Boundary Arm Exponents for SLE

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Abstract

We derive boundary arm exponents for SLE. Combining with the convergence of critical lattice models to SLE, these exponents would give the alternating half-plane arm exponents for the corresponding lattice models.
Keywords: Schramm Loewner Evolution, boundary arm exponents.

1 Introduction

Schramm-Loewner evolution (SLE) was introduced by Oded Schramm [Sch00] as the candidates for the scaling limits of interfaces in 2D critical lattice models. It is a one-parameter family of random fractal curves in simply connected domains from one boundary point to another boundary point, which is indexed by a positive real κ. Since its introduction, it has been proved to be the limits of several lattice models: SLE_2 is the limit of Loop Erased Random Walk and SLE_8 is the limit of the Peano curve of Uniform Spanning Tree [LSW04], SLE_3 is the limit of the interface in critical Ising model and $\mathrm{SLE}_{16/3}$ is the limit of the interface in FK-Ising model [CDCH$^+$14], SLE_4 is the limit of the level line of discrete Gaussian Free Field [SS09] and SLE_6 is the limit of the interface in critical Percolation [Smi01].

In the study of lattice models, arm exponents play an important role. Take percolation for instance, Kesten has shown that [Kes87] in order to understand the behavior of percolation near its critical point, it is sufficient to study what happens at the critical point, and many results would follow from the existence and values of the arm exponents. To be more precise, consider critical percolation with fixed mesh equal to 1, and for $n \geq 2$, consider the the event $E_n(z,r,R)$ that there exist n disjoint crossings of the annulus $A_z(r,R) := \{w \in \mathbb{C} : r < |w-z| < R\}$, not all of the same color. People would like to understand the decaying of the probability of $E_n(z,r,R)$ as $R \to \infty$. It turns out that this probability decays like a power in R, and the exponent is called plane arm exponents. There are another related quantities, called half-plane arm exponents. In this case, consider critical percolation in the upper-half plane $\mathbb{H}$, and for $n \geq 1, x \in \mathbb{R}$, define $H_n(x,r,R)$ to be the event that there exist n disjoint crossings of the semi-annulus $A_x^+(r,R) := \{w \in \mathbb{H} : r < |w-x| < R\}$. After the identification between SLE_6 and the limit of critical percolation on triangular lattice [Smi01], one could derive these exponents via the corresponding arm exponents for SLE_6 [SW01]:

$$\mathbb{P}[E_n(z,r,R)] = R^{-\alpha_n + o(1)}, \quad \mathbb{P}[H_n(x,r,R)] = R^{-\alpha_n^+ + o(1)}, \quad \text{as } R \to \infty,$$

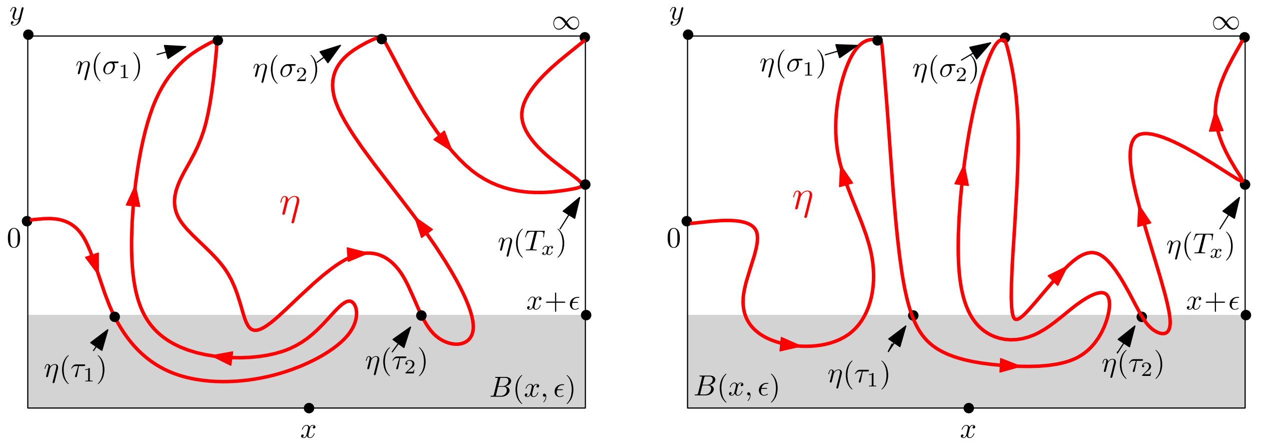
where

$$\alpha_n := (n^2-1)/12, \quad \alpha_n^+ := n(n+1)/6.$$

In this paper, we derive boundary arm exponents for SLE_κ. Combining with the identification between the limit of critical lattice model and SLE curves, these exponents for SLE would imply the arm exponents for the corresponding lattice models.

Fix $\kappa > 4$ and let η be an SLE_κ in $\mathbb{H}$ from 0 to ∞. Suppose that $y \leq 0 < \varepsilon \leq x$ and let T be the first time that η swallows the point x which is almost surely finite when $\kappa > 4$. We first define the crossing event H_{2n-1} (resp. $\hat{H}_{2n}$) that η crosses between the ball $B(x,\varepsilon)$ and the half-infinite line $(-\infty, y)$ at least $2n-1$ times (resp. at least $2n$ times) for $n \geq 1$. To be precise with the definition, we need to introduce a sequence of stopping times. Set $\tau_0 = \sigma_0 = 0$. Let τ_1 be the first time that η hits the ball $B(x,\varepsilon)$ and let σ_1 be the first time after τ_1 that η hits

$(-\infty,y)$. For $n \geq 1$, let τ_n be the first time after σ_{n-1} that η hits the connected component of $\partial B(x,\varepsilon)\setminus\eta[0,\sigma_{n-1}]$ containing $x+\varepsilon$ and let σ_n be the first time after τ_n that η hits $(-\infty,y)$. Define $H_{2n-1}(\varepsilon,x,y)$ to be the event that $\{\tau_n < T\}$. Define $\hat{H}_{2n}(\varepsilon,x,y)$ to be the event that $\{\sigma_n < T\}$. In the definition of $H_{2n-1}(\varepsilon,x,y)$ and $\hat{H}_{2n}(\varepsilon,x,y)$, we are particular interested in the case when x is large. Roughly speaking, the event $H_{2n-1}(\varepsilon,x,y)$ means that η makes at least $(2n-1)$ crossings between $B(x,\varepsilon)$ and $(-\infty,y)$. Imagine that η is the interface in the discrete model, then $H_{2n-1}(\varepsilon,x,y)$ interprets the event that there are $2n-1$ arms going from $B(x,\varepsilon)$ to far away place. The event $\hat{H}_{2n}(\varepsilon,x,y)$ means that η makes at least $2n$ crossings between $B(x,\varepsilon)$ and $(-\infty,y)$. Imagine that η is the interface in the discrete model, then $\hat{H}_{2n}(\varepsilon,x,y)$ interprets the event that there are $2n$ arms going from $B(x,\varepsilon)$ to far away place. See Figure 1.1(a).



(a) This figure indicates $\hat{H}_4$. The stopping times $\tau_1 < \sigma_1 < \tau_2 < \sigma_2 < T_x$ are indicated in the figure.

(b) This figure indicates H_4. The stopping times $\sigma_1 < \tau_1 < \sigma_2 < \tau_2 < T_x$ are indicated in the figure.

Fig. 1.1: The explanation of the definition of the crossing events. The gray part is the ball $B(x,\varepsilon)$.

Next, we define the crossing event H_{2n} (resp. $\hat{H}_{2n+1}$) that η crosses between the half-infinite line $(-\infty,y)$ and the ball $B(x,\varepsilon)$ at least $2n$ times (resp. at least $2n+1$ times) for $n \geq 0$. Set $\tau_0 = \sigma_0 = 0$. Let σ_1 be the first time that η hits $(-\infty,y)$ and τ_1 be the first time after σ_1 that η hits the connected component of $\partial B(x,\varepsilon)\setminus\eta[0,\sigma_1]$ containing $x+\varepsilon$. For $n \geq 1$, let σ_n be the first time after τ_{n-1} that η hits $(-\infty,y)$ and τ_n be the first time after σ_n that η hits the connected component of $\partial B(x,\varepsilon)\setminus\eta[0,\sigma_n]$ containing $x+\varepsilon$. Define $H_{2n}(\varepsilon,x,y)$ to be the event that $\{\tau_n < T\}$. Define $\hat{H}_{2n+1}(\varepsilon,x,y)$ to be the event that $\{\sigma_{n+1} < T\}$. In the definition of $H_{2n}(\varepsilon,x,y)$ and $\hat{H}_{2n+1}(\varepsilon,x,y)$. we are interested in the case when x is of the same size as ε and y is large. Roughly speaking, the event $H_{2n}(\varepsilon,x,y)$ means that η makes at least $2n$ crossings between $(-\infty,y)$ and $B(x,\varepsilon)$. Imagine that η is the interface in the discrete model, then $H_{2n}(\varepsilon,x,y)$ interprets the event that there are $2n$ arms going from $B(x,\varepsilon)$ to far away place. The event $\hat{H}_{2n+1}(\varepsilon,x,y)$ means that η makes at least $2n+1$ crossings between $(-\infty,y)$ and $B(x,\varepsilon)$. Imagine that η is the interface in the discrete model, then $\hat{H}_{2n+1}(\varepsilon,x,y)$ interprets the event that there are $2n+1$ arms going from $B(x,\varepsilon)$ to far away place. See Figure 1.1(b).

Note that in the definition of H_{2n-1} and $\hat{H}_{2n}$, we start from τ_1 and

$$H_{2n-1}(\varepsilon,x,y) = \{\tau_1 < \sigma_1 < \tau_2 < \cdots < \tau_n < T\}, \quad \hat{H}_{2n}(\varepsilon,x,y) = \{\tau_1 < \sigma_1 < \tau_2 < \cdots < \tau_n < \sigma_n < T\}.$$

In the definition of H_{2n} and $\hat{H}_{2n+1}$, we start from σ_1 and

$$H_{2n}(\varepsilon,x,y) = \{\sigma_1 < \tau_1 < \sigma_2 < \cdots < \tau_n < T\}, \quad \hat{H}_{2n+1}(\varepsilon,x,y) = \{\sigma_1 < \tau_1 < \sigma_2 < \cdots < \tau_n < \sigma_{n+1} < T\}.$$

The two sequences of stopping times are defined in different ways. Readers may wander why we do not define the events using the same sequence of stopping times. We realize that the definition using the same sequence of stopping times causes ambiguity. Therefore, we decide to define these events in the above way. The advantages of the current definition will become clear in the proofs.

We define the arm exponents as follows. Set $\alpha_0^+ = 0$. For $n \geq 1$ and $\kappa \in (0,8)$, define

$$\alpha_{2n-1}^+ = n(4n+4-\kappa)/\kappa, \quad \alpha_{2n}^+ = n(4n+8-\kappa)/\kappa. \tag{1.1}$$

For $n \geq 1$ and $\kappa \geq 8$, define

$$\alpha_{2n-1}^+ = (n-1)(4n+\kappa-8)/\kappa, \quad \alpha_{2n}^+ = n(4n+\kappa-8)/\kappa. \tag{1.2}$$

Theorem 1.1. *Fix $\kappa > 4$. The crossing events $H_{2n-1}(\varepsilon,x,y)$ and $H_{2n}(\varepsilon,x,y)$ are defined as above. Then, for any $y \leq 0 < \varepsilon \leq x$ and $n \geq 1$, we have*

$$\mathbb{P}[H_{2n-1}(\varepsilon,x,y)] \asymp \left(\frac{x}{x-y}\right)^{\alpha_{2n-2}^+} \left(\frac{\varepsilon}{x}\right)^{\alpha_{2n-1}^+}, \tag{1.3}$$

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \asymp \left(\frac{x}{x-y}\right)^{\alpha_{2n}^+} \left(\frac{\varepsilon}{x}\right)^{\alpha_{2n-1}^+}, \tag{1.4}$$

where the constants in $\asymp$ depend only on κ and n. In particular, fix some $\delta > 0$, we have

$$\mathbb{P}[H_{2n-1}(\varepsilon,x,y)] \asymp \varepsilon^{\alpha_{2n-1}^+}, \quad \textit{provided } \delta \leq x \leq 1/\delta, -1/\delta \leq y \leq 0,$$

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \asymp \varepsilon^{\alpha_{2n}^+}, \quad \textit{provided } \varepsilon \leq x \leq \varepsilon/\delta, -1/\delta \leq y \leq -\delta,$$

where the constants in $\asymp$ depend only on κ, n and δ.

By a similar proof, we could obtain a similar result as Theorem 1.1 for $\mathrm{SLE}_\kappa(\rho)$ curve in the case that x coincides with the force point. The exponents and a complete proof can be found in [Wu16b, Section 3], where the conditions are loosen such that the force point may not be equal to x. One may also study the arm exponents for $\kappa \in (0,4]$. Whereas, when $\kappa \leq 4$, the SLE curve does not touch the boundary, thus the above definition of the crossing events is not proper for $\kappa \leq 4$. In Section 4, we have Theorem 4.4 for the crossing events between a small circle and a half-infinite strip, where the arm exponents are defined in the same way as in (1.1). The proof of Theorem 4.4 also works for $\mathrm{SLE}_\kappa(\rho)$ when x coincides with the force point.

Theorem 1.2. *Fix $\kappa \in (4,8)$. Set $\hat{\alpha}_0^+ = 0$. The crossing events $\hat{H}_{2n}(\varepsilon,x,y)$ and $\hat{H}_{2n+1}(\varepsilon,x,y)$ are defined as above. For $n \geq 1$, define*

$$\hat{\alpha}_{2n-1}^+ = n(4n+\kappa-8)/\kappa, \quad \hat{\alpha}_{2n}^+ = n(4n+\kappa-4)/\kappa. \tag{1.5}$$

Then, for $y \leq 0 < \varepsilon \leq x$ and $n \geq 1$, we have

$$\mathbb{P}\left[\hat{H}_{2n-1}(\varepsilon,x,y)\right] \asymp \left(\frac{x}{x-y}\right)^{\hat{\alpha}_{2n-1}^+} \left(\frac{\varepsilon}{x}\right)^{\hat{\alpha}_{2n-2}^+}, \tag{1.6}$$

$$\mathbb{P}\left[\hat{H}_{2n}(\varepsilon,x,y)\right] \asymp \left(\frac{x}{x-y}\right)^{\hat{\alpha}_{2n-1}^+} \left(\frac{\varepsilon}{x}\right)^{\hat{\alpha}_{2n}^+}, \tag{1.7}$$

where the constants in $\asymp$ depend only on κ and n. In particular, fix some $\delta > 0$, we have

$$\mathbb{P}\left[\hat{H}_{2n-1}(\varepsilon,x,y)\right] \asymp \varepsilon^{\hat{\alpha}_{2n-1}^+}, \quad \textit{provided } \varepsilon \leq x \leq \varepsilon/\delta, -1/\delta \leq y \leq -\delta,$$

$$\mathbb{P}\left[\hat{H}_{2n}(\varepsilon,x,y)\right] \asymp \varepsilon^{\hat{\alpha}_{2n}^+}, \quad \textit{provided } \delta \leq x \leq 1/\delta, -1/\delta \leq y \leq 0,$$

where the constants in $\asymp$ depend only on κ, n and δ.

It is worthwhile to spend some more words on the relation between α_n^+ and $\hat{\alpha}_n^+$. In fact, we can also define the crossing events $\hat{H}_n(\varepsilon, x, y)$ for $\kappa \in [0,4]$ and $\kappa \geq 8$. When $\kappa \leq 4$, the SLE curve does not touch the boundary, thus the exponent $\hat{\alpha}_n^+$ coincides with α_{n-1}^+. When $\kappa \geq 8$, the SLE curve is space-filling, thus the exponent $\hat{\alpha}_n^+$ coincides with α_{n+1}^+. Whereas, when $\kappa \in (4,8)$, the exponent $\hat{\alpha}_n^+$ is distinct from α_n^+ in general. In terms of discrete model, both α_n^+ and $\hat{\alpha}_n^+$ interpret the boundary n-arm exponents, but their boundary conditions are different.

It is explained in [SW01] that combining the following three facts would imply the arm exponents for the discrete model: (1) Identification between SLE_κ and the limit of the interface in critical lattice model; (2) The arm exponents of SLE_κ; (3) Crossing probabilities enjoy (approximate) multiplicativity property. For critical Ising and FK-Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions, the convergence to SLE_3 and $\text{SLE}_{16/3}$ respectively is derived in [CS12, CDCH$^+$14], and the multiplicativity is derived in [CDCH13]. Therefore, we could derive the arm exponents for these two models. See more details in [Wu16b, Wu16a]. Moreover, the formula of α_{2n-1}^+ in (1.1) was predicted by KPZ in [Dup03, Equations (11.42), (11.44)].

Relation to previous results. The formula of α_n^+ and α_n for $\kappa = 6$ was obtained in [LSW01, SW01]. The exponent α_1^+ is related to the Hausdorff dimension of the intersection of SLE_κ with the real line which is $1-\alpha_1^+$ when $\kappa > 4$. This dimension was obtained in [AS08]. The most important ingredients in proving Theorem 1.1 is the Laplace transform of the derivatives of the conformal map in SLE evolution, which was obtained in [Law14].

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2 Preliminaries

Notations. We denote by $f \lesssim g$ if f/g is bounded from above by universal finite constants, by $f \gtrsim g$ if f/g is bounded from below by universal positive constants, and by $f \asymp g$ if $f \lesssim g$ and $f \gtrsim g$.
For $z \in \mathbb{C}, y \in \mathbb{R}, r > 0$.

$$B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}, \quad \mathbb{U} = B(0,1);$$

For two subsets $A, B \subset \mathbb{C}$,

$$\text{dist}(A,B) = \inf\{|x-y| : x \in A, y \in B\}.$$

Let Ω be an open set and let V_1, V_2 be two sets such that $V_1 \cap \overline{\Omega} \neq \emptyset$ and $V_2 \cap \overline{\Omega} \neq \emptyset$. We denote the extremal distance between V_1 and V_2 in Ω by $d_\Omega(V_1, V_2)$, see [Ahl10, Section 4] for the definition.

2.1 $\mathbb{H}$-hull and Loewner chain

We call a compact subset K of $\overline{\mathbb{H}}$ an $\mathbb{H}$-hull if $\mathbb{H} \setminus K$ is simple connected. Riemann's Mapping Theorem asserts that there exists a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ such that

$$\lim_{|z|\to\infty} |g_K(z) - z| = 0.$$

We call such g_K the conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ normalized at ∞. The limit $\text{hcap}(K) := \lim_{|z|\to\infty} z(g_K(z) - z)$ exists and is called the half-plane capacity of K.

Lemma 2.1. *Fix $x > 0$ and $\varepsilon > 0$. Let K be an $\mathbb{H}$-hull and let g_K be the conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ normalized at ∞. Assume that*

$$x > \max(K \cap \mathbb{R}).$$

Denote by γ the connected component of $\mathbb{H} \cap (\partial B(x, \varepsilon) \setminus K)$ whose closure contains $x + \varepsilon$. Then $g_K(\gamma)$ is contained in the ball with center $g_K(x+\varepsilon)$ and radius $3(g_K(x+3\varepsilon) - g_K(x+\varepsilon))$. Hence $g_K(\gamma)$ is also contained in the ball with center $g_K(x+3\varepsilon)$ and radius $8\varepsilon g_K'(x+3\varepsilon)$.

Proof. Define $r^* = \sup\{|z - g_K(x+\varepsilon)| : z \in g_K(\gamma)\}$. It is sufficient to show

$$r^* \leq 3(g_K(x+3\varepsilon) - g_K(x+\varepsilon)). \tag{2.1}$$

We will prove (2.1) by estimates on the extremal distance:

$$d_{\mathbb{H}}(g_K(\gamma), [g_K(x+3\varepsilon), \infty)).$$

By the conformal invariance and the comparison principle [Ahl10, Section 4.3], we can obtain the following lower bound.

$$\begin{aligned} d_{\mathbb{H}}(g_K(\gamma), [g_K(x+3\varepsilon), \infty)) &= d_{\mathbb{H}\setminus K}(\gamma, [x+3\varepsilon, \infty)) \\ &\geq d_{\mathbb{H}\setminus B(x,\varepsilon)}(B(x,\varepsilon), [x+3\varepsilon, \infty)) \\ &= d_{\mathbb{H}\setminus \mathbb{U}}(\mathbb{U}, [3,\infty)) = d_{\mathbb{H}}([-1,0],[1/3,\infty)). \end{aligned}$$

On the other hand, we will give an upper bound. Recall a fact for extremal distance: for $x < y$ and $r > 0$, the extremal distance in $\mathbb{H}$ between $[y,\infty)$ and a connected set $S \subset \overline{\mathbb{H}}$ with $x \in \overline{S} \subset \overline{B(x,r)}$ is maximized when $S = [x-r, x]$, see [Ahl06, Chapter I-E, Chapter III-A]. Since $g_K(\gamma)$ is connected and $g_K(x+\varepsilon) \in \mathbb{R} \cap \overline{g_K(\gamma)}$, by the above fact, we have the following upper bound.

$$\begin{aligned} d_{\mathbb{H}}(g_K(\gamma), [g_K(x+3\varepsilon), \infty)) &\leq d_{\mathbb{H}}([g_K(x+\varepsilon) - r^*, g_K(x+\varepsilon)], [g_K(x+3\varepsilon), \infty)) \\ &= d_{\mathbb{H}}\left([-r^*, 0], [g_K(x+3\varepsilon) - g_K(x+\varepsilon), \infty)\right). \end{aligned}$$

Combining the lower bound with the upper bound, we have

$$d_{\mathbb{H}}([-1,0],[1/3,\infty)) \leq d_{\mathbb{H}}\left([-r^*, 0], [g_K(x+3\varepsilon) - g_K(x+\varepsilon), \infty)\right).$$

This implies (2.1) and completes the proof. □

Lemma 2.2. *Fix $z \in \overline{\mathbb{H}}$ and $\varepsilon > 0$. Let K be an $\mathbb{H}$-hull and let g_K be the conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ normalized at ∞. Assume that*

$$\mathrm{dist}(K, z) \geq 16\varepsilon.$$

Then $g_K(B(z,\varepsilon))$ is contained in the ball with center $g_K(z)$ and radius $4\varepsilon |g_K'(z)|$.

Proof. By Koebe 1/4 theorem, we know that

$$\mathrm{dist}(g_K(K), g_K(z)) \geq d := 4\varepsilon |g_K'(z)|.$$

Let $h = g_K^{-1}$ restricted to $B(g_K(z), d)$. Applying Koebe 1/4 theorem to h, we know that

$$\mathrm{dist}(z, \partial h(B(g_K(z), d))) \geq d |h'(g_K(z))|/4 = \varepsilon.$$

Therefore $h(B(g_K(z),d))$ contains the ball $B(z,\varepsilon)$, and this implies that $B(g_K(z),d)$ contains the ball $g_K(B(z,\varepsilon))$ as desired. □

Loewner chain is a collection of $\mathbb{H}$-hulls $(K_t, t \geq 0)$ associated with the family of conformal maps $(g_t, t \geq 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \tag{2.2}$$

where $(W_t, t \geq 0)$ is a one-dimensional continuous function which we call the driving function. Let T_z be the swallowing time of z defined as $\sup\{t \geq 0 : \min_{s\in[0,t]} |g_s(z) - W_s| > 0\}$. Let $K_t := \overline{\{z \in \mathbb{H} : T_z \leq t\}}$. Then g_t is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ normalized at ∞.

Here we spend some words about the evolution of a point $y \in \mathbb{R}$ under g_t. We assume $y \le 0$, the case of $y \ge 0$ can be analyzed similarly. There are two possibilities: if y is not swallowed by K_t, then we define $Y_t = g_t(y)$; if y is swallowed by K_t, then we define Y_t to the be image of the leftmost of point of $K_t \cap \mathbb{R}$ under g_t. The process Y_t is decreasing in t, and it is uniquely characterized by the following equation:

$$Y_t = y + \int_0^t \frac{2ds}{Y_s - W_s}, \quad Y_t \le W_t, \quad \forall t \ge 0.$$

In this paper, we may write $g_t(y)$ for the process Y_t. Consider two points $x \ge 0 \ge y$ in $\mathbb{R}$. By the above fact, we have

$$g_t(x) = x + \int_0^t \frac{2ds}{g_s(x) - W_s}, \quad g_t(y) = y + \int_0^t \frac{2ds}{g_s(y) - W_s}, \quad g_t(y) \le W_t \le g_t(x).$$

Therefore, the quantity $g_t(x) - g_t(y)$ is increasing in t. We will use this fact in the paper without reference.

2.2 SLE processes

An SLE_κ is the random Loewner chain $(K_t, t \ge 0)$ driven by $W_t = \sqrt{\kappa} B_t$ where $(B_t, t \ge 0)$ is a standard one-dimensional Brownian motion. In [RS05], the authors prove that $(K_t, t \ge 0)$ is almost surely generated by a continuous transient curve, i.e. there almost surely exists a continuous curve η such that for each $t \ge 0$, H_t is the unbounded connected component of $\mathbb{H} \setminus \eta[0,t]$ and that $\lim_{t\to\infty} |\eta(t)| = \infty$.

We can define an $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ process with two force points $(x^L; x^R)$ where $x^L \le 0 \le x^R$. It is the Loewner chain driven by W_t which is the solution to the following systems of SDEs:

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho^L dt}{W_t - V_t^L} + \frac{\rho^R dt}{W_t - V_t^R}, \quad W_0 = 0;$$

$$dV_t^L = \frac{2dt}{V_t^L - W_t}, \quad V_0^L = x^L; \quad dV_t^R = \frac{2dt}{V_t^R - W_t}, \quad V_0^R = x^R.$$

The solution exists up to the first time that W hits V^L or V^R. When $\rho^L > -2$ and $\rho^R > -2$, the solution exists for all times $t \ge 0$, and the corresponding Loewner chain is almost surely generated by a continuous curve which is almost surely transient ([MS12, Section 2]). There are two special values of ρ: $\kappa/2 - 2$ and $\kappa/2 - 4$. When $\rho^R \ge \kappa/2 - 2$, then the curve will never hits $[x^R, \infty)$. When $\rho^R \le \kappa/2 - 4$, then the curve will almost surely accumulates at x^R at finite time. See [Dub09, Lemma 15].

From Girsanov Theorem, it follows that the law of an $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ process can be constructed by reweighting the law of an ordinary SLE_κ.

Lemma 2.3. *Suppose $x^L < 0 < x^R$, define*

$$\begin{aligned} M_t = & g_t'(x^L)^{\rho^L(\rho^L+4-\kappa)/(4\kappa)} (W_t - g_t(x^L))^{\rho^L/\kappa} \times g_t'(x^R)^{\rho^R(\rho^R+4-\kappa)/(4\kappa)} (g_t(x^R) - W_t)^{\rho^R/\kappa} \\ & \times (g_t(x^R) - g_t(x^L))^{\rho^L \rho^R/(2\kappa)}. \end{aligned}$$

Then M is a local martingale for SLE_κ and the law of SLE_κ weighted by M (up to the first time that W hits one of the force points) is equal to the law of $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ with force points $(x^L; x^R)$.

Proof. [SW05, Theorem 6]. □

Lemma 2.4. *Fix $\kappa > 0$ and $\nu \le \kappa/2 - 4$. Suppose $y \le 0 < x$. Let η be an $\mathrm{SLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with force point x. Since $\nu \le \kappa/2 - 4$, the curve η accumulates at the point x at almost surely finite time which is denoted by T. Then we have, for $\lambda \le 0$,*

$$\mathbb{E}\left[(g_T(x) - g_T(y))^\lambda \right] \asymp (x - y)^\lambda,$$

where the constants in $\asymp$ depend only κ, ν and λ.

Proof. Since the quantity $g_t(x)-g_t(y)$ is increasing in t, we have $g_T(x)-g_T(y)\ge(x-y)$. This implies the upper bound. We only need to show the lower bound. To this end, we will compare η with $\mathrm{SLE}_\kappa(\nu)$ with force point $x-y$ and show that the law of $(g_T(x)-g_T(y))/(x-y)$ is stochastically dominated by a random variable whose law depends only κ,ν. By the scaling invariance of $\mathrm{SLE}_\kappa(\nu)$, we may assume $x-y=1$.

Let $\tilde\eta$ be an $\mathrm{SLE}_\kappa(\nu)$ with force point 1, and define $\tilde W,\tilde g_t,\tilde T$ accordingly. Define $\tilde V_t$ to be the image of the leftmost point of $\tilde\eta[0,t]\cap\mathbb{R}$ under $\tilde g_t$. Set

$$\tilde J_t=\frac{\tilde W_t-\tilde V_t}{\tilde g_t(1)-\tilde V_t}.$$

Define the stopping time $\tau=\inf\{t:\tilde J_t=-y\}$. Note that $\tilde J_0=0,\tilde J_{\tilde T}=1$ and $\tilde J$ is continuous, we have that $0\le\tau\le\tilde T$. Given $\tilde\eta[0,\tau]$, the process $(\tilde\eta(t+\tau),0\le t\le\tilde T-\tau)$, under the map

$$f(z)=\frac{\tilde g_\tau(z)-\tilde W_\tau}{\tilde g_\tau(1)-\tilde V_\tau},$$

has the same law as $(\eta(t),0\le t\le T)$ after a linear time-change. Therefore, given $\tilde\eta[0,\tau]$, we have

$$\frac{\tilde g_{\tilde T}(1)-\tilde V_{\tilde T}}{\tilde g_\tau(1)-\tilde V_\tau}\overset{d}{=}g_T(x)-g_T(y).$$

Since $\tilde g_\tau(1)-\tilde V_\tau\ge1$, we may conclude that the quantity $(g_T(x)-g_T(y))$ is stochastically dominated from above by $(\tilde g_{\tilde T}(1)-\tilde V_{\tilde T})$. To complete the proof, it is sufficient to show

$$\tilde{\mathbb{E}}\left[\left(\tilde g_{\tilde T}(1)-\tilde V_{\tilde T}\right)^\lambda\right]\gtrsim1, \tag{2.3}$$

where $\tilde{\mathbb{P}}$ denotes the law of $\mathrm{SLE}_\kappa(\nu)$ with force point 1. Define the event

$$\tilde F=\{\tilde g_{\tilde T}(1)-\tilde V_{\tilde T}\le4\}.$$

It is clear that $\tilde{\mathbb{P}}[\tilde F]$ is strictly positive and depends only on κ and ν, thus

$$\tilde{\mathbb{E}}\left[\left(\tilde g_{\tilde T}(1)-\tilde V_{\tilde T}\right)^\lambda\right]\ge4^\lambda\tilde{\mathbb{P}}[\tilde F].$$

This implies (2.3) and completes the proof. □

Lemma 2.5. *Fix $\kappa>4$ and $\nu\ge\kappa/2-2$. Suppose $y<0<x$, let η be an $\mathrm{SLE}_\kappa(\nu)$ with force point x. For $c>0$ small, define*

$$\sigma=\inf\{t:\eta(t)\in(-\infty,y]\},\quad F=\{\mathrm{dist}(\eta[0,\sigma],x)\ge cx\}.$$

Then there exists a constant $c\in(0,1)$ depending only on κ and ν such that, for $\lambda\le0$,

$$\mathbb{E}\left[(g_\sigma(x)-g_\sigma(y))^\lambda 1_F\right]\asymp(x-y)^\lambda,$$

where the constants in $\asymp$ depend only on κ,ν and λ.

Proof. Since the quantity $g_t(x)-g_t(y)$ is increasing in t, we have $g_\sigma(x)-g_\sigma(y)\ge(x-y)$. This implies the upper bound. We only need to show the lower bound. We may assume that $x-y=1$. We first argue that

$$\mathbb{E}\left[(g_\sigma(x)-g_\sigma(y))^\lambda\right]\asymp(x-y)^\lambda. \tag{2.4}$$

The proof of (2.4) is similar to the proof of Lemma 2.4. Let $\tilde\eta$ be an $\mathrm{SLE}_\kappa(\nu)$ with force point 0^+. Define $\tilde W,\tilde g$ accordingly and let $\tilde\sigma$ be the first time that $\tilde\eta$ hits $(-\infty,-1)$. Let $\tilde V_t$ be the evolution of the force point. Define

$$\tilde J_t=\frac{\tilde V_t-\tilde W_t}{\tilde V_t-\tilde g_t(-1)},\quad \tau:=\inf\{t:\tilde J_t=x\}.$$

Given $\tilde{\eta}[0,\tau]$, the process $(\tilde{\eta}(t+\tau), 0 \le t \le \tilde{\sigma} - \tilde{\tau})$ under the map

$$f(z) = \frac{\tilde{g}_\tau(z) - \tilde{W}_\tau}{\tilde{V}_\tau - \tilde{g}_\tau(-1)}$$

has the same law as $(\eta(t), 0 \le t \le \sigma)$ after a linear time change. In particular,

$$\frac{\tilde{V}_{\tilde{\sigma}} - \tilde{g}_{\tilde{\sigma}}(-1)}{\tilde{V}_\tau - \tilde{g}_\tau(-1)} \overset{d}{=} g_\sigma(x) - g_\sigma(y).$$

Since $\tilde{V}_\tau - \tilde{g}_\tau(-1) \ge 1$, we know that $(g_\sigma(x) - g_\sigma(y))$ is stochastically dominated from above by $(\tilde{V}_{\tilde{\sigma}} - \tilde{g}_{\tilde{\sigma}}(-1))$, thus

$$\mathbb{E}\left[(g_\sigma(x) - g_\sigma(y))^\lambda\right] \ge \tilde{\mathbb{E}}\left[\left(\tilde{V}_{\tilde{\sigma}} - \tilde{g}_{\tilde{\sigma}}(-1)\right)^\lambda\right] \asymp 1.$$

This implies (2.4). Next, we prove the conclusion. By the scaling invariance of $\mathrm{SLE}_\kappa(\nu)$ process we know that the probability $\mathbb{P}[\mathrm{dist}(\eta, x) < cx]$ only depends on c. We denote this probability by $p(c)$. Since $\nu \ge \kappa/2 - 2$, we know that $p(c) \to 0$ as $c \to 0$. Therefore, by (2.4), we have

$$1 \asymp \mathbb{E}\left[(g_\sigma(x) - g_\sigma(y))^\lambda\right] \le \mathbb{E}\left[(g_\sigma(x) - g_\sigma(y))^\lambda 1_F\right] + p(c).$$

This implies the conclusion. □

3 Boundary Arm Exponents for $\kappa > 4$

3.1 Estimate on the derivative

Proposition 3.1. *Fix $\kappa > 0$ and let η be an SLE_κ in $\mathbb{H}$ from 0 to ∞. Let O_t be the image of the rightmost point of $K_t \cap \mathbb{R}$ under g_t. Set $\Upsilon_t = (g_1(1) - O_t)/g_t'(1)$. For $\varepsilon \in (0,1)$, define*

$$\hat{\tau}_\varepsilon = \inf\{t : \Upsilon_t = \varepsilon\}, \quad T_0 = \inf\{t : \eta(t) \in [1,\infty)\}.$$

For $\lambda \ge 0$, define

$$u_1(\lambda) = \frac{1}{\kappa}(4 - \kappa/2) + \frac{1}{\kappa}\sqrt{4\kappa\lambda + (4-\kappa/2)^2}.$$

For $b \in \mathbb{R}$, assume that

$$\kappa\lambda - \kappa u_1(\lambda) + 8 - 2\kappa < \kappa b \le \kappa\lambda + \kappa u_1(\lambda). \tag{3.1}$$

Then we have

$$\mathbb{E}\left[(g_{\hat{\tau}_\varepsilon}(1) - W_{\hat{\tau}_\varepsilon})^{\lambda - b} g'_{\hat{\tau}_\varepsilon}(1)^b 1_{\{\hat{\tau}_\varepsilon < T_0\}}\right] \asymp \varepsilon^{u_1(\lambda) + \lambda - b}, \tag{3.2}$$

where the constants in $\asymp$ depend only on κ and λ, b.

Attention that, in Proposition 3.1, we use the stopping time $\hat{\tau}_\varepsilon$ instead of τ_ε which is defined to be the first time that η hits $B(1,\varepsilon)$. Due to Koebe 1/4 thoerem, these two times are very close:

$$\tau_{4\varepsilon} \le \hat{\tau}_\varepsilon \le \tau_{\varepsilon/4}.$$

Due to technical reason, we only prove the conclusion in Proposition 3.1 for the time $\hat{\tau}_\varepsilon$, but this is sufficient for our purpose later in the paper.

Lemma 3.2. *Fix $\kappa > 0$ and $\nu \le \kappa/2 - 4$. Let η be an $\mathrm{SLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to ∞ with force point 1. Denote by W the driving function, V the evolution of the force point. Let O_t be the image of the rightmost point of $K_t \cap \mathbb{R}$ under g_t. Set $\Upsilon_t = (g_t(1) - O_t)/g_t'(1)$ and $\sigma(s) = \inf\{t : \Upsilon_t = e^{-2s}\}$. Set $J_t = (V_t - O_t)/(V_t - W_t)$. Let $T_0 = \inf\{t : \eta(t) \in [1,\infty)\}$. We have, for $\beta > 0$,*

$$\mathbb{E}\left[J_{\sigma(s)}^{-\beta} 1_{\{\sigma(s) < T_0\}}\right] \asymp 1, \quad \text{when } 8 + 2\nu + \kappa\beta < 2\kappa, \tag{3.3}$$

where the constants in $\asymp$ depend only on κ, ν, β.

Proof. Since $0 \le J_t \le 1$, we only need to show the upper bounds. Define $X_t = V_t - W_t$. We know that

$$dW_t = \sqrt{\kappa} dB_t + \frac{\nu dt}{W_t - V_t}, \quad dV_t = \frac{2dt}{V_t - W_t},$$

where B is a standard 1-dimensional Brownian motion. By Itô's formula, we have that

$$dJ_t = \frac{J_t}{X_t^2}\left(\kappa - \nu - 2 - \frac{2}{1-J_t}\right)dt + \frac{J_t}{X_t}\sqrt{\kappa} dB_t, \quad d\Upsilon_t = \Upsilon_t \frac{-2J_t dt}{X_t^2(1-J_t)}.$$

Recall that $\sigma(s) = \inf\{t : \Upsilon_t = e^{-2s}\}$, and denote by $\hat{X}, \hat{J}, \hat{\Upsilon}$ the processes indexed by $\sigma(s)$. Then we have that

$$d\sigma(s) = \hat{X}_s^2 \frac{1-\hat{J}_s}{\hat{J}_s} ds, \quad d\hat{J}_s = \left(\kappa - \nu - 4 - (\kappa - \nu - 2)\hat{J}_s\right) ds + \sqrt{\kappa \hat{J}_s (1-\hat{J}_s)} d\hat{B}_s,$$

where $\hat{B}$ is a standard 1-dimensional Brownian motion. By [Law14, Equations (56), (62)] and [Zha16, Appendix B], we know that $\hat{J}$ has an invariant density on $(0,1)$, which is proportional to $y^{1-(8+2\nu)/\kappa}(1-y)^{4/\kappa-1}$. Moreover, since $\hat{J}_0 = 1$, by a standard coupling argument, we may couple $(\hat{J}_s)$ with the stationary process $(\tilde{J}_s)$ that satisfies the same equation as $(\hat{J}_s)$, such that $\hat{J}_s \ge \tilde{J}_s$ for all $s \ge 0$. Then we get $\mathbb{E}[\hat{J}_s^{-\beta}] \le \mathbb{E}[\tilde{J}_s^{-\beta}]$, which is a finite constant if $8 + 2\nu + \kappa\beta < 2\kappa$. This gives the upper bound in (3.3) and completes the proof of (3.3). □

Proof of Proposition 3.1. Let O_t be the image of the rightmost point of $\eta[0,t] \cap \mathbb{R}$ under g_t. Define

$$\Upsilon_t = \frac{g_t(1) - O_t}{g_t'(1)}, \quad J_t = \frac{g_t(1) - O_t}{g_t(1) - W_t}.$$

Set

$$M_t = g_t'(1)^{\nu(\nu+4-\kappa)/(4\kappa)} (g_t(1) - W_t)^{\nu/\kappa}, \quad \text{where } \nu = -\kappa u_1(\lambda).$$

Then M is a local martingale for η, and from Lemma 2.3, the law of η weighted by M is the law of $\mathrm{SLE}_\kappa(\nu)$ with force point 1. Set $\beta = u_1(\lambda) + \lambda - b$. Then we have

$$M_t = (g_t(1) - W_t)^{\lambda - b} g_t'(1)^b \Upsilon_t^{-\beta} J_t^{\beta}.$$

At time $t = \hat{\tau}_\varepsilon < \infty$, we have $\Upsilon_t = \varepsilon$, thus

$$\mathbb{E}\left[(g_{\hat{\tau}_\varepsilon}(1) - W_{\hat{\tau}_\varepsilon})^{\lambda - b} g'_{\hat{\tau}_\varepsilon}(1)^b 1_{\{\hat{\tau}_\varepsilon < T_0\}}\right] \asymp \varepsilon^\beta \mathbb{E}^*\left[\left(J^*_{\hat{\tau}^*_\varepsilon}\right)^{-\beta} 1_{\{\hat{\tau}^*_\varepsilon < T_0^*\}}\right] \asymp \varepsilon^\beta,$$

where $\mathbb{P}^*$ is the law of $\mathrm{SLE}_\kappa(\nu)$ with force point x and $\eta^*, J^*, \hat{\tau}^*_\varepsilon, T_0^*$ are defined accordingly, and the last relation is due to (3.3). □

Remark 3.3. *Fix $\kappa > 0$ and let η be an* SLE_κ*. For $x > \varepsilon > 0$, let $u_1(\lambda)$ and b be as in Proposition 3.1. By the scaling invariance of* SLE*, we have*

$$\mathbb{E}\left[(g_{\hat{\tau}_\varepsilon}(x) - W_{\hat{\tau}_\varepsilon})^{\lambda - b} g'_{\hat{\tau}_\varepsilon}(x)^b 1_{\{\hat{\tau}_\varepsilon < T_0\}}\right] \asymp x^{-u_1(\lambda)} \varepsilon^{u_1(\lambda) + \lambda - b}, \tag{3.4}$$

where the constants in $\asymp$ depend only on κ, and λ, b. Taking $\lambda = b = 0$, we have

$$\mathbb{P}[\tau_\varepsilon < \infty] \asymp \mathbb{P}[\hat{\tau}_\varepsilon < \infty] \asymp \left(\frac{\varepsilon}{x}\right)^{\alpha_1^+}, \quad \textit{where } \alpha_1^+ = u_1(0) = 0 \vee (8/\kappa - 1).$$

This implies that (1.3) holds for $n = 1$.

3.2 From $2n-1$ to $2n$

Lemma 3.4. *Fix $\kappa > 4$ and let η be an* SLE_κ. *For $y < 0 < x$, define*

$$\sigma = \inf\{t : \eta(t) \in (-\infty, y]\}, \quad T = \inf\{t : \eta(t) \in [x, \infty)\}, \quad F = \{\mathrm{dist}(\eta[0,\sigma], x) \geq cx\},$$

where c is the constant decided in Lemma 2.5. For $\lambda \geq 0$, define

$$u_2(\lambda) = \frac{1}{\kappa}(\kappa/2-2) + \frac{1}{\kappa}\sqrt{4\kappa\lambda + (\kappa/2-2)^2}.$$

Then we have, for $\lambda \geq 0$ and $b \leq u_2(\lambda)$,

$$\mathbb{E}\left[g'_\sigma(x)^\lambda (g_\sigma(x) - W_\sigma)^b 1_{\{\sigma<T\}\cap F}\right] \gtrsim x^{u_2(\lambda)}(x-y)^{b-u_2(\lambda)},$$
$$\mathbb{E}\left[g'_\sigma(x)^\lambda (g_\sigma(x) - W_\sigma)^b 1_{\{\sigma<T\}}\right] \lesssim x^{u_2(\lambda)}(x-y)^{b-u_2(\lambda)},$$

where the constants in $\gtrsim$ and $\lesssim$ depend only on κ and λ, b.

Proof. Define

$$M_t = g'_t(x)^{\nu(\nu+4-\kappa)/(4\kappa)}(g_t(x) - W_t)^{\nu/\kappa}, \quad \text{where } \nu = \kappa u_2(\lambda).$$

Then M is a local martingale for η and the law of η weighted by M is the law of $\mathrm{SLE}_\kappa(\nu)$ with force point x. By the definition of u_2, we can also write

$$M_t = g'_t(x)^\lambda (g_t(x) - W_t)^{u_2(\lambda)}.$$

Thus

$$\mathbb{E}\left[g'_\sigma(x)^\lambda (g_\sigma(x) - W_\sigma)^b 1_{\{\sigma<T\}}\right] = M_0 \mathbb{E}^*\left[(g^*_{\sigma^*}(x) - g^*_{\sigma^*}(y))^{b-u_2(\lambda)} 1_{\{\sigma^*<T^*\}}\right],$$

where $\mathbb{P}^*$ denotes the law of $\mathrm{SLE}_\kappa(\nu)$ with force point x and η^*, g^*, σ^* and T^* are defined accordingly. Since $\nu \geq \kappa/2-2$, the curve will never swallows x, thus $T^* = \infty$. Note that $M_0 = x^{u_2(\lambda)}$. Therefore, proving the conclusion boils down to showing

$$\mathbb{E}^*\left[(g^*_{\sigma^*}(x) - g^*_{\sigma^*}(y))^{b-u_2(\lambda)} 1_{F^*}\right] \gtrsim (x-y)^{b-u_2(\lambda)}, \quad \text{where } F^* = \{\mathrm{dist}(\eta^*[0,\sigma^*], x) \geq cx\}; \tag{3.5}$$
$$\mathbb{E}^*\left[(g^*_{\sigma^*}(x) - g^*_{\sigma^*}(y))^{b-u_2(\lambda)}\right] \lesssim (x-y)^{b-u_2(\lambda)}. \tag{3.6}$$

Equation (3.5) is true by Lemma 2.5. Since the quantity $(g^*_t(x) - g^*_t(y))$ is increasing in t, we have

$$(g^*_{\sigma^*}(x) - g^*_{\sigma^*}(y)) \geq x - y.$$

Combining with the fact that $b - u_2(\lambda) \leq 0$, we obtain (3.6). □

Remark 3.5. *Taking $\lambda = b = 0$ in Lemma 3.4, we have*

$$\mathbb{P}[\sigma < T] \asymp x^{u_2(0)}.$$

This implies that (1.6) holds for $n = 1$ with

$$\hat{\alpha}_1^+ = u_2(0) = 1 - 4/\kappa.$$

Lemma 3.6. *Assume the same notations as in Theorem 1.1. Suppose that (1.3) holds for $2n-1$, then (1.4) holds for $2n$.*

Proof of Lemma 3.6, Upper Bound. Let η be an SLE_κ and define

$$\sigma = \inf\{t : \eta(t) \in (-\infty, y]\}, \quad T = \inf\{t : \eta(t) \in [x, \infty)\}.$$

We stop the curve at time σ. Let $\tilde{\eta}$ be the image of $\eta[\sigma, \infty)$ under the centered comformal map $f := g_\sigma - W_\sigma$. Then $\tilde{\eta}$ is an SLE_κ. Define $\tilde{H}_{2n-1}$ for $\tilde{\eta}$.

Given $\eta[0,\sigma]$ with $\sigma < T$, consider the event $H_{2n}(\varepsilon, x, y)$. Denote by γ the connected component of $B(x,\varepsilon) \setminus \eta[0,\sigma]$ whose boundary contains $x+\varepsilon$. We wish to control the image of $(-\infty, y]$ and the image of γ under f. We have the following observations.

- At time σ, we have $W_\sigma = g_\sigma(y)$, thus $f(y) = 0$.
- By Lemma 2.1, we know that $f(\gamma)$ is contained in the ball with center $f(x+3\varepsilon)$ and radius $8\varepsilon f'(x+3\varepsilon)$.

Combining these two facts, we know that, given $\eta[0,\sigma]$ with $\sigma < T$, the event $H_{2n}(\varepsilon, x, y)$ implies the event $\tilde{H}_{2n-1}(8\varepsilon f'(x+3\varepsilon), f(x+3\varepsilon), 0)$. If $f(x+3\varepsilon) \geq 8\varepsilon f'(x+3\varepsilon)$, by the assumption hypothesis, we have

$$\mathbb{P}[H_{2n}(\varepsilon,x,y) \,|\, \eta[0,\sigma], \sigma < T] \lesssim \left(\frac{\varepsilon g'_\sigma(x+3\varepsilon)}{g_\sigma(x+3\varepsilon) - W_\sigma} \right)^{\alpha^+_{2n-1}}.$$

If $f(x+3\varepsilon) \leq 8\varepsilon f'(x+3\varepsilon)$, the above upper bound is trivially true. Therefore, the above upper bound always holds. Then

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \lesssim \varepsilon^{\alpha^+_{2n-1}} \mathbb{E}\left[g'_\sigma(x+3\varepsilon)^{\alpha^+_{2n-1}} (g_\sigma(x+3\varepsilon) - W_\sigma)^{-\alpha^+_{2n-1}} 1_{\{\sigma<T\}} \right].$$

To apply Lemma 3.4, we only need to note that T is the first time that η swallows x which happens before the first time that η swallows $x+3\varepsilon$. Note further that

$$u_2(\alpha^+_{2n-1}) = \alpha^+_{2n} - \alpha^+_{2n-1}. \tag{3.7}$$

Thus, by Lemma 3.4, we have

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \lesssim \varepsilon^{\alpha^+_{2n-1}} x^{\alpha^+_{2n} - \alpha^+_{2n-1}} (x-y)^{-\alpha^+_{2n}} = \left(\frac{x}{x-y} \right)^{\alpha^+_{2n}} \left(\frac{\varepsilon}{x} \right)^{\alpha^+_{2n-1}}.$$

This completes the proof of the upper bound. □

Proof of Lemma 3.6, Lower Bound. Let η be an SLE_κ and assume the same notations as in the proof of the upper bound. Define $F = \{\mathrm{dist}(\eta[0,\sigma], x) \geq c\varepsilon\}$, where c is the constant decided in Lemma 2.5. We stop the curve at time σ. Let $\tilde{\eta}$ be the image of $\eta[\sigma,\infty)$ under the centered comformal map $f := g_\sigma - W_\sigma$. Then $\tilde{\eta}$ is an SLE_κ. Define $\tilde{H}_{2n-1}$ for $\tilde{\eta}$.

Given $\eta[0,\sigma]$ with $\{\sigma < T\} \cap F$, consider the event $H_{2n}(\varepsilon,x,y)$. We wish to control the image of $(-\infty, y]$ and the image of $\partial B(x,\varepsilon)$ under f. We have the following observations.

- At time σ, we have $W_\sigma = g_\sigma(y)$, thus $f(y) = 0$.
- On the event F, by Koebe 1/4 Theorem, we know that $f(B(x,\varepsilon))$ contains the ball with center $f(x)$ and radius $cf'(x)\varepsilon/4$.

Combining these two facts, we know that, given $\eta[0,\sigma]$ with $\{\sigma < T\} \cap F$, the event $H_{2n}(\varepsilon,x,y)$ contains the event $\tilde{H}_{2n-1}(f'(x)c\varepsilon/4, f(x), 0)$. By the assumption hypothesis, we have

$$\mathbb{P}[H_{2n}(\varepsilon,x,y) \,|\, \eta[0,\sigma], \{\sigma < T\} \cap F] \gtrsim \left(\frac{\varepsilon g'_\sigma(x)}{g_\sigma(x) - W_\sigma} \right)^{\alpha^+_{2n-1}}.$$

Therefore,

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \gtrsim \varepsilon^{\alpha^+_{2n-1}} \mathbb{E}\left[g'_\sigma(x)^{\alpha^+_{2n-1}} (g_\sigma(x) - W_\sigma)^{-\alpha^+_{2n-1}} 1_{\{\sigma<T\}\cap F} \right].$$

To apply Lemma 3.4, we only need to note that $x \geq \varepsilon$ and the event F contains the event $\{\mathrm{dist}(\eta[0,\sigma],x) \geq cx\}$. By (3.7) and Lemma 3.4, we have

$$\mathbb{P}[H_{2n}(\varepsilon,x,y)] \gtrsim \varepsilon^{\alpha^+_{2n-1}} x^{\alpha^+_{2n}-\alpha^+_{2n-1}} (x-y)^{-\alpha^+_{2n}} = \left(\frac{x}{x-y}\right)^{\alpha^+_{2n}} \left(\frac{\varepsilon}{x}\right)^{\alpha^+_{2n-1}}.$$

This completes the proof of the lower bound. □

3.3 From $2n$ to $2n+1$

Lemma 3.7. *Assume the same notations as in Theorem 1.1. Suppose that (1.4) holds for $2n$ with $n \geq 1$, then (1.3) holds for $2n+1$.*

Proof of Lemma 3.7, Upper Bound. If $\varepsilon \leq x \leq 64\varepsilon$, by the assumption hypothesis we have

$$\mathbb{P}[H_{2n+1}(\varepsilon,x,y)] \leq \mathbb{P}[H_{2n}(\varepsilon,x,y)] \lesssim \left(\frac{x}{x-y}\right)^{\alpha^+_{2n}},$$

which gives the upper bound in (1.3) for $2n+1$.

In the following, we assume that $x > 64\varepsilon$. Let η be an SLE$_\kappa$. Define T to be the first time that η swallows x. For $\varepsilon > 0$, let τ_ε be the first time that η hits $B(x,\varepsilon)$. Define O_t to be the image of the rightmost point of $\eta[0,t] \cap \mathbb{R}$ under g_t. Define

$$\hat{\tau}_\varepsilon = \inf\{t : \frac{g_t(x) - O_t}{g'_t(x)} = \varepsilon\}.$$

We stop the curve at time $\hat{\tau}_{64\varepsilon}$. Let $\tilde{\eta}$ be the image of $\eta[\hat{\tau}_{64\varepsilon},\infty)$ under the centered conformal map $f := g_{\hat{\tau}_{64\varepsilon}} - W_{\hat{\tau}_{64\varepsilon}}$. Then $\tilde{\eta}$ is an SLE$_\kappa$. Define the event $\tilde{H}_{2n}$ for $\tilde{\eta}$.

Given $\eta[0,\hat{\tau}_{64\varepsilon}]$, consider the event $H_{2n+1}(\varepsilon,x,y)$. We wish to control the image of the ball $B(x,\varepsilon)$ and the image of the half-infinite line $(-\infty,y)$ under f. We have the following observations.

- By Koebe 1/4 theorem, we know that $\hat{\tau}_{64\varepsilon} \leq \tau_{16\varepsilon}$. Combining with Lemma 2.2, we know that $f(B(x,\varepsilon))$ is contained in the ball $B(f(x), 4f'(x)\varepsilon)$.
- At time $\hat{\tau}_{64\varepsilon}$, there are two possibilities for the image of y under f: if y is not swallowed by $\eta[0,\hat{\tau}_{64\varepsilon}]$, then $f(y) = g_{\hat{\tau}_{64\varepsilon}}(y) - W_{\hat{\tau}_{64\varepsilon}}$ is the image of y under f; if y is swallowed by $\eta[0,\hat{\tau}_{64\varepsilon}]$, then the image of y under f is the image of leftmost point of $\eta[0,\hat{\tau}_{64\varepsilon}] \cap \mathbb{R}$ under f, in this case, we still write $f(y) = g_{\hat{\tau}_{64\varepsilon}}(y) - W_{\hat{\tau}_{64\varepsilon}}$ as explained in Section 2.

Combining these two facts, we know that, given $\eta[0,\hat{\tau}_{64\varepsilon}]$, $H_{2n+1}(\varepsilon,x,y)$ implies $\tilde{H}_{2n}(4f'(x)\varepsilon, f(x), f(y))$. By the assumption hypothesis, we have

$$\mathbb{P}[H_{2n+1}(\varepsilon,x,y) \,|\, \eta[0,\hat{\tau}_{64\varepsilon}], \hat{\tau}_{64\varepsilon} < T] \lesssim \left(\frac{g_{\hat{\tau}_{64\varepsilon}}(x) - W_{\hat{\tau}_{64\varepsilon}}}{g_{\hat{\tau}_{64\varepsilon}}(x) - g_{\hat{\tau}_{64\varepsilon}}(y)}\right)^{\alpha^+_{2n}} \left(\frac{g'_{\hat{\tau}_{64\varepsilon}}(x)\varepsilon}{g_{\hat{\tau}_{64\varepsilon}}(x) - W_{\hat{\tau}_{64\varepsilon}}}\right)^{\alpha^+_{2n-1}}.$$

For fixed x and y, the quantity $g_t(x) - g_t(y)$ is increasing in t, thus $g_t(x) - g_t(y) \geq x - y$. Plugging in the above inequality, we have

$$\mathbb{P}[H_{2n+1}(\varepsilon,x,y)] \lesssim (x-y)^{-\alpha^+_{2n}} \varepsilon^{\alpha^+_{2n-1}} \mathbb{E}\left[(g_{\hat{\tau}_{64\varepsilon}}(x) - W_{\hat{\tau}_{64\varepsilon}})^{\alpha^+_{2n}-\alpha^+_{2n-1}} g'_{\hat{\tau}_{64\varepsilon}}(x)^{\alpha^+_{2n-1}} 1_{\{\hat{\tau}_{64\varepsilon} < T\}}\right].$$

By Proposition 3.1 and (3.4), we have

$$\mathbb{P}[H_{2n+1}(\varepsilon,x,y)] \lesssim (x-y)^{-\alpha^+_{2n}} \varepsilon^{\alpha^+_{2n-1}} x^{-u_1(\alpha^+_{2n})} \varepsilon^{u_1(\alpha^+_{2n})+\alpha^+_{2n}-\alpha^+_{2n-1}}.$$

Note that

$$\alpha^+_{2n+1} = u_1(\alpha^+_{2n}) + \alpha^+_{2n}. \tag{3.8}$$

Therefore

$$\mathbb{P}\left[H_{2n+1}(\varepsilon,x,y)\right] \lesssim \left(\frac{x}{x-y}\right)^{\alpha^+_{2n}} \left(\frac{\varepsilon}{x}\right)^{\alpha^+_{2n+1}}$$

which completes the proof. □

Proof of Lemma 3.7, Lower Bound. Let η be an SLE_κ. Define T to be the first time that η swallows x. For $\varepsilon > 0$, let τ_ε be the first time that η hits $B(x,\varepsilon)$. We stop the curve at time τ_ε. Let $\tilde{\eta}$ be the image of $\eta[\tau_\varepsilon,\infty)$ under the centered conformal map $f := g_{\tau_\varepsilon} - W_{\tau_\varepsilon}$. Then $\tilde{\eta}$ is an SLE_κ. Define the event $\tilde{H}_{2n}$ for $\tilde{\eta}$.

Given $\eta[0,\tau_\varepsilon]$, consider the event $H_{2n+1}(\varepsilon,x,y)$. We wish to control the image of the ball $B(x,\varepsilon)$ and the image of the half-infinite line $(-\infty,y)$ under f. We have the following observations.

- Applying Koebe 1/4 Theorem to f, we know that $f(B(x,\varepsilon))$ contains the ball $B(f(x), f'(x)\varepsilon/4)$.
- At time τ_ε, we have $f(y) = g_{\tau_\varepsilon}(y) - W_{\tau_\varepsilon}$. Recall that if y is swallowed by $\eta[0,\tau_\varepsilon]$, then $f(y)$ should be understood as the image of the leftmost point of $\eta[0,\tau_\varepsilon]\cap\mathbb{R}$ under f.

Combining these two facts, we know that, given $\eta[0,\tau_\varepsilon]$, the event $H_{2n+1}(\varepsilon,x,y)$ contains $\tilde{H}_{2n}(f'(x)\varepsilon/4, f(x), f(y))$. By the assumption hypothesis, we have

$$\mathbb{P}\left[H_{2n+1}(\varepsilon,x,y) \,|\, \eta[0,\tau_\varepsilon], \tau_\varepsilon < T\right] \gtrsim \left(\frac{g_{\tau_\varepsilon}(x) - W_{\tau_\varepsilon}}{g_{\tau_\varepsilon}(x) - g_{\tau_\varepsilon}(y)}\right)^{\alpha^+_{2n}} \left(\frac{g'_{\tau_\varepsilon}(x)\varepsilon}{g_{\tau_\varepsilon}(x) - W_{\tau_\varepsilon}}\right)^{\alpha^+_{2n-1}}. \tag{3.9}$$

For $t \ge 0$, let O_t the image of the rightmost point of $\eta[0,t]\cap\mathbb{R}$ under g_t. Set

$$\Upsilon_t = \frac{g_t(x) - O_t}{g'_t(x)}, \quad J_t = \frac{g_t(x) - O_t}{g_t(x) - W_t}.$$

Define

$$M_t = g'_t(x)^{\nu(\nu+4-\kappa)/(4\kappa)} (g_t(x) - W_t)^{\nu/\kappa}, \quad \text{where } \nu = \kappa(\alpha^+_{2n} - \alpha^+_{2n+1}) \le \kappa/2 - 4.$$

Then M is a local martinagle and the law of η weighted by M becomes the law of $\mathrm{SLE}_\kappa(\nu)$ with force point x. By (3.8), we have

$$\nu(\nu+4-\kappa)/(4\kappa) = \alpha^+_{2n+1}.$$

The local martingale M can be written as

$$M_t = g'_t(x)^{\alpha^+_{2n+1}} (g_t(x) - W_t)^{\alpha^+_{2n} - \alpha^+_{2n+1}} = g'_t(x)^{\alpha^+_{2n-1}} (g_t(x) - W_t)^{\alpha^+_{2n} - \alpha^+_{2n-1}} \Upsilon_t^{\alpha^+_{2n-1} - \alpha^+_{2n+1}} J_t^{\alpha^+_{2n+1} - \alpha^+_{2n-1}}.$$

At time $t = \tau_\varepsilon < T$, by Koebe 1/4 Theorem, we have $\Upsilon_t \asymp \varepsilon$. Since $J_t \le 1$, we have

$$M_{\tau_\varepsilon} \varepsilon^{\alpha^+_{2n+1} - \alpha^+_{2n-1}} \lesssim g'_{\tau_\varepsilon}(x)^{\alpha^+_{2n-1}} (g_{\tau_\varepsilon}(x) - W_{\tau_\varepsilon})^{\alpha^+_{2n} - \alpha^+_{2n-1}}.$$

Combining with (3.9) and $M_0 = x^{\alpha^+_{2n} - \alpha^+_{2n+1}}$, we have

$$\mathbb{P}[H_{2n+1}(\varepsilon,x,y)] \gtrsim \varepsilon^{\alpha^+_{2n+1}} x^{\alpha^+_{2n} - \alpha^+_{2n+1}} \mathbb{E}^*\left[(g^*_{\tau^*_\varepsilon}(x) - g^*_{\tau^*_\varepsilon}(y))^{-\alpha^+_{2n}} 1_{\{\tau^* < T^*\}}\right],$$

where $\mathbb{P}^*$ denotes the law of $\mathrm{SLE}_\kappa(\nu)$ with force point x and $g^*, \tau^*_\varepsilon, T^*$ are defined for η^* whose law is $\mathbb{P}^*$ accordingly. Since $\nu \le \kappa/2 - 4$, the curve accumulates at the point x at almost surely finite time T^*, thus $\{\tau^*_\varepsilon < T^*\}$ always holds. To complete the proof, it is sufficient to show

$$\mathbb{E}^*\left[\left(g^*_{\tau^*_\varepsilon}(x) - g^*_{\tau^*_\varepsilon}(y)\right)^{-\alpha^+_{2n}}\right] \gtrsim (x-y)^{-\alpha^+_{2n}}. \tag{3.10}$$

Since the quantity $g^*_t(x) - g^*_t(y)$ is increasing t, we know that

$$x - y \le g^*_{\tau^*_\varepsilon}(x) - g^*_{\tau^*_\varepsilon}(y) \le g^*_{T^*}(x) - g^*_{T^*}(y).$$

Combining with Lemma 2.4, we obtain (3.10) and complete the proof. □

3.4 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Combining Remark 3.3 and Lemmas 3.7 and 3.6 implies the conclusion. □

Proof of Theorem 1.2. We have the following observations.

- By Remark 3.5, we know that (1.6) holds for $n = 1$.
- By the same arguments in Section 3.3, we could prove that, assume (1.6) holds for $2n-1$ with $n \ge 1$, then (1.7) holds for $2n$ where (3.8) should be replaced by

$$\hat{\alpha}^{+}_{2n} = u_1(\hat{\alpha}^{+}_{2n-1}) + \hat{\alpha}^{+}_{2n-1}.$$

- By the same arguments in Section 3.2, we could prove that, assume (1.7) holds for $2n$ with $n \ge 1$, then (1.6) holds for $2n+1$ where (3.7) should be replaced by

$$\hat{\alpha}^{+}_{2n+1} = u_2(\hat{\alpha}^{+}_{2n}) + \hat{\alpha}^{+}_{2n}.$$

Combining these three facts, we obtain the conclusion. □

4 Boundary Arm Exponents for $\kappa \le 4$

4.1 Definitions and Statements

In this section, we assume $\kappa \in (0,4]$, let η be a chordal SLE$_\kappa$ curve, and let g_t be the corresponding Loewner maps. Since η does not hit the boundary other than its end points, H_n and $\hat{H}_n$ defined in Section 1 are empty sets. So we need to modify their definitions.

For $y \in \mathbb{R}$ and $r > 0$, we define half strips:

$$L^{-}_{y;r} = \{z \in \mathbb{H} : \Im z \le r; \Re z \le y\}, \quad L^{+}_{y;r} = \{z \in \mathbb{H} : \Im z \le r; \Re z \ge y\};$$

and write $L^{\pm}_{y} = L^{\pm}_{y;\pi}$.

A crosscut in a domain D is an open simple curve in D, whose end points approach boundary points of D. Suppose S is a relatively closed subset of $\mathbb{H}$ such that $\partial S \cap \mathbb{H}$ is a crosscut of $\mathbb{H}$. Then we use $\partial^{+}_{\mathbb{H}} S$ (resp. $\partial^{-}_{\mathbb{H}} S$) to denote the curve $\partial S \cap \mathbb{H}$ oriented so that S lies to the left (resp. right) of the curve. For example, $\partial^{-}_{\mathbb{H}} L^{-}_{y;r}$ is from y to ∞; and for $x \in \mathbb{R}$, $\partial^{+}_{\mathbb{H}} B(x,r)$ is from $x-r$ to $x+r$.

Let $\xi_j : [0,T_j] \to \mathbb{C}$, $j = -1, 1$, and $\eta : [0,T) \to \mathbb{C}$ be three continuous curves. For $j = -1, 1$, define increasing functions $R_j(t) = \max(\{0\} \cup \{s \in [0,T_j] : \xi_j(s) \in \eta([0,t])\})$ for $t \in [0,T)$. Let $\tau_0 = 0$. After τ_n is defined for some $n \ge 0$, we define $\tau_{n+1} = \inf\{t \ge \tau_n : \eta(t) \in \xi_{(-1)^{n+1}}((R_{(-1)^{n+1}}(\tau_n), T_{(-1)^{n+1}}))\}$, where we set $\inf \emptyset = \infty$ by convention, and if any $\tau_{n_0} = \infty$, then $\tau_n = \infty$ for all $n \ge n_0$.

Definition 4.1. *If $\tau_{n_0} < \infty$ for some $n_0 \in \mathbb{N}$, then we say that η makes (at least) n_0 well-oriented (ξ_{-1}, ξ_1)-crossings.*

Remark 4.2. *The above name comes from the fact that the orientation-preserving reparametrizations of ξ_1, ξ_{-1}, η do not affect the event.*

Definition 4.3. *Let $x > y$, $x > 0$, and $\varepsilon > 0$. Let η be an SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. Define $H^{\pi}_{2n-1}(\varepsilon, x, y)$ to be the event that η makes at least $(2n-1)$ well-oriented $(\partial^{+}_{\mathbb{H}} B(x,\varepsilon), \partial^{-}_{\mathbb{H}} L^{-}_{y})$-crossings. Define $H^{\pi}_{2n}(\varepsilon, x, y)$ to be the event that η makes at least $2n$ well-oriented $(\partial^{-}_{\mathbb{H}} L^{-}_{y}, \partial^{+}_{\mathbb{H}} B(x,\varepsilon))$-crossings. Note that in either event, the last visit that counts is at the half circle $\partial^{+}_{\mathbb{H}} B(x,\varepsilon)$.*

The theorem below is our main theorem for $\kappa \le 4$. The function ϕ will be defined later in (4.7), and $\phi^{(k)}$ is the k times iteration of ϕ. The following estimate is useful to have a sense of $\phi^{(k)}$:

$$\phi^{(k)}(x) \ge \frac{x}{2}, \quad \text{if } x \ge 6k+3. \tag{4.1}$$

Theorem 4.4. *Let α^+_{2n} and α^+_{2n-1} be defined by (1.1). We have the following facts.*

(i) If (ε,x,y) satisfy $2^{5n-4}\varepsilon < \phi^{(2n-2)}(x-y)$, then

$$\mathbb{P}\left[H^\pi_{2n-1}(\varepsilon,x,y)\right] \lesssim \frac{x^{\alpha^+_{2n-2}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}}{\prod_{j=1}^{n-1}\phi^{(2n-2j-1)}(x-y)^{\alpha^+_{2j}-\alpha^+_{2j-2}}}. \tag{4.2}$$

If (ε,x,y) satisfy $2^{5n-1}\varepsilon < \phi^{(2n-1)}(x-y)$, and $\varepsilon \le x$, then

$$\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \lesssim \frac{x^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}}{\prod_{j=1}^{n}\phi^{(2n-2j)}(x-y)^{\alpha^+_{2j}-\alpha^+_{2j-2}}}. \tag{4.3}$$

Here the implicit constants depend only on κ,n.

(ii) For any $R>0$ and $n\in\mathbb{N}$, there is a constant $C_{n,R}$ depending only on κ,n,R such that

$$\mathbb{P}\left[H^\pi_{2n-1}(\varepsilon,x,y)\right] \ge C_{2n-1,R}x^{\alpha^+_{2n-2}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}, \quad \textit{provided } \varepsilon < x, \textit{and } \varepsilon < x-y \le R, \tag{4.4}$$
$$\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \ge C_{2n,R}x^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}, \quad \textit{provided } \varepsilon < x \le x-y \le R. \tag{4.5}$$

Remark 4.5. *Using (4.1), we see that, if $x-y\ge 12n$ and $2^{5n}\varepsilon < x-y$, then*

$$\mathbb{P}\left[H^\pi_{2n-1}(\varepsilon,x,y)\right] \lesssim \left(\frac{x}{x-y}\right)^{\alpha^+_{2n-2}}\left(\frac{\varepsilon}{x}\right)^{\alpha^+_{2n-1}}$$

and

$$\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \lesssim \left(\frac{x}{x-y}\right)^{\alpha^+_{2n}}\left(\frac{\varepsilon}{x}\right)^{\alpha^+_{2n-1}}.$$

So we get the same upper bound as in the case $\kappa>4$.

4.2 Comparison principle for well-oriented crossings

Let D be a simply connected domain. We say that $\eta:[0,T)\to\overline{D}$ is a non-self-crossing curve in D if $\eta(0)\in\partial D$, and for any $t_0\ge 0$, there is a unique connected component D_{t_0} of $D\setminus\eta[0,t_0]$ such that $\eta(t_0+\cdot)$ is the image of a continuous curve in $\overline{\mathbb{U}}$ under a continuous map from $\overline{\mathbb{U}}$ onto $\overline{D_{t_0}}$, which is an extension of a conformal map from $\mathbb{U}$ onto D_{t_0}. For example, an SLE curve is almost surely a non-self-crossing curve.

Lemma 4.6 (Comparison Principle)**.** *Let D be a simply connected domain, and η be a non-self-crossing curve in D. Let $\xi_j,\hat\xi_j:(0,1)\to\overline{D}$, $j=-1,1$, be crosscuts of D. Let (τ_n) and $R_j(t)$, $j=-1,1$ be as in the definition of oriented crossings for η and (ξ_{-1},ξ_1). Let $(\hat\tau_n)$ and $\hat R_j(t)$, $j=-1,1$, be the corresponding quantities for η and $(\hat\xi_{-1},\hat\xi_1)$. Assume the following. See Figure 4.1.*

(i) For $j=-1,1$, $\hat\xi_j$ disconnects ξ_j from both ξ_{-j} and $\hat\xi_{-j}$ in D; the distance between $\hat\xi_{-1}$ and $\hat\xi_1$ is positive; and $\hat\xi_{-1}$ disconnects ξ_{-1} from $\eta(0)$ in D. Here we allow the possibility that $\hat\xi_j$ touches ξ_j, or $\eta(0)\in\hat\xi_{-1}$.

(ii) If $\eta_{t_0}=\hat\xi_{(-1)^{n+1}}(\hat R_{(-1)^{n+1}}(\tau_n))$ or $\hat\xi_{(-1)^{n+1}}(1)$ for some $t_0\ge\tau_n$, then for any $\varepsilon>0$, there is $t_1\in[t_0,t_0+\varepsilon)$ such that $\eta(t_1)\in\hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n),1))$.

(iii) There is a closed boundary (prime end) arc I of D with end points $\xi_1(1)$ and $\xi_{-1}(1)$ such that $\hat\xi_j(1)\in I$, $j=-1,1$, and $\eta\cap I=\emptyset$.

If η makes n_0 well-oriented (ξ_{-1},ξ_1)-crossings, then it also makes n_0 well-oriented $(\hat\xi_{-1},\hat\xi_1)$-crossings.

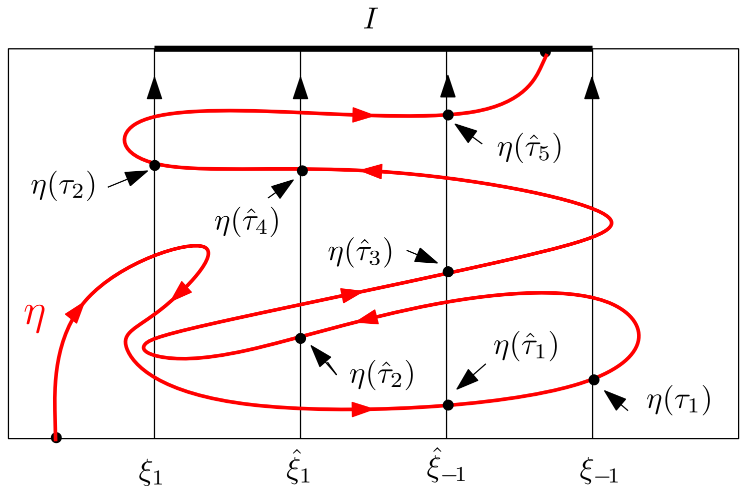

Fig. 4.1: The figure illustrates the definition of well-oriented crossings as well as the conditions of Lemma 4.6. The curve η totally makes 2 well-oriented (ξ_{-1}, ξ_1)-crossings and 5 well-oriented $(\hat\xi_{-1}, \hat\xi_1)$-crossings. The times τ_j, $1 \le j \le 2$, and $\hat\tau_j$, $1 \le j \le 5$, are indicated in the figure.

Remark 4.7. *The assumption that η is non-self-crossing forces $\eta(\tau_n + \cdot)$ to stay in the closure of the remaining domain D_{τ_n}. We need assumption (iii) to prevent $\eta(\tau_n + \cdot)$ to sneak into the region bounded by the crosscut $\hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n), 1))$ of D_{τ_n} through one of its endpoints without hitting the crosscut. This assumption is certainly satisfied if η is an SLE curve.*

Proof. Suppose η makes n_0 well-oriented (ξ_{-1}, ξ_1)-crossings. Then $\tau_{n_0} < \infty$. We will show that $\hat\tau_n \le \tau_n$ for $0 \le n \le n_0$. Especially, the inequality $\hat\tau_{n_0} < \infty$ is what we need.

First, we have $\tau_0 = \hat\tau_0 = \hat R_{-1}(0) = 0$. From assumptions (i) and (ii), we have

$$\hat\tau_1 = \inf\{t \ge 0 : \eta(t) \in \hat\xi_{-1}((0,1))\} \le \inf\{t \ge 0 : \eta(t) \in \xi_{-1}((0,1))\} = \tau_1.$$

Suppose we have proved that $\hat\tau_n \le \tau_n$ for some $n \in \{1, \dots, n_0 - 1\}$. Then $\eta(\tau_n) \in \xi_{(-1)^n}$, and for every $\varepsilon > 0$, there is $t \in [\tau_{n+1}, \tau_{n+1} + \varepsilon)$ such that $\eta(t) \in \xi_{(-1)^{n+1}}((R_{(-1)^{n+1}}(\tau_n), 1))$. Let D_{τ_n} be the connected component of $D \setminus \eta([0, \tau_n])$ such that $\eta[\tau_n, \infty) \subset \overline{D_{\tau_n}}$. Then $\xi_{(-1)^{n+1}}((R_{(-1)^{n+1}}(\tau_n), 1))$ is a crosscut of D_{τ_n} since it belongs to $D \setminus \eta([0, \tau_n])$ and is visited by η after τ_n. From assumption (iii) we know that $\hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n), 1))$ is also a crosscut of D_{τ_n}. Since D_{τ_n} is simply connected, this crosscut disconnects $\xi_{(-1)^{n+1}}((R_{(-1)^{n+1}}(\tau_n), 1))$ from η_{τ_n} in $D_{\hat\tau_n}$. From assumption (ii), we have

$$\inf\{t \ge \tau_n : \eta(t) \in \hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n), 1))\} \le \inf\{t \ge \tau_n : \eta(t) \in \xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n), 1))\} = \tau_{n+1}.$$

Since $\hat\tau_n \le \tau_n$ and $\hat R_{(-1)^{n+1}}(t)$ is increasing, we get $\hat R_{(-1)^{n+1}}(\hat\tau_n) \le \hat R_{(-1)^{n+1}}(\tau_n)$, and so

$$\hat\tau_{n+1} = \inf\{t \ge \tau_n : \eta(t) \in \hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\hat\tau_n), 1))\} \le \inf\{t \ge \tau_n : \eta(t) \in \hat\xi_{(-1)^{n+1}}((\hat R_{(-1)^{n+1}}(\tau_n), 1))\} \le \tau_{n+1}.$$

By induction, we conclude that $\hat\tau_n \le \tau_n$ for all $0 \le n \le n_0$, as desired. □

Remark 4.8. *The lemma also holds if we do not assume that ξ_{-1} and $\hat\xi_{-1}$ are crosscuts of D, but assume that they are the same curve in $\overline{D}$.*

4.3 Estimates on half strips

Given a nonempty $\mathbb{H}$-hull K, Let $a_K = \min(\overline{K} \cap \mathbb{R})$ and $b_K = \max(\overline{K} \cap \mathbb{R})$. Let $K^{\text{doub}} = K \cup [a_K, b_K] \cup \{\bar z : z \in K\}$. By Schwarz reflection principle, g_K extends to a conformal map from $\mathbb{C} \setminus K^{\text{doub}}$ onto $\mathbb{C} \setminus [c_K, d_K]$ for some $c_K < d_K \in \mathbb{R}$, and satisfies $g_K(\bar z) = \overline{g_K(z)}$. From [Zha08, (5.1)] we know that there is a positive measure μ_K supported by $[c_K, d_K]$ with total mass $|\mu_K| = \mathrm{hcap}(K)$ such that,

$$f_K(z) - z = \int \frac{-1}{z-x} d\mu_K(x), \quad z \in \mathbb{C} \setminus [c_K, d_K]. \tag{4.6}$$

For $x_0 \in \mathbb{R}$ and $r > 0$, let $\overline{B}^+(x_0, r)$ denote the special $\mathbb{H}$-hull $\overline{B(x_0,r)} \cap \mathbb{H}$. If an $\mathbb{H}$-hull K is contained in $\overline{B}^+(x_0,r)$, then $\mathrm{hcap}(K) \le \mathrm{hcap}(\overline{B}^+(x_0,r)) = r^2$ by the monotonicity of half-plane capacity, and $[c_K, d_K] \subset [c_{\overline{B}^+(x_0,r)}, d_{\overline{B}^+(x_0,r)}] = [x_0 - 2r, x_0 + 2r]$ by [Zha08, Lemma 5.3].

Lemma 4.9. *Let $x_0, y \in \mathbb{R}$ and $R, r > 0$. Suppose K is an $\mathbb{H}$-hull and $K \subset \overline{B}^+_{x_0,R}$. Then the unbounded connected component of $g_K(L^-_{y;r} \setminus K)$ contains $L^-_{y';r'}$ for $y' = \min\{x_0 - 2R - \frac{2R^2}{r}, y - \frac{r}{2}\}$ and $r' = r/2$.*

Proof. Let $z \in L^-_{y';r'}$. Since $\Re z \le x_0 - 2R - \frac{2R^2}{r}$ and $[c_K, d_K] \subset [x_0 - 2R, x_0 + 2R]$, we have $|z - x| \ge \frac{2R^2}{r}$ for any $x \in [c_K, d_K]$. From (4.6) and $|\mu_K| = \mathrm{hcap}(K) \le R^2$, we get $|f_K(z) - z| \le \frac{r}{2}$. Since $\Re z \le y' \le y - \frac{r}{2}$, we get $\Re f_K(z) \le y$. Since $0 < \Im z \le r' = r/2$, we get $0 < \Im f_K(z) \le r$ (f_K maps $\mathbb{H}$ into $\mathbb{H}$). Thus, we conclude that $f_K(L^-_{y';r'}) \subset L^-_{y;r}$. Since $f_K(L^-_{y';r'})$ is an unbounded domain contained in $\mathbb{H} \setminus K$, and $g_K = f_K^{-1}$, we get the conclusion. □

Now $L^-_{y;r}$ is not an $\mathbb{H}$-hull since it is not bounded. But we will still find a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus L^-_{y;r}$. By scaling and translation, it suffices to consider $L^-_0 = L^-_{0;\pi}$. We will use the map $f_{(0,i]}(z) = \sqrt{z^2 - 1}$ for the half open line segment $(0, i]$, and the map $f_{\overline{B}^+(0,1)}$ for the unit semi-disc. Recall that $f^{-1}_{\overline{B}^+(0,1)}(z) = g_{\overline{B}^+(0,1)}(z) = z + \frac{1}{z}$.

Lemma 4.10. *Let $f_{L^-_0}(z) = f_{(0,i]}(z) + \log(f_{\overline{B}^+(0,1)}(2z))$, where the branch of $\log$ is chosen so that it maps $\mathbb{H}$ onto $\{0 < \Im z < \pi\}$. Then $f_{L^-_0}$ maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus L^-_0$, and satisfies $f_{L^-_0}(z) = z + \log(2z) + O(1/z)$ as $z \to \infty$, and $f_{L^-_0}(1) = 0$, $f_{L^-_0}(-1) = \pi i$.*

Proof. We observe that $z \mapsto \log(f_{\overline{B}^+(0,1)}(2z))$ is a conformal map from $\mathbb{H}$ onto L^+_0, which takes 1 and -1 to 0 and πi respectively; and $f_{(0,i]}$ is a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus (0, i]$, which takes both 1 and -1 to 0. So the $f_{L^-_0}$ defined by the lemma satisfies $f_{L^-_0}(1) = 0, f_{L^-_0}(-1) = \pi i$. As $z \to \infty$, $f_{(0,i]}(z) = z + O(1/z)$ and $f_{\overline{B}^+(0,1)}(2z) = 2z + O(1/z)$. So $\log(f_{\overline{B}^+(0,1)}(2z)) = \log(2z) + O(1/z^2)$ as $z \to \infty$. Thus, $f_{L^-_0}(z) = z + \log(2z) + O(1/z)$ as $z \to \infty$.

It remains to show that $f_{L^-_0}$ maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus L^-_0$. It is easy to see that $f_{L^-_0}$ maps $(1, \infty)$ into $(0, \infty)$. By Schwarz-Christoffel transformation, it suffices to show that $f'_{L^-_0}(z) = \sqrt{\frac{z+1}{z-1}}$. Let $g(z) = g_{\overline{B}^+(0,1)}(z)/2 = \frac{z}{2} + \frac{1}{2z}$ and $f = g^{-1}$. Then $\log(f_{\overline{B}^+(0,1)}(2z)) = \log(f(z))$. We find that $\sqrt{g(z)^2 - 1} = \frac{z}{2} - \frac{1}{2z}$ and $g'(z) = \frac{1}{2} - \frac{1}{2z^2}$. So $\sqrt{g(z)^2 - 1} = z g'(z) = \frac{f(g(z))}{f'(g(z))}$, which implies that $\frac{f'(w)}{f(w)} = \sqrt{\frac{1}{w^2 - 1}}$. From this we get $\frac{d}{dz} \log(f_{\overline{B}^+(0,1)}(2z)) = \frac{f'(z)}{f(z)} = \frac{1}{\sqrt{z^2-1}}$. Since $f'_{(0,i]}(z) = \frac{z}{\sqrt{z^2-1}}$, we have $f'_{L^-_0}(z) = \frac{z}{\sqrt{z^2-1}} + \frac{1}{\sqrt{z^2-1}} = \sqrt{\frac{z+1}{z-1}}$, as desired. □

Define $f_{L^-_y}(z) = f_{L^-_0}(z - y) + y$, which maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus L^-_y$, and let $g_{L^-_y} = f^{-1}_{L^-_y}$. We will use $\mathrm{hm}(z, D; V)$ to denote the harmonic measure of V in a domain D seen from z, i.e., the probability that a planar Brownian motion started from $z \in D$ hits V before $\partial D \setminus V$.

Lemma 4.11. *For any $y, m \in \mathbb{R}$, and any boundary arc $I \subset \partial(\mathbb{H} \setminus L^-_y)$, we have $\lim_{h\to\infty} h \cdot \mathrm{hm}(m + ih, \mathbb{H} \setminus L^-_y; I) = |g_{L^-_y}(I)|/\pi$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$.*

Proof. From conformal invariance of the harmonic measure, we have

$$\mathrm{hm}(m + ih, \mathbb{H} \setminus L^-_y; I) = \mathrm{hm}(g_{L^-_y}(m + ih), \mathbb{H}; g_{L^-_y}(I).$$

Since $|f_{L^-_y}(z) - z|/|z| \to 0$ as $|z| \to \infty$, we get $|g_{L^-_y}(z) - z|/|z| \to 0$ as $|z| \to \infty$. From this we get

$$\lim_{h\to\infty} \mathrm{hm}(g_{L^-_y}(m + ih), \mathbb{H}; g_{L^-_y}(I)) / \mathrm{hm}(m + ih, \mathbb{H}; g_{L^-_y}(I)) = 1.$$

Since $\lim_{h\to\infty} h \cdot \mathrm{hm}(m + ih, \mathbb{H}; g_{L^-_y}(I)) = |g_{L^-_y}(I)|/\pi$, the proof is now finished. □

We will use $\mathrm{hm}(\infty,\mathbb{H}\setminus L_y^-;I)$ to denote $\lim_{h\to\infty}\pi\cdot h\cdot \mathrm{hm}(m+ih,\mathbb{H}\setminus L_y^-;I)$, which equals $|g_{L_y^-}(I)|$ by the above lemma. For example, we have $\mathrm{hm}(\infty,\mathbb{H}\setminus L_y^-;[y,y+i\pi])=2$, and

$$\mathrm{hm}(\infty,\mathbb{H}\setminus L_y^-;[y,y'])=g_{L_y^-}(y')-g_{L_y^-}(y)=g_{L_0^-}(y'-y)-1,\quad y'\ge y.$$

Note that $x\mapsto f_{L_0^-}(g_{L_0^-}(x)-2)$ is a homeomorphism from $[f_{L_0^-}(3),\infty)$ onto $[0,\infty)$. Now we define

$$\phi(x)=\begin{cases} f_{L_0^-}(g_{L_0^-}(x)-2), & \text{if } x\ge f_{L_0^-}(3);\\ 0, & \text{if } x\le f_{L_0^-}(3).\end{cases} \tag{4.7}$$

Lemma 4.12. *Let $x_0,y_0\in\mathbb{R}$. Let K be an $\mathbb{H}$-hull such that $x_0>b_K=\max(\overline{K}\cap\mathbb{R})$. Let γ denote the unbounded component of $\partial L_{y_0}^-\setminus(\mathbb{R}\cup K)$. If $x_0-y_0>f_{L_0^-}(3)$, then there is $y_1\in\mathbb{R}$ such that $g_K(\gamma)\subset L_{y_1}^-$ and $g_K(x_0)-y_1\ge \phi(x_0-y_0)$.*

Proof. Let L be the unbounded component of $L_{y_0}^-\setminus K$. Let $y_1=\sup\Re(g_K(\gamma))$. From (4.6) we see that $g_K=f_K^{-1}$ decreases the imaginary part of points in $\mathbb{H}$. So we have $g_K(\gamma)\subset L_{y_1}^-$.

Let $x_1=g_K(x_0)$. First, we prove that $x_1>y_1$. Choose $z_1\in\overline{g_K(\gamma)}$ such that $y_1=\Re z_1$. Suppose $x_1\le y_1$. Then $z_1\notin\mathbb{R}$ for otherwise z_1 is the image of $\overline{\gamma}\cap\partial K$ under g_K, which must lie to the left of the image of x_0. Let γ_v denote the vertical open line segment (y_1,z_1). It disconnects x_1 from ∞ in $\mathbb{H}\setminus g_K(L)$. Thus, $f_K(\gamma_v)$ is a crosscut in $\mathbb{H}\setminus(K\cup L)$, which connects $f_K(z_1)\in\gamma$ with $f_K(y_1)\ge x_0$, and separates $x_0=f_K(x_1)$ from ∞ in $\mathbb{H}\setminus(K\cup L)$. Then for big $h>0$,

$$\mathrm{hm}(ih,\mathbb{H}\setminus(K\cup L);f_K(\gamma_v))=\mathrm{hm}(ih,\mathbb{H}\setminus L;f_K(\gamma_v))\ge \mathrm{hm}(ih,\mathbb{H}\setminus L_{y_0}^-;f_K(\gamma_v))\ge \mathrm{hm}(ih,\mathbb{H}\setminus L_{y_0}^-;[y_0,x_0]). \tag{4.8}$$

Here the equality holds because $f_K(\gamma_v)$ disconnects K from ∞ in $\mathbb{H}\setminus L$ (here we use the fact that L is the unbounded component of $L_{y_0}^-\setminus K$); the first inequality holds because $\mathbb{H}\setminus L_{y_0}^-\subset\mathbb{H}\setminus L$; and the second inequality holds because $f_K(\gamma_v)$ disconnects $[y_0,x_0]$ from ∞ in $\mathbb{H}\setminus L_{y_0}^-$.

From conformal invariance of harmonic measure, $\mathbb{H}\setminus g_K(L)\supset\mathbb{H}\setminus L_{y_1}^-$, and $\gamma_v\subset[y_1,y_1+i\pi]$, we have

$$\mathrm{hm}(ih,\mathbb{H}\setminus(K\cup L);f_K(\gamma_v))=\mathrm{hm}(g_K(ih),\mathbb{H}\setminus g_K(L);\gamma_v)\le \mathrm{hm}(g_K(ih),\mathbb{H}\setminus L_{y_1}^-;[y_1,y_1+i\pi]).$$

Thus,

$$\mathrm{hm}(ih,\mathbb{H}\setminus L_{y_0}^-;[y_0,x_0])\le \mathrm{hm}(g_K(ih),\mathbb{H}\setminus L_{y_1}^-;[y_1,y_1+i\pi]).$$

Combining the above inequalities with (4.8) and letting $h\to\infty$, we get

$$\mathrm{hm}(\infty,\mathbb{H}\setminus L_{y_0}^-;[y_0,x_0])\le \mathrm{hm}(\infty,\mathbb{H}\setminus L_{y_1}^-;[y_1,y_1+i\pi]).$$

Then we get $g_{L_0^-}(x_0-y_0)-1\le 2$, which contradicts that $x_0-y_0>f_{L_0^-}(3)$. Thus, $g_K(x_0)=x_1>y_1$.

Finally, since $f_K([y_1,z_1]\cup[y_1,x_1])$ disconnects K from ∞ in $\mathbb{H}\setminus L$, and disconnects $[y_0,x_0]$ from ∞ in $\mathbb{H}\setminus L_{y_0}^-$, we get

$$\mathrm{hm}(\infty,\mathbb{H}\setminus L_{y_0}^-;[y_0,x_0])\le \mathrm{hm}(\infty,\mathbb{H}\setminus L_{y_1}^-;[y_1,y_1+i\pi]\cup[y_1,x_1]),$$

which implies that $g_{L_0^-}(x_0-y_0)-1\le 2+g_{L_0^-}(x_1-y_1)-1$. So the proof is finished. □

Let K_t, $0\le t\le t_0$, be chordal Loewner hulls driven by W_t, $0\le t\le t_0$. Recall that every K_t is an $\mathbb{H}$-hull with $\mathrm{hcap}(K_t)=2t$. From (2.2) it is easy to see that

$$\sup\{\Re z: z\in K_{t_0}\}\le\max\{W_t:0\le t\le t_0\},\quad \sup\{\Im z:z\in K_{t_0}\}\le\sqrt{4t_0}. \tag{4.9}$$

From [LSW01, Theorem 2.6] and [Zha08, Lemma 5.3], we know that

$$W_t\in[c_{K_{t_0}},d_{K_{t_0}}],\quad 0\le t\le t_0. \tag{4.10}$$

Lemma 4.13. *Let $R = L_y^- \cap L_x^+$ for some $x < y \in \mathbb{R}$. Then $c_R \ge x - 2$.*

Proof. Let $m = (x+y)/2$. Then R is symmetric w.r.t. $\{\Re z = m\}$. So $g_R(m+i\pi) = m$. By conformal invariance and comparison principle of harmonic measures, for any $h > \pi$, we get

$$h \cdot \mathrm{hm}(g_R(m+ih), \mathbb{H}; [g_R(x+i\pi), m]) = h \cdot \mathrm{hm}(m+ih, \mathbb{H} \setminus R; [x+i\pi, m+i\pi])$$
$$\le h \cdot \mathrm{hm}(m+ih, \{\Im z > \pi\}; [x+i\pi, m+i\pi]) = h \cdot \mathrm{hm}(m+i(h-\pi), \mathbb{H}; [x,m]).$$

Letting $h \to \infty$, we get $m - g_R(x+i\pi) \le m - x$, and so $g_R(x+i\pi) \ge x$. Similarly,

$$h \cdot \mathrm{hm}(g_R(m+ih), \mathbb{H}; [g_R(x), g_R(x+i\pi)]) = h \cdot \mathrm{hm}(m+ih, \mathbb{H} \setminus R; [x, x+i\pi]) \le h \cdot \mathrm{hm}(m+ih, \mathbb{H} \setminus L_x^+; [x, x+i\pi]).$$

Letting $h \to \infty$, and using Lemma 4.11 (applied to right half strips) and $(g_R(m+ih) - (m+ih))/h \to 1$ as $h \to \infty$, we get $g_R(x+i\pi) - g_R(x) \le 2$. Thus, $c_R = g_R(x) \ge g_R(x+i\pi) - 2 \ge x - 2$. ☐

Lemma 4.14. *Let $t_0 = \pi^2/4$. We have $K_{t_0} \cap L_y^- \ne \emptyset$ if $y > \min\{W_t : 0 \le t \le t_0\} + 2$.*

Proof. Let $l = \min\{W_t : 0 \le t \le t_0\}$ and $r = \max\{W_t : 0 \le t \le t_0\}$. From (4.9), we know that $K_{t_0} \subset L_r^-$. Suppose $K_{t_0} \cap L_y^- = \emptyset$ for some $y > l+2$. Then $K_{t_0} \subset R := L_y^+ \cap L_r^-$. From [Zha08, Lemma 5.3], we get $[c_{K_{t_0}}, d_{K_{t_0}}] \subset [c_R, d_R]$. From the above lemma, we get $c_{K_{t_0}} \ge c_R \ge y - 2 > l$, which contradicts (4.10). So the proof is finished. ☐

The above lemma means that, if $\min\{W_t : 0 \le t \le \pi^2/4\} < y - 2$, and if (W_t) generates a chordal Loewner curve η, then η visits L_y^- before $\frac{\pi^2}{4}$.

4.4 Estimate on the derivative

Proposition 4.15. *Assume the same setup as that in Proposition 3.1 except that (3.1) is replaced by*

$$4b \ge (\lambda - b)(\kappa\lambda - \kappa b + 4 - \kappa). \tag{4.11}$$

Let τ_ε be the first time that $|\eta(t) - 1| \le \varepsilon$. Then we have

$$\mathbb{E}\left[(g_{\tau_\varepsilon}(1) - W_{\tau_\varepsilon})^{\lambda-b} g'_{\tau_\varepsilon}(1)^b 1_{\{\tau_\varepsilon < T_0\}}\right] \asymp \varepsilon^{u_1(\lambda)+\lambda-b}, \tag{4.12}$$

where the constants in $\asymp$ depend only on κ, λ, b.

Proof. Let $X_t = (g_t(1) - W_t)^{\lambda-b} g'_t(1)^b 1_{\{t<T_0\}}$ and $\beta = u_1(\lambda) + \lambda - b$. First, (4.11) implies (3.1) and $\beta \ge 0$. By Proposition 3.1, we have

$$\mathbb{E}\left[X_{\hat\tau(\varepsilon)} 1_{\{\hat\tau(\varepsilon)<T_0\}}\right] \asymp \varepsilon^\beta.$$

From (4.11), we straightforwardly check that X_t is a super martingale using Itô's formula. In fact, if the equality in (4.11) holds, then X_t agrees with the local martingale in Lemma 2.3 with $\rho^L = 0$, $x^R = 1$, and $\rho^R = \kappa(\lambda - b)$. Also note that $g'_t(1)$ is decreasing. Thus, from $\hat\tau_\varepsilon \le \tau_\varepsilon$, we get

$$\mathbb{E}\left[X_{\tau(\varepsilon)} 1_{\{\tau(\varepsilon)<T_0\}}\right] \le \mathbb{E}\left[X_{\hat\tau(\varepsilon)} 1_{\{\hat\tau(\varepsilon)<T_0\}}\right] \asymp \varepsilon^\beta.$$

To prove the reverse inequality, we follow the proof of Proposition 3.1 to get

$$\mathbb{E}\left[X_{\tau(\varepsilon)} 1_{\{\hat\tau(\varepsilon)<T_0\}}\right] \asymp \varepsilon^\beta \mathbb{E}^*[J_{\tau_\varepsilon}^{-\beta}] \ge \varepsilon^\beta,$$

using $\Upsilon_{\tau_\varepsilon} \asymp \varepsilon$, $0 < J_t \le 1$ and $\beta \ge 0$. ☐

4.5 Proof of Theorem 4.4

Proof of Theorem 4.4. From Remark 3.3, we have (4.2) and (4.4) for $n=1$.
From $2n-1$ **to** $2n$: Suppose (4.2) and (4.4) hold. Let σ be the hitting time at L_y^-.
upper bound. If $y \ge 0$, then we use the estimate

$$\mathbb{P}[H_{2n}^{\pi}(\varepsilon,x,y)] \le \mathbb{P}[H_{2n-1}^{\pi}(\varepsilon,x,y)] \lesssim \frac{x^{\alpha_{2n-2}^+-\alpha_{2n-1}^+}\varepsilon^{\alpha_{2n-1}^+}}{\prod_{j=1}^{n-1}\phi^{(2n-2j-1)}(x-y)^{\alpha_{2j}^+-\alpha_{2j-2}^+}} \le \frac{x^{\alpha_{2n}^+-\alpha_{2n-1}^+}\varepsilon^{\alpha_{2n-1}^+}}{\prod_{j=1}^{n}\phi^{(2n-2j)}(x-y)^{\alpha_{2j}^+-\alpha_{2j-2}^+}},$$

where the last inequality follows from $\phi^{(2n-2j-1)}(x-y) \ge \phi^{(2n-2j)}(x-y)$, $x \ge x-y = \phi^{(0)}(x-y)$, and $\alpha_{2j}^+ \ge \alpha_{2j-2}^+$. So we get (4.3).

If $y<0$, then $\eta(\sigma) \in \partial_{\mathbb{H}}^- L_y^-$, and the righthand side of $\eta[0,\sigma]$ disconnects the union of $[\Re\eta(\sigma),0]$ and the righthand side of the line segment $[\Re\eta(\sigma),\eta(\sigma)]$ in $\mathbb{H}\setminus[\Re\eta(\sigma),\eta(\sigma)]$. From the comparison principal and conformal invariance of harmonic measure, we get

$$\begin{aligned}&\mathrm{hm}(\infty,\mathbb{H}\setminus\eta[0,\sigma];\text{RHS of }\eta[0,\sigma]) \ge \mathrm{hm}(\infty,\mathbb{H}\setminus(\eta[0,\sigma]\cup[\Re\eta(\sigma),\eta(\sigma)]);\text{RHS of }\eta[0,\sigma])\\ \ge&\mathrm{hm}(\infty,\mathbb{H}\setminus[\Re\eta(\sigma),\eta(\sigma)];[\Re\eta(\sigma),0]\cup\text{ RHS of }[\Re\eta(\sigma),\eta(\sigma)]).\end{aligned}$$

Since $\Re\eta(\sigma) \le y$, we get

$$g_\sigma(x)-W_\sigma \ge x-y. \tag{4.13}$$

The following local martingale is similar to the one used in the proof of Lemma 3.4 (recall (3.7)):

$$M_t = |g_t(x+3\varepsilon)-W_t|^{\alpha_{2n}^+-\alpha_{2n-1}^+} g_t'(x+3\varepsilon)^{\alpha_{2n-1}^+}.$$

The law of η weighted by M_t/M_0 is SLE$(\kappa;\nu)$ with force point at $x+3\varepsilon$, where $\nu = \kappa(\alpha_{2n}^+-\alpha_{2n-1}^+)$. Let $\mathbb{E}^*$ denote the expectation w.r.t. this SLE$(\kappa;\nu)$ process. Let $\varepsilon_1 = 4(g_\sigma(x+3\varepsilon)-g_\sigma(x+\varepsilon))$, $x_1 = g_\sigma(x+3\varepsilon)$, and $y_1 = \sup\{\Re g_\sigma(z) : z \in \partial_{\mathbb{H}}^{\sigma} L_y^-\}$, where we use $\partial_{\mathbb{H}}^{\sigma} L_y^-$ to denote the remaining part of $\partial_{\mathbb{H}}^- L_y^-$ at time σ in the positive direction, i.e., the unbounded component of $\partial_{\mathbb{H}}^- L_y^- \setminus \eta[0,\sigma]$. Then $g_\sigma(\partial_{\mathbb{H}}^{\sigma} L_y^-) \subset L_{y_1}^-$. From Lemma 2.1, the g_σ-image of the remaining part of $\partial_{\mathbb{H}}^+ B(x,\varepsilon)$ at time σ in the positive direction (which touches $x+\varepsilon$), denoted by $\partial_{\mathbb{H}}^{\sigma} B(x,\varepsilon)$ is enclosed by $\partial_{\mathbb{H}}^+ B(x_1,\varepsilon_1)$. From (4.13), we get

$$\varepsilon_1 \le 8\varepsilon \le 2^{5n-1}\varepsilon \le \phi^{(2n-1)}(x-y) \le x-y \le x_1 - W_\sigma.$$

This means that $\partial_{\mathbb{H}}^+ B(x_1,\varepsilon_1)$ disconnects W_σ from $g_\sigma(\partial_{\mathbb{H}}^{\sigma} B(x,\varepsilon))$. From Lemma 4.12, we have $x_1 - y_1 \ge \phi(x-y) \ge 2^4\varepsilon > \varepsilon_1$. So we may apply Lemma 4.6 and use DMP of SLE to get

$$\mathbb{P}[H_{2n}^{\pi}(\varepsilon,x,y)|\eta[0,\sigma]] \le H_{2n-1}^{\pi}(\varepsilon_1, x_1-W_\sigma, y_1-W_\sigma).$$

We assumed that (ε,x,y) satisfy $2^{5n-1}\varepsilon < \phi^{(2n-1)}(x-y)$. Since $g_\sigma' \le 1$ on $\mathbb{R}\setminus K_\sigma$, we have $\varepsilon_1 \le 8\varepsilon$. So we get

$$2^{5n-4}\varepsilon_1 \le 2^{5n-1}\varepsilon < \phi^{(2n-1)}(x-y) \le \phi^{(2n-2)}(x_1-y_1).$$

This means that $(\varepsilon_1, x_1-W_\sigma, y_1-W_\sigma)$ satisfy the conditions for (4.2). From the induction hypothesis, we get

$$\begin{aligned}&\mathbb{P}[H_{2n-1}^{\pi}(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma)] \lesssim f_n(x_1-y_1)(x_1-W_\sigma)^{\alpha_{2n-2}^+-\alpha_{2n-1}^+}\varepsilon_1^{\alpha_{2n-1}^+}\\ \le& f_n(x_1-y_1)(g_\sigma(x+3\varepsilon)-W_\sigma)^{\alpha_{2n-2}^+-\alpha_{2n-1}^+}(g_\sigma'(x+3\varepsilon)\varepsilon)^{\alpha_{2n-1}^+},\end{aligned}$$

where $f_n(x_1-y_1)$ is the factor coming from the denominator of (4.2), and the last inequality follows from $0 < g_\sigma(x+3\varepsilon)-g_\sigma(x+\varepsilon) \le g_\sigma(x+3\varepsilon)-V_\sigma \le 3g_\sigma'(x+3\varepsilon)\varepsilon$ and $\alpha_{2n-1}^+, \alpha_{2n-1}^+ \ge 0$. So we get

$$\begin{aligned}&\mathbb{P}[H_{2n}^{\pi}(\varepsilon,x,y)] = \mathbb{E}[\mathbb{P}[H_{2n}^{\pi}(\varepsilon,x,y)|\eta[0,\sigma]]] \le \mathbb{E}[H_{2n-1}^{\pi}(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma)]\\ \lesssim& f_n(x_1-y_1)\varepsilon^{\alpha_{2n-1}^+}\mathbb{E}[(g_\sigma(x+3\varepsilon)-W_\sigma)^{\alpha_{2n-2}^+-\alpha_{2n-1}^+}\cdot g_\sigma'(x+3\varepsilon)^{\alpha_{2n-1}^+}]\\ \le& f_n\circ\phi(x-y)\varepsilon^{\alpha_{2n-1}^+}M_0\mathbb{E}^*[(g_\sigma(x+3\varepsilon)-W_\sigma)^{\alpha_{2n-2}^+-\alpha_{2n}^+}]\\ \le& f_n\circ\phi(x-y)(x-y)^{\alpha_{2n-2}^+-\alpha_{2n}^+}(x+3\varepsilon)^{\alpha_{2n}^+-\alpha_{2n-1}^+}\varepsilon^{\alpha_{2n-1}^+},\end{aligned}$$

where in the second last inequality we used $x_1 - y_1 \ge \phi(x-y)$, and in the last inequality we used $\alpha^+_{2n-2} \le \alpha^+_{2n-1}$ and (4.13). Since $\varepsilon \le x$, we get (4.3).

Lower bound. We use the local martingale (similar to the one above):

$$M_t = g_t'(x)^{\alpha^+_{2n-1}} |g_t(x) - W_t|^{\alpha^+_{2n} - \alpha^+_{2n-1}}.$$

The law of η weighted by M_t/M_0 is SLE$(\kappa;\nu)$ with force point at x, where $\nu = \kappa(\alpha^+_{2n} - \alpha^+_{2n-1})$. Let $\mathbb{E}^*$ and $\mathbb{P}^*$ denote the expectation and probability w.r.t. this SLE$(\kappa;\nu)$ process.

Fix $R > 1 > \delta > 0$ and suppose $x - y \le R$. In the proof below, we use C to denote a positive constant, which depends only on κ, n, R, δ, and may change values between lines. Let $F(\delta)$ denote the event that $\eta[0,\sigma] \subset B(0, \frac{1}{\delta})$, η does not swallows x at σ, and $\mathrm{dist}(\eta[0,\sigma], x) \ge \delta x$. Suppose $F(\delta)$ occurs. From Lemma 4.9, the image of the unbounded connected component of $L^-_y \setminus \eta[0,\sigma]$ under g_σ contains $L^-_{y_1;\frac{\pi}{2}}$ for $y_1 := \min\{y - \frac{\pi}{2}, -\frac{2}{\delta} - \frac{2}{\pi\delta^2}\}$. Assume that $\varepsilon \le \frac{\delta x}{2}$. From Koebe's distortion theorem, the g_σ-image of $\partial^+_{\mathbb{H}} B(x,\varepsilon)$ encloses $\partial^+_{\mathbb{H}} B(x_1,\varepsilon_1)$, where $x_1 = g_\sigma(x)$ and $\varepsilon_1 = \frac{4}{9} g_\sigma'(x)\varepsilon$. Let $x_2 = 2(x_1 - W_\sigma)$, $y_2 = 2(y_1 - W_\sigma)$, and $\varepsilon_2 = 2\varepsilon_1$. From DMP and scaling property of SLE and Lemma 4.6, we get

$$\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y) | \eta[0,\sigma], F(\delta)] \ge H^\pi_{2n-1}(\varepsilon_2,x_2,y_2), \quad \text{if } \varepsilon \le \delta x/2.$$

From [Law05, (3.12)], we get $|x_1 - x| \le \frac{3}{\delta}$. So we have

$$x_1 - y_1 \le \max\{x - y + \frac{3}{\delta} + \frac{\pi}{2}, x + \frac{2}{\delta} + \frac{2}{\pi\delta^2}\} \le R + \frac{5}{\delta^2}. \tag{4.14}$$

Let $R_2 = 2(R + \frac{5}{\delta^2})$. Then $x_2 - y_2 \le R_2$, and R_2 depends only on R and δ. From the induction hypothesis, on the event $F(\delta)$, we have

$$\mathbb{P}[H^\pi_{2n-1}(\varepsilon_2,x_2,y_2)] \ge C x_2^{\alpha^+_{2n-2} - \alpha^+_{2n-1}} \varepsilon_2^{\alpha^+_{2n-1}} = C g_\sigma'(x)^{\alpha^+_{2n-1}} (g_\sigma(x) - W_\sigma)^{\alpha^+_{2n-2} - \alpha^+_{2n-1}} \varepsilon^{\alpha^+_{2n-1}}.$$

Thus, if $\varepsilon \le \delta x/2$, then

$$\begin{aligned}
&\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \ge \mathbb{E}[\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)|\eta[0,\sigma],F(\delta)]] \ge \mathbb{E}[1_{F(\delta)} H^\pi_{2n-1}(\varepsilon_2,x_2,y_2)] \\
\ge& C\varepsilon^{\alpha^+_{2n-1}} \mathbb{E}[1_{F(\delta)} g_\sigma'(x)^{\alpha^+_{2n-1}} (g_\sigma(x) - W_\sigma)^{\alpha^+_{2n-2} - \alpha^+_{2n-1}}] \\
=& C\varepsilon^{\alpha^+_{2n-1}} M_0 \mathbb{E}^*[1_{F(\delta)} (g_\sigma(x) - W_\sigma)^{\alpha^+_{2n-2} - \alpha^+_{2n}}] \ge C x^{\alpha^+_{2n} - \alpha^+_{2n-1}} \varepsilon^{\alpha^+_{2n-1}} \mathbb{P}^*[F(\delta)],
\end{aligned}$$

where we used $g_\sigma(x) - W_\sigma \le x_1 - y_1 \le R + \frac{5}{\delta^2}$ in the last inequality.

We now find some $\delta, C \in (0,1)$ depending only on κ, n, R such that $\mathbb{P}^*[F(\delta)] \ge C$. After choosing that δ, the constants C we had earlier also depend only on κ, n, R. Let η be a chordal SLE(κ,ν) curve started from 0 with force point x, and let W be the driving function. Since $\nu \ge (\frac{\kappa}{2} - 2) \vee 0$ and $x > 0$, W_t is stochastically bounded above by $\sqrt{\kappa} B_t$, η never swallows x, and $\mathrm{dist}(\eta[0,\infty), x) > 0$. Let E_W denote the event that $\min\{W_t : 0 \le t \le \pi^2/4\} < -R-2$ and $\max\{W_t : 0 \le t \le \pi^2/4\} \le R$, and let E_B denote a similar event with $\sqrt{\kappa} B_t$ in place of W_t. Then the probability of E_W is bounded below by the probability of E_B, which is bounded below by some $C_1 > 0$ depending only on κ, R. When E_W occurs, from Lemmas 4.9 and 4.14, we get $\sigma \le \pi^2/4$ and $\eta[0,\sigma] \subset [y,R] \times [0,\pi] \subset B(0, \frac{1}{\delta_1})$ for $\delta_1 = \frac{1}{R+\pi}$. By the scaling property of SLE(κ,ν) curve, we see that $\mathrm{dist}(\eta[0,\infty), x)/x$ is a positive random variable, whose distribution depends only on κ, n (but not on x). So there is $\delta_2 > 0$ depending only on κ, n, R such that the probability that $\mathrm{dist}(\eta[0,\infty), x) \le \delta_2 x$ is at most $C_1/2$. Let $\delta = \delta_1 \wedge \delta_2$ and $C = C_1/2$. Then $\mathbb{P}^*[F(\delta)] \ge C$. For such δ, if $\varepsilon \le \delta x/2$, then $\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \ge C x^{\alpha^+_{2n} - \alpha^+_{2n-1}} \varepsilon^{\alpha^+_{2n-1}}$. Finally, if $\varepsilon \ge \delta x/2$, then by comparison principle, we have

$$\mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \ge \mathbb{P}[H^\pi_{2n}(\delta x/2,x,y)] \ge C x^{\alpha^+_{2n}} \ge C x^{\alpha^+_{2n} - \alpha^+_{2n-1}} \varepsilon^{\alpha^+_{2n-1}},$$

where we used $\varepsilon \le x$ and $\alpha^+_{2n-1} \ge 0$ in the last inequality. So we get (4.5) as long as $\varepsilon \le x$.

From $2n$ **to** $2n+1$. Suppose (4.3) and (4.5) hold. We use the local martingale

$$M_t = g_t'(x)^{\alpha^+_{2n+1}}(g_t(x)-W_t)^{\alpha^+_{2n}-\alpha^+_{2n+1}} = g_t'(x)^{\alpha^+_{2n-1}}(g_t(x)-W_t)^{\alpha^+_{2n}-\alpha^+_{2n-1}}\Upsilon_t^{\alpha^+_{2n-1}-\alpha^+_{2n+1}}J_t^{\alpha^+_{2n+1}-\alpha^+_{2n-1}},$$

which is similar to the one used in the proof of Proposition 3.1 (recall (3.8)). The law of η weighted by M_t/M_0 is SLE$(\kappa;\nu)$ with force point at x, where $\nu=\kappa(\alpha^+_{2n}-\alpha^+_{2n+1})$. Let $\mathbb{E}^*$ and $\mathbb{P}^*$ denote the expectation and probability w.r.t. this SLE$(\kappa;\nu)$ process. Let τ_r be the hitting time at $\partial^+_{\mathbb{H}}B(x,r)$ for any $r>0$. Recall that $\Upsilon_{\tau_r}\asymp r$.

Upper bound. First, suppose $6\varepsilon \ge x$. Then we use the estimate

$$\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)] \le \mathbb{P}[H^\pi_{2n}(\varepsilon,x,y)] \lesssim \frac{x^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}}{\prod_{j=1}^n \phi^{(2n-2j)}(x-y)^{\alpha^+_{2j}-\alpha^+_{2j-2}}} \lesssim \frac{x^{\alpha^+_{2n}-\alpha^+_{2n+1}}\varepsilon^{\alpha^+_{2n+1}}}{\prod_{j=1}^n \phi^{(2n-2j-1)}(x-y)^{\alpha^+_{2j}-\alpha^+_{2j-2}}},$$

where we used $\alpha^+_{2j}\ge\alpha^+_{2j-2}$, $\phi^{(2n-2j)}(x-y)\le\phi^{(2n-2j-1)}(x-y)$, $\alpha^+_{2n+1}\ge\alpha^+_{2n-1}$, and $\varepsilon\gtrsim x$. So we get (4.2).

Now suppose $6\varepsilon<x$. Let $\sigma=\tau_{6\varepsilon}$. Then $\eta_\sigma\in\partial^+_{\mathbb{H}}B(x,6\varepsilon)$. Let $\varepsilon_1=g'_\sigma(x)\varepsilon/(1-1/6)^2$, $x_1=g_\sigma(x)$, $y_1=\sup\{\Re g_\sigma(z): z\in\partial^\sigma_{\mathbb{H}}L^-_y\}$, where $\partial^\sigma_{\mathbb{H}}L^-_y$ is the unbounded connected component of $\partial^-_{\mathbb{H}}L^-_y\setminus\eta[0,\sigma]$. Then $g_\sigma(\partial^\sigma_{\mathbb{H}}L^-_y)\subset L^-_{y_1}$ because g_σ decreases the imaginary part. From Koebe's distortion theorem, the image of $\partial^+_{\mathbb{H}}B(x,\varepsilon)$ under g_σ is enclosed by $\partial^+_{\mathbb{H}}B(x_1,\varepsilon_1)$.

Since the semicircle $\partial^+_{\mathbb{H}}B(x,6\varepsilon)$ disconnects the union of $[0,x)$ and the righthand side of $\eta[0,\sigma)$ from ∞ in $\mathbb{H}\setminus\eta[0,\sigma]$, by the conformal invariance and comparison principle for harmonic measure, we have

$$\begin{aligned}&\mathrm{hm}(\infty,\mathbb{H};[x-12\varepsilon,x+12\varepsilon]) = \mathrm{hm}(\infty,\mathbb{H},\partial^+_{\mathbb{H}}B(x,6\varepsilon)) \ge \mathrm{hm}(\infty,\mathbb{H}\setminus\eta[0,\sigma];\partial^+_{\mathbb{H}}B(x,6\varepsilon))\\ \ge&\mathrm{hm}(\infty,\mathbb{H}\setminus\eta[0,\sigma];[0,x]\cup\text{ RHS of }\eta[0,\sigma]) = \mathrm{hm}(\infty,\mathbb{H};[W_\sigma,x_1]).\end{aligned}$$

Thus, $x_1-W_\sigma\le 24\varepsilon$. Since $x_1-y_1\ge\phi(x-y)\ge\phi^{(2n)}(x-y)\ge 2^{5n}\varepsilon>24\varepsilon$, we get $y_1-W_\sigma<0$. This means that $\partial^-_{\mathbb{H}}L^-_{y_1}$ disconnects W_σ from $g_\sigma(\partial^\sigma_{\mathbb{H}}L^-_y)$. Besides, since $g'_\sigma(x)\in(0,1)$, we have $x_1-y_1>\varepsilon_1$. So we may apply Lemma 4.6 and use DMP of SLE to get

$$\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)|\eta[0,\sigma]] \le H^\pi_{2n}(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma).$$

We assumed that (ε,x,y) satisfy $2^{5n}\varepsilon<\phi^{(2n)}(x-y)$. Since $g'_\sigma\le 1$ on $\mathbb{R}\setminus K_\sigma$, we have $\varepsilon_1\le 4\varepsilon$. Thus,

$$2^{5n-2}\varepsilon_1\le 2^{5n}\varepsilon<\phi^{(2n)}(x-y)\le\phi^{(2n-1)}(x_1-y_1).$$

From Koebe's $1/4$ theorem, we get $x_1-W_\sigma\ge 6g'_\sigma(x)\varepsilon/4\ge g'_\sigma(x)\varepsilon/(1-1/6)^2=\varepsilon_1$. This means that $(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma)$ satisfy the conditions for (4.3). From the induction hypothesis, we get

$$\begin{aligned}&\mathbb{P}[H^\pi_{2n}(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma)] \lesssim f_n(x_1-y_1)(x_1-W_\sigma)^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon_1^{\alpha^+_{2n-1}}\\ \asymp& f_n(x_1-y_1)\varepsilon^{\alpha^+_{2n-1}}(g_\sigma(x)-W_\sigma)^{\alpha^+_{2n}-\alpha^+_{2n-1}}g'_\sigma(x)^{\alpha^+_{2n-1}},\end{aligned}$$

where $f_n(x_1-y_1)$ is the factor coming from the denominator of (4.3). Thus,

$$\begin{aligned}&\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)] = \mathbb{E}[\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)|\eta[0,\sigma]]] \le \mathbb{E}[H^\pi_{2n}(\varepsilon_1,x_1-W_\sigma,y_1-W_\sigma)]\\ \lesssim& f_n(x_1-y_1)\varepsilon^{\alpha^+_{2n-1}}\mathbb{E}[(g_\sigma(x)-W_\sigma)^{\alpha^+_{2n}-\alpha^+_{2n-1}}g'_\sigma(x)^{\alpha^+_{2n-1}}]\\ \lesssim& f_n\circ\phi(x-y)\varepsilon^{\alpha^+_{2n-1}}x^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{u_1(\alpha^+_{2n})+\alpha^+_{2n}-\alpha^+_{2n-1}} = f_n\circ\phi(x-y)x^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n+1}}\end{aligned}$$

where we used Proposition 4.15, the scaling invariance of SLE, and ((3.8)). Then we get (4.2) for $2n+1$.

Lower bound. We fix $R,\delta>0$ and suppose $x-y\le R$. In the proof below, we use C to denote a positive constant, which depends only on κ,n,R,δ, and may change values between lines. Let $\sigma=\tau_\varepsilon$. From Koebe's $1/4$ theorem, the g_σ-image of $\partial^+_{\mathbb{H}}B(x,\varepsilon)$ encloses $\partial^+_{\mathbb{H}}B(x_1,\varepsilon_1)$, where $x_1=g_\sigma(x)$ and $\varepsilon_1=g'_\sigma(x)\varepsilon/4$. Let $F(\delta)$ denote the event that $\sigma<\infty$, x is not swallowed at σ, and $\eta[0,\sigma]\subset B(0,\frac{1}{\delta})$. Suppose $F(\delta)$ occurs. From Lemma 4.9, the image of the unbounded connected component of $L_y^- \setminus \eta[0,\sigma]$ under g_σ contains $L^-_{y_1;\frac{\pi}{2}}$ for $y_1 := \min\{y - \frac{\pi}{2}, -\frac{2}{\delta} - \frac{2}{\pi\delta^2}\}$. Let $x_2 = 2(x_1 - W_\sigma)$, $y_2 = 2(y_1 - W_\sigma)$, and $\varepsilon_2 = 2\varepsilon_1$. From DMP and scaling property of SLE and Lemma 4.6, we get

$$\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)|\eta[0,\sigma],F(\delta)] \ge H^\pi_{2n}(\varepsilon_2,x_2,y_2).$$

Using the same argument as around (4.14), we get $x_2 - y_2 \le R_2 := 2(R + \frac{5}{\delta^2})$. From the induction hypothesis, on the event $F(\delta)$, we have

$$\mathbb{P}[H^\pi_{2n}(\varepsilon_2,x_2,y_2)] \ge Cx_2^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon_2^{\alpha^+_{2n-1}} = Cg'_\sigma(x)^{\alpha^+_{2n-1}}(g_\sigma(x)-W_\sigma)^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}.$$

Thus,

$$\begin{aligned}&\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)] \ge \mathbb{E}[\mathbb{P}[H^\pi_{2n+1}(\varepsilon,x,y)|\eta[0,\sigma],F(\delta)]] \ge \mathbb{E}[1_{F(\delta)}H^\pi_{2n}(\varepsilon_2,x_2,y_2)]\\ \ge& C\varepsilon^{\alpha^+_{2n-1}}\mathbb{E}[1_{F(\delta)}g'_\sigma(x)^{\alpha^+_{2n-1}}(g_\sigma(x)-W_\sigma)^{\alpha^+_{2n}-\alpha^+_{2n-1}}]\\ =& C\varepsilon^{\alpha^+_{2n-1}}M_0\mathbb{E}^*[1_{F(\delta)}J_\sigma^{\alpha^+_{2n-1}-\alpha^+_{2n+1}}\Upsilon_\sigma^{\alpha^+_{2n+1}-\alpha^+_{2n-1}}] \ge Cx^{\alpha^+_{2n}-\alpha^+_{2n-1}}\varepsilon^{\alpha^+_{2n-1}}\mathbb{P}^*[F(\delta)],\end{aligned} \tag{4.15}$$

where in the last inequality we used $\Upsilon_\sigma \asymp \varepsilon$, $J_\sigma \in (0,1]$, and $\alpha^+_{2n-1} - \alpha^+_{2n+1} \le 0$.

We now find some $\delta, C > 0$ depending only on κ, n, R such that $\mathbb{P}^*[F(\delta)] \ge C$. After choosing that δ, the constants C we had earlier also depend only on κ, n, R. Let η be a chordal SLE(κ,ν) curve started from 0 with force point x. Since $\nu \le \kappa/2 - 4$, the curve η goes all the way to x in finite time, and so is bounded. Moreover, η does not swallow x before it reaches x. By scaling property, diam$(\eta)/x$ is a bounded random variable, whose distribution depends only on κ, n. Thus, there are constants $\delta_1, C > 0$ depending only on κ, n, such that $\mathbb{P}^*[F(\delta_1/x)] \ge C$. Then we let $\delta = \delta_1/R$. Since $x \le x - y \le R$, we have $F(\delta_1/x) \subset F(\delta)$. Using such δ and applying (4.15), we get (4.4) for $2n+1$. □

References

[Ahl06] Lars V Ahlfors. Lectures on quasiconformal mappings. with supplemental chapters by cj earle, i. kra, m. shishikura and jh hubbard. university lecture series, 38. *American Mathematical Society, Providence, RI*, 1(4):12, 2006.

[Ahl10] Lars Valerian Ahlfors. *Conformal invariants: topics in geometric function theory*, volume 371. American Mathematical Soc., 2010.

[AS08] Tom Alberts and Scott Sheffield. Hausdorff dimension of the SLE curve intersected with the real line. *Electron. J. Probab.*, 13:no. 40, 1166–1188, 2008.

[CDCH13] Dmitry Chelkak, Hugo Duminil-Copin, and Clément Hongler. Crossing probabilities in topological rectangles for the critical planar fk-ising model. *arXiv preprint arXiv:1312.7785*, 2013.

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of ising interfaces to schramms sle curves. *Comptes Rendus Mathematique*, 352(2):157–161, 2014.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[Dub09] Julien Dubédat. Duality of Schramm-Loewner evolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):697–724, 2009.

[Dup03] Bertrand Duplantier. Conformal fractal geometry and boundary quantum gravity. *arXiv preprint math-ph/0303034*, 2003.

[Kes87] Harry Kesten. Scaling relations for 2d-percolation. *Communications in Mathematical Physics*, 109(1):109–156, 1987.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[Law14] Gregory F. Lawler. Minkowski content of the intersection of a Schramm-Loewner evolution (SLE) curve with the real line. *Available from http://www. math. uchicago. edu/lawler/papers. html*, 2014.

[LSW01] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[MS12] Jason Miller and Scott Sheffield. Imaginary Geometry I: Interacting SLEs. 2012.

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.*, 202(1):21–137, 2009.

[SW01] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[SW05] Oded Schramm and David B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669 (electronic), 2005.

[Wu16a] Hao Wu. Alternating arm exponents for the critical planar ising model. *arXiv preprint arXiv:1605.00985*, 2016.

[Wu16b] Hao Wu. Polychromatic Arm Expoentns for the Critical Planar FK-Ising Model. *arXiv:1604.06639*, 2016.

[Zha08] Dapeng Zhan. The scaling limits of planar lerw in finitely connected domains. *Ann. Probab.*, 36(2):467–529, 2008.

[Zha16] Dapeng Zhan. Ergodicity of the tip of an sle curve. *Probability Theory and Related Fields*, 164(1-2):333–360, 2016.

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