$. □

Lemma 2.7. *If $K_1 = K_2/K_0 \prec K_2$, then $S_{K_1} \subset S_{K_2}$, $f_{K_2} = f_{K_0} \circ f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$, and $g_{K_2} = g_{K_1} \circ g_{K_0}$ in $\widehat{\mathbb{C}} \setminus \widehat{K}_2$.*

Proof. Since $K_2 = K_0 \cdot K_1$, we have $f_{K_2} = f_{K_0} \circ f_{K_1}$ in $\mathbb{H}$, which implies that $g_{K_0} \circ f_{K_2} = f_{K_1}$ in $\mathbb{H}$. Since f_{K_2} maps $\widehat{\mathbb{C}} \setminus S_{K_2}$ conformally onto $\widehat{\mathbb{C}} \setminus \widehat{K}_2 \subset \widehat{\mathbb{C}} \setminus \widehat{K}_0$, and g_{K_0} is analytic in $\widehat{\mathbb{C}} \setminus \widehat{K}_2$, we see that $g_{K_0} \circ f_{K_2}$ is analytic in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Since $g_{K_0} \circ f_{K_2} = f_{K_1}$ in $\mathbb{H}$, from Lemma 2.6 we have $S_{K_1} \subset S_{K_2}$, and $g_{K_0} \circ f_{K_2} = f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Composing f_{K_0} to the left of both sides, we get $f_{K_2} = f_{K_0} \circ f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Taking inverse, we obtain the equality for g_K's. □

Definition 2.8. *$S \subset \widehat{\mathbb{C}}$ is called $\mathbb{R}$-symmetric if $I_{\mathbb{R}}(S) = S$. An $\mathbb{R}$-symmetric map W is a function defined in an $\mathbb{R}$-symmetric domain Ω, which commutes with $I_{\mathbb{R}}$, and maps $\Omega \cap \mathbb{H}$ into $\mathbb{H}$.*

Remarks.

1. For any $\mathbb{H}$-hull K, g_K and f_K are $\mathbb{R}$-symmetric conformal maps.
2. Let W be an R-symmetric conformal map defined in Ω. If an $\mathbb{H}$-hull K satisfies $\widehat{K} \subset \Omega$ and $\infty \notin W(\widehat{K})$, then $W(K)$ is also an $\mathbb{H}$-hull and $\widehat{W(K)} = W(\widehat{K})$.

Definition 2.9. *Let Ω be an $\mathbb{R}$-symmetric domain and K be an $\mathbb{H}$-hull. If $\widehat{K} \subset \Omega$, we write Ω_K or $(\Omega)_K$ for $S_K \cup g_K(\Omega \setminus \widehat{K})$, and call it the collapse of Ω via K. If $S_K \subset \Omega$, we write Ω^K or (Ω^K) for $\widehat{K} \cup f_K(\Omega \setminus S_K)$, and call it the lift of Ω via K.*

Remarks.

1. In the definition, Ω_K is an $\mathbb{R}$-symmetric domain containing S_K; Ω^K is an $\mathbb{R}$-symmetric domain containing $\widehat{K}$.
2. $(\Omega_K)^K = \Omega$ and $(\Omega^K)_K = \Omega$ if the lefthand sides are well defined.
3. $\Omega_{K_1 \cdot K_2} = (\Omega_{K_1})_{K_2}$ and $\Omega^{K_1 \cdot K_2} = (\Omega^{K_2})^{K_1}$ if either sides are well defined.

Definition 2.10. *Let W be an $\mathbb{R}$-symmetric conformal map with domain Ω. Let K be an $\mathbb{H}$-hull such that $\widehat{K} \subset \Omega$ and $\infty \notin W(\widehat{K})$. We write W_K or $(W)_K$ for the conformal extension of $g_{W(K)} \circ W \circ f_K$ to Ω_K, and call it the collapse of W via K.*

Remarks.

1. Since $g_{W(K)} \circ W \circ f_K : \Omega_K \setminus S_K \stackrel{\rm Conf}{\twoheadrightarrow} W(\Omega) \setminus S_{W(K)}$, the existence of W_K follows from the Schwarz reflection principle. W_K is an $\mathbb{R}$-symmetric conformal map, and $W_K(S_K) = S_{W(K)}$.
2. The g_K and f_K defined at the beginning of this section should not be understood as the collapse of g and f via K.
3. $W_{K_1 \cdot K_2} = (W_{K_1})_{K_2}$ if either side is well defined.
4. $V_{W(K)} \circ W_K = (V \circ W)_K$ if either side is well defined. In particular, $(W^{-1})_{W(K)} = (W_K)^{-1}$.

Let B_K^* and S_K^* be the convex hulls of B_K and S_K, respectively. Let $\widehat{K}^* = \widehat{K} \cup B_K^*$. Then $g_K : \widehat{\mathbb{C}} \setminus \widehat{K}^* \stackrel{\rm Conf}{\twoheadrightarrow} \widehat{\mathbb{C}} \setminus S_K^*$. If $K \neq \emptyset$, then S_K^* is a bounded closed interval, $\widehat{K}^*$ is a nondegenerate interior hull, and $\psi_{\widehat{K}^*} = f_K \circ \psi_{S_K^*}$. If $S_K^* \subset \Omega$, then $\Omega^K = \widehat{K}^* \cup f_K(\Omega \setminus S_K^*)$. The lemma below is a part of Lemma 5.3 in [17], where S_K^* was denoted by $[c_K, d_K]$.

Lemma 2.11. *If $K_1 \subset K_2$, then $S_{K_1}^* \subset S_{K_2}^*$.*

Theorem 2.12. *Let W be an $\mathbb{R}$-symmetric conformal map with domain Ω. Let K be an $\mathbb{H}$-hull such that $S_K \subset \Omega$ and $\infty \notin W(S_K)$. Then there is a unique $\mathbb{R}$-symmetric conformal map V defined in Ω^K such that $V_K = W$.*

Proof. We first consider the existence. If $K = \emptyset$, since $f_\emptyset = \mathrm{id}$ and $\Omega^\emptyset = \Omega$, $V = W$ is what we need. Now suppose $K \neq \emptyset$ and $S_K^* \subset \Omega$. Note that S_K^* is a bounded closed interval, and so is $W(S_K^*)$. Let $\Omega_{\mathbb{T}} = \psi_{S_K^*}^{-1}(\Omega \setminus S_K^*)$. Define a conformal map $W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$ by $W_{\mathbb{T}} = \psi_{W(S_K^*)}^{-1} \circ W \circ \psi_{S_K^*}$. Then $W_{\mathbb{T}}(z) \to \mathbb{T}$ as $\Omega_{\mathbb{T}} \ni z \to \mathbb{T}$. Thus, $W_{\mathbb{T}}$ extends conformally across $\mathbb{T}$, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Apply Theorem 2.2 to $W_{\mathbb{T}}$ and $\widehat{K}^*$. We find a conformal map $\widehat{V}$ defined in

$$\widehat{K}^* \cup \psi_{\widehat{K}^*}(\Omega_{\mathbb{T}}) = \widehat{K}^* \cup f_K(\Omega \setminus S_K^*) = \Omega^K$$

such that $\widehat{V} \circ \psi_{\widehat{K}^*} = \psi_{\widehat{V}(\widehat{K}^*)} \circ W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$. Let $\widetilde{V} = I_{\mathbb{R}} \circ \widehat{V} \circ I_{\mathbb{R}}$. Then $\widetilde{V}(\widehat{K}^*) = I_{\mathbb{R}} \circ \widehat{V}(\widehat{K}^*)$. So $\psi_{\widetilde{V}(\widehat{K}^*)} = I_{\mathbb{R}} \circ \psi_{\widehat{V}(\widehat{K}^*)} \circ I_{\mathbb{R}}$. Since $I_{\mathbb{R}}$ commutes with $\psi_{\widehat{K}^*}$ and $W_{\mathbb{T}}$, we see that $\widetilde{V}$ also satisfies the properties of $\widehat{V}$. So $\widetilde{V} = a\widehat{V} + b$ for some $a > 0$ and $b \in \mathbb{C}$. Thus, $I_{\mathbb{R}} \circ \widehat{V} \circ I_{\mathbb{R}} = a\widehat{V} + b$. Considering the values of $\widehat{V}$ on $\Omega^K \cap \mathbb{R}$, we find that $a = 1$ and $\mathrm{Re}\, b = 0$. Note that $\widehat{V} - \frac{b}{2}$ satisfies the property of $\widehat{V}$, and commutes with $I_{\mathbb{R}}$. By replacing $\widehat{V}$ with $\widehat{V} - \frac{b}{2}$, we may assume that $\widehat{V}$ is an $\mathbb{R}$-symmetric conformal map.

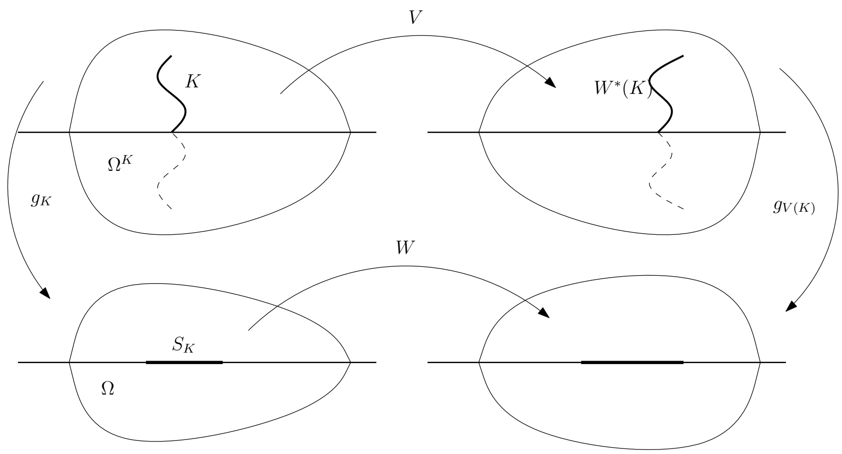

Figure 2: The situation of Theorem 2.12. Given K and W, there is a unique V, also denoted W^K, which is analytic across K and its reflection $I_{\mathbb{R}}(K)$, see Definition 2.13.

Since $\widehat{V}\circ\psi_{\widehat{K}^*}=\psi_{\widehat{V}(\widehat{K}^*)}\circ W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$, from $\psi_{\widehat{K}^*}=f_K\circ\psi_{S_K^*}$, $\psi_{\widehat{V}(\widehat{K}^*)}=f_{\widehat{V}(K)}\circ\psi_{S^*_{\widehat{V}(K)}}$, and the definitions of $W_{\mathbb{T}}$ and $\Omega_{\mathbb{T}}$, we have

$$\widehat{V}\circ f_K=f_{\widehat{V}(K)}\circ\psi_{S^*_{\widehat{V}(K)}}\circ\psi^{-1}_{W(S_K^*)}\circ W \tag{2.1}$$

on $\Omega\setminus S_K^*$. Let $h=\psi_{S^*_{\widehat{V}(K)}}\circ\psi^{-1}_{W(S_K^*)}$. Since $S^*_{\widehat{V}(K)}$ and $W(S_K^*)$ are both bounded closed intervals, we have $h(z)=az+b$ for some $a>0$ and $b\in\mathbb{R}$. Let $V=h^{-1}\circ\widehat{V}$. Then V is also an $\mathbb{R}$-symmetric conformal map defined on Ω^K, and $f_{V(K)}=h^{-1}\circ f_{\widehat{V}(K)}\circ h$. From (2.1) we have

$$f_{V(K)}\circ W=h^{-1}\circ f_{\widehat{V}(K)}\circ h\circ W=h^{-1}\circ\widehat{V}\circ f_K=V\circ f_K.$$

This finishes the existence part in the case that $K\ne\emptyset$ and $S_K^*\subset\Omega$.

Now we still assume that $K\ne\emptyset$ but do not assume that $S_K^*\subset\Omega$. Let $\Omega_0=\Omega$ and $W_0=W$. We will construct $\mathbb{H}$-hulls $K_1,\dots,K_n$ and $\mathbb{R}$-symmetric domains $\Omega_1,\dots,\Omega_n$ such that $K_n\cdot K_{n-1}\cdots K_1=K$, $\Omega_j=\Omega_{j-1}^{K_j}$, and $S^*_{K_j}\subset\Omega_{j-1}$, $1\le j\le n$. When they are constructed, using the above result, we can obtain $\mathbb{R}$-symmetric conformal maps W_j defined on Ω_j, $1\le j\le n$, such that $(W_j)_{K_j}=W_{j-1}$, $1\le j\le n$. Let $V=W_n$. Then V is defined in $\Omega_n=\Omega_0^{K_n\cdots K_1}=\Omega^K$, and $V_K=(W_n)_{K_n\cdots K_1}=W_0=W$. So V is what we need.

It remains to construct K_j and Ω_j with the desired properties. Since $\Omega\cap\mathbb{R}$ is a disjoint union of open intervals, and S_K is a compact subset of $\Omega\cap\mathbb{R}$, we may find finitely many components of $\Omega\cap\mathbb{R}$ which cover S_K. There exist mutually disjoint $\mathbb{R}$-symmetric Jordan curves $J_1,\dots,J_n$ in Ω such that their interiors $D_{J_1},\dots,D_{J_n}$ are mutually disjoint and contained in Ω, and $S_K\subset\bigcup_{k=1}^n D_{J_k}$. Then $J_j^K:=f_K(J_j)$, $1\le j\le n$ are $\mathbb{R}$-symmetric Jordan curves, which together with their interiors are mutually disjoint, and $\widehat{K}\subset\bigcup_{k=1}^n D_{f_K(J_k)}$. Let $H_j=K\cap\bigcup_{k=j}^n D_{J_j^K}$, $1\le j\le n$. Then each H_j is an $\mathbb{H}$-hull, and $K=H_1\supset H_2\supset\cdots\supset H_n$. Let $K_j=H_j/H_{j+1}$, $1\le j\le n-1$, and $K_n=H_n$. Then we have $K_n\cdots K_1=H_1=K$.

Construct Ω_j, $1 \le j \le n$, such that $\Omega_j = \Omega_{j-1}^{K_j}$, $1 \le j \le n$. Then

$$\Omega_{j-1} = (\Omega_0)^{K_{j-1}\cdots K_1} = (\Omega^{K_n\cdots K_1})_{K_n\cdots K_j} = (\Omega^K)_{H_j}, \quad 1 \le j \le n.$$

It suffices to show that $S^*_{K_j} \subset \Omega_{j-1}$. We have

$$K_j = H_j/H_{j+1} = g_{H_{j+1}}(H_j \setminus H_{j+1}) = g_{H_{j+1}}(K \cap D_{J_j^K}).$$

Thus, $\widehat{K}_j \subset D_{g_{H_{j+1}}(J_j^K)}$, which implies that $S_{K_j} \subset D_{g_{H_j}(J_j^K)}$. Since $\mathbb{R} \cap D_{g_{H_j}(J_j^K)}$ is an interval, we have $S^*_{K_j} \subset D_{g_{H_j}(J_j^K)}$. Since $\overline{D_{J_j^K}} \subset \Omega^K$, and J_j^K has positive distance from H_j, we have $D_{g_{H_j}(J_j^K)} \subset (\Omega^K)_{H_j} = \Omega_{j-1}$. So K_j and Ω_j satisfy the properties we need. This finishes the proof of the existence part.

Now we prove the uniqueness. Suppose $\widetilde{V}$ is another $\mathbb{R}$-symmetric conformal map defined on Ω^K such that $\widetilde{V}_K = W$. Then

$$g_{V(K)} \circ V = W \circ g_K = g_{\widetilde{V}(K)} \circ \widetilde{V}$$

on $\Omega \setminus \widehat{K}$. Thus, $\widetilde{V} \circ V^{-1} = f_{\widetilde{V}(K)} \circ g_{V(K)}$ on $V(\Omega \setminus \widehat{K}) = V(\Omega) \setminus V(\widehat{K})$. We know that $\widetilde{V} \circ V^{-1}$ is a conformal map defined on $V(\Omega)$, while $f_{\widetilde{V}(K)} \circ g_{V(K)}$ is a conformal map defined on $\widehat{\mathbb{C}} \setminus \widehat{V(K)} = \widehat{C} \setminus V(\widehat{K})$. Since $V(\Omega)$ and $\widehat{\mathbb{C}} \setminus V(\widehat{K})$ cover $\widehat{C}$, we may define an analytic function h on $\mathbb{C}$ such that $h = \widetilde{V} \circ V^{-1}$ on $V(\Omega)$ and $h = f_{\widetilde{V}(K)} \circ g_{V(K)}$ on $\widehat{\mathbb{C}} \setminus V(\widehat{K})$. From the properties of $f_{\widetilde{V}(K)}$ and $g_{V(K)}$, we see that $h(z) - z \to 0$ as $z \to \infty$. So $h = \mathrm{id}$, which implies that $\widetilde{V} = V$. So the uniqueness is proved. □

Definition 2.13. *We use W^K to denote the unique V in Theorem 2.12, and call it the lift of W via K. Let W^* be the map defined by $W^*(K) = W^K(K)$.*

Remarks.

1. $(W_K)^K = W$ and $(W^K)_K = W$.
2. The range of W^K is $W^K(\Omega^K) = (W(\Omega))^{W^*(K)}$.
3. $W^{K_1 \cdot K_2} = (W^{K_2})^{K_1}$, $V^{W^K(K)} \circ W^K = (V \circ W)^K$, and $(W^K)^{-1} = (W^{-1})^{W(K)}$.
4. The domain (resp. range) of W^* is the set of $\mathbb{H}$-hulls whose supports are contained in the domain (resp. range) of W; and $S_{W^*(K)} = W(S_K)$.
5. $V^* \circ W^* = (V \circ W)^*$; $(W^*)^{-1} = (W^{-1})^*$.

Lemma 2.14. *Suppose $K_1 \prec K_2$, S_{K_2} lies in the domain of an $\mathbb{R}$-symmetric conformal map W, and $\infty \notin W(S_{K_2})$. Then $W^*(K_1) \prec W^*(K_2)$, and*

$$W^*(K_2) : W^*(K_1) = W^{K_2}(K_2 : K_1). \tag{2.2}$$

Proof. From Lemma 2.7, $S_{K_1} \subset S_{K_2}$. So W^{K_1} and W^{K_2} exist. Let $K_0 = K_2 : K_1 \subset K_2$. Then $W^{K_2}(K_0) \subset W^{K_2}(K_2)$ and

$$W^{K_2}(K_2)/W^{K_2}(K_0) = g_{W^{K_2}(K_0)} \circ W^{K_2}(K_2 \setminus K_0)$$

$$= g_{W^{K_2}(K_0)} \circ W^{K_2} \circ f_{K_0}(K_2/K_0) = (W^{K_2})_{K_0}(K_2/K_0)$$
$$= (W^{K_0 \cdot K_1})_{K_0}(K_1) = W^{K_1}(K_1).$$

Thus, $W^{K_1}(K_1) \prec W^{K_2}(K_2)$ and $W^{K_2}(K_2) : W^{K_1}(K_1) = W^{K_2}(K_0)$. □

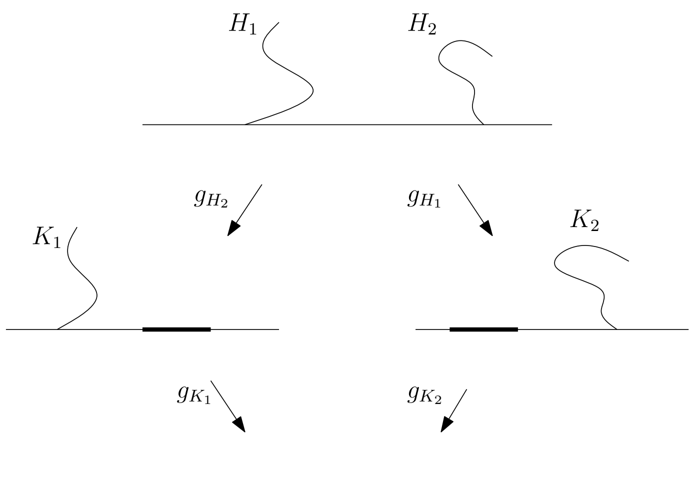

Figure 3: The pair (H_1, H_2) uniquely determines the pair (K_1, K_2), and vice versa, see Theorem 2.16 and Definition 2.15 .

Definition 2.15. *Let $\mathcal{P}^*$ denote the set of pair of $\mathbb{H}$-hulls (H_1, H_2) such that $\widehat{H}_1 \cap \widehat{H}_2 = \emptyset$. Let $\mathcal{P}_*$ denote the set of pair of $\mathbb{H}$-hulls (K_1, K_2) such that $S_{K_1} \cap S_{K_2} = \emptyset$. Define g_* on $\mathcal{P}^*$ by $g_*(H_1, H_2) = (g_{H_2}(H_1), g_{H_1}(H_2))$. Define f^* on $\mathcal{P}_*$ by $f^*(K_1, K_2) = (f^*_{K_2}(K_1), f^*_{K_1}(K_2))$.*

Remarks.

1. g_* is well defined on $\mathcal{P}^*$ because for $j = 1, 2$, $\widehat{K}_{3-j}$ is contained in the domain of g_{K_j}: $\widehat{\mathbb{C}} \setminus \widehat{K}_j$. The value of g_* is a pair of $\mathbb{H}$-hulls.
2. f^* is well defined on $\mathcal{P}_*$ because for $j = 1, 2$, $S_{K_{3-j}}$ is contained in the domain of f_{K_j}: $\widehat{\mathbb{C}} \setminus S_{K_j}$. The value of f^* is a pair of $\mathbb{H}$-hulls.

Theorem 2.16. *g_* and f^* are bijections between $\mathcal{P}^*$ and $\mathcal{P}_*$, and are inverse of each other. Moreover, if $(H_1, H_2) = f^*(K_1, K_2)$, then*

(i) $H_1 \cdot K_2 = H_2 \cdot K_1 = H_1 \cup H_2$;

(ii) $f_{K_2}(S_{K_1}) = S_{H_1}$ *and* $f_{K_1}(S_{K_2}) = S_{H_2}$;

(iii) $S_{H_1 \cup H_2} = S_{K_1} \cup S_{K_2}$.

Proof. Let $(H_1, H_2) \in \mathcal{P}^*$ and $(K_1, K_2) = g_*(H_1, H_2)$. Then $(\widehat{\mathbb{C}} \setminus \widehat{H}_1)_{H_2} = \widehat{\mathbb{C}} \setminus \widehat{K}_1$, $S_{H_1} \subset \widehat{\mathbb{C}} \setminus \widehat{K}_2$, and $(\widehat{\mathbb{C}} \setminus S_{H_1})_{K_2} = \widehat{\mathbb{C}} \setminus g_{K_2}(S_{H_1})$. Since $g_{H_1} : \widehat{\mathbb{C}} \setminus \widehat{H}_1 \overset{\text{Conf}}{\twoheadrightarrow} \widehat{\mathbb{C}} \setminus S_{H_1}$ and $g_{H_1}(H_2) = K_2$, we get $(g_{H_1})_{H_2}:\widehat{\mathbb{C}}\setminus\widehat{K}_1\overset{\rm Conf}{\twoheadrightarrow}\widehat{\mathbb{C}}\setminus g_{K_2}(S_{H_1})$. From the normalization of g_{H_1},g_{H_2},g_{K_2} at ∞, we conclude that

$$(g_{H_1})_{H_2}=g_{K_1},\quad g_{K_2}(S_{H_1})=S_{K_1}. \tag{2.3}$$

From $S_{H_1}\subset\widehat{\mathbb{C}}\setminus\widehat{K}_2$ and $g_{K_2}(S_{H_1})=S_{K_1}$, we see that $S_{K_1}\cap S_{K_2}=\emptyset$, i.e., $(K_1,K_2)\in\mathcal{P}_*$. Since $f_{H_1}=g_{H_1}^{-1}$, $f_{K_1}=g_{K_1}^{-1}$, and $g_{H_1}(H_2)=K_2$, from (2.3) we get $(f_{H_1})_{K_2}=f_{K_1}$, which implies that $(f_{K_1})^{K_2}=f_{H_1}$. Thus, $f_{K_1}^*(K_2)=f_{H_1}(K_2)=H_2$. Similarly, $f_{K_2}^*(K_1)=H_1$. Thus, $f^*(K_1,K_2)=(H_1,H_2)$. So $f^*\circ g_*=\mathrm{id}_{\mathcal{P}^*}$.

Let $(K_1,K_2)\in\mathcal{P}_*$ and $H_1=f_{K_2}^*(K_1)$. Then $S_{H_1}=f_{K_2}(S_{K_1})$ is disjoint from $\widehat{K}_2$. Thus, we may define another $\mathbb{H}$-hull $H_2:=f_{H_1}(K_2)$. Then $\widehat{H}_2\subset\widehat{\mathbb{C}}\setminus\widehat{H}_1$. So $(H_1,H_2)\in\mathcal{P}^*$. We have $(\widehat{\mathbb{C}}\setminus S_{K_2})^{K_1}=\widehat{\mathbb{C}}\setminus f_{K_1}(S_{K_2})$ and $(\widehat{\mathbb{C}}\setminus\widehat{K}_2)^{H_1}=\widehat{\mathbb{C}}\setminus\widehat{H}_2$. Since $f_{K_2}:\widehat{\mathbb{C}}\setminus S_{K_2}\overset{\rm Conf}{\twoheadrightarrow}\widehat{\mathbb{C}}\setminus\widehat{K}_2$ and $(f_{K_2})^*(K_1)=H_1$, we see that $(f_{K_2})^{K_1}:\widehat{\mathbb{C}}\setminus f_{K_1}(S_{K_2})\overset{\rm Conf}{\twoheadrightarrow}\widehat{\mathbb{C}}\setminus\widehat{H}_2$. From the normalization of f_{K_1},f_{H_1},f_{K_2} at ∞, we conclude that

$$(f_{K_2})^{K_1}=f_{H_2},\quad f_{K_1}(S_{K_2})=S_{H_2}. \tag{2.4}$$

Since $H_1=f_{K_2}^*(K_1)$, we get $f_{K_2}=g_{H_1}\circ f_{H_2}\circ f_{K_1}$ on $(\widehat{\mathbb{C}}\setminus S_{K_2})\setminus S_{K_1}$, which implies that $f_{H_1}\circ f_{K_2}=f_{H_2}\circ f_{K_1}$ on $\widehat{\mathbb{C}}\setminus(S_{K_1}\cup S_{K_2})$. So

$$H_2\cdot K_1=H_1\cdot K_2=H_1\cup f_{H_1}(K_2)=H_1\cup H_2. \tag{2.5}$$

Thus, $K_1=g_{H_2}(H_1)$ and $K_2=g_{H_1}(H_2)$, i.e., $(K_1,K_2)=g_*(H_1,H_2)$. This shows that the range of g_* is $\mathcal{P}_*$, which combining with $f^*\circ g_*=\mathrm{id}_{\mathcal{P}^*}$ shows that $f^*=(g_*)^{-1}$ and $g_*=(f^*)^{-1}$.

In the previous paragraph, since $(K_1,K_2)=g_*(H_1,H_2)$, $f^*(K_1,K_2)=(H_1,H_2)$. Thus, (i) follows from (2.5); the second parts of (ii) follow from (2.4), and the first part follows from symmetry. Finally, since $g_{K_2}\circ g_{H_1}=g_{H_1\cdot K_2}=g_{H_1\cup H_2}$, from $g_{H_1}:\widehat{\mathbb{C}}\setminus(\widehat{H}_1\cup\widehat{H}_2)\overset{\rm Conf}{\twoheadrightarrow}\widehat{\mathbb{C}}\setminus(S_{H_1}\cup\widehat{K}_2)$, $g_{K_2}:\widehat{\mathbb{C}}\setminus(S_{H_1}\cup\widehat{K}_2)\overset{\rm Conf}{\twoheadrightarrow}\widehat{\mathbb{C}}\setminus(g_{K_2}(S_{H_1})\cup S_{K_2})$, and (2.3), we get (iii). $\square$

Definition 2.17. *For $(K_1,K_2)\in\mathcal{P}_*$, we define the quotient union of K_1 and K_2 to be $K_1\vee K_2=H_1\cup H_2$, where $(H_1,H_2)=f^*(K_1,K_2)$.*

Remark. From Theorem 2.16, $K_1,K_2\prec K_1\vee K_2$ and $S_{K_1\vee K_2}=S_{K_1}\cup S_{K_2}$.

The space of $\mathbb{H}$-hulls has a natural metric $d_{\mathcal{H}}$ described in Appendix C. Let $\mathcal{H}_S$ denote the set of $\mathbb{H}$-hulls whose supports are contained in S. From Lemma C.2, if F is compact, $(\mathcal{H}_F,d_{\mathcal{H}})$ is compact, and $H_n\to H$ in $\mathcal{H}_F$ implies that $f_{H_n}\overset{\rm l.u.}{\longrightarrow}f_H$ in $\mathbb{C}\setminus F$.

Theorem 2.18. *(i) Let $F\subset\mathbb{R}$ be compact. Let W be an $\mathbb{R}$-symmetric conformal map whose domain contains F. Then $W^*:\mathcal{H}_F\to\mathcal{H}_{W(F)}$ is continuous.*

(ii) Let E and F be two nonempty compact subsets of $\mathbb{R}$ with $E\cap F=\emptyset$. Then f^* and $(K_1,K_2)\mapsto K_1\vee K_2$ are continuous on $\mathcal{H}_E\times\mathcal{H}_F$.

Proof. (i). First, W^* is well defined on $\mathcal{H}_F$, and the range of W^* is $\mathcal{H}_{W(F)}$. Suppose (H_n) is a sequence in $\mathcal{H}_F$ and $H_n\to H_0\in\mathcal{H}_F$. To prove the continuity of W^*, we need to show that $W^*(H_n)\to W^*(H_0)$. Suppose this is not true. Since $\mathcal{H}_{W(F)}$ is compact, by passing to a subsequence, we may assume that $W^*(H_n)\to K_0\neq W^*(H_0)$. For each n_k,

$W^{H_{n_k}} = f_{W^*(H_{n_k})} \circ W \circ g_{H_{n_k}}$ on $f_{H_{n_k}}(\Omega \setminus F)$. We have $g_{H_{n_k}} \stackrel{\text{l.u.}}{\longrightarrow} g_{H_0}$ in $f_{H_0}(\Omega \setminus F)$ and $f_{W^*(H_{n_k})} \stackrel{\text{l.u.}}{\longrightarrow} f_{K_0}$ in $W(\Omega) \setminus W(F)$. Thus, $W^{H_{n_k}} \stackrel{\text{l.u.}}{\longrightarrow} f_{K_0} \circ W \circ g_{H_0} =: V$ in $f_{H_0}(\Omega \setminus F)$. The domain of $W^{H_{n_k}}$ is $\Omega^{H_{n_k}} = \widehat{H}_{n_k} \cup f_{H_{n_k}}(\Omega \setminus S_{H_{n_k}})$, which converges to $\Omega^{H_0} = \widehat{H}_0 \cup f_{H_0}(\Omega \setminus S_{H_0}) \supset f_{H_0}(\Omega \setminus F)$. It is clear that $\Omega^{H_0} \setminus f_{H_0}(\Omega \setminus F)$ is compact. Since $W^{H_{n_k}} \stackrel{\text{l.u.}}{\longrightarrow} V$ in $f_{H_0}(\Omega \setminus F)$, from the maximum principle, $W^{H_{n_k}}$ converges locally uniformly in Ω^{H_0}. We still let V denote the limit function. Since $H_{n_k} \to H_0$ and $W^{H_{n_k}}(H_{n_k}) \to K_0$, we have $V(H_0) = K_0$. Since $f_{K_0} \circ W \circ g_{H_0} = V$ in $f_{H_0}(\Omega \setminus F)$, we see that $f_{V(H_0)} \circ W \circ g_{H_0} = V$ in $f_{H_0}(\Omega \setminus S_{H_0})$. Thus, $V = W^{H_0}$. So $K_0 = W^{H_0}(H_0) = W^*(H_0)$. This is the contradiction we need.

(ii). To show f^* is continuous, it suffices to show that, if (K_1^n, K_2^n) is a sequence in $\mathcal{H}_E \times \mathcal{H}_F$ which converges to $(K_1^0, K_2^0) \in \mathcal{H}_E \times \mathcal{H}_F$, then it has a subsequence $(K_1^{(n_k)}, K_2^{(n_k)})$ such that $f^*(K_1^{(n_k)}, K_2^{(n_k)}) \to f^*(K_1^0, K_2^0)$. Let $(H_1^n, H_2^n) = f^*(K_1^n, K_2^n)$, $n \in \mathbb{N}$. From Theorem 2.16 (iii), $S_{H_1^n \cup H_2^n} = S_{K_1^n} \cup S_{K_2^n} \subset E \cup F$. From Lemma C.2, $(H_1^n \cup H_2^n)$ has a convergent subsequence with limit in $\mathcal{H}_{E \cup F}$. From Lemma 2.11, $S_{H_1^n} \subset S^*_{H_1^n \cup H_2^n} \subset A$, where A is the convex hull of $E \cup F$. From Lemma C.2, (H_1^n) has a convergent subsequence. For the same reason, (H_2^n) also has a convergent subsequence. By passing to subsequences, we may assume that $H_1^n \cup H_2^n \to M^0 \in \mathcal{H}_{E \cup F}$ and $H_j^n \to H_j^0$, $j = 1, 2$.

From Theorem 2.16 (i) and the continuity of the dot product, we get $H_1^0 \cdot K_2^0 = H_2^0 \cdot K_1^0 = M^0$. This implies that $M^0 = H_1^0 \cup f_{H_1^0}(K_2^0)$. The measures $(\mu_{H_1^n})$ (see Appendix C) converges to $\mu_{H_1^0}$ weakly. Each $\mu_{H_1^n}$ is supported by $S_{H_1^n}$. From Theorem 2.16 (ii), $S_{H_1^n} = f_{K_2^n}(S_{K_1^n}) \subset f_{K_2^n}(E)$. Since E is a compact subset of $\mathbb{C} \setminus F$, we have $f_{K_2^n} \to f_{K_2^0}$ uniformly on E. Thus, $f_{K_2^n}(E) \to f_{K_2^0}(E)$ in the Hausdorff metric. So $\mu_{H_1^0}$ is supported by $f_{K_2^0}(E)$, which implies that $S_{H_1^0} \subset f_{K_2^0}(E)$. Hence $f_{H_1^0}(K_2^0)$ is another $\mathbb{H}$-hull, which is bounded away from H_1^0. From $K_2^n \to K_2^0$ we have $\mathbb{H} \setminus K_2^n \stackrel{\text{Cara}}{\longrightarrow} \mathbb{H} \setminus K_2^0$. From (C.1) we get $f_{H_1^n} \stackrel{\text{l.u.}}{\longrightarrow} f_{H_1^0}$ in $\mathbb{C} \setminus S_{H_1^0}$. Thus, $\mathbb{H} \setminus f_{H_1^n}(K_2^n) \stackrel{\text{Cara}}{\longrightarrow} \mathbb{H} \setminus f_{H_1^0}(K_2^0)$. Since $H_2^n = f_{H_1^n}(K_2^n)$, we have $\mathbb{H} \setminus H_2^n \stackrel{\text{Cara}}{\longrightarrow} \mathbb{H} \setminus f_{H_1^0}(K_2^0)$. On the other hand, $\mathbb{H} \setminus H_2^n \stackrel{\text{Cara}}{\longrightarrow} \mathbb{H} \setminus H_2^0$. Since $\mathbb{H} \setminus H_2^0$ and $\mathbb{H} \setminus f_{H_1^0}(K_2^0)$ both contain a neighborhood of ∞ in $\mathbb{H}$, they must be the same domain. Thus, $H_2^0 = f_{H_1^0}(K_2^0)$ is bounded away from H_1^0, i.e., $(H_1^n, H_2^n) \in \mathcal{P}^*$. For the same reason, $H_1^0 = f_{H_2^0}(K_1^0)$. Thus, $(H_1^n, H_2^n) \to (H_1^0, H_2^0) = f^*(K_1^0, K_2^0)$. This shows that f^* is continuous. Finally, since $K_1 \vee K_2 = H_1 \cdot K_2$ if $(H_1, H_2) = f^*(K_1, K_2)$, we see that $(K_1, K_2) \mapsto K_1 \vee K_2$ is also continuous. □

Corollary 2.19. *(i) Let W be an $\mathbb{R}$-symmetric conformal map with domain Ω. Then W^* is measurable on $\mathcal{H}_{\Omega \cap \mathbb{R}}$.*

(ii) f^* and $(K_1, K_2) \mapsto K_1 \vee K_2$ are measurable on $\mathcal{P}_*$.

Proof. (i) We may find an increasing sequence of compact subsets (F_n) of $\Omega \cap \mathbb{R}$ such that $\mathcal{H}_{\Omega \cap \mathbb{R}} = \bigcup_{n=1}^\infty \mathcal{H}_{F_n}$. From Theorem 2.18 (i), W^* is continuous on each $\mathcal{H}_{F_n}$. Thus, W^* is measurable on $\mathcal{H}_{\Omega \cap \mathbb{R}}$.

(ii) We may find a sequence of pairs of disjoint bounded closed intervals of $\mathbb{R}$: (E_n, F_n), $n \in \mathbb{N}$, such that $\mathcal{P}_* = \bigcup_{n=1}^\infty \mathcal{H}_{E_n} \times \mathcal{H}_{F_n}$. From Theorem 2.18 (ii), f^* and $(K_1, K_2) \mapsto K_1 \vee K_2$ are continuous on each $\mathcal{H}_{E_n} \times \mathcal{H}_{F_n}$, and so they are measurable on $\mathcal{P}_*$. □

2.3 Hulls in the unit disc

A subset K of $\mathbb{D} = \{|z| < 1\}$ is called a $\mathbb{D}$-hull if $\mathbb{D} \setminus K$ is a simply connected domain containing 0. For every $\mathbb{D}$-hull K, there is a unique $g_K : \mathbb{D} \setminus K \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{D}$ such that $g_K(0) = 0$ and $g_K'(0) > 0$. Then $\ln g_K'(0) \ge 0$ is called the $\mathbb{D}$-capacity of K, and is denoted by $\mathrm{dcap}(K)$. Let $f_K = g_K^{-1}$.

We may define $K_1 \cdot K_2$, K_2/K_1 (when $K_1 \subset K_2$), and $K_1 \prec K_2$ on the space of $\mathbb{D}$-hulls as in Definition 2.4. Then the remarks after Definition 2.4 still hold if $\mathbb{H}$ is replaced by $\mathbb{D}$ and hcap is replaced by dcap. Then we may define $K_2 : K_1$ (when $K_1 \prec K_2$) as in Definition 2.5. For a $\mathbb{D}$-hull K, the base B_K of K is $\overline{K} \cap \mathbb{T}$, and the double of K is $\widehat{K} = K \cup I_{\mathbb{T}}(K) \cup B_K$, where $I_{\mathbb{T}}(z) := 1/\overline{z}$. Then g_K extends to a conformal map (still denoted by g_K) on $\widehat{\mathbb{C}} \setminus \widehat{K}$, which commutes with $I_{\mathbb{T}}$. Moreover, $g_K(\widehat{\mathbb{C}} \setminus \widehat{K}) = \widehat{\mathbb{C}} \setminus S_K$ for some compact $S_K \subset \mathbb{T}$, which is called the support of K. So f_K extends to a conformal map from $\widehat{\mathbb{C}} \setminus S_K$ onto $\widehat{\mathbb{C}} \setminus \widehat{K}$, which commutes with $I_{\mathbb{T}}$. Then Lemma 2.6 and Lemma 2.7 still hold here.

We may define $\mathbb{T}$-symmetric sets and $\mathbb{T}$-symmetric conformal maps using Definition 2.8 with $\mathbb{R}$ and $\mathbb{H}$ replaced by $\mathbb{T}$ and $\mathbb{D}$, respectively. For a $\mathbb{T}$-symmetric domain Ω and a $\mathbb{D}$-hull K, we may define domains Ω_K (when $\widehat{K} \subset \Omega$) and Ω^K (when $S_K \subset \Omega$) using Definition 2.9. If W is a $\mathbb{T}$-symmetric conformal map with domain Ω, and if Ω_K is defined, we may then define W_K using Definition 2.10, which is a $\mathbb{T}$-symmetric conformal map on Ω_K. The remarks after Definition 2.8, Definition 2.9, and Definition 2.10 hold here with minor modifications. We claim that Theorem 2.12 holds here with modifications. We need several lemmas.

The lemma below relates the $\mathbb{H}$-hulls with $\mathbb{D}$-hulls. To distinguish the two set of symbols, we use $f_K^{\mathbb{H}}$, $g_K^{\mathbb{H}}$, $B_K^{\mathbb{R}}$, $S_K^{\mathbb{R}}$, and $\widehat{K}^{\mathbb{R}}$ for $\mathbb{H}$-hulls, and $f_K^{\mathbb{D}}$, $g_K^{\mathbb{D}}$, $B_K^{\mathbb{T}}$, $S_K^{\mathbb{T}}$, and $\widehat{K}^{\mathbb{T}}$ for $\mathbb{D}$-hulls.

Theorem 2.20. *(i) Let W be a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$, and K be a $\mathbb{D}$-hull such that $W^{-1}(\infty) \notin S_K^{\mathbb{T}}$. Then there is a unique Möbius transformation W^K that maps $\mathbb{D}$ onto $\mathbb{H}$ such that $W^K(K)$ is an $\mathbb{H}$-hull, $g^{\mathbb{H}}_{W^K(K)} \circ W^K \circ f_K^{\mathbb{D}} = W$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{T}}$, and $S^{\mathbb{R}}_{W^K(K)} = W(S_K^{\mathbb{T}})$.*

(ii) Let W be a Möbius transformation that maps $\mathbb{H}$ onto $\mathbb{D}$, and K be an $\mathbb{H}$-hull. Then there is a unique Möbius transformation W^K that maps $\mathbb{H}$ onto $\mathbb{D}$ such that $W^K(K)$ is a $\mathbb{D}$-hull, $g^{\mathbb{D}}_{W^K(K)} \circ W^K \circ f_K^{\mathbb{H}} = W$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{R}}$, and $S^{\mathbb{T}}_{W^K(K)} = W(S_K^{\mathbb{R}})$.

Proof. (i) Let $z_0 = W^{-1}(\infty) \in \mathbb{T} \setminus S_K^{\mathbb{T}}$. Then $w_0 := f_K^{\mathbb{D}}(z_0) \in \mathbb{T} \setminus B_K^{\mathbb{T}}$ is well defined. Let $W_0^K(z) = i\frac{w_0+z}{w_0-z}$. Then W_0^K is a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$ and takes w_0 to ∞. Let $L_0 = W_0^K(K)$. Since w_0 is bounded away from K, we see that L_0 is an $\mathbb{H}$-hull. We have $W_0^K : \widehat{\mathbb{C}} \setminus \widehat{K} \overset{\mathrm{Conf}}{\twoheadrightarrow} \widehat{\mathbb{C}} \setminus \widehat{L}_0$. Define $G = g^{\mathbb{H}}_{L_0} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1}$ on $\widehat{\mathbb{C}} \setminus W(S_K^{\mathbb{T}})$. Then $G : \widehat{\mathbb{C}} \setminus W(S_K^{\mathbb{T}}) \overset{\mathrm{Conf}}{\twoheadrightarrow} \widehat{\mathbb{C}} \setminus S^{\mathbb{R}}_{L_0}$, fixes ∞, and maps $\mathbb{H}$ onto $\mathbb{H}$. So $G(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. Let $W^K = G^{-1} \circ W_0^K$. Then W^K is also a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$, and $W^K(K)$ is also an $\mathbb{H}$-hull with $S^{\mathbb{R}}_{W^K(K)} = G^{-1}(S^{\mathbb{R}}_{L_0}) = W(S_K^{\mathbb{T}})$ and $g^{\mathbb{H}}_{W^K(K)} = G^{-1} \circ g^{\mathbb{H}}_{L_0} \circ G$. Thus,

$$g^{\mathbb{H}}_{W^K(K)} \circ W^K \circ f_K^{\mathbb{D}} \circ W^{-1} = G^{-1} \circ g^{\mathbb{H}}_{L_0} \circ G \circ G^{-1} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1}$$

$$= G^{-1} \circ g^{\mathbb{H}}_{L_0} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1} = G^{-1} \circ G = \mathrm{id}_{\widehat{\mathbb{C}} \setminus W(S_K)}.$$

This implies that $g_L^{\mathbb{H}} \circ W^K \circ f_K^{\mathbb{D}} = W$ in $\widehat{\mathbb{C}} \setminus S_K$. So we proved the existence. On the other hand, if W^K satisfies the desired property, then from $W^K = f_L^{\mathbb{H}} \circ W \circ g_K^{\mathbb{D}}$ we get $W^K(w_0) = \infty$. So $W^K = G_0 \circ W_0^K$, where $G_0(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. The above argument shows that $G_0 = G^{-1}$. So we get the uniqueness.

(ii) We may use the proof of (i) with slight modifications: replace ∞ by 0, swap $\mathbb{H}$ and $\mathbb{D}$, swap $\mathbb{R}$ and $\mathbb{T}$, and define $W_0^K(z) = \frac{z-w_0}{z-\overline{w_0}}$. □

We also use $W^*(K)$ to denote the hull $W^K(K)$ in the above lemma. The following lemma is similar to Lemma 2.14.

Lemma 2.21. *Let K_1 and K_2 be two $\mathbb{H}$-(resp. $\mathbb{D}$-)hulls such that $K_1 \prec K_2$. Let W be a Möbius transformation that maps $\mathbb{H}$ onto $\mathbb{D}$ (resp. maps $\mathbb{D}$ onto $\mathbb{H}$) such that $\infty \notin W(S_{K_2})$. Then $W^*(K_1) \prec W^*(K_2)$ and (2.2) still holds.*

The following lemma is used to treat the case $S_K = \mathbb{T}$ in Theorem 2.23.

Lemma 2.22. *Let W be a $\mathbb{T}$-symmetric conformal map with domain $\Omega \supset \mathbb{T}$. Let (K_n) be a sequence of $\mathbb{D}$-hulls which converges to K. Suppose that for each n, there is a $\mathbb{T}$-symmetric conformal map $V^{\langle n\rangle}$ defined on Ω^{K_n} such that $V^{\langle n\rangle}_{K_n} = W$. Then there is a $\mathbb{T}$-symmetric conformal map V defined on Ω^K such that $V_K = W$. Moreover, $V(K)$ is a subsequential limit of $(V^{\langle n\rangle}(K_n))$.*

Proof. Since $K_n \to K$, $\Omega^{K_n} \xrightarrow{\text{Cara}} \Omega^K$. Since $V^{\langle n\rangle}$ maps $\Omega^{K_n} \cap \mathbb{D}$ into $\mathbb{D}$, the family $(V^{\langle n\rangle}|_{\Omega^{K_n}\cap\mathbb{D}})$ is uniformly bounded. Thus, $(V^{\langle n\rangle})$ contains a subsequence, which convergence locally uniformly in $\Omega^K \cap \mathbb{D}$. To save the symbols, we assume that $(V^{\langle n\rangle})$ itself converges locally uniformly in $\Omega^K \cap \mathbb{D}$. Since each $V^{\langle n\rangle}$ is $\mathbb{T}$-symmetric, the sequence also converges locally uniformly in $\Omega^K \cap \mathbb{D}^*$. From the maximum principle, $(V^{\langle n\rangle})$ converges locally uniformly in Ω^K. Let V be the limit function. Since each $V^{\langle n\rangle}$ maps $\mathbb{T}$ onto $\mathbb{T}$, and $V^{\langle n\rangle} \to V$ uniformly on $\mathbb{T}$, V can not be constant. From Lemma A.2, V is a conformal map. It is $\mathbb{T}$-symmetric because each $V^{\langle n\rangle}$ is $\mathbb{T}$-symetric. Since $K_n \to K$, we have $V^{\langle n\rangle}(K_n) \to V(K)$. From $V^{\langle n\rangle}_{K_n} = W$ we have $g_{V^{\langle n\rangle}(K_n)} \circ V^{\langle n\rangle} \circ f_{K_n} = W$ in $\Omega \setminus \mathbb{T}$. Letting $n \to \infty$ we get $g_{V(K)} \circ V \circ f_K = W$ in $\Omega \setminus \mathbb{T}$. By continuation, this equality also holds on $\Omega \setminus S_K$. Thus, $V_K = W$. □

Theorem 2.23. *Let W be a $\mathbb{T}$-symmetric conformal map with domain Ω. Let K be a $\mathbb{D}$-hull such that $S_K \subset \Omega$. Then there is a unique $\mathbb{T}$-symmetric conformal map V defined on Ω^K such that $V_K = W$.*

Proof. We first consider the existence. Case 1. $S_K^{\mathbb{T}} \neq \mathbb{T}$. We will apply Theorems 2.12 and 2.20 for this case. Pick $z_0 \in \mathbb{T} \setminus S_K^{\mathbb{T}}$ and let $h(z) = i\frac{z_0+z}{z_0-z}$. From Theorem 2.20 (i), there is a Möbius transformation h^K that maps $\mathbb{D}$ onto $\mathbb{H}$ such that $L := h^K(K)$ is an $\mathbb{H}$-hull, and $g_L^{\mathbb{H}} \circ h^K \circ f_K^{\mathbb{D}} = h$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{T}}$. Since W is a homeomorphism on S_K, $W(S_K) \neq \mathbb{T}$. So there is $z_W \in \mathbb{T} \setminus W(S_K)$. Let $h_W(z) = z_W \cdot \frac{z-i}{z+i}$. Then h_W is a Möbius transformation that maps $\mathbb{H}$ onto $\mathbb{D}$ and takes ∞ to z_W. Let $\widetilde{W} = h_W^{-1} \circ W \circ h^{-1}$. Then $\widetilde{W}$ is an $\mathbb{R}$-symmetric conformal map with domain $h(\Omega)$, and $\widetilde{W}(S_L^{\mathbb{R}}) = h_W^{-1} \circ W(S_K^{\mathbb{T}}) \not\ni \infty$. From Theorem 2.12, there is an $\mathbb{R}$-symmetric conformal map $\widetilde{V}$ with domain $\widehat{L}^{\mathbb{R}} \cup f_L^{\mathbb{R}}(h(\Omega) \setminus S_L^{\mathbb{R}})$ such that $L^* := \widetilde{V}(L)$ is an $\mathbb{H}$-hull, and $\widetilde{V} = f_{L^*}^{\mathbb{H}} \circ \widetilde{W} \circ g_L^{\mathbb{H}}$ in $\widehat{\mathbb{C}} \setminus \widehat{L}^{\mathbb{R}}$. From Theorem 2.20

(ii), there is a Möbius transformation $h_W^{L^*}$ that maps $\mathbb{H}$ onto $\mathbb{D}$ such that $K^* := h_W^{L^*}(L^*)$ is a $\mathbb{D}$-hull, and $g_{K^*}^{\mathbb{D}} \circ h_W^{L^*} \circ f_{L^*}^{\mathbb{H}} = h_W$ in $\widehat{\mathbb{C}} \setminus S_{L^*}^{\mathbb{R}}$. Finally, let $V = h_W^{L^*} \circ \widetilde{V} \circ h^K$. Then

$$V(K) = h_W^{L^*} \circ \widetilde{V}(L) = h_W^{L^*}(L^*) = K^*,$$

and

$$\begin{aligned} g_{K^*} \circ V \circ f_K &= g_{K^*} \circ h_W^{L^*} \circ \widetilde{V} \circ h^K \circ f_K \\ &= g_{K^*} \circ h_W^{L^*} \circ f_{L^*}^{\mathbb{H}} \circ \widetilde{W} \circ g_L^{\mathbb{H}} \circ h^K \circ f_K \\ &= h_W \circ \widetilde{W} \circ h = W \end{aligned}$$

in $\widehat{\mathbb{C}} \setminus \widehat{K}$. This finishes the existence part for Case 1.

Case 2. $S_K = \mathbb{T}$. First, we may approximate K using $\mathbb{D}$-hulls bounded by $\mathbb{T}$ and a Jordan curve in $\mathbb{D}$. For example, let $J_n = f_K(\{|z| = 1 - 1/(2n)\})$, and let $K_n = \mathbb{D} \setminus D_{J_n}$. Then each K_n is a $\mathbb{D}$-hull, and $K_n \to K$. Second, if K' has the form of $\mathbb{D} \setminus D_J$ for some Jordan curve J, then we may define a curve β, which starts from $\beta(0) = z_0 \in \mathbb{T}$, then follows a simple curve in $\mathbb{D} \cap D_J^*$ to a point on J, and then follows J in the clockwise direction, and ends when it finishes one round. Suppose the domain of β is $[0, 1]$. Then β is simple on $[0, 1 - \varepsilon]$ for any $\varepsilon > 0$. Let $K_n = \beta((0, 1 - 1/n])$, $n \in \mathbb{N}$. Then each K_n is a $\mathbb{D}$-hull with $S_{K_n} \neq \mathbb{T}$, and $K_n \to K'$. Thus, K can be approximated by a sequence of $\mathbb{D}$-hulls (K_n) such that $S_{K_n} \neq \mathbb{T}$ for each K_n. Then the existence of V follows from Case 1 and Lemma 2.22.

Now we prove the uniqueness. Suppose $\widetilde{V}$ is another $\mathbb{T}$-symmetric conformal map defined on Ω^K such that $\widetilde{V}_K = W$. We may use the argument in the proof of Theorem 2.12 to construct an analytic function h on $\mathbb{C}$ such that $h = \widetilde{V} \circ V^{-1}$ on $V(\Omega)$ and $h = f_{\widetilde{V}(K)} \circ g_{V(K)}$ on $\mathbb{C} \setminus V(\widehat{K})$. Then h is $\mathbb{T}$-symmetric. From the properties of $f_{\widetilde{V}(K)}$ and $g_{V(K)}$, we see that $h(0) = 0$ and $h'(0) > 0$. So $h = \text{id}$, which implies that $\widetilde{V} = V$. □

We may then define W^K and W^* using Definition 2.13 with Theorem 2.23 in place of Theorem 2.12 and $\mathbb{D}$ in place of $\mathbb{H}$. The remarks after Definition 2.13 hold here with minor modifications, and so does Lemma 2.14. Then we define $\mathcal{P}^*$, $\mathcal{P}_*$, g_*, and f^* using Definition 2.15 with $\mathbb{H}$ replaced by $\mathbb{D}$. Then Theorem 2.16 still holds here, and we may define the quotient union $K_1 \vee K_2$ for $(K_1, K_2) \in \mathcal{P}_*$.

The space of $\mathbb{D}$-hulls has a natural metric $d_{\mathcal{H}}$ described in Appendix D. Let $\mathcal{H}_S$ denote the set of $\mathbb{D}$-hulls whose supports are contained in S. We claim that Theorem 2.18 still holds here if every $\mathbb{R}$ is replaced by $\mathbb{T}$. For part (i), if $F \neq \mathbb{T}$, then the proof of Theorem 2.18 (i) still goes through with Lemma D.2 in place of Lemma C.2; if $F = \mathbb{T}$, then the continuity of W^* follows from Lemma 2.22. For part (ii), the proof of Theorem 2.18 still goes through with some modifications. The relatively compactness of $(H_n \cup J_n)$ follows from Lemma D.2 instead of Lemma C.2 because $S_{H_n \cup J_n} \subset E \cup F \subsetneqq \mathbb{T}$. To show the relatively compactness of (H_n) and (J_n), instead of applying Lemma 2.11, we now apply Lemma D.1, and use the relatively compactness of $(H_n \cup J_n)$ and the inequalities $\text{dcap}(H_n), \text{dcap}(J_n) \leq \text{dcap}(H_n \cup J_n)$. In addition, (D.2) will be used in place of (C.1). This finishes the proof of Theorem 2.18 in the radial case. Then Corollary 2.19 in the radial case immediately follows.

The proof of Theorem 2.18 (i) may also be used to show that the map $K \mapsto W^K(K)$ in Theorem 2.20 (i) (resp. (ii)) is continuous if restricted to $\mathcal{H}_F^{\mathbb{D}}$ (resp. $\mathcal{H}_F^{\mathbb{H}}$), where F is a compact subset of $\mathbb{T} \setminus W^{-1}(\infty)$ (resp. $\mathbb{R}$). We then can conclude that the maps $K \mapsto W^K(K)$ in Theorem 2.20 (i) and (ii) are both measurable.

3 Loewner Equations and Loewner Chains

3.1 Forward Loewner equations

We review the definitions and basic facts about (forward) Loewner equations. The reader is referred to [6] for details. Let $\lambda \in C([0,T))$, where $T \in (0,\infty]$. The chordal Loewner equation driven by λ is

$$\partial_t g_t(z) = \frac{2}{g_t(z)-\lambda(t)}, \quad g_0(z)=z.$$

We assume that $g_t(\infty)=\infty$ for $0 \le t < \infty$. For $z \in \mathbb{C}$, suppose that the maximal interval for $t \mapsto g_t(z)$ is $[0,\tau_z)$. Let $K_t = \{z \in \mathbb{H} : \tau_z \le t\}$, i.e., the set of $z \in \mathbb{H}$ such that $g_t(z)$ is not defined. Then g_t and K_t, $0 \le t < T$, are called the chordal Loewner maps and hulls driven by λ. It is known that each K_t is an $\mathbb{H}$-hull with $\mathrm{hcap}(K_t)=2t$, and for $0<t<T$, $g_t = g_{K_t}$ with exactly the same domain: $\widehat{\mathbb{C}} \setminus \widehat{K}_t$. At $t=0$, $K_0=\emptyset$ and $g_0 = \mathrm{id}_{\widehat{\mathbb{C}}\setminus\{\lambda(0)\}}$.

We say that λ generates a chordal trace β if

$$\beta(t) := \lim_{\mathbb{H}\ni z\to\lambda(t)} g_t^{-1}(z) \in \overline{\mathbb{H}}$$

exists for $0 \le t < T$, and β is a continuous curve. We call such β the chordal trace driven by λ. If the chordal trace β exists, then for each t, $\mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\beta((0,t])$, and f_t extends continuously from $\mathbb{H}$ to $\mathbb{H}\cup\mathbb{R}$. The trace β is called simple if it is a simple curve and $\beta(t) \in \mathbb{H}$ for $0<t<T$, in which case $K_t = \beta((0,t])$ for $0 \le t < T$.

The radial Loewner equation driven by λ is

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\lambda(t)}+g_t(z)}{e^{i\lambda(t)}-g_t(z)}, \quad 0 \le t < T; \quad g_0(z)=z.$$

We assume that $g_t(\infty)=\infty$ for $0 \le t < \infty$. For each $t \in [0,T)$, let K_t be the set of $z \in \mathbb{D} := \{|z|<1\}$ at which g_t is not defined. Then g_t and K_t, $0 \le t < T$, are called the radial Loewner maps and hulls driven by λ. It is known that, each K_t is a $\mathbb{D}$-hull with $\mathrm{dcap}(K_t)=t$, and for $0<t<T$, $g_t = g_{K_t}$ with exactly the same domain: $\widehat{\mathbb{C}}\setminus\widehat{K}_t$. At $t=0$, $K_0=\emptyset$ and $g_0=\mathrm{id}_{\widehat{\mathbb{C}}\setminus\{e^{i\lambda(0)}\}}$.

We say that λ generates a radial trace β if

$$\beta(t) := \lim_{\mathbb{D}\ni z\to e^{i\lambda(t)}} g_t^{-1}(z) \in \overline{\mathbb{D}}$$

exists for $0 \le t < T$, and β is a continuous curve. We call such β the radial trace driven by λ. If the radial trace β exists, then for each t, $\mathbb{D}\setminus K_t$ is the component of $\mathbb{D}\setminus\beta((0,t])$ that contains 0. The trace β is called simple if it is a simple curve and $\beta(t)\in\mathbb{D}$ for $0<t<T$, in which case $K_t=\beta((0,t])$ for $0 \le t < T$.

Let $\cot_2(z)=\cot(z/2)$. The covering radial Loewner equation driven by λ is

$$\partial \widetilde{g}_t(z) = \cot_2(\widetilde{g}_t(z)-\lambda(t)), \quad 0 \le t < T, \quad \widetilde{g}_0(z)=z.$$

For each $t\in[0,T)$, let $\widetilde{K}_t$ be the set of all $z\in\mathbb{H}$ at which $\widetilde{g}_t$ is not defined. Then $\widetilde{g}_t$ and $\widetilde{K}_t$, $0\le t<T$, are called the covering radial Loewner maps and hulls driven by λ. We have $\widetilde{g}_t : \mathbb{H}\setminus\widetilde{K}_t \stackrel{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{H}$. If g_t and K_t, $0\le t<T$, are the radial Loewner maps and hulls driven by λ, then $\widetilde{K}_t = (e^i)^{-1}(K_t)$ and $e^i\circ\widetilde{g}_t = g_t\circ e^i$, where e^i denotes the map $z\mapsto e^{iz}$.

For $\kappa > 0$, chordal (resp. radial) SLE_κ is defined by solving the chordal (resp. radial) Loewner equation with $\lambda(t) = \sqrt{\kappa}B(t)$. Such driving function a.s. generates a chordal (resp. radial) trace, which is simple if $\kappa \in (0, 4]$.

3.2 Backward Loewner equations

Let $\lambda \in C([0, T))$. The backward chordal Loewner equation driven by λ is

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda(t)}, \quad f_0(z) = z. \tag{3.1}$$

We assume that $f_t(\infty) = \infty$ for $0 \le t < T$. Let $L_t = \mathbb{H} \setminus f_t(\mathbb{H})$. We call f_t and L_t, $0 \le t < T$, the backward chordal Loewner maps and hulls driven by λ.

Define a family of maps f_{t_2,t_1}, $t_1, t_2 \in [0, T)$, such that, for any fixed $t_1 \in [0, T)$ and $z \in \widehat{\mathbb{C}} \setminus \{\lambda(t_1)\}$, the function $t_2 \mapsto f_{t_2,t_1}(z)$ is the maximal solution of the ODE

$$\partial_{t_2} f_{t_2,t_1}(z) = \frac{-2}{f_{t_2,t_1}(z) - \lambda(t_2)}, \quad f_{t_1,t_1}(z) = z.$$

Note that $f_{t,0} = f_t$ and $f_{t,t} = \text{id}_{\widehat{\mathbb{C}} \setminus \{\lambda(t)\}}$, $0 \le t < T$. If $t_1 \in (0, T)$, then t_2 could be bigger or smaller than t_1. Some simple observations give the following lemma.

Lemma 3.1. *(i) For any $t_1, t_2, t_3 \in [0, T)$, $f_{t_3,t_2} \circ f_{t_2,t_1}$ is a restriction of f_{t_3,t_1}. In particular, this implies that $f_{t_1,t_2} = f_{t_2,t_1}^{-1}$.*

(ii) For any fixed $t_0 \in [0, T)$, f_{t_0+t,t_0}, $0 \le t < T - t_0$, are the backward chordal Loewner maps driven by $\lambda(t_0 + t)$, $0 \le t < T - t_0$.

(iii) For any fixed $t_0 \in [0, T)$, f_{t_0-t,t_0}, $0 \le t \le t_0$, are the (forward) chordal Loewner maps driven by $\lambda(t_0 - t)$, $0 \le t \le t_0$.

Let $L_{t_2,t_1} = \mathbb{H} \setminus f_{t_2,t_1}(\mathbb{H})$ for $0 \le t_1 \le t_2 < T$. From (i), (iii), and the properties of forward chordal Loewner maps, we see that, if $0 \le t_1 < t_2 < T$, then L_{t_2,t_1} is an $\mathbb{H}$-hull with $\text{hcap}(L_{t_2,t_1}) = 2(t_2 - t_1)$, and $f_{t_2,t_1} = f_{L_{t_2,t_1}}$. If $t_1 = t_2$, this is almost still true except that $f_{t_1,t_1} = \text{id}_{\widehat{\mathbb{C}} \setminus \{\lambda(t_1)\}}$ and $f_{L_{t_1,t_1}} = f_\emptyset = \text{id}_{\widehat{\mathbb{C}}}$. Since $L_{t,0} = L_t$, and $\lambda(t) \in \mathbb{R}$ does not lie in the range of f_t, which is $\widehat{\mathbb{C}} \setminus \widehat{L}_t$ for $t > 0$, we get the following lemma.

Lemma 3.2. *For $0 \le t < T$, L_t is an $\mathbb{H}$-hull with $\text{hcap}(L_t) = 2t$. If $t \in (0, T)$, then $f_t = f_{L_t}$ with the same domain: $\widehat{\mathbb{C}} \setminus S_{L_t}$, and $\lambda(t) \in B_{L_t}$.*

If $t_2 \ge t_1 \ge t_0$, from $f_{t_2,t_1} \circ f_{t_1,t_0} = f_{t_2,t_0}$ we get $L_{t_2,t_0} = L_{t_2,t_1} \cdot L_{t_1,t_0}$. From Lemmas 2.7 and 3.1, we obtain the following lemma.

Lemma 3.3. *For any $0 \le t_1 < t_2 < T$, $L_{t_1} \prec L_{t_2}$ and $S_{L_{t_1}} \subset S_{L_2(t_2)}$. For any fixed $t_0 \in [0, T)$, the family $L_{t_0} : L_{t_0-t} = L_{t_0,t_0-t}$, $0 \le t \le t_0$, are the chordal Loewner hulls driven by $\lambda(t_0 - t)$, $0 \le t \le t_0$.*

Note that $S_{L_0} = S_\emptyset = \emptyset$, and its is easy to see that, for $0 < t_0 < T$, $S_{L_{t_0}}$ is the set of $x \in \mathbb{R}$ such that the solution $f_t(x)$ to (3.1) blows up before or at t_0, i.e., $S_{L_{t_0}} = \{x \in \mathbb{R} : \tau_x \le t_0\}$. So every S_{L_t}, $0 < t < T$, is a real interval, and $\bigcap_{0<t<T} S_{L_t} = \{\lambda(0)\}$.

If for every $t_0 \in [0, T)$, $\lambda(t_0 - t)$, $0 \le t \le t_0$, generates a (forward) chordal trace, which we denote by $\beta_{t_0}(t_0 - t)$, $0 \le t \le t_0$, then we say that λ generates backward chordal traces

β_{t_0}, $0 \le t_0 < T$. If this happens, then for any $0 \le t_1 \le t_2 < T$, $\mathbb{H} \setminus L_{t_2,t_1}$ is the unbounded component of $\mathbb{H} \setminus \beta_{t_2}([t_1,t_2))$, and f_{t_2,t_1} extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$ such that

$$f_{t_2,t_1}(\lambda(t_1)) = \beta_{t_2}(t_1), \quad 0 \le t_1 \le t_2 < T. \tag{3.2}$$

Here we still use f_{t_2,t_1} to denote the continuation if there is no confusion. For $0 \le t_0 \le t_1 \le t_2 < T$, the equality $f_{t_2,t_0} = f_{t_2,t_1} \circ f_{t_1,t_0}$ still holds after continuation, which together with (3.2) implies that

$$f_{t_2,t_1}(\beta_{t_1}(t)) = \beta_{t_2}(t), \quad 0 \le t \le t_1 \le t_2 < T. \tag{3.3}$$

Remark. One should keep in mind that each β_t is a continuous function defined on $[0,t]$, $\beta_t(0)$ is the tip of β_t, and $\beta_t(t)$ is the root of β_t, which lies on $\mathbb{R}$. The parametrization is different from a forward chordal trace β, of which $\beta(0)$ is the root.

The backward radial Loewner equations and the backward covering radial Loewner equation driven by $\lambda \in C([0,T))$ are the following two equations respectively:

$$\partial_t f_t(z) = -f_t(z)\frac{e^{i\lambda(t)} + f_t(z)}{e^{i\lambda(t)} - f_t(z)}, \quad f_0(z) = z;$$

$$\partial_t \widetilde{f}_t(z) = -\cot_2(\widetilde{f}_t(z) - \lambda(t)), \quad \widetilde{f}_0(z) = z.$$

We have $f_t \circ e^i = e^i \circ \widetilde{f}_t$. Let $L_t = \mathbb{D} \setminus f_t(\mathbb{D})$. We call f_t and L_t, $0 \le t < T$, the backward radial Loewner maps and hulls driven by λ, and call $\widetilde{f}_t$, $0 \le t < T$, the backward covering radial Loewner maps driven by λ.

By introducing f_{t_2,t_1} in the radial setting, we find that Lemma 3.1 holds if the word "chordal" is replaced by "radial". The following lemma is similar to Lemma 3.2.

Lemma 3.4. *For $0 \le t < T$, L_t is a $\mathbb{D}$-hull with $\operatorname{dcap}(L_t) = t$. If $t \in (0,T)$, then $f_t = f_{L_t}$ with the same domain: $\widehat{\mathbb{C}} \setminus S_{L_t}$, and $e^{i\lambda(t)} \in B_{L_t}$.*

We find that Lemma 3.3 holds here if the word "chordal" is replaced by "radial". So we may define backward radial traces β_t, $0 \le t < T$, in a similar manner.

The following lemma holds only in the radial case.

Lemma 3.5. *If $T = \infty$, then $\mathbb{T} \setminus \bigcup_{0<t<\infty} S_{L_t}$ contains at most one point.*

Proof. Let $S_\infty = \bigcup_{0<t<\infty} S_{L_t}$. From Koebe's 1/4 theorem, as $t \to \infty$, $\operatorname{dist}(0, L_t) \to 0$, which implies that the harmonic measure of $\mathbb{T} \setminus B_{L_t}$ in $\mathbb{D} \setminus L_t$ seen from 0 tends to 0. Since $f_t : \mathbb{D} \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{D} \setminus L_t$, $f_t(0) = 0$, and $f_t(\mathbb{T} \setminus S_{L_t}) = \mathbb{T} \setminus B_{L_t}$, the above harmonic measure at time t equals to $|\mathbb{T} \setminus S_{L_t}|/(2\pi)$. Thus, $|\mathbb{T} \setminus S_\infty| = \lim_{t\to\infty} |\mathbb{T} \setminus S_{L_t}| = 0$. $\square$

For $\kappa > 0$, the backward chordal (resp. radial) SLE_κ is defined by solving backward chordal (resp. radial) Loewner equation with $\lambda(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$. Since for any fixed $t_0 > 0$, $(\lambda(t_0 - t) - \lambda(t_0), 0 \le t \le t_0)$ has the distribution of $(\sqrt{\kappa}B(t), 0 \le t \le t_0)$, using the existence of forward chordal (resp. radial) SLE_κ traces, we conclude that λ a.s. generates a family of backward chordal (resp. radial) traces.

3.3 Normalized global backward trace

First we consider a backward chordal Loewner process generated by $\lambda(t)$, $0\le t<T$. Let $S_t=S_{L_t}$, $0\le t<T$, and $S_T=\bigcup_{0\le t<T}S_t$. Then (S_t) is an increasing family, and S_T is an interval. The following Lemma is similar in spirit to Proposition 5.1 in [16].

Lemma 3.6. *There exists a family of conformal maps $F_{T,t}$, $0\le t<T$, on $\mathbb{H}$ such that $F_{T,t_1}=F_{T,t_2}\circ f_{t_2,t_1}$ in $\mathbb{H}$ if $0\le t_1\le t_2<T$. Let $D_t=F_{T,t}(\mathbb{H})$, $0\le t<T$, and $D_T=\bigcup_{t<T}D_t$. If $(\widehat F_{T,t})$ satisfies the same property as $(F_{T,t})$, then there is a conformal map h_T defined on D_T such that $\widehat F_{T,t}=h_T\circ F_{T,t}$, $0\le t<T$. If there is $z_0\in\mathbb{H}$ such that*

$$\lim_{t\to T}\frac{Im\, f_t(z_0)}{|f_t'(z_0)|}=\infty, \tag{3.4}$$

then we may construct $(F_{T,t})$ such that $D_T=\mathbb{C}$, and we have $S_T=\mathbb{R}$.

Proof. Fix $z_0\in\mathbb{H}$. Let $z_t=f_t(z_0)$ and $u_t=f_t'(z_0)$, $0\le t<T$. For $t\in[0,T)$, let $M_t(z)=\frac{z-z_t}{u_t}$ and $F_t=M_t\circ f_t$. Then F_t maps z_0 to 0 and has derivative 1 at z_0. For $0\le t_1\le t_2<T$, define $F_{t_2,t_1}=M_{t_2}\circ f_{t_2,t_1}$. Then $F_{t_2,t_1}\circ f_{t_1,t_0}=F_{t_2,t_0}$ if $t_0\le t_1\le t_2$. Setting $t_0=0$ we get $F_{t_2,t_1}\circ f_{t_1}=F_{t_2}$. Thus, F_{t_2,t_1} is a conformal map on $\mathbb{H}$ with $F_{t_2,t_1}(z_{t_1})=0$ and $F_{t_2,t_1}'(z_{t_1})=1/u_{t_1}$. By Koebe's distortion theorem, for any $t_1\in[0,T)$, $\{F_{t_2,t_1}:t_2\in[t_1,T)\}$ is uniformly bounded on each compact subset of $\mathbb{H}$. This implies that every sequence in this family contains a subsequence which converges locally uniformly, and the limit function is also conformal on $\mathbb{H}$, maps z_{t_1} to 0, and has derivative $1/u_{t_1}$ at z_{t_1}.

From a diagonal argument, we can find a sequence (t_n) in $[0,T)$ such that $t_n\to T$ and for any $q\in\mathbb{Q}\cap[0,T)$, $(F_{t_n,q})$ converges locally uniformly on $\mathbb{H}$. Let $F_{T,q}$, $q\in\mathbb{Q}\cap[0,T)$, denote the limit functions, which are conformal on $\mathbb{H}$. Since $F_{t_n,q_2}\circ f_{q_2,q_1}=F_{t_n,q_1}$ for each n, we have $F_{T,q_2}\circ f_{q_2,q_1}=F_{T,q_1}$. For $t\in[0,T)$, choose $q\in\mathbb{Q}\cap[t,T)$ and define the conformal map $F_{T,t}=F_{T,q}\circ f_{q,t}$ on $\mathbb{H}$. If $q_1\le q_2\in\mathbb{Q}\cap[t,T)$, then $F_{T,q_1}\circ f_{q_1,t}=F_{T,q_2}\circ f_{q_2,q_1}\circ f_{q_1,t}=F_{T,q_2}\circ f_{q_2,t}$. Thus, the definition of $F_{T,t}$ does not depend on the choice of q. If $0\le t_1\le t_2<T$, by choosing $q\in\mathbb{Q}\cap[0,T)$ with $q\ge t_1\vee t_2$, we get $F_{T,t_2}\circ f_{t_2,t_1}=F_{T,q}\circ f_{q,t_2}\circ f_{t_2,t_1}=F_{T,q}\circ f_{q,t_1}=F_{T,t_1}$.

If (3.4) holds, then we start the construction of $(F_{T,t})$ with such z_0. Since $F_{T,t}:(\mathbb{H};z_t)\overset{\text{Conf}}{\twoheadrightarrow}(D_t;0)$ and $F_{T,t}'(z_t)=1/u_t$, Koebe's 1/4 theorem implies that $\operatorname{dist}(0,\partial D_t)\ge\frac14\operatorname{Im} z_t/|u_t|=\frac14\frac{\operatorname{Im} f_t(z_0)}{|f_t'(z_0)|}$, which tends to ∞ as $t\to T$. So D_T has to be $\mathbb{C}$.

Suppose $\widehat F_{T,t}$, $0\le t<T$, satisfies the same property as $F_{T,t}$, $0\le t<T$. Let $h_t=\widehat F_{T,t}\circ F_{T,t}^{-1}$, $0\le t<T$. Then each h_t is a conformal map defined on D_t. If $0\le t_1<t_2<T$, then

$$h_{t_1}\circ F_{T,t_1}=\widehat F_{T,t_1}(z)=\widehat F_{T,t_2}\circ f_{t_2,t_1}=h_{t_2}\circ F_{T,t_2}\circ f_{t_2,t_1}=h_{t_2}\circ F_{T,t_1}$$

in $\mathbb{H}$, which implies that $h_{t_1}=h_{t_2}|_{D_{t_1}}$. So we may define a conformal map h_T on D_T such that $h_t=h_T|_{D_t}$ for $0\le t<T$. Such h_T is what we need.

Suppose that (3.4) holds but $S_T\ne\mathbb{R}$. Since S_T is an interval, $\overline{S_T}\ne\mathbb{R}$. Choose $\widehat z_0\in\mathbb{R}\setminus\overline{S_T}$, and start the construction with $\widehat z_0$ in place of z_0 at the beginning of this proof. Let $\widehat F_{T,t}$, $0\le t<T$, denote the family of maps constructed in this way. Then each $\widehat F_{T,t}$ is an $\mathbb{R}$-symmetric conformal map, which implies that $\widehat D_T\subset\mathbb{H}$. However, now $D_T=\mathbb{C}$ and $h_T:D_T\overset{\text{Conf}}{\twoheadrightarrow}\widehat D_T$, which is impossible. Thus, $S_T=\mathbb{R}$ when (3.4) holds. □

Let $(F_{T,t})$, D_t, and D_T be as in Lemma 3.6. Let $F_T = F_{T,0}$. Suppose λ generates backward chordal traces β_{t_0}, $0 \le t_0 < T$, which satisfy

$$\forall t_0 \in [0,T), \quad \exists t_1 \in (t_0,T), \quad \beta_{t_1}([0,t_0]) \subset \mathbb{H}. \tag{3.5}$$

We may define $\beta(t)$, $0 \le t < T$, as follows. For every $t \in [0,T)$, pick $t_0 \in (t,T)$ such that $\beta_{t_0}(t) \in \mathbb{H}$, which is possible by (3.5), and define

$$\beta(t) = F_{T,t_0}\beta_{t_0}(t) \in D_{t_0} \subset D_T. \tag{3.6}$$

Since $F_{T,t_1} = F_{T,t_2} \circ f_{t_2,t_1}$ in $\mathbb{H}$, from (3.3) we see that the definition of β does not depend on the choice of t_0. Let $t_0 \in [0,T)$. From (3.5), there is $t_1 > t_0$ such that $\beta_{t_1}([0,t]) \in \mathbb{H}$. Since $\beta(t) = F_{T,t_0}(\beta_{t_0}(t))$, $0 \le t \le t_0$, we see that β is continuous on $[0,t_0]$. Thus, $\beta(t)$, $0 \le t < T$, is a continuous curve in D_T.

Fix any $x \in S_T$. Then $x \in S_{t_0}$ for some $t_0 \in (0,T)$. So $f_{t_0}(x)$ lies on the outer boundary of L_{t_0}, which implies that $f_{t_0}(x) \in \beta_{t_0}(t)$ for some $t \in [0,t_0]$. From (3.5), there is $t_1 \in (t_0,T)$ such that $\beta_{t_1}([0,t_0]) \subset \mathbb{H}$. Then $f_{t_1}(x) = f_{t_1,t_0}(\beta_{t_0}(t)) = \beta_{t_1}(t) \in \mathbb{H}$. From the continuity of f_{t_1} on $\mathbb{H} \cup \mathbb{R}$, there is a neighborhood U of x in $\mathbb{H} \cup \mathbb{R}$ such that $f_{t_1}(U) \subset \mathbb{H}$. This shows that $U \cap \mathbb{R} \subset S_{t_1} \subset S_T$. Since $F_T = F_{T,t_1} \circ f_{t_1}$ in $\mathbb{H}$, we find that F_T has continuation on U. Since $x \in S_T$ is arbitrary, we conclude that S_T is an open interval, and F_T has continuation to $\mathbb{H} \cup S_T$.

Now we assume that λ generates backward chordal traces, and both (3.4) and (3.5) hold. Then $D_T = \mathbb{C}$, $S_T = \mathbb{R}$, a continuous curve $\beta(t)$, $0 \le t < T$, is well defined, and F_T extends continuously to $\mathbb{H} \cup \mathbb{R}$. Moreover, F_T is unique up to a Möbius transformation that fixes ∞. With some suitable normalization condition, the family $F_{T,t}$ and the curve β will be determined by λ. We will use the following normalization:

$$F_T(\lambda(0)) = \lambda(0), \quad F_T(\lambda(0)+i) = \lambda(0)+i. \tag{3.7}$$

If (3.7) holds, we call β the normalized global backward chordal trace generated by λ. From (3.7) we see that $\beta(0) = \lambda(0)$, and β does not pass through $\lambda(0)+i$.

For the radial case, Lemma 3.6 still holds with $\mathbb{H}$ replaced by $\mathbb{D}$, and (3.4) replaced by $T = \infty$. For the construction, we choose $z_0 = 0$ and let $F_{t_2,t_1}(z) = e^{t_2} f_{t_2,t_1}(z)$. If λ generates backward radial traces β_t, $0 \le t < T$, which satisfy

$$\forall t_1 \in [0,T), \quad \exists t_2 \in (t_1,T), \quad \beta_{t_2}(t_1) \in \mathbb{D}, \tag{3.8}$$

then we may define a continuous curve $\beta(t)$, $0 \le t < T$, in D_T using (3.6). If $T = \infty$, then $D_T = \mathbb{C}$, and such β is determined by λ up to a Möbius transformation that fixes ∞, which means that we may define a normalized global backward radial trace once a normalization condition is fixed.

3.4 Forward and backward Loewner chains

In this section, we review a condition on a family of hulls that corresponds to continuously driven (forward) Loewner hulls, and discuss the corresponding condition for backward Loewner chains.

Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain such that $\widehat{\mathbb{C}} \setminus D$ contains more than one point. A relatively closed subset H of D is called a (boundary) hull in D if $D \setminus H$ is simply connected. For example, a hull in $\mathbb{H}$ is an $\mathbb{H}$-hull iff it is bounded; a hull in $\mathbb{D}$ is a $\mathbb{D}$-hull iff it does not contain 0. Let $T \in (0, \infty]$. A family of hulls in D: K_t, $0 \le t < T$, is called a Loewner chain in D if

1. $K_0 = \emptyset$ and $K_{t_1} \subsetneqq K_{t_2}$ whenever $0 \le t_1 < t_2 < T$;
2. for any fixed $a \in [0, T)$ and a compact set $F \subset D \setminus K_a$, the conjugate extremal distance (c.f. [1]) between $K_{t+\varepsilon} \setminus K_t$ and F in $D \setminus K_t$ tends to 0 as $\varepsilon \to 0$, uniformly in $t \in [0, a]$.

If K_t, $0 \le t < T$, is a Loewner chain in D, and $a \in [0, T)$, then we also call the restriction K_t, $0 \le t \le a$, a Loewner chain in D.

There are two important properties for Loewner chains. If K_t, $0 \le t < T$, is a Loewner chain in D, and u is a continuous increasing function defined on $[0, T)$ with $u(0) = 0$, then $K_{u^{-1}(t)}$, $0 \le t < u(T)$, is also a Loewner chain in D, which is called a time-change of (K_t) via u. If W maps D conformally onto E, then $W(K_t)$, $0 \le t < T$, is a Loewner chain in E.

An $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain is a Loewner chain in $\mathbb{H}$ (resp. $\mathbb{D}$) such that each hull is an $\mathbb{H}$-(resp. $\mathbb{D}$-)hull. An $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain (K_t) is said to be normalized if $\mathrm{hcap}(K_t) = 2t$ (resp. $\mathrm{dcap}(K_t) = t$) for each t.

The conditions for the conformal invariance property of $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chains can be slightly weakened as below.

Proposition 3.7. *If K_t, $0 \le t < T$, is an $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, and W is an $\mathbb{R}$-(resp. $\mathbb{T}$-)symmetric conformal map, whose domain contains $\widehat{K}_t$ for each t and whose image does not contain ∞ (resp. 0), then $W(K_t)$, $0 \le t < T$, is also an $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain.*

The following proposition combines some results in [7] and [10].

Proposition 3.8. *Let $T \in (0, \infty]$. The following are equivalent.*

(i) K_t, $0 \le t < T$, are chordal (resp. radial) Loewner hulls driven by some $\lambda \in C([0, T))$.

(ii) K_t, $0 \le t < T$, is a normalized $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain.

If either of the above holds, with $\mathring{\lambda}(t) = \lambda(t)$ (resp. $\mathring{\lambda}(t) = e^{i\lambda(t)}$ in the radial case) we have

$$\{\mathring{\lambda}(t)\} = \bigcap_{\varepsilon > 0} \overline{K_{t+\varepsilon}/K_t}, \quad 0 \le t < T.$$

In addition, if K_t, $0 \le t < T$, is any $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, then the function $u(t) := \mathrm{hcap}(K_t)/2$ (resp. $u(t) := \mathrm{dcap}(K_t)$), $0 \le t < T$, is continuous increasing with $u(0) = 0$, which implies that $K_{u^{-1}(t)}$, $0 \le t < u(T)$, is a normalized $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain.

Definition 3.9. *A family of $\mathbb{H}$-(resp. $\mathbb{D}$-)hulls: L_t, $0 \le t < T$, is called a backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain if they satisfy*

1. *$L_0 = \emptyset$ and $L_{t_1} \prec L_{t_2}$ if $0 \le t_1 \le t_2 < T$;*
2. *$L_{t_0} : L_{t_0 - t}$, $0 \le t \le t_0$, is an $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain for any $t_0 \in (0, T)$.*

If u is a continuous increasing function defined on $[0, T)$ with $u(0) = 0$, then $L_{u^{-1}(t)}$, $0 \le t < u(T)$, is also a backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, and is called a time-change of (L_t) via u. A backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain (L_t) is said to be normalized if $\mathrm{hcap}(L_t) = 2t$ (resp. $\mathrm{dcap}(L_t) = t$) for any $t \in [0, T)$.

Using Lemma 3.3 and Proposition 3.8, we obtain the following.

Proposition 3.10. *Let $T \in (0, \infty]$. The following are equivalent.*

(i) L_t, $0 \le t < T$, *are backward chordal (resp. radial) Loewner hulls driven by some* $\lambda \in C([0,T))$.

(ii) L_t, $0 \le t < T$, *is a normalized backward* $\mathbb{H}$*-(resp.* $\mathbb{D}$*-)Loewner chain.*

If either of the above holds, with $\mathring{\lambda}(t) = \lambda(t)$ *(resp.* $\mathring{\lambda}(t) = e^{i\lambda(t)}$ *in the radial case) we have*

$$\{\mathring{\lambda}(t)\} = \bigcap_{\varepsilon>0} \overline{L_t : L_{t-\varepsilon}}, \quad 0 < t < T, \tag{3.9}$$

In addition, if L_t, $0 \le t < T$, *is any backward* $\mathbb{H}$*-(resp.* $\mathbb{D}$*-)Loewner chain, then the function* $u(t) := \text{hcap}(K_t)/2$ *(resp.* $u(t) := \text{dcap}(K_t)$*),* $0 \le t < T$, *is continuous increasing with* $u(0) = 0$, *which implies that* $L_{u^{-1}(t)}$, $0 \le t < u(T)$, *is a normalized backward* $\mathbb{H}$*-(resp.* $\mathbb{D}$*-)Loewner chain.*

We say that f_t and L_t, $0 \le t < T$, are backward chordal (resp. radial) Loewner maps and hulls, via a time-change u, driven by λ, if u is a continuous increasing function defined on $[0,T)$ with $u(0) = 0$, such that $f_{u^{-1}(t)}$ and $L_{u^{-1}(t)}$, $0 \le t < u(T)$, are backward chordal (resp. radial) Loewner maps and hulls driven by $\lambda \circ u^{-1}$. From the above proposition, if (L_t) is any $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, then L_t, $0 \le t < T$, are backward chordal (resp. radial) Loewner hulls, via a time-change $u(t) := \text{hcap}(L_t)/2$ (resp. $\text{dcap}(L_t)$), driven by λ, which satisfies (3.9).

3.5 Simple curves and weldings

An $\mathbb{H}$-simple (resp. $\mathbb{D}$-simple) curve is a half-open simple curve in $\mathbb{H}$ (resp. $\mathbb{D} \setminus \{0\}$), whose open side approaches a single point on $\mathbb{R}$ (resp. $\mathbb{T}$). Every $\mathbb{H}$(resp. $\mathbb{D}$)-simple curve β is an $\mathbb{H}$(resp. $\mathbb{D}$)-hull, whose base B_β is a single point, and whose support S_β is an $\mathbb{R}$(resp. $\mathbb{T}$-)interval. Here an $\mathbb{T}$-interval is an arc on $\mathbb{T}$. The function f_β extends continuously from $\mathbb{H}$ (resp. $\mathbb{D}$) to $\overline{\mathbb{H}}$ (resp. $\overline{\mathbb{D}}$), which maps S_β onto $\overline{\beta}$, sends the two ends of S_β to B_β, and sends a unique point, say $z_\beta \in S_\beta$ to the tip of β. The point z_β divides S_β into two $\mathbb{R}$(resp. $\mathbb{T}$-)intervals such that the restriction of f_β to either interval is a homeomorphism onto $\overline{\beta}$. Thus, there is a unique involution (an auto homeomorphism whose inverse is itself) ϕ_β of S_β, which fixes only one point: z_β, swaps the two end points of S_β, and satisfies that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. We call ϕ_β the welding induced by β.

Suppose K is an $\mathbb{H}$- or $\mathbb{D}$-simple curve. Let W be as in Theorems 2.12, 2.20, or 2.23. Then $W^*(K)$ is also an $\mathbb{H}$- or $\mathbb{D}$-simple curve. The equality $W^K \circ f_K = f_{W^*(K)} \circ W$ holds after continuous extension from $\mathbb{H}$ or $\mathbb{D}$ to its closure. So the weldings induced by K and $W^*(K)$ satisfy $\phi_{W^*(K)} = W \circ \phi_K \circ W^{-1}$.

Suppose the hulls (L_t) generated by a backward chordal (resp. radial) Loewner process driven by λ are all $\mathbb{H}$(resp. $\mathbb{D}$)-simple curves. Then the process generates backward chordal (resp. radial) traces (β_t) such that every β_t is a simple curve, and $L_t = \beta_t([0,t))$, $0 \le t < T$. Let ϕ_t be the welding induced by L_t, which is an involution of $S_t := S_{L_t}$. Recall that (S_t) is an increasing family because $L_{t_1} \prec L_{t_2}$ for $t_1 < t_2$. If $0 \le t_1 < t_2 < T$, then from $f_{t_2,t_1} \circ f_{t_1} = f_{t_2}$ we see that $\phi_{t_2}|_{S_{t_1}} = \phi_{t_1}$. Thus, there is a unique involution ϕ of $S_T := \bigcup_t S_t$ such that $\phi|_{S_t} = \phi_t$ for each $t \in [0,T)$. In other words, $y = \phi(x)$ implies that $f_t(x) = f_t(y)$ for some $t \ge 0$, where f_t is the continuous extension of the Loewner map at time t from $\mathbb{H}$(resp. $\mathbb{D}$) to $\overline{\mathbb{H}}$ (resp. $\overline{\mathbb{D}}$). We say that ϕ is the welding induced by this process. In the case that $S_T = \mathbb{R}$ (resp. $\mathbb{T} \setminus \{z_0\}$ for some $z_0 \in \mathbb{T}$), we will extend ϕ to an involution of $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (resp. $\mathbb{T}$) such that ∞ (resp. z_0) is the other fixed point of ϕ.

Here is another way to view the welding ϕ. For every $t \in (0,T)$, ϕ swaps the two end points of S_t. Let $\mathring{\lambda}(0) = \lambda(0)$ (resp. $e^{i\lambda(0)}$). Since $f_t(\mathring{\lambda}(0)) = \beta_t(0)$ is the tip of L_t for each t, we see that $\mathring{\lambda}(0)$ is the only fixed point of ϕ. On the other hand, it is easy to see that, x and y are end points of S_t if and only if $\tau_x = \tau_y = t$, $0 < t < T$; and every point on $S_T \setminus \{\mathring{\lambda}(0)\}$ is an end point of some S_t, $0 < t < T$. Thus, for $x \neq y \in S_T \setminus \{\mathring{\lambda}(0)\}$, $y = \phi(x)$ if and only if $\tau_x = \tau_y$, i.e., x and y are swallowed at the same time.

Let $\kappa \in (0, 4]$. Since the backward chordal (resp. radial) SLE$_\kappa$ traces are $\mathbb{H}$(resp. $\mathbb{D}$-)simple curves, so the process induces a random welding, which we call a backward chordal (resp. radial) SLE$_\kappa$ welding. In the chordal case, For any $x \in \mathbb{R} \setminus \{\lambda(0)\} = \mathbb{R} \setminus \{0\}$, the process $X_t^x := \lambda(t) - f_t(x)$ is a rescaled Bessel process of dimension $1 - \frac{4}{\kappa} < 1$, which implies that a.s. $X_t^x \to 0$ at some finite time. Thus, $S_\infty = \mathbb{R}$. which implies that a chordal SLE$_\kappa$ welding is an involution of $\widehat{\mathbb{R}}$ with two fixed points: $\lambda(0) = 0$ and ∞. In the radial case, since $T = \infty$, Lemma 3.5 says that $S_\infty = \mathbb{T}$ or $\mathbb{T} \setminus \{z_0\}$ for some $z_0 \in \mathbb{T}$. The first case can not happen since ϕ has only one fixed point on S_∞. Thus, a radial SLE$_\kappa$ welding is an involution of $\mathbb{T}$ with two fixed points, one of which is $e^{i\lambda(0)} = 1$.

Suppose a backward chordal (resp. radial) Loewner process generates $\mathbb{H}$(resp. $\mathbb{D}$)-simple backward traces β_t, $0 \le t < T$. Then (3.5) (resp. (3.8)) is satisfied because $\beta_{t_2}(t_1)$ lies in $\mathbb{H}$ (resp. $\mathbb{D}$) if $t_2 > t_1$. It is clear that the curve β defined by (3.6) is simple, and $D_t = D_T \setminus \beta([t, T))$ for $0 \le t < T$. Let ϕ be the welding induced by the process. If $y = \phi(x)$, there is $t \in [0, T)$ such that $y, x \in S_t$ and $f_t(y) = f_t(x)$. From $F_{T,t} \circ f_t = F_T$, we get $F_T(y) = F_T(x)$. This means that ϕ can be realized by the conformal map F_T.

If a backward chordal (resp. radial) Loewner chain (L_t) is composed of $\mathbb{H}$(resp. $\mathbb{D}$)-simple curves, then (L_t) induces a welding ϕ, which is an involution of $\bigcup S_{L_t}$, and agrees with ϕ_{L_t} on S_{L_t} for each t. To see this, one may first normalized the backward Loewner chain so that it is generated by a backward Loewner process.

4 Conformal Transformations

Proposition 4.1. *Suppose L_t, $0 \le t < T$, is a backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, W is an $\mathbb{R}$-(resp. $\mathbb{T}$-)symmetric conformal map whose domain contains every S_{L_t}, and $\infty \notin W(S_{L_t})$ for $0 \le t < T$. Then $W^*(L_t)$, $0 \le t < T$, is also a backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain.*

Proof. From Theorem 2.12, W^{L_t} and $W^*(L_t)$ are well defined. Since $L_0 = \emptyset$, $W^*(L_0) = \emptyset$. Let $0 \le t_1 \le t_2 < T$. Since $L_{t_1} \prec L_{t_2}$, from Lemma 2.14, $W^*(L_{t_1}) \prec W^*(L_{t_2})$. Fix $t_0 \in (0, T)$. Since $L_{t_0} : L_{t_0 - t}$, $0 \le t \le t_0$, is an $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, from Lemma 2.14 and Proposition 3.7 we see that

$$W^*(L_{t_0}) : W^*(L_{t_0 - t}) = W^{L_{t_0}}(L_{t_0} : L_{t_0 - t}), \quad 0 \le t \le t_0,$$

is also an $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain. This finishes the proof. □

Using Lemma 2.21 instead of Lemma 2.14, we can show that a similar proposition holds.

Proposition 4.2. *Suppose L_t, $0 \le t < T$, is a backward $\mathbb{H}$-(resp. $\mathbb{D}$-)Loewner chain, W is a Mobius transform that maps $\mathbb{H}$ onto $\mathbb{D}$ (resp. maps $\mathbb{D}$ onto $\mathbb{H}$) such that $\infty \notin W(S_{L_t})$ for $0 \le t < T$. Then $W^*(L_t)$, $0 \le t < T$, is a backward $\mathbb{D}$-(resp. $\mathbb{H}$-)Loewner chain.*

Suppose (L_t) is composed of $\mathbb{H}$- or $\mathbb{D}$-simple curves. Then $(W^*(L_t))$ is also composed of $\mathbb{H}$ or $\mathbb{D}$-simple curves. Let ϕ and ϕ_W be the weldings induced by these two chains, which are involutions of $\bigcup S_{L_t}$ and $\bigcup S_{W^*(L_t)}$, respectively. Since for each $t \in (0,T)$, $\phi|_{S_{L_t}} = \phi_{L_t}$, $\phi_W|_{S_{W^*(L_t)}} = \phi_{W^*(L_t)}$, $S_{W^*(L_t)} = W(S_{L_t})$, and $\phi_{W^*(L_t)} = W \circ \phi_{L_t} \circ W^{-1}$, we see that $\bigcup S_{W^*(L_t)} = W(\bigcup S_{L_t})$ and

$$\phi_W = W \circ \phi \circ W^{-1}. \tag{4.1}$$

This means that the conformal transformation preserves the welding.

The following proposition is essentially Lemma 2.8 in [7].

Proposition 4.3. *Let W be an $\mathbb{R}$-symmetric conformal map, whose domain contains $z_0 \in \mathbb{R}$, such that $W(z_0) \neq \infty$. Then*

$$\lim_{H \to z_0} \frac{\text{hcap}(W(H))}{\text{hcap}(H)} = |W'(z_0)|^2,$$

where $H \to z_0$ means that $\text{diam}(H \cup \{z_0\}) \to 0$ *with H being a nonempty $\mathbb{H}$-hull.*

Using the integral formulas for capacities of $\mathbb{H}$-hulls and $\mathbb{D}$-hulls, it is not hard to derive the following similar proposition.

Proposition 4.4. *(i) Let W be a conformal map on a $\mathbb{T}$-symmetric domain Ω, which satisfies $I_{\mathbb{R}} \circ W = W \circ I_{\mathbb{T}}$ and $W(\Omega \cap \mathbb{D}) \subset \mathbb{H}$. Let $z_0 \in \Omega \cap \mathbb{T}$ be such that $W(z_0) \neq \infty$. Then*

$$\lim_{H \to z_0} \frac{\text{hcap}(W(H))}{\text{dcap}(H)} = 2|W'(z_0)|^2,$$

where $H \to z_0$ means that $\text{diam}(H \cup \{z_0\}) \to 0$ *with H being a nonempty $\mathbb{D}$-hull.*

(ii) Proposition 4.3 holds with $\mathbb{R}$ replaced by $\mathbb{T}$, hcap *replaced by* dcap, *and $H \to z_0$ understood as in (i).*

4.1 Transformations between backward $\mathbb{H}$-Loewner chains

Suppose L_t and f_t, $0 \le t < T$, are backward chordal Loewner hulls and maps driven by $\lambda \in C([0,T))$. From Proposition 3.10, (L_t) is a backward $\mathbb{H}$-Loewner chain. Let W be an $\mathbb{R}$-symmetric conformal map, whose domain Ω contains the support of every L_t. Write W_t for W^{L_t}. The domain of W_t is Ω^{L_t}, which contains $\widehat{L}_t$. If $t > 0$, $\lambda(t) \in \widehat{L}_t$, so $\lambda(t)$ is contained in the domain of W_t. This is also true for $t = 0$ because $W_0 = W$ and $\{\lambda(0)\} = S_0 \subset S_t = S_{L_t} \subset \Omega$ for any $t \in (0,T)$. Let $L_t^* = W^*(L_t) = W_t(L_t)$, $0 \le t < T$. From Proposition 4.1, (L_t^*) is a backward $\mathbb{H}$-Loewner chain, and

$$W_t(L_t : L_{t-\varepsilon}) = L_t^* : L_{t-\varepsilon}^*, \quad 0 \le t - \varepsilon < t < T. \tag{4.2}$$

From Proposition 3.10, L_t^*, $0 \le t < T$, are backward chordal Loewner hulls via a time-change $u(t) := \text{hcap}(L_t^*)/2$, driven by some λ^*, which satisfies

$$\{\lambda^*(t)\} = \bigcap_{\varepsilon > 0} \overline{L_t^* : L_{t-\varepsilon}^*}, \quad 0 < t < T.$$

From (3.9), (4.2), and continuity, we find that

$$\lambda^*(t) = W_t(\lambda(t)), \quad 0 \le t < T. \tag{4.3}$$

Since (L_t) and $(L^*_{u^{-1}(t)})$ are normalized, we know that $\text{hcap}(L_t : L_{t-\varepsilon}) = 2\varepsilon$ and $\text{hcap}(L^*_t : L^*_{t-\varepsilon}) = 2u(t) - 2u(t-\varepsilon)$. From (4.2) and Proposition 4.3, we find that

$$u'(t) = W_t'(\lambda(t))^2, \quad 0 \le t < T. \tag{4.4}$$

Let $f^*_t = f_{L^*_t}$. From the definition of $W_t = W^{L_t}$, we have the equality

$$W_t \circ f_t = f^*_t \circ W, \tag{4.5}$$

which holds in $\Omega \setminus S_{L_t}$. Differentiating (4.5) w.r.t. t, and using (4.3) and (4.4), we get

$$\partial_t W_t(f_t(z)) + W_t'(f_t(z)) \frac{-2}{f_t(z) - \lambda(t)} = \frac{-2u'(t)}{f^*_t(W(z)) - \lambda^*(t)}$$

$$= \frac{-2W_t'(\lambda(t))^2}{W_t(f_t(z)) - W_t(\lambda(t))}.$$

Thus, for any $w = f_t(z) \in f_t(\Omega \setminus S_{L_t}) = \Omega^{L_t} \setminus \widehat{L_t}$,

$$\partial_t W_t(w) = \frac{-2W_t'(\lambda(t))^2}{W_t(w) - W_t(\lambda(t))} - W_t'(w) \frac{-2}{w - \lambda(t)}. \tag{4.6}$$

By analytic extension, the above equality holds for any $w \in \Omega^{L_t} \setminus \{\lambda(t)\}$. Letting $w \to \lambda(t)$, we find that

$$\partial_t W_t(\lambda(t)) = 3W_t''(\lambda(t)), \quad 0 \le t < T. \tag{4.7}$$

Differentiating (4.6) w.r.t. w and letting $w \to \lambda(t)$, we get

$$\frac{\partial_t W_t'(\lambda(t))}{W_t'(\lambda(t))} = -\frac{1}{2}\Big(\frac{W_t''(\lambda(t))}{W_t'(\lambda(t))}\Big)^2 + \frac{4}{3} \frac{W_t'''(\lambda(t))}{W_t'(\lambda(t))}. \tag{4.8}$$

4.2 Transformations involving backward $\mathbb{D}$-Loewner chains

Now suppose L_t, $0 \le t < T$, are backward radial Loewner hulls driven by λ. Let f_t and $\widetilde{f}_t$ be the corresponding radial Loewner maps and covering maps. Suppose W is a $\mathbb{T}$-symmetric conformal map, whose domain Ω contains the support of every L_t. Let $W_t = W^{L_t}$, $L^*_t = W_t(L_t) = W^*(L_t)$, and $u(t) = \text{dcap}(L^*_t)$, $0 \le t < T$. Then L^*_t, $0 \le t < T$, are backward radial Loewner hulls via a time-change $u(t) := \text{dcap}(L^*_t)$, driven by some λ^*, which satisfies

$$\{e^{i\lambda^*(t)}\} = \bigcap_{\varepsilon > 0} \overline{L^*_t : L^*_{t-\varepsilon}}, \quad 0 < t < T.$$

Let f^*_t (resp. $\widetilde{f}^*_t$), $0 \le t < T$, denote the backward radial (resp. covering radial) Loewner hulls via the time-change u driven by λ^*. The argument in the last subsection still works with Proposition 4.4 in place of Proposition 4.3. We can conclude that $e^{i\lambda(t)}$ lies in the domain of W_t for $0 \le t < T$; $W_t(e^{i\lambda(t)}) = e^{i\lambda^*(t)}$; $u'(t) = |W_t'(e^{i\lambda(t)})|^2$; and (4.5) still holds. Suppose $\widetilde{W}$ is an $\mathbb{R}$-symmetric conformal map defined on $\widetilde{\Omega} = (e^i)^{-1}(\Omega)$, which satisfies $e^i \circ \widetilde{W} = W \circ e^i$. Define $\widetilde{W}_t$ to be the analytic extension of $\widetilde{f}^*_t \circ \widetilde{W} \circ \widetilde{f}_t^{-1}$ to $\widetilde{\Omega}_t := (e^i)^{-1}(\Omega^{L_t})$. Then we get

$$\widetilde{W}_t \circ \widetilde{f}_t = \widetilde{f}^*_t \circ \widetilde{W}; \tag{4.9}$$

Comparing this with (4.5) we find $e^i\circ\widetilde{W}_t=W_t\circ e^i$. So $\lambda(t)$ lies in the domain of $\widetilde{W}_t$, and

$$u'(t)=\widetilde{W}_t'(\lambda(t))^2,\quad 0\le t<T. \tag{4.10}$$

Since $W_t(e^{i\lambda(t)})=e^{i\lambda^*(t)}$, from the continuity, there is $n\in\mathbb{N}$ such that $\widetilde{W}_t(\lambda(t))=\lambda^*(t)+2n\pi$ for $0\le t<T$. Since λ^* and $\lambda^*+2n\pi$ generate the same backward radial Loewner objects via the time-change u, by replacing λ^* with $\lambda^*+2n\pi$, we may assume that

$$\widetilde{W}_t(\lambda(t))=\lambda^*(t),\quad 0\le t<T. \tag{4.11}$$

Differentiating (4.9) w.r.t. t and letting $w=\widetilde{f}_t(z)$, we get

$$\partial_t\widetilde{W}_t(w)=-\widetilde{W}_t'(\lambda(t))^2\cot_2(\widetilde{W}_t(w)-\widetilde{W}_t(\lambda(t)))+\widetilde{W}_t'(w)\cot_2(w-\lambda(t)), \tag{4.12}$$

which holds for $w\in(e^i)^{-1}(\Omega^{L_t}\setminus\{e^{i\lambda(t)}\})$. Letting $w\to\lambda(t)$, we get

$$\partial_t\widetilde{W}_t(\lambda(t))=3\widetilde{W}_t''(\lambda(t)),\quad 0\le t<T. \tag{4.13}$$

Differentiating (4.12) w.r.t. w and letting $w\to\lambda(t)$, we get

$$\frac{\partial_t\widetilde{W}_t'(\lambda(t))}{\widetilde{W}_t'(\lambda(t))}=-\frac12\Big(\frac{W_t''(\lambda(t))}{W_t'(\lambda(t))}\Big)^2+\frac43\,\frac{W_t'''(\lambda(t))}{W_t'(\lambda(t))}+\frac16\widetilde{W}_t'(\lambda(t))^2-\frac16. \tag{4.14}$$

The number $\frac16$ comes from the Laurent series of $\cot_2(z)$: $\frac{2}{z}-\frac{z}{6}+O(z^3)$.

Let (L_t), (f_t), and $(\widetilde{f}_t)$ be as above. Now suppose W is a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$ such that $W^{-1}(\infty)\not\in S_{L_t}$ for every t. Let W^{L_t} be as in Theorem 2.20. Let $W_t=W^{L_t}$ and $L_t^*=W_t(L_t)=W^*(L_t)$, $0\le t<T$. Then L_t^*, $0\le t<T$, are backward chordal Loewner hulls via a time-change $u(t):=\operatorname{hcap}(L_t^*)/2$, driven by some λ^*. Let $f_t^*=f_{L_t^*}$. Then (4.5) still holds, and we have $u'(t)=|W_t'(e^{i\lambda(t)})|^2$ and $W_t(e^{i\lambda(t)})=e^{i\lambda^*(t)}$. Let $\widetilde{W}=W\circ e^i$ and $\widetilde{W}_t=W_t\circ e^i$. We get (4.10), (4.11), and $\widetilde{W}_t\circ\widetilde{f}_t=f_t^*\circ\widetilde{W}$. Differentiating this equality w.r.t. t and letting $w=\widetilde{f}_t(z)$ tend to $\lambda(t)$, we find that (4.13) still holds.

4.3 Conformal invariance of backward SLE$(\kappa;\rho)$ processes

We now define backward chordal and radial SLE$(\kappa;\vec{\rho})$ processes, where $\vec{\rho}=(\rho_1,\dots,\rho_n)\in\mathbb{R}^n$. Let $x_0,q_1,\dots,q_n\in\mathbb{R}$ such that $q_k\ne x_0$ for all k. Let $\lambda(t)$, $0\le t<T$, be the maximal solution of the equation

$$d\lambda(t)=\sqrt{\kappa}dB(t)+\sum_{k=1}^n\frac{-\rho_k}{\lambda(t)-f_t^\lambda(q_k)}\,dt;\quad \lambda(0)=x_0. \tag{4.15}$$

Here f_t^λ, $0\le t<T$, are the backward chordal Loewner maps driven by λ. Then we call the backward chordal Loewner process driven by λ the chordal SLE$(\kappa;\vec{\rho})$ process started from x_0 with force points $(q_1,\dots,q_n)$, or simply started from $(x_0;q_1,\dots,q_n)$. We allow some q_k to be ∞. In that case, $f_t^\lambda(q_k)$ is always ∞, and the term $\frac{-\rho_k}{\lambda(t)-f_t^\lambda(q_k)}$ vanishes.

Let $x_0,q_1,\dots,q_n\in\mathbb{R}$ be such that $q_k\not\in x_0+2\pi\mathbb{Z}$ for all k. Let $\lambda(t)$, $0\le t<T$, be the maximal solution of the equation

$$d\lambda(t)=\sqrt{\kappa}dB(t)+\sum_{k=1}^n\frac{-\rho_k}{2}\cot_2(\lambda(t)-\widetilde{f}_t^\lambda(q_k))\,dt;\quad \lambda(0)=x_0. \tag{4.16}$$

Here $\widetilde{f}_t^\lambda$, $0 \le t < T$, are the covering backward radial Loewner maps driven by λ. Then the backward radial Loewner process driven by λ is called the radial SLE$(\kappa;\vec{\rho})$ process started from e^{ix_0} with marked points $(e^{iq_1},\dots,e^{iq_n})$, or simply started from $(e^{ix_0};e^{iq_1},\dots,e^{iq_n})$.

The existence of backward chordal (resp. radial) SLE$_\kappa$ traces and Girsanov's Theorem imply the existence of a backward chordal (resp. radial) SLE$(\kappa;\vec{\rho})$ traces. The traces are $\mathbb{H}$(resp. $\mathbb{D}$)-simple curves if $\kappa \in (0,4]$.

The following lemma is easy to check.

Lemma 4.5. *Let W be a Möbius transformation. Then the following hold.*

(i) For any $z \in \mathbb{C} \cap W^{-1}(\mathbb{C})$ and $w \in \widehat{\mathbb{C}}$,

$$\frac{2W'(z)}{W(z)-W(w)} - \frac{2}{z-w} = \frac{W''(z)}{W'(z)}.$$

(ii) Let $\widetilde{W} = W \circ e^i$. For any $z \in \mathbb{C} \cap \widetilde{W}^{-1}(\mathbb{C})$ and $w \in \mathbb{C}$,

$$\frac{2\widetilde{W}'(z)}{\widetilde{W}(z)-\widetilde{W}(w)} - \cot_2(z-w) = \frac{\widetilde{W}''(z)}{\widetilde{W}'(z)}.$$

(iii) Suppose an analytic function $\widetilde{W} : \Omega \to \mathbb{C}$ satisfies $e^i \circ \widetilde{W} = W \circ e^i$ in Ω. Then for any $z, w \in \Omega$,

$$\widetilde{W}'(z)\cot_2(\widetilde{W}(z)-\widetilde{W}(w)) - \cot_2(z-w) = \frac{\widetilde{W}''(z)}{\widetilde{W}'(z)}.$$

Theorem 4.6. *Let L_t, $0 \le t < T$, be the backward chordal SLE$(\kappa;\vec{\rho})$ hulls started from $(x_0;q_1,\dots,q_n)$. Suppose $\sum \rho_k = -\kappa - 6$. Let W be a Möbius transformation from $\mathbb{H}$ onto $\mathbb{H}$ such that $\{\infty, W^{-1}(\infty)\} \subset \{q_1,\dots,q_n\}$. Then, after a time-change, $W^*(L_t)$, $0 \le t < T$, are the backward chordal SLE$(\kappa;\vec{\rho})$ hulls started from $(W(x_0);W(q_1),\dots,W(q_n))$.*

Proof. Since $W^{-1}(\infty)$ is a force point, it is not contained in the support of any L_t. So $\infty \notin W(S_{L_t})$, $0 \le t < T$. Let λ be the driving function, and $f_t = f_t^\lambda$, $0 \le t < T$, be the corresponding maps. We may and now adopt the notation in Section 4.1. Let $(\mathcal{F}_t)$ be the complete filtration generated by $B(t)$ in (4.15). Then (λ_t) and (L_t) are $(\mathcal{F}_t)$-adapted. From Corollary 2.19 (i), $(W^*(L_t))$ is also $(\mathcal{F}_t)$-adapted. Since $W_t = W^{L_t} = f_{W^*(L_t)} \circ W \circ g_{L_t}$ on $\Omega^{L_t} \setminus \widehat{L_t}$, (W_t) is $(\mathcal{F}_t)$-adapted. So we may apply Itô's formula (c.f. [12]). From (4.3) and (4.7), we get

$$d\lambda^*(t) = W_t'(\lambda(t))d\lambda(t) + \Big(\frac{\kappa}{2}+3\Big)W_t''(\lambda(t))dt, \quad 0 \le t < T.$$

Applying (4.15) and Lemma 4.5 (i), and using the condition that $\sum \rho_k = -\kappa - 6$, we find that

$$d\lambda^*(t) = W_t'(\lambda(t))\sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k W_t'(\lambda(t))^2}{W_t(\lambda(t)) - W_t \circ f_t^\lambda(q_k)}\,dt$$

$$= W_t'(\lambda(t))\sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k W_t'(\lambda(t))^2}{\lambda^*(t) - f_t^* \circ W(q_k)}\,dt, \quad 0 \le t < T.$$

From (4.3) we get $\lambda^*(0) = W_0(\lambda(0)) = W(x_0)$. Since $L_t^* = W^*(L_t)$ and f_t^* are backward chordal Loewner hulls and maps via the time-change u driven by λ^*, from (4.4) and the above equation, we conclude that, after a time-change, $W^*(L_t)$, $0 \le t < T$, are the backward chordal SLE$(\kappa; \vec{\rho})$ hulls started from $(W(x_0); W(q_1), \dots, W(q_n))$ and stopped at some time.

It remains to show that the above process is completed. If not, the process can be extended without swallowing the force points $W(q_1), \dots, W(q_n)$. From the condition, $W(\infty)$ is among these force points. So $(W^{-1})^*$ is well defined at the hulls of the extended process. From Propositions 4.1 and 3.10, this implies that the backward chordal Loewner hulls L_t, $0 \le t < T$, can be extended without swallowing any of $q_1, \dots, q_n$, which is a contradiction. $\square$

The following theorem can be proved using the above proof with minor modifications: we now use the argument in Section 4.2 instead of that in Section 4.1, apply Lemma 4.5 (ii) and (iii) instead of (i), and use Proposition 4.2 in addition to Proposition 4.1.

Theorem 4.7. *Suppose $\sum \rho_k = -\kappa - 6$. Let (L_t) be the backward radial SLE$(\kappa; \vec{\rho})$ hulls started from $(e^{ix_0}; e^{iq_1}, \dots, e^{iq_n})$. Let W map $\mathbb{D}$ conformally onto $\mathbb{H}$ (resp. $\mathbb{D}$) such that $\{W^{-1}(\infty)\} \cap \mathbb{T} \subset \{e^{iq_1}, \dots, e^{iq_n}\}$. Then, after a time-change, $(W^*(L_t))$ are the backward chordal (resp. radial) SLE$(\kappa; \vec{\rho})$ hulls started from $(W(e^{ix_0}); W(e^{iq_1}), \dots, W(e^{iq_n}))$.*

Corollary 4.8. *Let (L_t) be the backward radial SLE$(\kappa; -\kappa-6)$ hulls started from $(e^{ix_0}; e^{iq_0})$. Let W map $\mathbb{D}$ conformally onto $\mathbb{H}$ such that $W(e^{ix_0}) = 0$ and $W(e^{iq_0}) = \infty$. Then, after a time-change, $(W^*(L_t))$ are the backward chordal SLE$_\kappa$ hulls started from 0.*

Remarks.

1. The above theorems resemble the work in [15] for forward SLE$(\kappa; \vec{\rho})$ processes. The condition in their paper is $\sum \rho_k = \kappa - 6$. This is one reason why we may view backward SLE$_\kappa$ as SLE$_{-\kappa}$.
2. The definition of backward SLE$(\kappa; \vec{\rho})$ process differ from Sheffield's definition in [16] by a minus sign in (4.15) and (4.16) before the ρ_k's. If Sheffield's definition were used, the condition for conformal invariance would be $\sum \rho_k = \kappa + 6$ instead of $\sum \rho_k = -\kappa - 6$.
3. We may allow interior force points as in [15]. For the chordal (resp. radial) SLE$(\kappa; \vec{\rho})$ process, if $q_k \in \mathbb{H}$ (resp. $e^{iq_k} \in \mathbb{D}$) is a force point, we use $\mathrm{Re}\, f_t^\lambda(q_k)$ (resp. $\mathrm{Re}\, \widetilde{f}_t^\lambda(q_k)$) instead of $f_t^\lambda(q_k)$ (resp. $\widetilde{f}_t^\lambda(q_k)$) in (4.15) (resp. (4.16)). In the radial case, adding 0 to be a force point or change the force for 0 does not affect the process. Theorems 4.6 and Theorem 4.7 still hold if some or all force points lie inside $\mathbb{H}$ or $\mathbb{D}$. For the proofs, we apply Lemma 4.5 with real parts taken on the displayed formulas. One particular example is the following corollary.

   Corollary 4.9. *Let L_t, $0 \le t < \infty$, be a backward radial SLE$_\kappa$ process. Let W be a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$ such that $W(1) \ne \infty$. Let T be the maximum number such that $W^{-1}(\infty) \notin S_{L_t}$, $0 \le t < T$. Then, after a time-change, $W^*(L_t)$, $0 \le t < T$, are the backward chordal SLE$(\kappa; -\kappa - 6)$ hulls started from $(W(1); W(0))$.*

4. Using the properties of Bessel process and applying Girsanov's theorem, one may define backward chordal or radial SLE$(\kappa; \vec{\rho})$ processes with exactly one degenerate force point, if the corresponding force ρ_1 satisfies $\rho_1 < -2$ (which corresponds to a Bessel or Bessel-like process of dimension $d = 1 - \frac{4+2\rho_1}{\kappa} > 1$). Theorems 4.6 and 4.7 still hold when a degenerate force point exists. Unlike forward SLE$(\kappa; \vec{\rho})$ process, it is

impossible to define a backward SLE$(\kappa;\vec{\rho})$ process with two different degenerate force points.

5. Consider the radial case with one force point. Suppose the force $\rho_1 \le -\frac{\kappa}{2} - 2$. Let $X_t = \lambda(t) - \widetilde{f}_t^\lambda(q_1)$. Then X_t is a Bessel-like process with dimension $d = 1 - \frac{4+2\rho_1}{\kappa} \ge 2$, which implies that X_t never hits $2\pi\mathbb{Z}$. So $T = \infty$ and $e^{iq_1} \notin S_t$ for any t. From Lemma 3.5, $S_\infty = \mathbb{T} \setminus \{e^{iq_1}\}$. If, in addition, $\kappa \in (0,4]$, then a backward radial SLE$(\kappa;\rho_1)$ process induces a random welding ϕ, which is a involution of $\mathbb{T}$ with exactly two fixed point, $e^{i\lambda(0)}$ and e^{iq_1}, which are the initial point and the force point of the process.

5 Commutation Relations

Definition 5.1. *Let $\kappa_1, \kappa_2 > 0$, $n \in \mathbb{N}$, and $\vec{\rho}_1, \vec{\rho}_2 \in \mathbb{R}^n$. Let z_1, z_2, w_k, $2 \le k \le n$, be distinct points on $\mathbb{R}$ (resp. $\mathbb{T}$). We say that a backward chordal (resp. radial) SLE$(\kappa_1;\vec{\rho}_1)$ started from $(z_1; z_2, w_2, \dots, w_n)$ commutes with a backward chordal (resp. radial) SLE$(\kappa_2;\vec{\rho}_2)$ started from $(z_2; z_1, w_2, \dots, w_n)$ if there exists a coupling of two processes $(L_1(t); 0 \le t < T_1)$ and $(L_2(t); 0 \le t < T_2)$ such that*

(i) For $j = 1,2$, $(L_j(t), 0 \le t < T_j)$ is a complete backward chordal (resp. radial) SLE$(\kappa_j;\vec{\rho}_j)$ process started from $(z_j; z_{3-j}, w_2, \dots, w_n)$.

(ii) For $j \ne k \in \{1,2\}$, if $\bar{t}_k < T_k$ is a stopping time w.r.t. the complete filtration $(\mathcal{F}_t^k)$ generated by $(L_k(t))$, then conditioned on $\mathcal{F}_{\bar{t}_k}^k$, after a time-change, $f_k(\bar{t}_k,\cdot)^*(L_j(t_j))$, $0 \le t_j < T_j(\bar{t}_k)$, has the distribution of a partial backward chordal (resp. radial) SLE$(\kappa_j;\vec{\rho}_j)$ process started from

$$(f_k(\bar{t}_k,(z_j)); \mathring{\lambda}_k(\bar{t}_k), f_k(\bar{t}_k,w_2), \dots, f_k(\bar{t}_k,w_n)),$$

where $f_k(\bar{t}_k,\cdot) := f_{L_k(\bar{t}_k)}$, $T_j(\bar{t}_k) := \sup\{t_j < T_j : S_{L_j(t_j)} \cap S_{L_k(\bar{t}_k)} = \emptyset\}$, $\mathring{\lambda}_k(\bar{t}_k) = \lambda_k(\bar{t}_k)$ in the chordal case (resp. $e^{i\lambda_k(\bar{t}_k)}$ in the radial case), and λ_k is the driving function for $(L_k(t))$.

Here a partial backward SLE$(\kappa;\vec{\rho}_j)$ process is a complete SLE$(\kappa;\vec{\rho}_j)$ process stopped at a positive stopping time. If the commutation holds for any distinct points z_1, z_2, w_k, $2 \le k \le n$ on $\mathbb{R}$ (resp. $\mathbb{T}$), then we simply say that backward chordal (resp. radial) SLE$(\kappa_1;\vec{\rho}_1)$ commutes with backward chordal (resp. radial) SLE$(\kappa_2;\vec{\rho}_2)$.

Theorem 5.2. *For any $\kappa > 0$, backward chordal (resp. radial) SLE$(\kappa;-\kappa-6)$ commutes with backward chordal (resp. radial) SLE$(\kappa;-\kappa-6)$.*

We will prove this theorem in the next two subsections.

5.1 Ensemble

We first consider the radial case. Fix $\kappa > 0$ and $z_1 \ne z_2 \in \mathbb{T}$. Write $z_j = e^{i\widetilde{z}_j}$, $j = 1,2$. For $j = 1,2$, let $L_j(t)$, $0 \le t < T_j$, be a backward radial SLE$(\kappa;-\kappa-6)$ process started from $(z_j; z_{3-j})$; let λ_j be the driving function, and let $f_j(t,\cdot)$ and $\widetilde{f}_j(t,\cdot)$, $0 \le t < T_j$, be the corresponding maps and covering maps. At first, we suppose that the two processes are independent. Then for $j = 1,2$, λ_j satisfies $\lambda_j(0) = \widetilde{z}_j$ and the SDE:

$$d\lambda_j(t) = \sqrt{\kappa}dB_j(t) - \frac{-\kappa-6}{2}\cot_2(\lambda_j(t) - \widetilde{f}_j(t,\widetilde{z}_{3-j}))dt, \quad 0 \le t < T_j, \tag{5.1}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions. For $j=1,2$, let $(\mathcal{F}^j_t)$ denote the complete filtration generated by $B_j(t)$.

Define $\mathcal{D}=\{(t_1,t_2)\in[0,T_1)\times[0,T_2): S_{L_1(t_1)}\cap S_{L_2(t_2)}=\emptyset\}$. Then for $(t_1,t_2)\in\mathcal{D}$, we have $(L_1(t_1),L_2(t_2))\in\mathcal{P}_*$. So we may define

$$(L_{1,t_2}(t_1),L_{2,t_1}(t_2))=f^*(L_1(t_1),L_2(t_2)).$$

Let $f_{1,t_2}(t_1,\cdot)=f_{L_{1,t_2}(t_1)}$ and $f_{2,t_1}(t_2,\cdot)=f_{L_{2,t_1}(t_2)}$. From a radial version of Theorem 2.16, we see that

$$f_{1,t_2}(t_1,\cdot)\circ f_2(t_2,\cdot)=f_{L_1(t_1)\vee L_2(t_2)}=f_{2,t_1}(t_2,\cdot)\circ f_1(t_1,\cdot). \tag{5.2}$$

Fix $j\ne k\in\{1,2\}$. From a radial version of Corollary 2.19 (ii), the random map $f_{j,t_k}(t_j,\cdot)$ is $\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k}$-measurable. Let $u_{j,t_k}(t_j)=\operatorname{dcap}(L_{j,t_k}(t_j))$. From Propositions 3.10 and 4.1, for any fixed $t_k\in[0,T_k)$, $f_{j,t_k}(t_j,\cdot)$ are backward radial Loewner maps via the time-change u_{j,t_k}. Let $\widetilde{f}_{j,t_k}(t_j,\cdot)$ be the corresponding covering maps. So $e^i\circ\widetilde{f}_{j,t_k}(t_j,\cdot)=f_{j,t_k}(t_j,\cdot)\circ e^i$. From continuity, we see that $\widetilde{f}_{j,t_k}(t_j,\cdot)$ is also $\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k}$-measurable, and from (5.2) we have

$$\widetilde{f}_{1,t_2}(t_1,\cdot)\circ\widetilde{f}_2(t_2,\cdot)=\widetilde{f}_{2,t_1}(t_2,\cdot)\circ\widetilde{f}_1(t_1,\cdot). \tag{5.3}$$

Define m on $\mathcal{D}$ by $\mathrm{m}(t_1,t_2)=\operatorname{dcap}(L_1(t_1)\vee L_2(t_2))$. From (5.2) we get

$$\mathrm{m}(t_1,t_2)=u_{1,t_2}(t_1)+t_2=u_{2,t_1}(t_2)+t_1. \tag{5.4}$$

Apply the argument in the first paragraph of Section 4.2 with $\lambda=\lambda_j$, $L_{t_j}=L_j(t_j)$, $W=f_k(t_k,\cdot)$, and $\widetilde{W}=\widetilde{f}_k(t_k,\cdot)$, where $t_k\in[0,T_k)$ is fixed. Then we have correspondence: $L^*_{t_j}=L_{j,t_k}(t_j)$, $u=u_{j,t_k}$, and $\widetilde{f}^*_{t_j}=\widetilde{f}_{j,t_k}(t_j,\cdot)$. Since $\widetilde{W}_{t_j}$ is an analytic extension of $\widetilde{f}^*_{t_j}\circ\widetilde{W}\circ\widetilde{f}_{t_j}^{-1}$, from (5.3), we find that $\widetilde{W}_{t_j}=\widetilde{f}_{k,t_j}(t_k,\cdot)$. Thus, $e^{i\lambda_j(t_j)}$ (resp. $\lambda_j(t_j)$) lies in the domain of $f_{k,t_j}(t_k,\cdot)$ (resp. $\widetilde{f}_{k,t_j}(t_k,\cdot)$) as long as $(t_1,t_2)\in\mathcal{D}$.

Write $\widetilde{F}_{k,t_k}(t_j,\cdot)=\widetilde{f}_{k,t_j}(t_k,\cdot)$. We will use ∂_t to denote the partial derivative w.r.t. the first variable inside the parentheses, and use $'$ and the superscript (h) to denote the partial derivatives w.r.t. the second variable inside the parentheses. For $h=0,1,2,3$, define $A_{j,h}$ on $\mathcal{D}$ by

$$A_{j,h}(t_1,t_2)=\widetilde{f}^{(h)}_{k,t_j}(t_k,\lambda_j(t_j))=\widetilde{F}^{(h)}_{k,t_k}(t_j,\lambda_j(t_j)). \tag{5.5}$$

Use the superscript (S) to denote the (partial) Schwarzian derivative. Define $A_{j,S}$ on $\mathcal{D}$ by

$$A_{j,S}(t_1,t_2)=\widetilde{f}^{(S)}_{k,t_j}(t_k,\lambda_j(t_j))=\widetilde{F}^{(S)}_{k,t_k}(t_j,\lambda_j(t_j)) \tag{5.6}$$

From Section 4.2, we know that $L_{j,t_k}(t_j)$ are backward radial Loewner hulls via the time-change u_{j,t_k} driven by λ_{j,t_k}, which can be chosen such that

$$\lambda_{j,t_k}(t_j)=A_{j,0}(t_1,t_2). \tag{5.7}$$

Moreover, from (4.10), (4.13), and (4.14), we have

$$u'_{j,t_k}(t_j)=A_{j,1}^2, \tag{5.8}$$

$$\partial_t\widetilde{F}_{k,t_k}(t_j,\lambda_j(t_j))=3A_{j,2}, \tag{5.9}$$

$$\frac{\partial_t \widetilde{F}'_{k,t_k}(t_j,\lambda_j(t_j))}{\widetilde{F}'_{k,t_k}(t_j,\lambda_j(t_j))} = -\frac{1}{2}\Big(\frac{A_{j,2}}{A_{j,1}}\Big)^2 + \frac{4}{3}\frac{A_{j,3}}{A_{j,1}} + \frac{1}{6}A_{j,1}^2 - \frac{1}{6}, \tag{5.10}$$

where all $A_{j,h}$ are valued at (t_1,t_2).

From now on, we fix an $(\mathcal{F}^k_t)$-stopping time t_k with $t_k < T_k$. Then the process of conformal maps $(\widetilde{F}_{k,t_k}(t_j,\cdot))$ is $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-adapted. Let $T_j(t_k)$ be the maximal number such that for any $t_j < T_j(t_k)$, we have $(t_1,t_2)\in\mathcal{D}$. Then $T_j(t_k)$ is an $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-stopping time. Recall that $(\lambda_j(t))$ is an $(\mathcal{F}^j_{t_j})$-adapted local martingale with $\langle\lambda_j\rangle_t = \kappa t$. From now on, we will apply Itô's formula repeatedly. All SDEs below are $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-adapted, and t_j runs in the interval $[0,T_j(t_k))$.

From (5.7), (5.5), and (5.9), we get

$$d\lambda_{j,t_k}(t_j) = A_{j,1}d\lambda_j(t_j) + \Big(\frac{\kappa}{2}+3\Big)A_{j,2}dt, \quad 0\le t < T_j(t_k). \tag{5.11}$$

From (5.5) and (5.10) we get

$$\frac{\partial_{t_j}A_{j,h}}{A_{j,h}} = \frac{A_{j,2}}{A_{j,1}}d\lambda_j + \Big[-\frac{1}{2}\Big(\frac{A_{j,2}}{A_{j,1}}\Big)^2 + \Big(\frac{\kappa}{2}+\frac{4}{3}\Big)\frac{A_{j,3}}{A_{j,1}} + \frac{1}{6}A_{j,1}^2 - \frac{1}{6}\Big]dt_j. \tag{5.12}$$

Let

$$\alpha = \frac{6-(-\kappa)}{2(-\kappa)}, \quad \mathrm{c} = \frac{(8-3(-\kappa))(-\kappa-6)}{2(-\kappa)}.$$

Note that if $-\kappa$ is replaced by κ, then c becomes the central charge for forward SLE_κ. So we view the c here the central charge for backward SLE_κ, which runs in the interval $[25,\infty)$. Since $A_{j,S} = \frac{A_{j,3}}{A_{j,1}} - \frac{3}{2}(\frac{A_{j,2}}{A_{j,1}})^2$, from (5.12) we get

$$\frac{\partial_{t_j}A_{j,1}^\alpha}{A_{j,1}^\alpha} = \alpha\frac{A_{j,2}}{A_{j,1}}d\lambda_j + \Big[-\frac{\mathrm{c}}{6}A_{j,S} + \frac{\alpha}{6}A_{j,1}^2 - \frac{\alpha}{6}\Big]dt_j. \tag{5.13}$$

Now we study $\partial_{t_j}A_{k,h}$ and $\partial_{t_j}A_{k,S}$. From (5.5) we have $A_{k,h}(t_1,t_2) = \widetilde{F}^{(h)}_{j,t_j}(t_k,\lambda_k(t_k))$. Recall that $\widetilde{F}_{j,t_j}(t_k,\cdot) = \widetilde{f}_{j,t_k}(t_j,\cdot)$, and $\widetilde{f}_{j,t_k}(t_j,\cdot)$ are backward covering radial Loewner maps via the time-change u_{j,t_k} driven by λ_{j,t_k}. From (5.7) and (5.8), we get

$$\partial_t\widetilde{f}_{j,t_k}(t_j,z) = -A_{j,1}^2\cot_2(\widetilde{f}_{j,t_k}(t_j,z)-A_{j,0}). \tag{5.14}$$

Differentiate the above formula w.r.t. z, we get

$$\frac{\partial_t\widetilde{f}'_{j,t_k}(t_j,z)}{\widetilde{f}'_{j,t_k}(t_j,z)} = -A_{j,1}^2\cot_2'(\widetilde{f}_{j,t_k}(t_j,z)-A_{j,0}). \tag{5.15}$$

Differentiating the above formula w.r.t. z, we get

$$\partial_t\frac{\widetilde{f}''_{j,t_k}(t_j,z)}{\widetilde{f}'_{j,t_k}(t_j,z)} = -A_{j,1}^2\cot_2''(\widetilde{f}_{j,t_k}(t_j,z)-A_{j,0})\widetilde{f}'_{j,t_k}(t_j,z).$$

Since $f^{(S)} = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$, from the above formula, we get

$$\partial_t\widetilde{f}^{(S)}_{j,t_k}(t_j,z) = -A_{j,1}^2\cot_2'''(\widetilde{f}_{j,t_k}(t_j,z)-A_{j,0})\widetilde{f}'_{j,t_k}(t_j,z)^2. \tag{5.16}$$

Letting $z = \lambda_k(t_k)$ in (5.14), (5.15), and (5.16), we get

$$\partial_{t_j} A_{k,0} = -A_{j,1}^2 \cot_2(A_{k,0} - A_{j,0}) dt_j; \tag{5.17}$$

$$\frac{\partial_{t_j} A_{k,1}}{A_{k,1}} = -A_{j,1}^2 \cot_2'(A_{k,0} - A_{j,0}) dt_j; \tag{5.18}$$

$$\partial_{t_j} A_{k,S} = -A_{j,1}^2 A_{k,1}^2 \cot_2'''(A_{k,0} - A_{j,0}) dt_j. \tag{5.19}$$

Define X_j on $\mathcal{D}$ such that $X_j = A_{j,0} - A_{k,0}$. Then $X_1 + X_2 = 0$. Since $e^{i\lambda_j(t_j)}$ lies in the domain of $f_{k,t_j}(t_k, \cdot)$, $e^{iA_{j,0}} = f_{k,t_j}(t_k, e^{i\lambda_j(t_j)})$ lies in the range of $f_{k,t_j}(t_k, \cdot)$, i.e., $\widehat{\mathbb{C}} \setminus L_{k,t_j}(t_k)$. On the other hand, since via a time-change, $L_{k,t_j}(t_k)$ are backward radial Loewner hulls driven by $\lambda_{k,t_j}(t_k) = A_{k,0}$, from Lemma 3.4 we have $e^{iA_{k,0}} \in L_{k,t_j}(t_k)$ when $t_k > 0$. Thus, $e^{iA_{j,0}} \neq e^{iA_{k,0}}$ if $t_k > 0$. Switching j and k, the inequality also holds if $t_j > 0$. If $t_j = t_k = 0$, then $e^{iA_{j,0}} = e^{i\widetilde{z}_j} \neq e^{i\widetilde{z}_k} = e^{iA_{k,0}}$. Thus, $X_j, X_k \notin 2\pi\mathbb{Z}$. So we may define

$$Y = |\sin_2(X_1)|^{-2\alpha} = |\sin_2(X_2)|^{-2\alpha}.$$

From (5.7), (5.11), and (5.17), we get

$$\partial_{t_j} X_j = A_{j,1} d\lambda_j + \Big(\frac{\kappa}{2} + 3\Big) A_{j,2} dt - A_{j,1}^2 \cot_2(X_j) dt.$$

From Itô's formula, we get

$$\frac{\partial_{t_j} Y}{Y} = -\alpha \cot_2(X_j) A_{j,1} d\lambda_j - \alpha(\frac{\kappa}{2} + 3) A_{j,2} \cot_2(X_j) dt_j$$
$$- \frac{\alpha}{2} A_{j,1}^2 \cot_2^2(X_j) dt_j + \frac{\alpha\kappa}{4} A_{j,1}^2 dt_j. \tag{5.20}$$

Define Q and F on $\mathcal{D}$ such that $Q = \cot_2'''(X_1) = \cot_2'''(X_2)$ and

$$F(t_1, t_2) = \exp\Big(\int_0^{t_2}\int_0^{t_1} A_{1,1}(s_1, s_2)^2 A_{2,1}(s_1, s_2)^2 Q(s_1, s_2) ds_1 ds_2\Big). \tag{5.21}$$

Since $\widetilde{F}_{k,t_k}^{(S)}(0, \cdot) = \mathrm{id}$, from (5.6) we have $A_{j,S} = 0$ when $t_j = 0$. From (5.19) we get

$$\frac{\partial_{t_j} F}{F} = -A_{j,S} dt_j. \tag{5.22}$$

Define a positive function $\widehat{M}$ on $\mathcal{D}$ by

$$\widehat{M} = A_{1,1}^\alpha A_{2,1}^\alpha Y F^{-\frac{c}{6}} e^{\frac{c}{12}\mathrm{m}}. \tag{5.23}$$

From (5.4), (5.8), (5.13), (5.18), (5.20), and (5.22), we have

$$\frac{\partial_{t_j} \widehat{M}}{\widehat{M}} = \alpha \frac{A_{j,2}}{A_{j,1}} d\lambda_j - \alpha \cot_2(X_j) A_{j,1} d\lambda_j - \frac{\alpha}{6} dt_j. \tag{5.24}$$

When $t_k = 0$, we have $A_{j,1} = 1$, $A_{j,2} = 0$, $\mathrm{m} = t_j$, and $X_j = \lambda_j(t_j) - \widetilde{f}_j(t_j, \widetilde{z}_k)$, so the RHS of (5.24) becomes

$$\frac{1}{\kappa}\Big(\frac{\kappa}{2} + 3\Big) \cot_2(\lambda_j(t_j) - \widetilde{f}_j(t_j, \widetilde{z}_k)) d\lambda_j - \frac{\alpha}{6} dt_j. \tag{5.25}$$

Define another positive function M on $\mathcal{D}$ by

$$M(t_1,t_2)=\frac{\widehat{M}(t_1,t_2)\widehat{M}(0,0)}{\widehat{M}(t_1,0)\widehat{M}(0,t_2)}. \tag{5.26}$$

Then $M(\cdot,0)\equiv M(0,\cdot)\equiv 1$. From (5.1), (5.24), and (5.25), we have

$$\frac{\partial_{t_j}M}{M}=\Big[-\Big(3+\frac{\kappa}{2}\Big)\frac{A_{j,2}}{A_{j,1}}-\frac{-\kappa-6}{2}\cot_2(X_j)A_{j,1}$$
$$+\frac{-\kappa-6}{2}\cot_2(\lambda_j(t_j)-\widetilde{f}_j(t_j,\widetilde{z}_k))\Big]\frac{dB_j(t_j)}{\sqrt{\kappa}}. \tag{5.27}$$

So when $t_k\in[0,p)$ is a fixed $(\mathcal{F}^k_t)$-stopping time, M as a function in t_j is an $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-local martingale.

5.2 Coupling measures

Let JP denote the set of disjoint pairs of closed arcs (J_1,J_2) on $\mathbb{T}$ such that $z_j=e^{i\widetilde{z}_j}$ is contained in the interior of J_j, $j=1,2$. Let $T_j(J_j)$ denote the first time that $S_{L_j(t)}$ intersects $\overline{\mathbb{T}\setminus J_j}$. Then for every $(J_1,J_2)\in$ JP, if $t_j\le T_j(J_j)$, then $S_{L_j(t_j)}\subset J_j$, which implies that $L_j(t_j)\in\mathcal{H}_{J_j}$. So $[0,T_1(J_1)]\times[0,T_2(J_2)]\subset\mathcal{D}$.

Proposition 5.3. *(Boundedness) For any $(J_1,J_2)\in$ JP, $|\ln(M)|$ is uniformly bounded on $[0,T_1(J_1)]\times[0,T_2(J_2)]$ by a constant depending only on J_1 and J_2.*

Proof. Fix $(J_1,J_2)\in$ JP. In this proof, all constants depend only on (J_1,J_2), and we say a function is uniformly bounded if its values on $[0,T_1(J_1)]\times[0,T_2(J_2)]$ are bounded in absolute value by a constant. From (5.23) and (5.26), it suffices to show that $\ln(A_{1,1})$, $\ln(A_{2,1})$, $\ln(Y)$, $\ln(F)$, and m are all uniformly bounded.

Note that if $t_j\le T_j(J_j)$, then $L_j(t_j)\in\mathcal{H}_{J_j}$. From a radial version of Theorem 2.16 (iii), we have

$$\{L_1(t_1)\vee L_2(t_2):t_j\in[0,T_j(J_j)],\,j=1,2\}\subset\mathcal{H}_{J_1\cup J_2}. \tag{5.28}$$

Since $J_1\cup J_2\subsetneqq\mathbb{T}$, from Lemma D.2, the righthand side is a compact set. So the lefthand side is relatively compact. Since $H\mapsto \mathrm{dcap}(H)$ is continuous, and $\mathrm{m}(t_1,t_2)=\mathrm{dcap}(L_1(t_1)\vee L_2(t_2))$, we see that m is uniformly bounded. For $j=1,2$, since $T_j(J_j)\le \mathrm{m}$, $T_j(J_j)$ is also uniformly bounded.

Let S_1 and S_2 be the two components of $\mathbb{T}\setminus(J_1\cup J_2)$. For $s=1,2$, let $E_s\subset S_s$ be a compact arc. From Lemma D.3, $L_n\to L$ in $\mathcal{H}_{J_1\cup J_2}$ implies that $f_{L_n}\xrightarrow{\text{l.u.}} f_L$ in $\mathbb{C}\setminus(J_1\cup J_2)$, which then implies that $f'_{L_n}\xrightarrow{\text{l.u.}} f'_L$ in $\mathbb{C}\setminus(J_1\cup J_2)$. From (5.28), the compactness of $\mathcal{H}_{J_1\cup J_2}$, and that $E_1\cup E_2$ are compact subsets of $\mathbb{C}\setminus(J_1\cup J_2)$, we conclude that there is a constant $c_1>0$ such that $|f'_{L_1(t_1)\vee L_2(t_2)}(z)|\ge c$ for any $t_j\le T_j(J_j)$, $j=1,2$, and $z\in E_1\cup E_2$. Thus, for $t_j\in[0,T_j(J_j)]$, $j=1,2$, the length of $f_{L_1(t_1)\vee L_2(t_2)}(E_s)$, $s=1,2$, is bounded below by a constant $c_2>0$. Suppose $t_j\in(0,T_j(t_j)]$, $j=1,2$. From Lemma 3.4, $e^{iA_{j,0}}\in B_{L_{j,t_{3-j}}(t_j)}$, $j=1,2$. Note that $f_{L_1(t_1)\vee L_2(t_2)}(E_1\cup E_2)$ disconnects $B_{L_{1,t_2}(t_1)}$ from $B_{L_{2,t_1}(t_2)}$ on $\mathbb{T}$. Thus, there is a constant $c_3>0$ such that $|e^{iA_{1,0}(t_1,t_2)}-e^{iA_{2,0}(t_1,t_2)}|\ge c_3$ for $t_j\in(0,T_j(t_j)]$, $j=1,2$. From continuity, this still holds if $t_j\in[0,T_j(J_j)]$, $j=1,2$. Thus, $\ln(Y)=-2\alpha\ln|\sin_2(X_j)|$, $|\cot_2'(X_j)|$, and $|\cot_2'''(X_j)|$, $j=1,2$, are all uniformly bounded.

We may find a Jordan curve σ, which is disjoint from $J_1 \cup J_2$, such that its interior contains J_1 and its exterior contains J_2. From compactness, $\sup_{z\in\sigma} \ln |f_j'(t_j, z)|$ and $\sup_{z\in\sigma} \ln |f'_{L_1(t_1)\vee L_2(t_2)}(z)|$ are both uniformly bounded. From (5.2) we see that the value $\sup_{w\in f_j(t_j,\sigma)} \ln |f'_{3-j,t_j}(t_k, w)|$ is also uniformly bounded. Note that the interior of $f_j(t_j, \sigma)$ contains $\widehat{L_j(t_j)}$, which contains $e^{i\lambda_j(t_j)}$ if $t_j > 0$. From maximal principle, there is $c_4 \in (0, \infty)$ such that $A_{j,1}(t_1, t_2) = |f'_{3-j,t_j}(t_{3-j}, e^{i\lambda_j(t_j)})| \le c_4$ if $t_j \in (0, T_j(J_j)]$ and $t_{3-j} \in [0, T_{3-j}(J_{3-j})]$. From continuity, $A_{j,1}$ is uniformly bounded, $j = 1, 2$. From (5.18) and the uniformly boundedness of $|\cot_2'(X_j)|$ we see that $\ln(A_{j,1})$ is uniformly bounded, $j = 1, 2$. From (5.21) and the uniformly boundedness of $|\cot_2'''(X_j)|$ we see that $\ln(F)$ is also uniformly bounded, which completes the proof. □

Let μ_j denote the distribution of (λ_j), $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then μ is the joint distribution of (λ_1) and (λ_2), since λ_1 and λ_2 are independent. Fix $(J_1, J_2) \in \mathrm{JP}$. From the local martingale property of M and Proposition 5.3, we have $\mathbf{E}_\mu[M(T_1(J_1), T_2(J_2))] = M(0, 0) = 1$. Define ν_{J_1,J_2} by $d\nu_{J_1,J_2}/d\mu = M(T_1(J_1), T_2(J_2))$. Then ν_{J_1,J_2} is a probability measure. Let ν_1 and ν_2 be the two marginal measures of ν_{J_1,J_2}. Then $d\nu_1/d\mu_1 = M(T_1(J_1), 0) = 1$ and $d\nu_2/d\mu_2 = M(0, T_2(J_2)) = 1$, so $\nu_j = \mu_j$, $j = 1, 2$. Suppose temporarily that the joint distribution of (λ_1) and (λ_2) is ν_{J_1,J_2} instead of μ. Then the distribution of each (λ_j) is still μ_j.

Fix an $(\mathcal{F}_t^2)$-stopping time $t_2 \le T_2(J_2)$. From (5.1), (5.27), and Girsanov theorem (c.f. [12]), under the probability measure ν_{J_1,J_2}, there is an $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{t_2})_{t_1\ge 0}$-Brownian motion $\widetilde{B}_{1,t_2}(t_1)$ such that $\lambda_1(t_1)$, $0 \le t_1 \le T_1(J_1)$, satisfies the $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{t_2})_{t_1\ge 0}$-adapted SDE:

$$d\lambda_1(t_1) = \sqrt{\kappa} d\widetilde{B}_{1,t_2}(t_1) - \Big(3 + \frac{\kappa}{2}\Big)\frac{A_{1,2}}{A_{1,1}} dt_1 - \frac{-\kappa - 6}{2} \cot_2(X_1) A_{1,1} dt_1,$$

which together with (5.5), (5.7), (5.9), and Itô's formula, implies that

$$d\lambda_{1,t_2}(t_1) = A_{1,1}\sqrt{\kappa} d\widetilde{B}_{1,t_2}(t_1) - \frac{-\kappa - 6}{2} \cot_2(X_1) A_{1,1}^2 dt_1.$$

From (5.5) and (5.7) we get $X_1 = A_{1,1} - A_{2,1} = \lambda_{1,t_2}(t_1) - \widetilde{f}_{1,t_2}(t_1, \lambda_2(t_2))$. Note that $\lambda_{1,t_2}(0) = \widetilde{f}_{2,0}(t_2, \widetilde{z}_1) = \widetilde{f}_2(t_2, \widetilde{z}_1)$. Since $L_{1,t_2}(t_1)$ and $\widetilde{f}_{1,t_2}(t_1, \cdot)$ are backward radial Loewner hulls and covering maps via the time-change u_{1,t_2}, from (5.8) and the above equation, we find that, under the measure ν_{J_1,J_2}, conditioned on $\mathcal{F}^1_{t_2}$ for any $(\mathcal{F}^2_t)$-stopping time $t_2 \le T_2(J_2)$, via the time-change u_{1,t_2}, $L_{1,t_2}(t_1) = f_2(t_2, \cdot)^*(L_1(t_1))$, $0 \le t_1 \le T_1(J_1)$, is a partial backward radial SLE$(\kappa; \frac{-\kappa-6}{2})$ process started from $e^i \circ \widetilde{f}_2(t_2, \widetilde{z}_1) = f_2(t_2, z_1)$ with marked point $e^i(\lambda_2(t_2))$. Similarly, the above statement holds true if the subscripts "1" and "2" are exchanged.

The joint distribution ν_{J_1,J_2} is a local coupling such that the desired properties in the statement of Theorem 5.2 holds true up to the stopping times $T_1(J_1)$ and $T_2(J_2)$. Then we can apply the maximum coupling technique developed in [18] to construct a global coupling using the local couplings within different pairs (J_1, J_2). The reader is referred to Theorem 4.5 and Section 4.3 in [19] for the construction of a global coupling between two forward SLE processes. For the coupling of backward SLE processes, the method is essentially the same. A slight difference is that for the forward SLE processes, a pair of hulls were used to control the growth of $M(\cdot, \cdot)$, which stays uniformly bounded up to the time that the SLE hulls grow out of the given hulls; while for the backward SLE processes, we here used a pair of arcs to control the growth of $M(\cdot,\cdot)$. One fact that is worth mentioning is that here we may choose a dense countable set $\mathrm{JP}^*\subset \mathrm{JP}$ such that, when $S_{L_1(t_1)}\cap S_{L_2(t_2)}=\emptyset$, there exists $(J_1,J_2)\in \mathrm{JP}^*$ with $S_{L_j(t_j)}\subset J_j$, $j=1,2$, from which follows that

$$T_j(t_k)=\sup\{T_j(J_j):(J_1,J_2)\in \mathrm{JP}^*,t_k\le T_k(J_k)\},\quad j\ne k\in\{1,2\}. \tag{5.29}$$

This finishes the proof of Theorem 5.2 in the radial case.

Now we briefly describe the proof for the chordal case. The proof in this case is simpler because there are no covering maps. Suppose the two backward chordal SLE$(\kappa;-\kappa-6)$ processes start from $(z_j;z_k)$, where $z_1\ne z_2\in\mathbb{R}$. Formula (5.1) holds with all tildes removed and the function $\cot_2$ replaced by $z\mapsto \frac{2}{z}$. The domain $\mathcal{D}$ and the $\mathbb{H}$-hulls $L_{1,t_2}(t_1)$ and $L_{2,t_1}(t_2)$ are defined in the same way. Then (5.2) still holds. From Corollary 2.19 (ii), $f_{1,t_2}(t_1,\cdot)$ and $f_{2,t_1}(t_2,\cdot)$ are $\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2}$-measurable. Define $\mathrm{m}(t_1,t_2)=\mathrm{hcap}(L_1(t_1)\vee L_2(t_2))/2$. Then (5.4) holds with $u_{j,t_k}(t_j):=\mathrm{hcap}(L_{j,t_k}(t_j))/2$.

Now we apply the argument in Section 4.1 with $W=f_k(t_k,\cdot)$. Then $W_t=f_{k,t_j}(t_k,\cdot)$. Let $F_{k,t_k}(t_j,\cdot)=f_{k,t_j}(t_k,\cdot)$, and define $A_{j,h}$ and $A_{j,S}$ using (5.5) and (5.6) with all tildes removed. Using (4.3), (4.4), (4.7), and (4.8), we see that (5.7) still holds here; (5.8) and (5.9) hold with all tildes removed; and (5.10) holds without the tildes and the terms $+\frac16 A_{j,1}^2-\frac16$. Then we get the SDEs (5.11) and (5.13) without the terms $+\frac{\alpha}{6}A_{j,1}^2-\frac{\alpha}{6}$. Formulas (5.17), (5.18), and (5.19) hold with $\cot_2$ replaced by $z\mapsto\frac{2}{z}$. We still define $X_j=A_{j,1}-A_{k,1}$. Then $X_j\ne 0$ in $\mathcal{D}$. Define Y on $\mathcal{D}$ by $Y=|X_1|^{-2\alpha}=|X_2|^{-2\alpha}$. Then (5.20) holds with $\cot_2$ replaced by $z\mapsto\frac{2}{z}$ and the term $+\frac{\alpha\kappa}{4}A_{j,1}^2dt_j$ removed. Define F using (5.21) with $Q=-\frac{12}{X_1^4}=-\frac{12}{X_2^4}$. Then (5.22) still holds. Define $\widehat{M}$ using (5.23) without the factor $e^{\frac{\mathrm{c}}{12}\mathrm{m}}$. Then (5.24) holds with $\cot_2$ replaced by $z\mapsto\frac{2}{z}$ and the term $-\frac{\alpha}{6}dt_j$ removed. Define M using (5.26). Then (5.27) holds with all tildes removed and $\cot_2$ replaced by $z\mapsto\frac{2}{z}$.

We define JP to be the set of disjoint pairs of closed real intervals (J_1,J_2) such that z_j is contained in the interior of J_j. Then Proposition 5.3 holds with a similar proof, where Lemma C.2 is applied here, and we can show that $|X_1|$ is uniformly bounded away from 0. The argument on the local couplings hold with all tildes and e^i removed and $\cot_2$ replaced by $z\mapsto\frac{2}{z}$. Finally, we may apply the maximum coupling technique to construct a global coupling with the desired properties. Formula (5.29) still holds here and is used in the construction. This finishes the proof in the chordal case.

5.3 Other results

Besides Theorem 5.2, one may also prove the following two theorems, which are similar to the couplings for forward SLE that appear in [4] and [19].

Theorem 5.4. *Let $\kappa_1,\kappa_2>0$ satisfy $\kappa_1\kappa_2=16$, and $c_1,\dots,c_n\in\mathbb{R}$ satisfy $\sum_{k=1}^n c_k=\frac32$. Let $\vec\rho_j=(\frac{\kappa_j}{2},c_1(-\kappa_j-4),\dots,c_n(-\kappa_j-4))$, $j=1,2$. Then backward chordal (resp. radial) SLE$(\kappa_1;\vec\rho_1)$ commutes with backward chordal (resp. radial) SLE$(\kappa_2;\vec\rho_2)$.*

Theorem 5.5. *Let $\kappa>0$ and $\vec\rho\in\mathbb{R}^n$, whose first coordinate is 2. Then backward chordal (resp. radial) SLE$(\kappa;\vec\rho)$ commutes with backward chordal (resp. radial) SLE$(\kappa;\vec\rho)$.*

6 Reversibility of Backward Chordal SLE

Theorem 6.1. *Let $\kappa \in (0,4]$ and $z_1 \ne z_2 \in \mathbb{T}$. Suppose a backward radial SLE$(\kappa; -\kappa-6)$ process $(L_1(t))$ started from $(z_1; z_2)$ commutes with a backward radial SLE$(\kappa;-\kappa-6)$ process $(L_2(t))$ started from $(z_2; z_1)$. Then a.s. they induce the same welding.*

Proof. For $j = 1,2$, let $S^j_t = S_{L_j(t)}$ and $f^j_t = f_{L_j(t)}$. Let $T_j(\cdot)$, $j=1,2$, be as in Definition 5.1. Let ϕ_j be the welding induced by $(L_j(t))$. Since $-\kappa - 6 \le -\kappa/2 - 2$, from the last remark in Section 4.3, we see that, for $j = 1,2$, a.s. $T_j = \infty$, $S^j_\infty = \mathbb{T} \setminus \{z_{3-j}\}$, and ϕ_j is an involution of $\mathbb{T}$ with exactly two fixed points: z_1 and z_2.

Fix $t_2 > 0$. Since $(L_1(t))$ and $(L_2(t))$ commute, the following is true. Conditioned on $(L_2(t))_{t \le t_2}$, $(f^2_{t_2})^*(L_1(t_1))$, $0 \le t_1 < T_1(t_2)$, is a partial backward radial SLE$(\kappa; -\kappa-6)$ process, after a time-change, started from $(f^2_{t_2}(z_1); B_{L_2(t_2)})$. Here we use $B_{L_2(t_2)}$ also to denote the unique point in the base of $L_2(t_2)$, which is equal to $e^{i\lambda_2(t_2)}$, where λ_2 is a driving function for $(L_2(t_2))$. We have

$$S := \bigcup_{0 \le t_1 < T_1(t_2)} S_{(f^2_{t_2})^*(L_1(t_1))} = f^2_{t_2}\Big(\bigcup_{0 \le t_1 < T_1(t_2)} S^1_{t_1}\Big) = f^2_{t_2}(S^1_{T_1(t_2)-}). \tag{6.1}$$

Recall that $f^2_{t_2}$ is a homeomorphism from $\mathbb{T} \setminus S^2_{t_2}$ onto $\mathbb{T} \setminus B_{L_2(t_2)}$. From the definition of $T_1(t_2)$, we see that $S^1_{T_1(t_2)}$ intersects $S^2_{t_2} \ne \emptyset$ at one or two end points of both arcs. If they intersect at only one point, then $S^1_{T_1(t_2)-}$ is a proper subset of $\mathbb{T} \setminus S^2_{t_2}$, and these two arcs share an end point. From (6.1), this then implies that the arc S is a proper subset of $\mathbb{T} \setminus B_{L_2(t_2)}$, and $B_{L_2(t_2)}$ is an end point of S. Recall that, after a time-change, $(f^2_{t_2})^*(L_1(t_1))$, $0 \le t_1 < T_1(t_2)$, is a partial backward radial SLE$(\kappa; -\kappa-6)$ process. Since $S \ne \mathbb{T} \setminus B_{L_2(t_2)}$, the process is not complete. Then we conclude that S is contained in a closed arc on $\mathbb{T}$ that does not contain $B_{L_2(t_2)}$ because the force point is not swallowed by the process at any finite time, which contradicts that $B_{L_2(t_2)}$ is an end point of S. Thus, a.s. $S^1_{T_1(t_2)}$ and $S^2_{t_2}$ share two end points. Since ϕ_j swaps the two end points of any S^j_t, $j=1,2$, we see that a.s. $\phi_2 = \phi_1$ on $\partial_{\mathbb{T}} S^2_{t_2}$. Let $t_2 > 0$ vary in the set of rational numbers, we see that a.s. $\phi_2 = \phi_1$ on $\bigcup_{t \in \mathbb{Q}_{>0}} \partial_{\mathbb{T}} S^2_{t_2}$, which is a dense subset of $\mathbb{T}$. The conclusion follows since ϕ_1 and ϕ_2 are continuous. $\square$

We now state the reversibility of backward chordal SLE$_\kappa$ for $\kappa \in (0,4]$ in terms of its welding. Recall that a backward chordal SLE$_\kappa$ welding is an involution of $\widehat{\mathbb{R}}$ with two fixed points: 0 and ∞.

Theorem 6.2. *Let $\kappa \in (0,4]$, and ϕ be a backward chordal SLE$_\kappa$ welding. Let $h(z) = -1/z$. Then $h \circ \phi \circ h$ has the same distribution as ϕ.*

Proof. Let $(L_1(t))$ and $(L_2(t))$ be commuting backward radial SLE$(\kappa; -\kappa-6)$ procesees as in Theorem 5.2, which induce the weldings ψ_1 and ψ_2, respectively. The above theorem implies that a.s. $\psi_1 = \psi_2$. For $j = 1,2$, let W_j be a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{H}$ such that $W_j(z_j) = 0$ and $W_j(z_{3-j}) = \infty$, and $W_2 = h \circ W_1$. From Corollary 4.8, $K_j(t) := W_j^*(L_j(t))$, $0 \le t < \infty$, is a backward chordal SLE$_\kappa$, after a time-change, which then induces backward chordal SLE$_\kappa$ welding ϕ_j, $j=1,2$. Then ϕ_1 and ϕ_2 have the same law as ϕ. From (4.1), we get $\phi_j = W_j \circ \psi_j \circ W_j^{-1}$, $j = 1,2$, which implies that a.s. $\phi_2 = h \circ \phi_1 \circ h$. The conclusion follows since ϕ_1 and ϕ_2 has the same distribution as ϕ. $\square$

Lemma 6.3. *Let $\kappa > 0$. Let f_t, $0 \le t < \infty$, be backward chordal SLE$_\kappa$ maps. Then for every $z_0 \in \mathbb{H}$, a.s. (3.4) holds.*

Proof. Let $Z_t = f_t(z_0)$, $X_t = \operatorname{Re} z_t$, and $Y_t = \operatorname{Im} z_t$. Then

$$dX_t = -\sqrt{\kappa} dB(t) - \frac{2X_t}{X_t^2 + Y_t^2}\, dt, \quad dY_t = \frac{2Y_t}{X_t^2 + Y_t^2}\, dt$$

Let $R_t = |f_t'(z_0)|$. Then $\frac{dR_t}{R_t} = \frac{2(X_t^2 - Y_t^2)}{(X_t^2+Y_t^2)^2} dt$. Let $N_t = Y_t/R_t$ and $A_t = X_t/Y_t$. Then

$$\frac{dN_t}{N_t} = \frac{4Y_t^2}{(X_t^2 + Y_t^2)^2}\, dt, \quad dA_t = -\frac{\sqrt{\kappa} dB(t)}{Y_t} - \frac{4A_t}{X_t^2 + Y_t^2}\, dt.$$

Let $u(t) = \ln(Y_t)$. Then $u'(t) = \frac{2}{X_t^2+Y_t^2}$. Let $T = \sup u([0,\infty))$ and define $\widehat{N}_s = N_{u^{-1}(s)}$ and $\widehat{A}_s = A_{u^{-1}(s)}$ for $0 \le s < T$. Then

$$\frac{d\widehat{N}_s}{\widehat{N}_s} = \frac{2}{\widehat{A}_s^2 + 1}\, ds, \quad d\widehat{A}_s = -\sqrt{1 + \widehat{A}_s^2}\sqrt{\kappa/2} d\widehat{B}(s) - 2\widehat{A}_s ds,$$

where $\widehat{B}(s)$ is another Brownian motion. We claim that $T = \infty$. Suppose $T < \infty$. Then $\lim_{t\to\infty} Y(t) = e^T \in \mathbb{R}$. From the SDE for A_s, we see that a.s. $\lim_{s\to T} A_s \in \mathbb{R}$, which implies that $\lim_{t\to\infty} A_t \in \mathbb{R}$ and $\lim_{t\to\infty} X_t \in \mathbb{R}$ as $X_t = Y_t A_t$. Then we have a.s. $s'(t) = \frac{2}{X_t^2+Y_t^2}$ tends to a finite positive number as $t \to \infty$, which contradicts that $T = \sup\{s(t), 0 \le t < \infty\} < \infty$. So the claim is proved. Using Itô's formula, we see that $\widehat{A}_s$, $0 \le s < \infty$, is recurrent. Since $(\ln(\widehat{N}_s))' = \frac{2}{\widehat{A}_s^2+1}$, we see that a.s. $\widehat{N}_s \to \infty$ as $s \to \infty$. So a.s. $N_t = \frac{\operatorname{Im} f_t(z_0)}{|f_t'(z_0)|} \to \infty$ as $t \to \infty$, i.e., (3.4) holds. □

If $\kappa \in (0, 4]$, then since the backward chordal traces are simple, (3.5) holds. From the above lemma and Section 3.3, we see that, for $\kappa \in (0, 4]$, the backward chordal SLE$_\kappa$ a.s. generates a normalized global backward chordal trace β, which we call a normalized global backward chordal SLE$_\kappa$ trace. Recall that $\beta(t)$, $0 \le t < \infty$, is simple with $\beta(0) = 0$, and $i \notin \beta$; and there is $F_\infty : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{C} \setminus \beta$, whose continuation maps $\mathbb{R}$ onto β such that (3.7) holds, and for any $x \in \mathbb{R}$, $F_\infty(x) = F_\infty(\phi(x)) \in \beta$. Now we state the reversibility of the backward chordal SLE$_\kappa$ for $\kappa \in (0, 4)$ in terms of β.

Theorem 6.4. *Let $\kappa \in (0, 4)$, and β be a normalized global backward chordal SLE$_\kappa$ trace. Let $h(z) = -1/z$. Then $h(\beta \setminus \{0\})$ has the same distribution as $\beta \setminus \{0\}$ as random sets.*

Proof. For $j = 1, 2$, let ϕ_j be a backward chordal SLE$_\kappa$ welding and β_j be the corresponding normalized global trace. Then β_j is a simple curve with one end point 0, and there exists $F_j : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{C} \setminus \beta_j$ such that $F_j(i) = i$, $F_j(0) = 0$, and $F_j(x) = F_j(\phi_j(x))$ for $x \in \mathbb{R}$. From Theorem 6.2 we may assume that $\phi_2 = h \circ \phi_1 \circ h^{-1}$. Now it suffices to show that $h(\beta_2 \setminus \{0\}) = \beta_1 \setminus \{0\}$.

Define $G = h \circ F_2 \circ h \circ F_1^{-1}$. Then G is a conformal map defined on $\mathbb{C} \setminus \beta_1$. It has continuation to $\beta_1 \setminus \{0\}$. In fact, if $z \in \mathbb{C} \setminus \beta_1$ and $z \to z_0 \in \beta_1 \setminus \{0\}$, then $F_1^{-1}(z) \to \{x, \phi_1(x)\}$ for some $x \in \mathbb{R} \setminus \{0\}$, which then implies that $h \circ F_1^{-1}(z) \to \{h(x), h \circ \phi_1(x)\}$; since

$\phi_2 \circ h = h \circ \phi_1$, we find that $F_2 \circ h \circ F_1^{-1}(z)$ tends to some point on $\beta_2 \setminus \{0\}$, so $G(z)$ tends to some point on $h(\beta_2 \setminus \{0\})$. It was proved in [13] that a forward SLE_κ trace is the boundary of a Hölder domain. Then the same is true for backward chordal SLE_κ traces and the normalized global trace. From the results in [5], we see that $\beta_1 \setminus \{0\}$ is conformally removable, which means that G extends to a conformal map from $(\mathbb{C}\setminus\beta_1)\cup(\beta_1\setminus\{0\}) = \mathbb{C}\setminus\{0\}$ onto $\mathbb{C} \setminus \{0\}$, and maps $\beta_1 \setminus \{0\}$ to $h(\beta_2 \setminus \{0\})$. Since $G(i) = i$, either $G = \mathrm{id}$ or $G = h$. Suppose $G = h$. Then $F_1 = F_2 \circ h$. Since $F_1(0) = F_2(0) = 0$, for $j = 1, 2$, F_j maps a neighborhood of 0 in $\mathbb{H}$ onto a neighborhood of 0 in $\mathbb{C}$ without a simple curve. Since $F_1 = F_2 \circ h$, F_1 also maps a neighborhood of ∞ in $\mathbb{H}$ onto a neighborhood of 0 without a simple curve, which contradicts the univalent property of F_1. Thus, $G = \mathrm{id}$, and we get $h(\beta_2 \setminus \{0\}) = G(\beta_1 \setminus \{0\}) = \beta_1 \setminus \{0\}$, as desired. □

Now we propose a couple of questions. First, let's consider backward chordal SLE_κ for $\kappa > 4$. Since the process does not generate simple backward chordal traces, the random welding ϕ can not be defined. However, the lemma below and the discussion in Section 3.3 show that we can still define a global backward chordal SLE_κ trace.

Lemma 6.5. *Let $\kappa \in (0, \infty)$. Suppose β_t, $0 \le t < \infty$, are backward chordal traces driven by $\lambda(t) = \sqrt{\kappa}B(t)$. Then a.s. (3.5) holds.*

Proof. If $\kappa \in (0, 4]$, a.s. the traces are simple, so (3.5) holds. Now suppose $\kappa > 4$. Let f_t and L_t be the corresponding maps and hulls. It suffices to show that, for any $t_0 > 0$, a.s. there exists $t_1 > t_0$ such that $\beta_{t_1}([0, t_0]) \subset \mathbb{H}$.

Let g_t and K_t, $0 \le t < \infty$, be the forward chordal Loewner maps and hulls driven by $\sqrt{\kappa}B(t)$. From Theorem 6.1 in [20], for any deterministic time $t_1 \in (0, \infty)$, the continuation of $g_{t_1}^{-1}$ a.s. maps the interior of $S_{K_{t_1}}$ into $\mathbb{H}$. From Lemma 3.1 and the property of Brownian motion, we see that, for any $t_1 \in (0, \infty)$, f_{t_1} has the same distribution as $\lambda(t_1)+g_{t_1}^{-1}(\cdot-\lambda(t_1))$, which implies that the continuation of f_{t_1} a.s. maps the interior of $S_{L_{t_1}}$ into $\mathbb{H}$.

Since a.s. $\bigcup_{n=1}^\infty S_n = S_\infty = \mathbb{R} \supset \lambda([0, t_0])$, and (S_t) is an increasing family of intervals, we see that a.s. there is $N \in \mathbb{N}$ such that the interior of S_N contains $\lambda([0, t_0])$. Let $t_1 = N$. Then f_{t_1} maps $\lambda([0, t_0])$ into $\mathbb{H}$, which implies that $\beta_{t_1}(t) = f_{t_1}(\lambda(t)) \in \mathbb{H}$ for $0 \le t \le t_0$. □

Question 6.6. *Do we have the reversibility of the global backward chordal SLE_κ trace for $\kappa > 4$?*

Second, let's consider backward radial SLE_κ processes. One can show that (3.8) a.s. holds. Since $T = \infty$, we may define a global backward radial SLE_κ trace.

Question 6.7. *Does a global backward radial SLE_κ trace satisfy some reversibility property of any kind?*

Recall that the forward radial SLE_κ trace does not satisfy the reversibility property in the usual sense. However, it's proved in [22] that, for $\kappa \in (0, 4]$, the whole-plane SLE_κ, as a close relative of radial SLE_κ, satisfies reversibility.

Finally, it is worth mentioning the following simple fact. Recall that, if $\kappa \in (0, 4]$, a backward radial SLE_κ welding is an involution of $\mathbb{T}$ with two fixed points, one of which is 1. The following theorem gives the distribution of the other fixed point ζ, and says that a backward radial SLE_κ process conditioned on ζ is a backward radial $\mathrm{SLE}(\kappa; -4)$ process with force point ζ. It is similar to Theorem 3.1 in [20], and we omit its proof.

Theorem 6.8. *Let $\kappa \in (0,4]$. Let μ denote the distribution of a backward radial SLE$_\kappa$ process. For $\theta \in (0,2\pi)$, let ν_θ denote the distribution of a backward radial SLE$(\kappa;-4)$ process started from $(1;e^{i\theta})$. Let $f(\theta) = C\sin_2(\theta)^{4/\kappa}$, where $C>0$ is such that $\int_0^{2\pi} f(\theta)d\theta = 1$. Then*

$$\mu = \int_0^{2\pi} \nu_\theta f(\theta) d\theta.$$

Appendices

A Carathéodory Topology

Definition A.1. *Let $(D_n)_{n=1}^\infty$ and D be domains in $\mathbb{C}$. We say that (D_n) converges to D, and write $D_n \stackrel{\rm Cara}{\longrightarrow} D$, if for every $z \in D$, $\mathrm{dist}(z, \mathbb{C}\setminus D_n) \to \mathrm{dist}(z, \mathbb{C}\setminus D)$. This is equivalent to the following:*

(i) every compact subset of D is contained in all but finitely many D_n's;

(ii) for every point $z_0 \in \partial D$, there exists $z_n \in \partial D_n$ for each n such that $z_n \to z_0$.

Remark. A sequence of domains may converge to two different domains. For example, let $D_n = \mathbb{C}\setminus((-\infty, n])$. Then $D_n \stackrel{\rm Cara}{\longrightarrow} \mathbb{H}$, and $D_n \stackrel{\rm Cara}{\longrightarrow} -\mathbb{H}$ as well. But two different limit domains of the same domain sequence must be disjoint from each other, because if they have nonempty intersection, then one contains some boundary point of the other, which implies a contradiction.

Lemma A.2. *Suppose $D_n \stackrel{\rm Cara}{\longrightarrow} D$, $f_n : D_n \stackrel{\rm Conf}{\twoheadrightarrow} E_n$, $n \in \mathbb{N}$, and $f_n \stackrel{\rm l.u.}{\longrightarrow} f$ in D. Then either f is constant on D, or f is a conformal map on D. In the latter case, let $E = f(D)$. Then $E_n \stackrel{\rm Cara}{\longrightarrow} E$ and $f_n^{-1} \stackrel{\rm l.u.}{\longrightarrow} f^{-1}$ in E.*

Remark. The above lemma resembles the Carathéodory kernel theorem (Theorem 1.8, [11]), but the domains here don't have to be simply connected. The main ingredients in the proof are Rouché's theorem and Koebe's 1/4 theorem. The lemma also holds in the case that D_n and D are domains of any Riemann surface, if the metric in the underlying space is used in place of the Euclidean metric for Definition A.1 and locally uniformly convergence. In particular, if we use the spherical metric, then Lemma A.2 holds for domains of $\widehat{\mathbb{C}}$.

B Topology on Interior Hulls

Let $\mathcal{H}$ denote the set of all interior hulls in $\mathbb{C}$. Recall that for any $H \in \mathcal{H}$, ϕ_H^{-1} is defined on $\{|z| > \mathrm{rad}(H)\}$, and for a nondegenerate interior hull, $\psi_H(z) = \varphi_H^{-1}(z) = \phi_H^{-1}(\mathrm{rad}(H)z)$ is defined on $\{|z|>1\}$. It's shown in Section 2.5 of [21] that there is a metric $d_{\mathcal{H}}$ on $\mathcal{H}$ such that for any $H_n, H \in \mathcal{H}$, the followings are equivalent:

1. $d_{\mathcal{H}}(H_n, H) \to 0$
2. $\mathrm{rad}(H_n) \to \mathrm{rad}(H)$ and $\phi_{H_n}^{-1} \stackrel{\rm l.u.}{\longrightarrow} \phi_H^{-1}$ in $\{|z| > \mathrm{rad}(H)\}$.
3. $\mathbb{C}\setminus H_n \stackrel{\rm Cara}{\longrightarrow} \mathbb{C}\setminus H$.

In particular, we see that rad is a continuous function on $(\mathcal{H}, d_{\mathcal{H}})$. Thus, for nondegenerate interior hulls, $d_{\mathcal{H}}(H_n, H) \to 0$ iff $\psi_{H_n} \xrightarrow{\text{l.u.}} \psi_H$ in $\{|z| > 1\}$. The following lemma is Lemma 2.2 in [21].

Lemma B.1. *For any $F \in \mathcal{H}$, the set $\{H \in \mathcal{H} : H \subset F\}$ is compact.*

Corollary B.2. *For any $F \in \mathcal{H}$ and $r > 0$, the set $\{H \in \mathcal{H} : H \subset F, \text{rad}(H) \geq r\}$ is compact.*

C Topology on $\mathbb{H}$-hulls

From Section 5.2 in [17], there is a metric $d_{\mathcal{H}}$ on the space of $\mathbb{H}$-hulls such that $d_{\mathcal{H}}(H_n \to H_\infty)$ iff $f_{H_n} \xrightarrow{\text{l.u.}} f_{H_\infty}$ in $\mathbb{H}$. From Lemma A.2, this implies that $\mathbb{H} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{H} \setminus H_\infty$. But $\mathbb{H} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{H} \setminus H_\infty$ does not imply $d_{\mathcal{H}}(H_n \to H_\infty)$. A counterexample is $H_n = \{z \in \mathbb{H} : |z - 2n| \leq n\}$ and $H_\infty = \emptyset$. Since $H_1 \cdot H_2 = H_3$ iff $f_{H_1} \circ f_{H_2} = f_{H_3}$, the dot product is continuous.

Formula (5.1) in [17] states that for any $\mathbb{H}$-hull H, there is a positive measure μ_H supported by S_H^*, the convex hull of S_H, such that for any $z \in \mathbb{C} \setminus S_H^*$,

$$f_H(z) = z + \int \frac{-1}{z - x} d\mu_H(x). \tag{C.1}$$

In particular, if H is bounded by a crosscut, then μ_H is absolutely continuous w.r.t. the Lebesgue measure, and $d\mu_H/dx = \frac{1}{\pi}\text{Im}\, f_H(x)$, where the value of f_H on S_H^* is the continuation of f_H from $\mathbb{H}$. If H is approximated by a sequence of $\mathbb{H}$-hulls (H_n), then μ_H is the weak limit of (μ_{H_n}). We may choose each H_n to be bounded by a crosscut, whose height is not bigger than $h + 1/n$, where h is the height of H. Then each μ_{H_n} has a density function, whose L^∞ norm is not bigger than $(h + 1/n)/\pi$. Thus, μ_H also has a density function, whose L^∞ norm is not bigger than h/π. We use ρ_H to denote the density function of μ_H. Since $f_H : \mathbb{C} \setminus S_H^* \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{C} \setminus \widehat{H}^*$ and $f_H'(\infty) = 1$, we see that $\text{rad}(\widehat{H}^*) = \text{rad}(S_H^*) = |S_H^*|/4$. Thus, $\text{diam}(\widehat{H}^*) \leq 4\,\text{rad}(\widehat{H}^*) = |S_H^*|$. On the other hand, the diameter of $\widehat{H}^*$ is at least twice the height of H. So $\|\rho_H\|_\infty \leq \frac{|S_H^*|}{2\pi}$.

By approximating any $\mathbb{H}$-hull H using a sequence of $\mathbb{H}$-hulls (H_n), each of which is the union of finitely many mutually disjoint $\mathbb{H}$-hulls bounded by crosscuts in $\mathbb{H}$, we see that μ_H is in fact supported by S_H. By continuation, (C.1) holds for any $z \in \mathbb{C} \setminus S_H$. Furthermore, the support of μ_H is exactly S_H because from (C.1) f_H extends analytically to the complement of the support of μ_H, while from Lemma 2.6f_H can not be extended analytically beyond $\mathbb{C} \setminus S_H$. So we obtain the following lemma.

Lemma C.1. *For any $\mathbb{H}$-hull H, μ_H has a density function ρ_H, whose support is S_H, and whose L^∞ norm is no more than $\frac{|S_H^*|}{2\pi}$. Moreover, (C.1) holds for any $z \in \mathbb{C} \setminus S_H$.*

The following lemma extends Lemma 5.4 in [17], and we now give a proof.

Lemma C.2. *For any compact $F \subset \mathbb{R}$, $\mathcal{H}_F := \{H : S_H \subset F\}$ is compact, and $H_n \to H$ in $\mathcal{H}_F$ implies that $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C} \setminus F$.*

Proof. Suppose (H_n) is a sequence in $\mathcal{H}_F$. Let $|F^*|$ denote the length of the convex hull of F. Then for each n, ρ_{H_n} is supported by $S_{H_n} \subset F$, and the L^∞ norm of ρ_{H_n} is no more than $\frac{|S^*_{H_n}|}{2\pi} \leq \frac{|F^*|}{2\pi}$. Thus, (ρ_{H_n}) contains a subsequence $(\rho_{H_{n_k}})$, which converges in the weak-$*$ topology to a function ρ supported by F. From (C.1) we see that $f_{H_{n_k}}$ converges uniformly on each compact subset of $\mathbb{C}\setminus F$, and if f is the limit function, then $f(z)-z = \int_F \frac{-1}{z-x}\rho(x)dx$, $z \in \mathbb{C} \setminus F$. So $f(z) - z \to 0$ as $z \to \infty$. This means that f can not be constant. From Lemma A.2, f is a conformal map on $\mathbb{C} \setminus F$. Since $f(z) - z \to 0$ as $z \to \infty$, ∞ is a simple pole of f. Thus, $f(\mathbb{C} \setminus F)$ contains a neighborhood of ∞. Let $G = \mathbb{C} \setminus f(\mathbb{C} \setminus F)$. Then G is compact. Since every $f_{H_{n_k}}$ is $\mathbb{R}$-symmetric, so is f. Let $H = G \cap \mathbb{H}$. Then f maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus H$. This implies that H is an $\mathbb{H}$-hull and $f = f_H$ on $\mathbb{H}$ because $f(z) - z \to 0$ as $z \to \infty$. Since f extends $f_H|_{\mathbb{H}}$, from Lemma 2.6, we see that $S_H \subset F$ and $f = f_H$ in $\mathbb{C} \setminus F$. Since $f_{H_{n_k}} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{H}$, we get $H_{n_k} \to H \in \mathcal{H}_F$. This shows that $\mathcal{H}_F$ is compact. The above argument also gives $f_{H_{n_k}} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C} \setminus F$. If $H_n \to H$, then any subsequence (H_{n_k}) of (H_n) contains a subsequence $(H_{n_{k_l}})$ such that $f_{H_{n_{k_l}}} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C}\setminus F$, which implies that $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C} \setminus F$. □

D Topology on $\mathbb{D}$-hulls

Define a metric $d_\mathcal{H}$ on the space of $\mathbb{D}$-hulls such that

$$d_\mathcal{H}(H_1, H_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{|z|\leq 1-1/n} \{|f_{H_1}(z) - f_{H_2}(z)|\}. \tag{D.1}$$

It is clear that $d_\mathcal{H}(H_n, H) \to 0$ iff $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{D}$. From Lemma A.2, this implies that $\mathbb{D}\setminus H_n \xrightarrow{\text{Cara}} \mathbb{D}\setminus H$. On the other hand, from Lemma D.1 below, one see that $\mathbb{D}\setminus H_n \xrightarrow{\text{Cara}} \mathbb{D}\setminus H$ also implies that $H_n \to H$. Since $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{D}$ implies that $f'_{H_n}(0) \to f'_H(0)$, we see that dcap is a continuous function. Moreover, the dot product is also continuous.

Lemma D.1. *For any $M < \infty$, $\{H : \operatorname{dcap}(H) \leq M\}$ is compact.*

Proof. Suppose (H_n) is a sequence of $\mathbb{D}$-hulls with $\operatorname{dcap}(H_n) \leq M$ for each n. Then $f'_{H_n}(0) = e^{-\operatorname{dcap}(H_n)} \geq e^{-M}$. Since (f_{H_n}) is uniformly bounded in $\mathbb{D}$, it contains a subsequence $(f_{H_{n_k}})$, which converges locally uniformly in $\mathbb{D}$. Let f be the limit. Then $f'(0) = \lim_{k\to\infty} f'_{H_{n_k}}(0) \geq e^{-M}$. Thus, f is not constant. From Lemma A.2, f is conformal in $\mathbb{D}$. Since $f(0) = \lim_{k\to\infty} f_{H_{n_k}}(0)$ and $f'(0) > 0$, we see that $f = f_H|_\mathbb{D}$ for some $\mathbb{D}$-hull H. Since $f'(0) \geq e^{-M}$, $\operatorname{dcap}(H) \leq M$. From $f_{H_{n_k}} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{D}$ we get $H_{n_k} \to H$. □

Remark. We may compactify the space of $\mathbb{D}$-hulls by adding one element H_∞ with the associated function $f_{H_\infty} \equiv 0$ in $\mathbb{D}$, and defining the metric $d_\mathcal{H}$ in the extended space using (D.1).

Lemma D.2. *For any compact $F \subsetneq \mathbb{T}$, $\mathcal{H}_F := \{H : S_H \subset F\}$ is compact.*

Proof. Let $H \in \mathcal{H}_F$. From conformal invariance, the harmonic measure of $\mathbb{T} \setminus \widehat{H}$ in $\mathbb{D} \setminus H$ seen from 0 equals to the harmonic measure of $\mathbb{T} \setminus S_H$ in $\mathbb{D}$ seen from 0, which is bounded below by $|\mathbb{T}\setminus F|/|\mathbb{T}|>0$. This implies that the distance between 0 and H is bounded below by a positive constant r depending on F, which then implies that $\mathrm{dcap}(H)$ is bounded above by $-\ln(r)<\infty$. From Lemma D.1, we see that $\mathcal{H}_F$ is relatively compact.

It remains to show that $\mathcal{H}_F$ is bounded. Let (H_n) be a sequence in $\mathcal{H}_F$, which converges to H. We need to show that $H\in\mathcal{H}_F$. Since $H_n\in\mathcal{H}_F$, each f_{H_n} is analytic in $\widehat{\mathbb{C}}\setminus F$. We have $f_{H_n}\stackrel{\text{l.u.}}{\longrightarrow} f_H$ in $\mathbb{D}$. From $\mathbb{T}$-symmetry, $f_{H_n}\stackrel{\text{l.u.}}{\longrightarrow} f_H$ in $\mathbb{D}^*$. Let $J=\{|z|=2\}\subset\mathbb{D}^*$. Then $f_{H_{n_k}}\to f_H$ uniformly on J. Sine f_{H_n} maps $\{|z|<2\}\setminus F$ into the Jordan domain bounded by $f_{H_n}(J)$, we see that the family (f_{H_n}) is uniformly bounded in $\{|z|<2\}\setminus F$. So it contains a subsequence $(f_{H_{n_k}})$, which converges locally uniformly in $\{|z|<2\}\setminus F$. The limit function is analytic in $\{|z|<2\}\setminus F$ and agrees with f_H on $\mathbb{D}$, which implies that f_H extends analytically across $\mathbb{T}\setminus F$. So $S_H\subset F$, i.e., $H\in\mathcal{H}_F$. □

There is an integral formula for $\mathbb{D}$-hulls which is similar to (C.1). For any $\mathbb{D}$-hull H, there is a positive measure μ_H with support S_H such that

$$f(z)=z\cdot\exp\Big(\int_{\mathbb{T}}-\frac{x+z}{x-z}d\mu_H(x)\Big),\quad z\in\mathbb{C}\setminus S_H, \tag{D.2}$$

and $H_n\to H$ iff $\mu_{H_n}\to\mu_H$ weakly. Moreover, μ_H is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{T}$, and the density function is bounded. From this integral formula, it is easy to get the following lemma.

Lemma D.3. *For any compact $F\subset\mathbb{T}$, $H_n\to H$ in $\mathcal{H}_F$ implies that $f_{H_n}\stackrel{\text{l.u.}}{\longrightarrow} f_H$ in $\mathbb{C}\setminus F$.*

References

[1] Lars V. Ahlfors. *Conformal invariants: topics in geometric function theory.* McGraw-Hill Book Co., New York, 1973.

[2] Kari Astala, Peter Jones, Antti Kupiainen, and Eero Saksman. Random Conformal Weldings. *Acta. Math.*, 207(2):203-254, 2011.

[3] Christopher Bishop, Conformal welding and Koebe's theorem, Annals of Mathematics, 166: 613–656, 2007

[4] Julien Dubédat. Commutation relations for SLE, *Communications on Pure and Applied Mathematics*, 60(12):1792-1847, 2007.

[5] Peter W. Jones and Stanislav K. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, 38(2):263-279, 2000.

[6] Gregory F. Lawler. *Conformally Invariant Processes in the Plane.* Am. Math. Soc., Providence, RI, 2005.

[7] Gregory F. Lawler, Oded Schramm and Wendelin Werner. Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.*, 187(2):237-273, 2001.

[8] Olli Lehto and Kalle Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.

[9] Jason Miller and Scott Sheffield, Imaginary geometry III, arXiv:1201.1498, 2012.

[10] Christian Pommerenke. On the Löwner differential equation. *Mich. Math. J.* 13:435-443, 1968.

[11] Christian Pommerenke. *Boundary behaviour of conformal maps.* Springer-Verlag, Berlin Heidelberg New York, 1991.

[12] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion.* Springer, Berlin, 1991.

[13] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. Math.*, 161(2):883-924, 2005.

[14] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221-288, 2000.

[15] Oded Schramm and David B. Wilson. SLE coordinate changes. *New York Journal of Mathematics*, 11:659–669, 2005.

[16] Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. In preprint, arXiv:1012.4797v1.

[17] Dapeng Zhan. The Scaling Limits of Planar LERW in Finitely Connected Domains. *Ann. Probab.*, 36(2):467-529, 2008.

[18] Dapeng Zhan. Reversibility of chordal SLE. *Ann. Probab.*, 36(4):1472-1494, 2008.

[19] Dapeng Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309-353, 2008.

[20] Dapeng Zhan. Duality of chordal SLE, II. *Ann. I. H. Poincare-Pr.*, 46(3):740-759, 2010.

[21] Dapeng Zhan. Continuous LERW started from interior points. *Stoch. Proc. Appl.*, 120:1267-1316, 2010.

[22] Dapeng Zhan. Reversibility of whole-plane SLE. In preprint, arXiv:1004.1865.

[23] Dapeng Zhan. Ergodicity of the tip of an SLE curve. In preprint, arXiv:1310.2573.