# Some Properties of Annulus SLE

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February 2, 2008

**Abstract**

An annulus $\mathrm{SLE}_\kappa$ trace tends to a single point on the target circle, and the density function of the end point satisfies some differential equation. Some martingales or local martingales are found for annulus $\mathrm{SLE}_4$, $\mathrm{SLE}_8$ and $\mathrm{SLE}_{8/3}$. From the local martingale for annulus $\mathrm{SLE}_4$ we find a candidate of discrete lattice model that may have annulus $\mathrm{SLE}_4$ as its scaling limit. The local martingale for annulus $\mathrm{SLE}_{8/3}$ is similar to those for chordal and radial $\mathrm{SLE}_{8/3}$. But it seems that annulus $\mathrm{SLE}_{8/3}$ does not satisfy the restriction property.

## 1 Introduction

Schramm-Loewner evolution (SLE) is a family of random growth processes invented by O. Schramm in [12] by connecting Loewner differential equation with a one-dimentional Browinian motion. SLE depend on a single parameter $\kappa \geq 0$, and behaves differently for different value of $\kappa$. Schramm conjectured that SLE(2) is the scaling limit of some loop-erased random walks (LERW) and proved his conjuecture with some additional assumptions. He also suggested that SLE(6) and SLE(8) should be the scaling limits of certain discrete lattice models.

After Schramm's paper, there were many papers working on SLE. In the series of papers [4][5][6], the locality property of SLE(6) was used to compute the intersection exponent of plane Brwonian motion. In [14], SLE(6) was proved to be the scaling limit of the cite percolation explorer on the triangle lattice. It was proved in [7] that SLE(2) is the scaling limit of the corresponding loop-erased random walk (LERW), and SLE(8) is the scaling limit of some uniform spanning tree (UST) Peano curve. SLE(4) was proved to be the scaling limit of the harmonic exploer in [13]. SLE(8/3) satisfies restriction property, and was conjectured in [8] to be the scaling limit of some self avoiding walk (SAW). Chordal SLE$(\kappa, \rho)$ processes were also invented in [8], and they satisfy one-sided restriction property. For basic properties of SLE, see [11], [3], [16], [15].

The SLE invented by O. Schramm has a chordal and a radial version. They are all defined in simply connected domains. In [17], a new version of SLE, called annulus SLE, was defined in doubly connected domains as follows.

For $p > 0$, let the annulus

$$\mathbb{A}_p = \{z \in \mathbb{C} : e^{-p} < |z| < 1\},$$

and the circle

$$\mathbf{C}_p = \{z \in \mathbb{C} : |z| = e^{-p}\}.$$

Then $\mathbb{A}_p$ is bounded by $\mathbf{C}_p$ and $\mathbf{C}_0$. Let $\xi(t)$, $0 \le t < p$, be a real valued continuous function. For $z \in \mathbb{A}_p$, solve the annulus Loewner differential equation

$$\partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}_{p-t}(\varphi_t(z)/\exp(i\xi(t))), \;\; 0 \le t < p, \;\; \varphi_0(z) = z, \tag{1}$$

where for $r > 0$,

$$\mathbf{S}_r(z) = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{e^{2kr} + z}{e^{2kr} - z}.$$

For $0 \le t < p$, let $K_t$ be the set of $z \in \mathbb{A}_p$ such that the solution $\varphi_s(z)$ blows up before or at time $t$. Then for each $0 \le t < p$, $\varphi_t$ maps $\mathbb{A}_p \setminus K_t$ conformally onto $\mathbb{A}_{p-t}$, and maps $\mathbf{C}_p$ onto $\mathbf{C}_{p-t}$. We call $K_t$ and $\varphi_t$, respectively, $0 \le t < p$, the annulus LE hulls and maps, respectively, of modulus $p$, driven by $\xi(t)$, $0 \le t < p$. If $(\xi(t)) = \sqrt{\kappa}B(t)$, $0 \le t < p$, where $\kappa \ge 0$ and $B(t)$ is a standard linear Brownian motion, then $K_t$ and $\varphi_t$, $0 \le t < p$, are called standard annulus $\mathrm{SLE}_\kappa$ hulls and maps, respectively, of modulus $p$. Suppose $D$ is a doubly connected domain with finite modulus $p$, $a$ is a boundary point and $C$ is a boundary component of $D$ that does not contain $a$. Then there is $f$ that maps $\mathbb{A}_p$ conformally onto $D$ such that $f(1) = a$ and $f(\mathbf{C}_p) = C$. Let $K_t$, $0 \le t < p$, be standard annulus $\mathrm{SLE}_\kappa$ hulls. Then $(f(K_t), 0 \le t < p)$ is called an annulus $\mathrm{SLE}_\kappa(D; a \to C)$ chain.

It is known in [17] that annulus $\mathrm{SLE}_\kappa$ is weakly equivalent to radial $\mathrm{SLE}_\kappa$. so from the existence of radial $\mathrm{SLE}_\kappa$ trace, we know the existence of a standard annulus $\mathrm{SLE}_\kappa$ trace, which is $\beta(t) = \varphi_t^{-1}(\exp(i\xi(t)))$, $0 \le t < p$. Almost surely $\beta$ is a continuous curve in $\overline{\mathbb{A}_p}$, and for each $t \in [0, p)$, $K_t$ is the hull generated by $\beta((0, t])$, i.e., the complement of the component of $\mathbb{A}_p \setminus \beta((0, t])$ whose boundary contains $\mathbf{C}_p$. It is known that when $\kappa = 2$ or $\kappa = 6$, $\lim_{t\to p} \beta(t)$ exists and lies on $\mathbf{C}_p$ almost surely. In this paper, we prove that this is true for any $\kappa > 0$. And we discuss the density function of the distribution of the limit point. The density function should satisfy some differential equation.

When $\kappa = 2$, $8/3$, $4$, $6$, or $8$, radial and chordal $\mathrm{SLE}_\kappa$ satisfy some special properties. Radial $\mathrm{SLE}_6$ satisfies locality property. Since annulus $\mathrm{SLE}_6$ is (strongly) equivalent to radial $\mathrm{SLE}_6$, so annulus $\mathrm{SLE}_6$ also satisfies the locality property. Annulus $\mathrm{SLE}_2$ is the scaling limit of the corresponding loop-erased random walk. In this paper, we discuss the cases $\kappa = 4$, 8, and 8/3. We find martingales or local martingales for annulus SLE$_\kappa$ in each of these cases. From the local martingale for annulus SLE$_4$, we may construct a harmonic explorer whose scaling limit is annulus SLE$_4$. The martingales for annulus SLE$_{8/3}$ are similar to the martingales for radial and chordal SLE$_{8/3}$, which are used to show that radial and chordal SLE$_{8/3}$ satisfy the restriction property. However, the martingales for annulus SLE$_{8/3}$ does not help us to prove that annulus SLE$_{8/3}$ satisfies the restriction property. On the contrary, it seems that annulus SLE$_{8/3}$ does not satisfy the restriction property.

# 2 Annulus Loewner Evolution in the Covering Space

We often lift the annulus Loewner evolution to the covering space. Let $e^i$ denote the map $z \mapsto e^{iz}$. For $p > 0$, let $\mathbb{S}_p = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < p\}$, $\mathbb{R}_p = ip + \mathbb{R}$, and $\mathbf{H}_p(z) = \frac{1}{i}\mathbf{S}_p(e^i(z))$. Then $\mathbb{S}_p = (e^i)^{-1}(\mathbb{A}_p)$ and $\mathbb{R}_p = (e^i)^{-1}(\mathbf{C}_p)$. Solve

$$\partial_t \widetilde{\varphi}_t(z) = \mathbf{H}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)), \ \ \widetilde{\varphi}_0(z) = z. \tag{2}$$

For $0 \le t < p$, let $\widetilde{K}_t$ be the set of $z \in \mathbb{S}_p$ such that $\widetilde{\varphi}_t(z)$ blows up before or at time $t$. Then for each $0 \le t < p$, $\widetilde{\varphi}_t$ maps $\mathbb{S}_p \setminus \widetilde{K}_t$ conformally onto $\mathbb{S}_{p-t}$, and maps $\mathbb{R}_p$ onto $\mathbb{R}_{p-t}$. And for any $k \in \mathbb{Z}$, $\widetilde{\varphi}_t(z + 2k\pi) = \widetilde{\varphi}_t(z) + 2k\pi$. We call $\widetilde{K}_t$ and $\widetilde{\varphi}_t$, $0 \le t < p$, the annulus LE hulls and maps, respectively, of modulus $p$ in the covering space, driven by $\xi(t)$, $0 \le t < p$. Then we have $\widetilde{K}_t = (e^i)^{-1}(K_t)$ and $e^i \circ \widetilde{\varphi}_t = \varphi_t \circ e^i$. If $(\xi(t))_{0 \le t < p}$ has the law of $(\sqrt{\kappa}B(t))_{0 \le t < p}$, then $\widetilde{K}_t$ and $\widetilde{\varphi}_t$, $0 \le t < p$, are called standard annulus SLE$_\kappa$ hulls and maps, respectively, of modulus $p$ in the covering space.

It is clear that $\mathbf{H}_r$ is an odd function. It is analytic in $\mathbb{C}$ except at the set of simple poles $\{2k\pi + i2mr : k, m \in \mathbb{Z}\}$. And at each pole $z_0$, the principle part is $\frac{2}{z - z_0}$. For each $z \in \mathbb{C}$, $\mathbf{H}_r(z + 2\pi) = \mathbf{H}_r(z)$, and $\mathbf{H}_r(z + i2r) = \mathbf{H}_r(z) - 2i$.

Let

$$f_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z).$$

Then $f_r$ is an odd function. It is analytic in $\mathbb{C}$ except at the set of simple poles $\{z \in \mathbb{C} : i\frac{\pi}{r}z = 2k\pi + i2m\pi^2/r$, for some $k, m \in \mathbb{Z}\} = \{2m\pi - i2kr : m, k \in \mathbb{Z}\}$. And at each pole $z_0$, the principle part is $\frac{2}{z - z_0}$. We then compute

$$f_r(z + 2\pi) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z + i2\pi^2/r) = i\frac{\pi}{r}(\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - 2i) = f_r(z) + 2\frac{\pi}{r};$$

$$f_r(z + i2r) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z - 2\pi) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) = f_r(z).$$

Let $g_r(z) = f_r(z) - \mathbf{H}_r(z)$. Then $g_r$ is an odd entire function, and satisfies

$$g_r(z+2\pi) = g_r(z) + 2\pi/r, \quad g_r(z+i2r) = g_r(z) + 2i,$$

for any $z \in \mathbb{C}$. Thus $g_r(z) = z/r$. So we have

$$\mathbf{H}_r(z) = f_r(z) - g_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - \frac{z}{r}. \tag{3}$$

# 3 Long Term Behaviors of Annulus SLE Trace

In this section we fix $\kappa > 0$ and $p > 0$. Let $\varphi_t$ and $K_t$, $0 \le t < p$, be the annulus LE maps and hulls, respectively, of modulus $p$ driven by $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < p$. Let $\widetilde{\varphi}_t$ and $\widetilde{K}_t$ be the corresponding annulus LE maps and hulls in the covering space. Let $\beta(t)$ be the corresponding annulus $\mathrm{SLE}_\kappa$ trace.

Let $Z_t(z) = \widetilde{\varphi}_t(z) - \xi(t)$. Then we have

$$dZ_t(z) = \mathbf{H}_{p-t}(Z_t(z))dt - \sqrt{\kappa}dB(t).$$

Let $W_t(z) = \frac{\pi}{p-t}Z_t(z)$. Then $W_t$ maps $(\mathbb{S}_p \setminus \widetilde{K}_t, \mathbb{R}_p)$ conformally onto $(\mathbb{S}_\pi, \mathbb{R}_\pi)$. From Ito's formula and equation (3) we have

$$\begin{aligned} dW_t(z) = \frac{\pi dZ_t(z)}{p-t} + \frac{\pi Z_t(z)}{(p-t)^2} &= -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{\pi}{p-t}(\mathbf{H}_{p-t}(Z_t(z)) + \frac{Z_t(z)}{p-t})dt \\ &= -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{\pi}{p-t}i\frac{\pi}{p-t}\mathbf{H}_{\pi^2/(p-t)}(i\frac{\pi}{p-t}Z_t(z))dt \\ &= -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{i\pi^2}{(p-t)^2}\mathbf{H}_{\pi^2/(p-t)}(iW_t(z))dt. \end{aligned}$$

Now we change variables as follows. Let $s = u(t) = \pi^2/(p-t)$. Then $u'(t) = \pi^2/(p-t)^2$. For $\pi^2/p \le s < \infty$, let $\widehat{W}_s(z) = W_{u^{-1}(s)}(z)$. Then there is a standard one dimensional Brownian motion $(B_1(s), s \ge \pi^2/p)$ such that

$$d\widehat{W}_s(z) = \sqrt{\kappa}dB_1(s) + i\mathbf{H}_s(i\widehat{W}_s(z))ds,$$

Let $\widehat{\varphi}_s(z) = \widehat{W}_s(z) - \sqrt{\kappa}B_1(s)$. Then $\partial_s\widehat{\varphi}_s(z) = i\mathbf{H}_s(i\widehat{W}_s(z))$. Let $X_s(z) = \mathrm{Re}\,\widehat{W}_s(z)$. For $z \in \mathbb{R}_p$, we have $\widehat{W}_s(z), \widehat{\varphi}_s(z) \in \mathbb{R}_\pi$, so $\widehat{W}_s(z) = X_s(z) + i\pi$. Thus for $z \in \mathbb{R}_p$,

$$\partial_s \mathrm{Re}\,\widehat{\varphi}_s(z) = \mathrm{Re}\,\partial_s\widehat{\varphi}_s(z) = \mathrm{Re}\,(i\mathbf{H}_s(i(X_s(z)+i\pi))) = \lim_{M\to\infty}\sum_{k=-M}^{M}\frac{e^{X_s(z)} - e^{2ks}}{e^{X_s(z)} + e^{2ks}}. \tag{4}$$

Note that $\widehat{W}_s'(z)=\widehat{\varphi}_s'(z)$. So for $z\in\mathbb{R}_p$,

$$\partial_s\widehat{\varphi}_s'(z)=\sum_{k=-\infty}^{\infty}\frac{2e^{X_s(z)}e^{2ks}}{(e^{X_s(z)}+e^{2ks})^2}\widehat{\varphi}_s'(z),$$

which implies that

$$\partial_s\ln|\widehat{\varphi}_s'(z)|=\sum_{k=-\infty}^{+\infty}\frac{2e^{X_s(z)}e^{2ks}}{(e^{X_s(z)}+e^{2ks})^2}. \tag{5}$$

**Lemma 3.1** *For every $z\in\mathbb{R}_p$, $X_s(z)$ is not bounded on $[\pi^2/p,\infty)$ almost surely.*

**Proof.** Suppose the lemma is not true. Then there is $z_0\in\mathbb{R}_p$ and $a>0$ such that the probability that $|X_s(z_0)|<a$ for all $s\in[\pi^2/p,\infty)$ is positive. Let $X_s$ denote $X_s(z_0)$. Then we have

$$dX_s=\sqrt{\kappa}dB_1(s)+\left(\lim_{M\to\infty}\sum_{k=-M}^{M}\frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}}\right)ds.$$

Let $T_a$ be the first time that $|X_s|=a$. If such time does not exist, then let $T_a=\infty$. Let $f(x)=\int_{-\infty}^x\cosh(s/2)^{-4/\kappa}ds$. Then $f$ maps $\mathbb{R}$ onto $(0,C(\kappa))$ for some $C(\kappa)<\infty$, and $f'(x)=\cosh(x/2)^{-4/k}$. So $f'(x)\frac{e^x-1}{e^x+1}+\frac{\kappa}{2}f''(x)=0$. Let $U_s=f(X_s)$. Then

$$dU_s=f'(X_s)dX_s+\frac{\kappa}{2}f''(X_s)ds$$

$$=f'(X_s)\sqrt{\kappa}dB_1(s)+f'(X_s)\lim_{M\to\infty}\left(\sum_{k=-M}^{1}\frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}}+\sum_{k=1}^{M}\frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}}\right)ds$$

$$=f'(X_s)\sqrt{\kappa}dB_1(s)+f'(X_s)\sum_{k=1}^{\infty}\frac{2\sinh(X_s)}{\cosh(2ks)+\cosh(X_s)}ds.$$

Let $v(s)=\int_{\pi^2/p}^s f'(X_t)^2dt$ for $\pi^2/p\le s<T_a$. Let $\widehat{T}_a=v(T_a)$. For $0\le r<\widehat{T}_a$, let $\widehat{U}_r=U_{v^{-1}(r)}$. Then

$$d\widehat{U}_r=\sqrt{\kappa}dB_2(r)+f'(X_{v^{-1}(r)})^{-1}\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})}dr,$$

where $B_2(r)$ is another standard one dimensional Brownian motion. And $\widehat{T}_a$ is a stopping time w.r.t. $B_2(r)$. Let

$$A(r)=f'(X_{v^{-1}(r)})^{-1}\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})};$$

$$M(r)=\exp\left(-\int_0^r A(s)\sqrt{\kappa}dB_2(s)-\frac{\kappa}{2}\int_0^r A(s)^2ds\right).$$

For $0\le r<\widehat{T}_a$, $|X_{v^{-1}(r)}|<a$, so $|f'(X_{v^{-1}(r)})^{-1}|\le\cosh(a/2)^{4/\kappa}$. And

$$\left|\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})}\right|\le\sum_{k=1}^{\infty}\frac{2\sinh(a)}{e^{2ks}/2}=\frac{4\sinh(a)}{e^{2s}-1}.$$

Thus the Nivikov's condition

$$\mathbf{E}\,[\exp(\frac{\kappa}{2}\int_0^{\widehat{T}_a}A(s)^2ds)]<\infty$$

is satisfied. Let $\mathbf{P}$ denote the original measure for $B_2(r)$. Define $\mathbf{Q}$ on $\widehat{\mathcal{F}}_{\widehat{T}_a}$ such that $d\mathbf{Q}\,(\omega)=M_{\widehat{T}_a}(\omega)d\mathbf{P}\,(\omega)$. Then $(\widehat{U}_r,0\le r<\widetilde{T}_a)$ is a one dimensional Brownian motion started from 0 and stopped at time $\widehat{T}_a$ w.r.t. the probability law $\mathbf{Q}$. For $0\le s<T_a$, $|X_s|\le a$, so $|f'(X_s|\ge\cosh(a/2)^{-4/\kappa}$. Thus if $T_a=\infty$, then $\widehat{T}_a=\infty$ too. From the hypothesis of the proof, $\mathbf{P}\,\{T_a=\infty\}>0$, so $\mathbf{P}\,\{\widehat{T}_a=\infty\}>0$. Since $(\widehat{U}_r,0\le r<\widehat{T}_a)$ is a one dimensional Brownian motion w.r.t. $\mathbf{Q}$, so on the event that $\widehat{T}_a=\infty$, $\mathbf{Q}\,\{\limsup_{r\to\infty}|\widehat{U}_r|<\infty\}=0$. Thus $\mathbf{Q}\,\{\limsup_{r\to\infty}|\widehat{U}_r|=\infty\}>0$. Since $\mathbf{P}$ and $\mathbf{Q}$ are equivalent probability measures, so $\mathbf{P}\,\{\limsup_{r\to\widehat{T}_a}|\widehat{U}_r|=\infty\}>0$. Thus $\mathbf{P}\,\{\limsup_{s\to T_a}|U_s|=\infty\}>0$. This contradicts the fact that for all $s\in[\pi^2/p,\infty)$, $U_s\in(0,C(\kappa))$ and $C(\kappa)<\infty$. Thus the hypothesis is wrong, and the proof is completed. $\square$

From this lemma and the definition of $X_t$, we know that for any $z\in\mathbb{R}_p$, $(\mathrm{Re}\,\widetilde{\varphi}_t(z)-\sqrt{\kappa}B(t))/(p-t)$ is not bounded on $t\in[0,p)$ a.s.. Since for any $k\in\mathbb{Z}$ and $z\in\mathbb{R}_p$, $\widetilde{\varphi}_t(z)-2k\pi=\widetilde{\varphi}_t(z-2k\pi)$, so $(\mathrm{Re}\,\widetilde{\varphi}_t(z)-2k\pi-\sqrt{\kappa}B(t)))/(p-t)=(\mathrm{Re}\,\widetilde{\varphi}_t(z-2k\pi)-\sqrt{\kappa}B(t))/(p-t)$ is not bounded on $t\in[0,p)$ a.s., which implies that $X_s(z)-2ks$ is not bounded on $s\in[\pi^2/p,\infty)$ a.s..

**Lemma 3.2** *For every $z\in\mathbb{R}_p$, almost surely $\lim_{s\to\infty}X_s(z)/s$ exists and the limit is an odd integer.*

**Proof.** Fix $\varepsilon_0\in(0,1/2)$ and $z_0\in\mathbb{R}_p$. Let $X_s$ denote $X_s(z_0)$. There is $b>0$ such that the probability that $|\sqrt{\kappa}B(t)|\le b+\varepsilon_0 t$ for any $t\ge0$ is greater than $1-\varepsilon_0$. Since $\coth(x/2)\to\pm1$ as $x\in\mathbb{R}$ and $x\to\pm\infty$, so there is $R>0$ such that when $\pm x\ge R$, $\pm\coth(x/2)\ge1-\varepsilon_0$. Let $T=R+b+1$. If for any $s\ge0$, $|X_s-2ks|<T$ for some $k=k(s)\in\mathbb{Z}$, then there is $k_0\in\mathbb{Z}$ such that $|X_s-2k_0s|<T$ for all $s\ge T$. From the argument after Lemma 3.1, the probability of this event is 0. Let $s_0$ be the first time

that $|X_s - 2ks| \ge T$ for all $k \in \mathbb{Z}$. Then $s_0$ is finite almost surely. There is $k_0 \in \mathbb{Z}$ such that $2k_0 s_0 + T \le X_{s_0} \le 2(k_0+1)s_0 - T$. Let $s_1$ be the first time after $s_0$ such that $X_s = 2k_0 s + R$ or $X_s = 2(k_0+1)s - R$. Let $s_1 = \infty$ if such time does not exist. For $s \in [s_0, s_1)$, we have $X_s \in [2k_0 s + R, 2(k_0+1)s - R]$. Note that $(e^x - e^{2ks})/(e^x + e^{2ks}) \to \mp 1$ as $k \to \pm\infty$. So

$$\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}} = 2k_0 + \lim_{M\to\infty} \sum_{k=k_0-M}^{k_0+M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}}$$

$$= 2k_0 + \lim_{M\to\infty} \sum_{j=-M}^{M} \frac{e^{X_s - 2k_0 s} - e^{2js}}{e^{X_s - 2k_0 s} + e^{2js}}$$

$$= 2k_0 + \coth(\frac{X_s - 2k_0 s}{2}) + \sum_{j=1}^{\infty} \frac{2\sinh(X_s - 2k_0 s)}{\cosh(2js) + \cosh(X_s - 2k_0 s)}$$

$$\ge 2k_0 + \coth(\frac{X_s - 2k_0 s}{2}) \ge 2k_0 + 1 - \varepsilon_0;$$

and

$$\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}} = 2(k_0+1) + \lim_{M\to\infty} \sum_{j=-M}^{M} \frac{e^{X_s - 2(k_0+1)s} - e^{2js}}{e^{X_s - 2(k_0+1)s} + e^{2js}}$$

$$= 2k_0 + 2 + \coth(\frac{X_s - 2(k_0+1)s}{2}) + \sum_{j=1}^{\infty} \frac{2\sinh(X_s - 2(k_0+1)s)}{\cosh(2js) + \cosh(X_s - 2(k_0+1)s)}$$

$$\le 2k_0 + 2 + \coth(\frac{X_s - 2(k_0+1)s}{2}) \le 2k_0 + 2 + (-1 + \varepsilon_0) = 2k_0 + 1 + \varepsilon_0.$$

From equation (4), we have that for $s \in [s_0, s_1)$,

$$(2k_0 + 1 - \varepsilon_0)(s - s_0) \le \mathrm{Re}\,\widehat{\varphi}_s(z_0) - \mathrm{Re}\,\widehat{\varphi}_{s_0}(z_0) \le (2k_0 + 1 + \varepsilon_0)(s - s_0).$$

Note that $X_s = \mathrm{Re}\,\widehat{\varphi}_s(z_0) - \sqrt{\kappa}B_1(s)$, and $(\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0), s \ge s_0)$ has the same distribution as $(\sqrt{\kappa}B(s - s_0), s \ge s_0)$. Let $E_b$ denote the event that $|\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0)| \le b + \varepsilon_0(s - s_0)$ for all $s \ge s_0$. Then $\mathbf{P}(E) > 1 - \varepsilon_0$. And on the event $E_b$, we have

$$(2k_0 + 1 - \varepsilon_0)(s - s_0) - b - \varepsilon_0(s - s_0) \le X_s - X_{s_0}$$

$$\le (2k_0 + 1 + \varepsilon_0)(s - s_0) + b + \varepsilon_0(s - s_0),$$

from which follows that

$$X_s \le X_{s_0} + (2k_0 + 1 + \varepsilon_0)(s - s_0) + b + \varepsilon_0(s - s_0)$$

$$\leq 2(k_0+1)s_0 - T + (2k_0+1+\varepsilon_0)(s-s_0) + b + \varepsilon_0(s-s_0)$$
$$= 2(k_0+1)s - T + b - (1-2\varepsilon_0)(s-s_0) \leq 2(k_0+1)s - R - 1$$

and

$$X_s \geq X_{s_0} + (2k_0+1-\varepsilon_0)(s-s_0) - b - \varepsilon_0(s-s_0)$$
$$\geq 2k_0 s_0 + T + (2k_0+1-\varepsilon_0)(s-s_0) - b - \varepsilon_0(s-s_0)$$
$$= 2k_0 s + T - b + (1-2\varepsilon_0)(s-s_0) \geq 2k_0 s + R + 1.$$

So on the event $E_b$ we have $s_1 = \infty$, which implies that $2k_0 s + R \leq X_s \leq 2(k_0+1)s - R$ for all $s \geq s_0$, and so $\partial_s \mathrm{Re}\,\widehat{\varphi}_s(z_0) \in (2k_0+1-\varepsilon_0, 2k_0+1+\varepsilon_0)$ for all $s \geq s_0$. Thus the event that

$$2k_0+1-\varepsilon_0 \leq \liminf_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s \leq \limsup_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s \leq 2k_0+1+\varepsilon_0$$

has probability greater than $1-\varepsilon_0$. Since we may choose $\varepsilon_0 > 0$ arbitrarily small, so a.s. $\lim_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s$ exists and the limit is $2k_0+1$ for some $k_0 \in \mathbb{Z}$. The proof is now finished by the facts that $X_s(z_0) = \mathrm{Re}\,\widehat{\varphi}_s(z_0) + \sqrt{\kappa}B_1(s)$ and $\lim_{s\to\infty} B_1(s)/s = 0$. $\square$

Let
$$m_- = \sup\{x \in \mathbb{R} : \lim_{s\to\infty} X_s(x+ip)/s \leq -1\}$$
and
$$m_+ = \inf\{x \in \mathbb{R} : \lim_{s\to\infty} X_s(x+ip)/s \geq 1\}.$$

Since $X_s(x_1+ip) < X_s(x_2+ip)$ if $x_1 < x_2$, so we have $m_- \leq m_+$. If the event that $m_- < m_+$ has a positive probability, then there is $a \in \mathbb{R}$ such that the event that $m_- < a < m_+$ has a positive probability. From the definitions, $m_- < a < m_+$ implies that $\lim_{s\to\infty} X_s(a+ip)/s \in (-1,1)$, which is an event with probability 0 by Lemma 3.2. This contradiction shows that $m_- = m_+$ a.s.. Let $m = m_+$. For any $t \in [0,p)$, $z \in \mathbb{S}_p \setminus \widetilde{K}_t$ and $k \in \mathbb{Z}$, since $\widetilde{\varphi}_t(z+2k\pi) = \widetilde{\varphi}_t(z) + 2k\pi$, so $Z_t(z+2k\pi) = Z_t(z) + 2k\pi$, then we have $W_t(z+2k\pi) = W_t(z) + 2k\pi^2/(p-t)$. Thus $X_s(z+2k\pi) = X_s(z) + 2ks$ for any $s \in [\pi^2/p, \infty)$, $z \in \mathbb{S}_p \setminus \widetilde{K}_{p-\pi^2/s}$ and $k \in \mathbb{Z}$. If $x \in (m+2k\pi, m+2(k+1)\pi)$ for some $k \in \mathbb{Z}$, then $x - 2k\pi > m$ and $x - 2(k+1)\pi < m$. So

$$\lim_{s\to\infty} X_s(x+ip)/s = \lim_{s\to\infty} (X_s(x-2k\pi+ip) + 2ks)/s \geq 2k+1$$

and

$$\lim_{s\to\infty} X_s(x+ip)/s = \lim_{s\to\infty} (X_s(x-2(k+1)\pi+ip) + 2(k+1)s)/s \leq 2k+1.$$

Therefore $\lim_{s\to\infty} X_s(x+ip)/s = 2k+1$.

Let $K_p = \cup_{0\leq t<p} K_t$ and $\widetilde{K}_p = \cup_{0\leq t<p} \widetilde{K}_t$. Then $K_p = e^i(\widetilde{K}_p)$, and so $\overline{K_p} = e^i(\overline{\widetilde{K}_p})$.

**Lemma 3.3** $\overline{K_p} \cap \mathbf{C}_p = \{e^{-p+im}\}$ *almost surely.*

**Proof.** We first show that $m + ip \in \overline{\widetilde{K}_p}$. If this is not true, then there is $a, b > 0$ such that the distance between $[m - a + ip, m + a + ip]$ and $\widetilde{K}_t$ is greater than $b$ for all $t \in [0, p)$. From the definition of $m$, we have $X_s(m \pm a + ip) \to \pm\infty$ as $s \to \infty$. Thus $\mathrm{Re}\,\widehat{\varphi}_s(m+a+ip) - \mathrm{Re}\,\widehat{\varphi}_s(m-a+ip) \to \infty$ as $s \to \infty$. So there is $c(s) \in (m-a, m+a)$ such that $\widehat{\varphi}_s'(c(s) + ip) \to \infty$ as $s \to \infty$. Since $\widehat{\varphi}_s$ maps $(\mathbb{S}_p \setminus \widetilde{K}_{p-\pi^2/s}, \mathbb{R}_p)$ conformally onto $(\mathbb{S}_\pi, \mathbb{R}_\pi)$, so by Koebe's 1/4 theorem, the distance between $c(s) + ip$ and $\widetilde{K}_{p-\pi^2/s}$ tends to 0 as $s \to \infty$. This is a contradiction. Thus $m + ip \in \overline{\widetilde{K}_p}$.

Now fix $x_1 < x_2 \in (m, m + 2\pi)$. Then $X_s(x_j + ip)/s \to 1$ as $s \to \infty$ for $j = 1, 2$. So there is $s_0$ such that $X_s(x_j + ip) \in (s/2, 3s/2)$ for $s \geq s_0$ and $j = 1, 2$. So if $x_0 \in [x_1, x_2]$ and $s \geq s_0$, then $X_s(x_0 + ip) \in (s/2, 3s/2)$, and so

$$\sum_{k=-\infty}^{+\infty} \frac{e^{X_s(x_0+ip)}e^{2ks}}{(e^{X_s(x_0+ip)} + e^{2ks})^2} \leq \sum_{k=-\infty}^{0} e^{2ks - X_s(x_0+ip)} + \sum_{k=1}^{+\infty} e^{X_s(x_0+ip)-2ks}$$

$$\leq \sum_{k=-\infty}^{0} e^{2ks-s/2} + \sum_{k=1}^{+\infty} e^{3s/2-2ks} = \frac{2e^{-s/2}}{1 - e^{-2s}} \leq \frac{2e^{-s/2}}{1 - e^{-2\pi^2/p}}.$$

From equation (5), for all $s \geq s_0$,

$$\partial_s \ln |\widehat{\varphi}_s'(x_0 + ip)| \leq \frac{4e^{-s/2}}{1 - e^{-2\pi^2/p}},$$

which implies that

$$\ln |\widehat{\varphi}_s'(x_0 + ip)| \leq \ln |\widehat{\varphi}_{s_0}'(x_0 + ip)| + \frac{8e^{-s_0/2}}{1 - e^{-2\pi^2/p}}.$$

So there is $M < \infty$ such that $|\widehat{\varphi}_s'(x_0 + ip)| \leq M$ for all $x_0 \in [x_1, x_2]$ and $s \geq s_0$. From Koebe's 1/4 theorem, we see that $\widetilde{K}_t$ is uniformly bounded away from $[x_1 + ip, x_2 + ip]$ for $t \in [0, p)$. Thus $[x_1 + ip, x_2 + ip] \cap \overline{\widetilde{K}_p} = \emptyset$. Since $x_1 < x_2$ are chosen arbitrarily from $(m, m+2\pi)$, so $(m+ip, m+2\pi+ip) \cap \overline{\widetilde{K}_p} = \emptyset$. Thus $\overline{\widetilde{K}_p} \cap [m+ip, m+2\pi+ip) = \{m+ip\}$. Since $\mathbf{C}_p = e^i([m + ip, m + 2\pi + ip))$, so $\overline{K_p} \cap \mathbf{C}_p = \{e^i(m + ip)\} = \{e^{-p+im}\}$. $\square$

**Lemma 3.4** *For every $\varepsilon \in (0, 1)$, there is $C_0 > 0$ depending on $\varepsilon$ such that if $q \in (0, \frac{2\pi^2}{\ln(2)}]$, and $L_t$, $0 \leq t < q$, are standard annulus $SLE_\kappa$ hulls of modulus $q$, then the probability that $\cup_{0 \leq t < q} L_t \subset \{e^{iz} : |Re\, z| \leq C_0 q\}$ is greater than $1 - \varepsilon$.*

**Proof.** Let $q_0 = \frac{2\pi^2}{\ln(2)}$. Suppose $q \in (0, q_0]$. Let $L_t$ and $\psi_t$, $0 \le t < q$, be the annulus LE hulls and maps of modulus $q$ driven by $\sqrt{\kappa}B(t)$, $0 \le t < q$. Let $\widetilde{L}_t$ and $\widetilde{\psi}_t$, $0 \le t < q$, be the corresponding annulus LE hulls and maps in the covering space. There is $b = b(\varepsilon) > 0$ such that the probability that $|\sqrt{\kappa}B(t)| \le b + t/4$ for all $t \ge 0$ is greater than $1 - \varepsilon$. Let $R = \ln(64)$ and $C_0 = (R + b + 1)/\pi$. Let $s_0 = \pi^2/q$. Let $Z_t(z) = \widetilde{\psi}_t(z) - \sqrt{\kappa}B(t)$, $W_t(z) = \pi Z_t(z)/(q - t)$ for $0 \le t < q$. Let $\widehat{W}_s(z) = W_{q-\pi^2/s}(z)$ for $s_0 \le s < \infty$. Then there is another standard one dimensional Brownian motion $B_1(s)$, $s \ge s_0$, such that $\widehat{\psi}_s$ defined by $\widehat{\psi}_s(z) = \widehat{W}_s(z) + \sqrt{\kappa}B_1(s)$ satisfies

$$\partial_s \widehat{\psi}_s(z) = \lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{\widehat{W}_s(z)} + e^{2ks}}{e^{\widehat{W}_s(z)} - e^{2ks}}$$

for $s_0 \le s < \infty$. Let $E_\varepsilon$ be the event that $|\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0)| \le b + (s - s_0)/4$ for all $s \ge s_0$. Then $\mathbf{P}(E_\varepsilon) > 1 - \varepsilon$. Fix $z_0 \in \mathbb{S}_q$ with $C_0 q < \operatorname{Re} z_0 < 2\pi - C_0 q$. We claim that in the event $E_\varepsilon$, $\widetilde{\psi}_t(z_0)$ never blows up for $0 \le t < q$. If this claim is justified, then on the event $E_\varepsilon$, $z_0 \notin \widetilde{L}_t$ for any $0 \le t < q$ and $z_0 \in \mathbb{S}_q$ with $C_0 q < \operatorname{Re} z_0 < 2\pi - C_0 q$. So $\cup_{0\le t<q} \widetilde{L}_t$ is disjoint from $\{z \in \mathbb{C} : C_0 q < \operatorname{Re} z < 2\pi - C_0 q\}$. Since $L_t = e^i(\widetilde{L}_t)$, so $\cup_{0\le t<q} L_q$ is disjoint from $\{e^{iz} : C_0 q < \operatorname{Re} z < 2\pi - C_0 q\}$ on the event $E_\varepsilon$. Then we are done.

Assume the event $E_\varepsilon$. Let $Z_t$ denote $Z_t(z_0)$, $W_t$ denote $W_t(z_0)$, $\widehat{W}_s$ denote $\widehat{W}_s(z_0)$, and $\widehat{\psi}_s$ denote $\widehat{\psi}_s(z_0)$. If $\widetilde{\psi}_t(z_0)$ blows up at time $t_* < q$, then $Z_t \to 2k\pi$ for some $k \in \mathbb{Z}$ as $t \to t_*$. Then $\widehat{W}_s - 2ks \to 0$ as $s \to \pi^2/(q - t_*)$. Since $\operatorname{Re} Z_0 = \operatorname{Re} z_0 \in [C_0 q, 2\pi - C_0 q]$, so $\widehat{W}_{s_0} = W_0 \in [C_0\pi, 2s_0 - C_0\pi] \subset (R, 2s_0 - R)$, and so there is a first time $s_1 > s_0$ such that $\operatorname{Re} \widehat{W}_{s_1} \in \{R, 2s_1 - R\}$. Then for $s \in [s_0, s_1]$, we have $\operatorname{Re} \widehat{W}_s \in [R, 2s - R]$. Then

$$\Big|\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} - 1\Big| \le \sum_{k=-\infty}^{0} \Big|\frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} - 1\Big| + \sum_{k=1}^{\infty} \Big|\frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} + 1\Big|$$

$$\le \sum_{k=-\infty}^{0} \frac{2}{|e^{\widehat{W}_s - 2ks}| - 1} + \sum_{k=1}^{\infty} \frac{2}{|e^{2ks - \widehat{W}_s}| - 1} \le \sum_{k=-\infty}^{0} \frac{4}{e^{R-2ks}} + \sum_{k=1}^{\infty} \frac{4}{e^{2ks-(2s-R)}}$$

$$\le \frac{8e^{-R}}{1 - e^{-2s}} \le 16e^{-R} \le \frac{1}{4},$$

where we use the fact that $e^{-R} \le \frac{1}{64}$ and $e^{-2s} \le e^{-2s_0} = e^{-2\pi^2/q} \le e^{-2\pi^2/q_0} \le \frac{1}{2}$. Thus

$$|(\widehat{W}_{s_1} - \widehat{W}_{s_0}) - (s_1 - s_0)| \le |(\widehat{\psi}_{s_1} - \widehat{\psi}_{s_0}) - (s_1 - s_0)| + |\sqrt{\kappa}B_1(s_1) - \sqrt{\kappa}B_1(s_0)|$$

$$\le (s_1 - s_0)/4 + b + (s_1 - s_0)/4 = b + (s_1 - s_0)/2.$$

Then we have

$$\operatorname{Re}\widehat{W}_{s_1} \geq \operatorname{Re}\widehat{W}_{s_0} + (s_1 - s_0) - b - (s_1 - s_0)/2 \geq C_0\pi + (s_1 - s_0)/2 - b > R$$

and

$$\operatorname{Re}\widehat{W}_{s_1} \leq \operatorname{Re}\widehat{W}_{s_0} + (s_1 - s_0) + b + (s_1 - s_0)/2 \leq 2s_0 - C_0\pi + b + 3(s_1 - s_0)/2$$

$$= 2s_1 - (s_1 - s_0)/2 - C_0\pi + b < 2s_1 - R.$$

This contradicts that $\operatorname{Re}\widehat{W}_{s_1} \in \{R, 2s_1 - R\}$. Thus $\widetilde{\psi}_t(z_0)$ does not blow up for $t \in [0, q)$. Then the claim is justified, and the proof is finished. $\square$

For two nonempty sets $A_1, A_2 \subset \mathbb{A}_p$, we define the angular distance between $A_1$ and $A_2$ to be $d_a(A_1, A_2) = \inf\{|\operatorname{Re} z_1 - \operatorname{Re} z_2| : e^{iz_1} \in A_1, e^{iz_2} \in A_2\}$. For a nonempty set $A \subset \mathbb{A}_p$, we define the angular diameter of $A$ to be $diam_a(A) = \sup\{d_a(z_1, z_2) : z_1, z_2 \in A\}$. If $A$ intersects both $A_1$ and $A_2$, then $d_a(A_1, A_2) \leq diam_a(A)$. In the above lemma, $\cup_{0 \leq t < q} L_t \subset \{e^{iz} : |\operatorname{Re} z| \leq C_0 q\}$ implies that $diam_a(\cup_{0 \leq t < q} L_t) \leq 2C_0 q$. Form conformal invariance and comparison principle of extremal distance, we have that for any $d > 0$, there is $h(d) > 0$ such that for any $p > 0$, if for $j = 1, 2$, $A_j$ is a union of connected subsets of $\mathbb{A}_p$, each of which touches both $\mathbf{C}_p$ and $\mathbf{C}_0$, and the extremal distance between $A_1$ and $A_2$ in $\mathbb{A}_p$ is greater than $h(d)$, then $d_a(A_1, A_2) > dp$.

**Theorem 3.1** $\lim_{t \to p} \beta(t) = e^{-p+im}$ *almost surely.*

**Proof.** From Lemma 3.3, the distance from $e^{-p+im}$ to $K_t$ tends to 0 as $t \to p$ a.s.. Since $K_t$ is the hull generated by $\beta((0, t])$, so the distance from $e^{-p+im}$ to $\beta((0, t])$ tends to 0 as $t \to p$ a.s.. Suppose the theorem does not hold. Then there is $a, \delta > 0$ such that the event that $\limsup_{t \to p} |e^{-p+im} - \beta(t)| > a$ has probability greater than $\delta$. Let $E_1$ denote this event. Let $\varepsilon = \delta/4$. Let $C_0$ depending on $\varepsilon$ be as in Lemma 3.4. Let $R = \min\{a, e^{-p}\}$ and $r = \min\{1 - e^{-p}, R\exp(-2\pi h(2C_0 + 1))\}$, where $h$ is the function in the argument before this theorem. Since $K_t$ is generated by $\beta((0, t])$, and $e^{-p+im} \in \overline{K_p}$ a.s., so the distance between $e^{-p+im}$ and $\beta((0, t])$ tends to 0 a.s. as $t \to p$. So there is $t_0 \in (0, p)$ such that the event that the distance between $e^{-p+im}$ and $\beta((0, t_0])$ is less than $r$ has probability greater than $1 - \varepsilon$. Let $E_2$ denote this event. Let $q_0 = \frac{2\pi^2}{\ln(2)}$, $T = \max\{t_0, p - q_0, -\ln(r + e^{-p})\}$, $p_T = p - T$, and $\xi_T(t) = \xi(T + t) - \xi(T)$ for $0 \leq t < p_T$. Let $K_{T,t} = \varphi_T(K_{T+t} \setminus K_T)/e^{i\xi(T)}$ and $\varphi_{T,t}(z) = \varphi_{T+t} \circ \varphi_T^{-1}(\exp(i\xi(T))z)/\exp(i\xi(T))$ for $0 \leq t < p_T$. Then one may check that $K_{T,t}$ and $\varphi_{T,t}$, $0 \leq t < p_T$, are the annulus LE hulls and maps of modulus $p_T$ driven by $\xi_T$. Since $\xi_T(t)$ has the same law as $\sqrt{\kappa}B(t)$ and $p_T = p - T \leq q_0$, so from Lemma 3.4, the event that $diam_a(\cup_{0 \leq t < p_T} K_{T,t}) \leq 2C_0 p_T$ has probability greater than $1 - \varepsilon$. Let $E_3$ denote this event. Since $\mathbf{P}(E_1^c) + \mathbf{P}(E_2^c) + \mathbf{P}(E_3^c) < (1 - \delta) + \varepsilon + \varepsilon < 1$, so $\mathbf{P}(E_1 \cap E_2 \cap E_3) > 0$. This means that the events $E_1$, $E_2$ and $E_3$ can happen at the same time. We will prove that this is a contradiction. Then the theorem is proved.

Assume the event $E_1 \cap E_2 \cap E_3$. Let $A_r$ ($A_R$, resp.) be the union of connected components of $\{z \in \mathbb{C} : |z - e^{-p+im}| = r\} \cap (\mathbb{A}_p \setminus K_T)$ ($\{z \in \mathbb{C} : |z - e^{-p+im}| = R\} \cap (\mathbb{A}_p \setminus K_T)$, resp.) that touch $\mathbf{C}_p$. From the properties of $\beta$ in the event $E_1$ and $E_2$, we see that $A_r$ and $A_R$ both intersect $K_p \setminus K_T$. Since the distance between $e^{-p+im}$ and $K_T$ is less than $r$, and $r < R$, so both $A_r$ and $A_R$ are unions of two curves which touch both $\mathbf{C}_p$ and $\mathbf{C}_0 \cup K_T$. Let $B_r = e^{-i\xi(T)}\varphi_T(A_r)$ and $B_R = e^{-i\xi(T)}\varphi_T(A_R)$. Then both $B_r$ and $B_R$ are unions of two curves in $\mathbb{A}_{p_T}$ that touch both $\mathbf{C}_{p_T}$ and $\mathbf{C}_0$.

The extremal distance between $A_r$ and $A_R$ in $\mathbb{A}_p \setminus K_T$ is at least $\ln(R/r)/(2\pi) \geq h(2C_0+1)$. Thus the extremal distance between $B_r$ and $B_R$ in $\mathbb{A}_{p_T}$ is at least $h(2C_0+1)$. So the angular distance between $B_r$ and $B_R$ is at least $(2C_0+1)p_T$. Since $A_R$ and $A_r$ both intersect $K_p \setminus K_T$, so $B_R$ and $B_r$ both intersect $\varphi_T(K_p \setminus K_T)/e^{i\xi(T)} = \cup_{0 \leq t < p_T} K_{T,t}$, which implies that $diam_a(\cup_{0\leq t<p_T} K_{T,t}) \geq (2C_0+1)p_T$. However, in the event $E_3$, $diam_a(\cup_{0\leq t<p_T} K_{T,t}) \leq 2C_0 p_T$. This contradiction finishes the proof. $\square$

Let's see what can we say about the distribution of $\lim_{t\to p}\beta(t)$. Let $\widetilde{\beta}(t) = \widetilde{\varphi}_t^{-1}(\xi(t))$. Then $\widetilde{\beta}$ is a simple curve in $\mathbb{S}_p$ started from 0, and $\beta(t) = e^i(\widetilde{\beta}(t))$. From Theorem 3.1, $\lim_{t\to p}\widetilde{\beta}(t)$ exists and lies on $\mathbb{R}_p$. We call $\widetilde{\beta}$ an annulus SLE$_\kappa$ trace in the covering space. Let $m_p + ip$ denote the limit point, where $m_p$ is a real valued random variable.

Suppose the distribution of $m_p$ is absolutely continuous w.r.t. the Lebesgue measure, and the density function $\widetilde{\lambda}(p,x)$ is $C^{1,2}$ continuous. This hypothesis is very likely to be true, but the proof is still missing now. We then have $\int_{\mathbb{R}} \widetilde{\lambda}(p,x)dx = 1$ for any $p > 0$. Since the distribution of $\widetilde{\beta}$ is symmetric w.r.t. the imaginary axis, so is the distribution of $\lim_{t\to p}\widetilde{\beta}(t)$. Thus $\widetilde{\lambda}(p,-x) = \widetilde{\lambda}(p,x)$. Moreover, we expect that when $p \to 0$ the distribution of $(m_p + ip) * \frac{\pi}{p}$ tends to the distribution of the limit point of a strip SLE$_\kappa$ trace introduced in [18], whose density is $\cosh(x/2)^{-4/\kappa}/C(\kappa)$ for some $C(\kappa) > 0$. If this is true, then the distribution of $m_p$ tends to the point mass at 0 as $p \to 0$.

For $0 \leq t < p$, let $\mathcal{F}_t$ be the $\sigma-$algebra generated by $\xi(s)$, $0 \leq s \leq t$. Fix $T \in [0,p)$. Let $p_T = p - T$. For $0 \leq t < p_T$, let $\xi_T(t) = \xi(T+t) - \xi(T)$. Then $\xi_T(t)$ has the same distribution as $\sqrt{\kappa}B(t)$, and is independent of $\mathcal{F}_T$. For $0 \leq t < T$, let

$$\widetilde{\varphi}_{T,t}(z) = \widetilde{\varphi}_{T+t} \circ \widetilde{\varphi}_T^{-1}(z + \xi(T)) - \xi(T).$$

Then $\partial_t \widetilde{\varphi}_{T,t}(z) = \mathbf{H}_{p_T - t}(\widetilde{\varphi}_{T,}(z) - \xi_T(t))$, and $\widetilde{\varphi}_{T,0}(z) = z$. Thus $\widetilde{\varphi}_{T,t}(z)$, $0 \leq t < p_T$, are annulus LE maps of modulus $p_T$ in the covering space driven by $\xi_T(t)$, $0 \leq t < p_T$, and so are independent of $\mathcal{F}_T$. Let

$$\widetilde{\beta}_T(t) = \widetilde{\varphi}_{T,t}^{-1}(\xi_T(t)) = \widetilde{\varphi}_T \circ \widetilde{\varphi}_{T+t}^{-1}(\xi(T_t)) - \xi(T) = \widetilde{\varphi}_T(\widetilde{\beta}(T+t)) - \xi(T), \tag{6}$$

for $0 \le t < p_T$. Then $\widetilde{\beta}_T(t)$, $0 \le t < p_T$, is a standard annulus SLE$_\kappa$ trace of modulus $p_T$ in the covering space, and is independent of $\mathcal{F}_T$. Thus $\lim_{t\to p_T} \beta_T(t)$ exists and lies on $\mathbb{R}_{p_T}$ a.s.. Let $m_{p_T} + ip_T$ denote the limit point. Then $m_{p_T}$ is independent of $\mathcal{F}_T$, and the density of $m_{p_T}$ w.r.t. the Lebesgue measure is $\widetilde{\lambda}(p_T,\cdot)$. From equation (6), we see $m_{p_T} = \widetilde{\varphi}_T(m_p+ip) - ip_T - \xi(T)$. For $0 \le t < p$, let $\widetilde{\psi}_t(z) = \widetilde{\varphi}_t(z+ip) - i(p-t)$. Then $\widetilde{\psi}_t$ takes real values on $\mathbb{R}$, and $\partial_t \widetilde{\psi}_t(z) = \widehat{\mathbf{H}}_{p-t}(\widetilde{\psi}_t(z) - \xi(t))$, where $\widehat{\mathbf{H}}_r(z) = \mathbf{H}(z+ir) + i$. Let $X_t(z) = \widetilde{\psi}_t(z) - \xi(t)$ for $0 \le t < p_T$. So $m_{p_T} = X_T(m_p)$. From the differential equation for $\widetilde{\psi}_t$, we get

$$dX_t(x) = \widehat{\mathbf{H}}_{p-t}(X_t(x))dt - d\xi(t);$$

and

$$dX_t'(x) = \widehat{\mathbf{H}}_{p-t}'(X_t(x))X_t'(x)dt.$$

Let $a < b \in \mathbb{R}$. Then $\{m_p \in [a,b]\} = \{m_{p_T} \in [X_T(a), X_T(b)]\}$. Since $m_{p_T}$ has density $\widetilde{\lambda}(p_T,\cdot)$ and is independent of $\mathcal{F}_T$, and $X_T$ is $\mathcal{F}_T$ measurable, so

$$\mathbf{E}\left[\mathbf{1}_{\{m_p\in[a,b]\}}|\mathcal{F}_T\right] = \int_{X_T(a)}^{X_T(b)} \widetilde{\lambda}(p-T,x)dx = \int_a^b \widetilde{\lambda}(p-T,X_T(x))X_T'(x)dx.$$

Thus $(\int_a^b \widetilde{\lambda}(p-t,X_t(x))X_t'(x)dx, 0 \le t < p)$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t=0}^p$. Fix $x \in \mathbb{R}$. Choose $a < x < b$ and let $a,b \to x$. Then $(\widetilde{\lambda}(p-t,X_t(x))X_t'(x), 0 \le t < p)$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t=0}^p$. From Ito's formula, we have

$$-\partial_1\widetilde{\lambda}(r,x) + \widehat{\mathbf{H}}_r'(x)\widetilde{\lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\lambda}(r,x) = 0, \tag{7}$$

where $\partial_1$ and $\partial_2$ are partial derivatives w.r.t. the first and second variable, respectively.

Let $\widetilde{\Lambda}(p,x) = \int_0^x \widetilde{\lambda}(p,s)ds$ for $p > 0$ and $x \in \mathbb{R}$. Then for any $p > 0$, $\widetilde{\Lambda}(p,\cdot)$ is an odd and increasing function, $\lim_{x\to\pm\infty}\widetilde{\Lambda}(p,x) = \pm\frac{1}{2}$, and $\widetilde{\lambda}(p,x) = \partial_2\widetilde{\Lambda}(p,x)$. Thus for any $r > 0$ and $x \in \mathbb{R}$,

$$\partial_2(-\partial_1\widetilde{\Lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\Lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,x)) = 0.$$

Since $\widetilde{\Lambda}(r,\cdot)$ is an odd function and $\widehat{\mathbf{H}}_r(0) = 0$, so

$$-\partial_1\widetilde{\Lambda}(r,0) + \widehat{\mathbf{H}}_r(0)\partial_2\widetilde{\Lambda}(r,0) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,0) = 0.$$

Thus for any $r > 0$ and $x \in \mathbb{R}$, we have

$$-\partial_1\widetilde{\Lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\Lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,x) = 0. \tag{8}$$

And we expect that for any $x \in \mathbb{R} \setminus \{0\}$, $\lim_{r\to 0} \widetilde{\Lambda}(r,x) \to \text{sign}\frac{1}{2}$. On the other hand, if $\widetilde{\Lambda}(r,x)$ satisfies (8), then $\widetilde{\lambda}(r,x) := \partial_2 \widetilde{\Lambda}(r,x)$ satisfies (7).

Let $\lambda(r,x) = \sum_{k\in\mathbb{Z}} \widetilde{\lambda}(r, x+2k\pi)$. Then $\lambda(r,\cdot)$ has a period $2\pi$, and is the density function of the distribution of the argument of $\lim_{t\to r}\beta(t)$, where $\beta$ is a standard annulus $\mathrm{SLE}_\kappa$ trace of modulus $r$. So it satisfies $\int_{-\pi}^{\pi} \lambda(r,x)dx = 1$. And $\lambda(r,\cdot)$ is an even function for any $r > 0$. Since $\widehat{\mathbf{H}}_r$ has a period $2\pi$, so $\lambda(r,x)$ also satisfies equation (7). Let $\Lambda(r,x) = \int_0^x \lambda(r,s)ds$. Then $\Lambda(r,x)$ satisfies (8). But $\Lambda(r,x)$ does not satisfies $\lim_{x\to\pm\infty} \Lambda(r,x) = \pm 1$. Instead, we have $\Lambda(r, x+2\pi) = \Lambda(r,x) + 1$. In the case that $\kappa = 2$, we have some nontrivial solutions to (8). From Lemma 3.1 in [17], we see $-\partial_r \mathbf{H}_r + \mathbf{H}_r\mathbf{H}_r' + \mathbf{H}_r'' = 0$, where the function $\widetilde{\mathbf{S}}_r$ in [17] is the function $\mathbf{H}_r$ here. From the definition of $\widehat{\mathbf{H}}_r$, we may compute that $-\partial_r \widehat{\mathbf{H}}_r + \widehat{\mathbf{H}}_r \widehat{\mathbf{H}}_r' + \widehat{\mathbf{H}}_r'' = 0$. Thus $\Lambda_1(r,x) = \widehat{\mathbf{H}}_r(x)$ and $\Lambda_2(r,x) = r\mathbf{H}_r(x) + x$ satisfy equation (8). So $\lambda_1(r,x) = \widehat{\mathbf{H}}_r'(x)$ and $\lambda_2(r,x) = r\mathbf{H}_r'(x) + 1$ are solutions to (7). In fact, $\lambda_2(r,x)/(2\pi)$ is the distribution of the argument of the end point of a Brownian Excursion in $\mathbb{A}_r$ started from 1 conditioned to hit $\mathbf{C}_r$. From Corollary 3.1 in [17], this is also the distribution of the argument of the limit point of a standard annulus $\mathrm{SLE}_2$ trace of modulus $r$. So we justified equation (7) in the case $\kappa = 2$.

We may change variables in the following way. For $-\infty < s < 0$, let $\widetilde{\mathrm{P}}(s,y) = \widetilde{\Lambda}(-\frac{\pi^2}{s}, -\frac{\pi}{s}y)$ and $\mathrm{P}(s,y) = \Lambda(-\frac{\pi^2}{s}, -\frac{\pi}{s}y)$. Then for any $s<0$, $\lim_{y\to\pm\infty} \widetilde{\mathrm{P}}(s,y) = \pm\frac{1}{2}$ and $\mathrm{P}(s, y+2s) = \mathrm{P}(s,y) - 1$. And we expect that $\lim_{s\to-\infty} \widetilde{\mathrm{P}}(s,y) = \int_0^y \cosh(\frac{s}{2})^{-4/\kappa} ds / C(\kappa)$. Let $\mathbf{G}_s(y) = i\mathbf{H}_{-s}(iy - \pi)$ for $s < 0$ and $y \in \mathbb{R}$. From formula (3), we may compute that $\widetilde{\Lambda}(r,x)$ ($\Lambda(r,x)$, resp.) satisfies equation (8) iff $\widetilde{\mathrm{P}}(s,y)$ ($\mathrm{P}(s,y)$, resp.) satisfies

$$-\partial_1 \widetilde{\mathrm{P}}(s,y) + \mathbf{G}_s(y)\partial_2 \widetilde{\mathrm{P}}(s,y) + \frac{\kappa}{2}\partial_2^2 \widetilde{\mathrm{P}}(s,y) = 0. \tag{9}$$

From the equation for $\mathbf{H}_r$ and the definition of $\mathbf{G}_s$, we have $-\partial_s \mathbf{G}_s + \mathbf{G}_s\mathbf{G}_s' + \mathbf{G}_s'' = 0$. Thus $\mathrm{P}_1(s,y) = \mathbf{G}_s(y)$ and $\mathrm{P}_2(s,y) = s\mathbf{G}_s(y) + y$ are solutions to (9). In fact, $\mathrm{P}_1(s,y)$ corresponds to $-\Lambda_2(r,x)/\pi$, and $\mathrm{P}_2(s,y)$ corresponds to $-\pi\Lambda_1(r,x)$.

# 4 Local Martingales for Annulus $\mathrm{SLE}_4$ and $\mathrm{SLE}_8$

## 4.1 Annulus $\mathrm{SLE}_4$

Fix $\kappa = 4$. Let $K_t$ and $\varphi_t$, $0 \le t < p$, be the annulus LE hulls and maps of modulus $p$, respectively, driven by $\xi(t) = \sqrt{\kappa}B(t)$. Let $\beta(t)$, $0 \le t < p$, be the trace. For $r > 0$, let $\mathbf{T}_r^{(2)}(z) = \frac{1}{2}\mathbf{S}_r(z^2)$ and $\widetilde{\mathbf{T}}_r^{(2)}(z) = \frac{1}{i}\mathbf{T}_r^{(2)}(e^{iz})$. Solve the differential equations:

$$\partial_t \psi_t(z) = \psi_t(z)\mathbf{T}_{p-t}^{(2)}(\psi_t(z)/e^{i\xi(t)/2}), \;\; \psi_0(z) = z;$$

$$\partial_t\widetilde{\psi}_t(z) = \widetilde{\mathbf{T}}^{(2)}_{p-t}(\widetilde{\psi}_t(z) - \xi(t)/2), \;\; \widetilde{\psi}_0(z) = z.$$

Let $P_2$ be the square map: $z \mapsto z^2$. Then we have $P_2 \circ \psi_t = \varphi_t \circ P_2$ and $e^i \circ \widetilde{\psi}_t = \psi_t \circ e^i$. Let $L_t := P_2^{-1}(K_t)$ and $\widetilde{L}_t = (e^i)^{-1}(L_t)$. Then $\psi_t$ maps $\mathbb{A}_{p/2} \setminus L_t$ conformally onto $\mathbb{A}_{(p-t)/2}$, and $\widetilde{\psi}_t$ maps $\mathbb{S}_{p/2} \setminus \widetilde{L}_t$ conformally onto $\mathbb{S}_{(p-t)/2}$. Since $K_t = \beta(0,t]$, and $\beta$ is a simple curve in $\mathbb{A}_p$ with $\beta(0) = 1$, so $L_t$ is the union of two disjoint simple curves opposite to each other, started from $1$ and $-1$, respectively. Let $\alpha_\pm(t)$ denote the curve started from $\pm 1$. Then $\psi_t(\alpha_\pm(t)) = e^i(\pm\xi(t)/2)$.

For each $r > 0$, suppose $J_r$ is the conformal map from $\mathbb{A}_{r/2}$ onto $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1\} \setminus [-a_r, a_r]$ for some $a_r > 0$ such that $\pm 1$ is mapped to $\pm\infty$. This $J_r$ is symmetric w.r.t. both $x$-axis and $y$-axis, i.e., $J_r(\overline{z}) = \overline{J_r(z)}$, and $J_r(-z) = -J_r(z)$. And $\mathrm{Im}\, J_r$ is the unique bounded harmonic function in $\mathbb{A}_{r/2}$ that satisfies (i) $\mathrm{Im}\, J_r \equiv \pm 1$ on the open arc of $\mathbf{C}_0$ from $\pm 1$ to $\mp 1$ in the ccw direction; and (ii) $\mathrm{Im}\, J_r \equiv 0$ on $\mathbf{C}_{r/2}$. Let $\widetilde{J}_r = J_r \circ e^i$.

**Lemma 4.1** $-\partial_r \widetilde{J}_r + \widetilde{J}_r' \widetilde{\mathbf{T}}^{(2)}_r + \frac{1}{2}\widetilde{J}_r'' \equiv 0$ *in* $\widetilde{\mathbb{A}}_{r/2}$.

**Proof.** Since $\mathrm{Im}\, \widetilde{J}_r \equiv 0$ on $\mathbb{R}_{r/2}$, by reflection principle, $\widetilde{J}_r$ can be extended analytically across $\mathbb{R}_{r/2}$. And we have $\mathrm{Im}\, \widetilde{J}_r' = \partial_x \mathrm{Im}\, \widetilde{J}_r \equiv 0$ and $\mathrm{Im}\, \widetilde{J}_r'' = \partial_x^2 \mathrm{Im}\, \widetilde{J}_r \equiv 0$ on $\mathbb{R}_{r/2}$. From the equality $\mathrm{Im}\, \widetilde{J}_r(x + ir/2) = 0$, we have $\partial_r \mathrm{Im}\, \widetilde{J}_r + \partial_y \mathrm{Im}\, \widetilde{J}_r/2 \equiv 0$ on $\mathbb{R}_{r/2}$. On $\mathbb{R}_{r/2}$, note that $\mathrm{Im}\, \widetilde{\mathbf{T}}^{(2)}_r \equiv -1/2$, so

$$\mathrm{Im}\, (\widetilde{J}_r' \widetilde{\mathbf{T}}^{(2)}_r) = \mathrm{Re}\, \widetilde{J}_r' \mathrm{Im}\, \widetilde{\mathbf{T}}^{(2)}_r + \mathrm{Im}\, \widetilde{J}_r' \mathrm{Re}\, \widetilde{\mathbf{T}}^{(2)}_r$$

$$= -1/2 \mathrm{Re}\, \widetilde{J}_r' = -1/2 \partial_y \mathrm{Im}\, \widetilde{J}_r = \partial_r \mathrm{Im}\, \widetilde{J}_r.$$

Let $F_r := -\partial_r \widetilde{J}_r + \widetilde{J}_r' \widetilde{\mathbf{T}}^{(2)}_r + \frac{1}{2}\widetilde{J}_r''$. Then $\mathrm{Im}\, F_r \equiv 0$ on $\mathbb{R}_{r/2}$.

For any $k \in \mathbb{Z}$, we see that $\widetilde{J}_r(z)$ is equal to $(-1)^{k+1}\frac{2}{\pi}\ln(z - k\pi)$ plus some analytic function for $z \in \widetilde{\mathbb{A}}_{r/2}$ near $k\pi$. So we may extend $\mathrm{Re}\, \widetilde{J}_r(z)$ harmonically across $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\mathrm{Im}\, \widetilde{J}_r$ takes constant value $(-1)^k$ on each interval $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, we have $\mathrm{Re}\, \widetilde{J}_r(\overline{z}) = \mathrm{Re}\, \widetilde{J}_r(z)$. Moreover, the following properties hold: $\partial_r \widetilde{J}_r$ is analytic in a neighborhood of $\mathbb{R}$, $\widetilde{J}_r'$ and $\widetilde{J}_r''$ are analytic in a neighborhood of $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$.

The fact that $\mathrm{Im}\, \widetilde{J}_r$ takes constant value $(-1)^k$ on each $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, implies that $\mathrm{Im}\, \partial_r \widetilde{J}_r$, $\mathrm{Im}\, \widetilde{J}_r'$ and $\mathrm{Im}\, \widetilde{J}_r''$ vanishes on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\mathrm{Im}\, \widetilde{\mathbf{T}}^{(2)}_r$ also vanishes on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$, so we compute $\mathrm{Im}\, F_r \equiv 0$ on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$.

From $J_r(\overline{z}) = \overline{J_r(z)}$, we find that $\widetilde{J}_r(-\overline{z}) = \overline{\widetilde{J}_r(z)}$. So $\mathrm{Re}\, \widetilde{J}_r(z) = \mathrm{Re}\, \widetilde{J}_r(-\overline{z}) = \mathrm{Re}\, \widetilde{J}_r(-z)$. This means that $\mathrm{Re}\, \widetilde{J}_r$ is an even function, so is $\partial_r \widetilde{J}_r$ and $\widetilde{J}_r''$. And $\widetilde{J}_r'$ is an odd function. Note that $\widetilde{\mathbf{T}}^{(2)}_r$ is an odd function, so $F_r$ is an even function. Since $\widetilde{\mathbf{T}}^{(2)}_r(z)$

is equal to $1/(2z)$ plus some analytic function for $z$ near 0, so the pole of $F_r$ at 0 has order at most 2. However, the coefficient of $1/z^2$ is equal to $2/\pi * 1/2 - 1/2 * 2/\pi = 0$. And 0 is not a simple pole of $F_r$ because $F_r$ is even. So 0 is a removable pole of $F_r$. Similarly, $\pi$ is also a removable pole of $F_r$. Since $F_r$ has period $2\pi$, so every $k\pi$, $k \in \mathbb{Z}$, is a removable pole of $F_r$. So $F_r$ can be extended analytically across $\mathbb{R}$, and $\operatorname{Im} F_r \equiv 0$ on $\mathbb{R}$. Thus $\operatorname{Im} F_r \equiv 0$ in $\mathbb{S}_{r/2}$, which implies that $F_r \equiv C$ for some constant $C \in \mathbb{R}$.

Finally, $J_r(-z) = -J_r(z)$ implies that $\widetilde{J}_r(z+\pi) = -\widetilde{J}_r(z)$. Since $\pi$ is a period of $\widetilde{\mathbf{T}}_r^{(2)}$, we compute $F_r(z+\pi) = -F_r(z)$. So $C$ has to be 0. □

**Proposition 4.1** *For any $z \in \mathbb{A}_{p/2}$, $J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$ is a local martingale, from which follows that $\operatorname{Im} J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$ is a bounded martingale.*

**Proof.** Fix $z_0 \in \mathbb{S}_{p/2}$, let $Z_t := \widetilde{\psi}_t(z_0) - \xi(t)/2$, then

$$dZ_t = \widetilde{\mathbf{T}}_{p-t}^{(2)}(Z_t)dt - d\xi(t)/2.$$

Note that $\xi(t)/2 = B(t)$. From Ito's formula and the last lemma, we have

$$d\widetilde{J}_{p-t}(Z_t) = -\partial_r \widetilde{J}_{p-t}(Z_t)dt + \widetilde{J}'_{p-t}(Z_t)dZ_t + \frac{1}{2}\widetilde{J}''_{p-t}(Z_t)dt$$

$$= (-\partial_r \widetilde{J}_{p-t}(Z_t) + \widetilde{J}_{p-t}(Z_t)\widetilde{\mathbf{T}}_{p-t}^{(2)}(Z_t) + \frac{1}{2}\widetilde{J}''_{p-t}(Z_t))dt - \widetilde{J}'_{p-t}(Z_t)d\xi(t)/2 = -\widetilde{J}'_{p-t}(Z_t)d\xi(t)/2.$$

Thus $\widetilde{J}_{p-t}(Z_t)$, $0 \le t < p$, is a local martingale. For any $z \in \mathbb{A}_{p/2}$, there is $z_0 \in \mathbb{S}_{p/2}$ such that $z = e^i(z_0)$. Then

$$J_{p-t}(\psi_t(z)/e^{i\xi(t)/2}) = J_{p-t}(\psi_t(e^i(z_0))/e^{i\xi(t)/2})$$

$$= J_{p-t}(e^i(\widetilde{\psi}_t(z_0) - \xi(t)/2)) = \widetilde{J}_{p-t}(\widetilde{\psi}_t(z_0) - \xi(t)/2.$$

So $J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, $0 \le t < p$, is a local martingale. Since $|\operatorname{Im} J_r(z)| \le 1$ for any $r > 0$ and $z \in \mathbb{A}_{r/2}$, so $\operatorname{Im} J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, $0 \le t < p$, is a bounded martingale. □

Let $h_t(z) = J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$. Then $h_t$ maps $\mathbb{A}_{(p-t)/2} \setminus L_t$ conformally onto $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\} \setminus [-a_{p-t}, a_{p-t}]$ so that $\alpha_\pm(t)$ is mapped to $\pm\infty$. So $\operatorname{Im} h_t$ is the unique bounded harmonic function in $\mathbb{A}_{p/2} \setminus L_t$ that vanishes on $\mathbf{C}_{p/2}$, equals to 1 on the arc of $\mathbf{C}_0$ from 1 to $-1$ in the ccw direction and the north side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, and equals to $-1$ on the arc of $\mathbf{C}_0$ from $-1$ to 1 in the ccw direction and the south side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$.

We have another choice of $J_r$. Let $J_r$ be the conformal map of $\mathbb{A}_{r/2}$ onto the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\} \setminus [-ib_r, ib_r]$ for some $b_r > 0$ so that $\pm 1$ is mapped to $\pm\infty$. Then

$\operatorname{Im} J_r$ is the bounded harmonic function in $\mathbb{A}_{r/2}$ determined by the following properties: (i) $\operatorname{Im} J_r \equiv \pm 1$ on the open arc of $\mathbf{C}_0$ from $\pm 1$ to $\mp 1$ in the ccw direction; and (ii) the normal derivative of $\operatorname{Im} J_r$ vanishes on $\mathbf{C}_{r/2}$. Let $\widetilde{J}_r = J_r \circ e^i$. The proposition and lemma in this subsection still hold for $J_r$ and $\widetilde{J}_r$ defined here. The proofs are almost the same. The only difference is at the step when we prove $\operatorname{Im} F_r \equiv 0$ on $\mathbb{R}_{r/2}$. Here we have $\operatorname{Re} \widetilde{J}_r' = \partial_y \operatorname{Im} \widetilde{J}_r$ vanishes on $\mathbb{R}_{r/2}$. Use an argument similar to the proof of the lemma, we can show that $\operatorname{Re} F_r'$ vanishes on $\mathbb{R}_{r/2}$. So $\partial_y \operatorname{Im} F_r$ vanishes on $\mathbb{R}_{r/2}$. Since $\operatorname{Im} F_r$ vanishes on $\mathbb{R}$, $\operatorname{Im} F_r$ has to vanish in $\mathbb{S}_{r/2}$. Use $J_r(-z) = -J_r(z)$, we then conclude that $F_r \equiv 0$. If we let $h_t(z) = J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, then $\operatorname{Im} h_t$ is the unique bounded harmonic function in $\mathbb{A}_{p/2} \setminus L_t$ that satisfies the following properties: equals to 1 on the arc of $\mathbf{C}_0$ from 1 to $-1$ in the ccw direction, and the north side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, equals to $-1$ on the arc of $\mathbf{C}_0$ from $-1$ to 1 in the ccw direction, and the south side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, and the normal derivatives vanish on $\mathbf{C}_{p/2}$.

Suppose $E$ is a doubly connected domain such that 0 lies in the bounded component of $\mathbb{C} \setminus E$. Fix $v \in \partial_o E$, the outside boundary component of $E$. Let $\beta(t)$, $0 \le t < p$, be an $\mathrm{SLE}_4(E; v \to \partial_i E)$ trace, where $\partial_i E$ is the inside boundary component of $E$. So $\beta(t)$ is the image of a standard annulus $\mathrm{SLE}_4$ trace of modulus $p$ under the conformal map from $(\mathbb{A}_p, 1)$ onto $(E, v)$, where $p$ is the modulus of $E$. Let $D = P_2^{-1}(E)$ and $\{v_+, v_-\} = P_2^{-1}(v)$. $P_2^{-1}(\{\beta(t) : 0 \le t < p\})$ is the union of two disjoint simple curve started from $v_+$ and $v_-$, respectively. Let $\alpha_\pm$ denote the curve started from $v_\pm$. Then $D$ is a symmetric ($-D = D$) doubly connected domain, and $\alpha_-(t) = -\alpha_+(t)$ for $0 \le t < p$. Let $D_t = D \setminus \alpha_-([0,t]) \setminus \alpha_+([0,t])$. Let $\gamma_t^\pm$ denote the boundary arc of $\partial_o D_t$ from $\alpha_\pm(t)$ to $\alpha_\mp(t)$ in the ccw direction. Then $\gamma_t^\pm$ contains a boundary arc of $\partial_o D$, one side of $\alpha_+([0,t])$ and one side of $\alpha_-([0,t])$. Let $H_t$ be the bounded harmonic function in $D_t$ which has continuations at $\partial_i D$ and $\gamma_t^\pm$ such that $H_t \equiv 0$ on $\partial_i D$ and $H_t \equiv \pm 1$ on $\gamma_t^\pm$. By the definition of $\mathrm{SLE}_4(E; v \to \partial_i E)$ and conformal invariance of harmonic functions, for any fixed $z_0 \in D$, $H_t(z_0)$, $0 \le t < p$, is a bounded martingale. This $H_t$ corresponds to $\operatorname{Im} h_t$ defined right after Proposition 4.1. We may replace the condition $H_t \equiv 0$ on $\partial_i D$ by $\partial_{\mathbf{n}} H_t \equiv 0$ on $\partial_i D$. Then this $H_t$ corresponds to the $\operatorname{Im} h_t$ defined in the last paragraph. So for any fixed $z_0 \in D$, it is still true that $H_t(z_0)$, $0 \le t < p$, is a bounded martingale.

## 4.2 Harmonic Explorers for Annulus $\mathrm{SLE}_4$

Let $D$ be a symmetric ($-D = D$) doubly connected subset of hexagonal faces in the planar honeycomb lattice. Two faces of $D$ are considered adjacent if they share an edge. Let $\partial_o D$ and $\partial_i D$ denote the outside and inside component of $\partial D$, respectively. Suppose $v_+$ and $v_-$ are vertices that lie on $\partial_o D$, and are opposite to each other, i.e., $v_- = -v_+$. Suppose $\operatorname{Re} v_+ > 0$. Then $v_+$ and $v_-$ partition the boundary faces of $D$ near $\partial_o D$ into an "upper" boundary component, colored black, and a "lower" boundary component, colored white. All other hexagons in $D$ are uncolored.

Now we construct two curves $\alpha_+$ and $\alpha_-$ as follows. Let $\alpha_\pm(0) = v_\pm$. Let $\alpha_\pm(1)$ be a neighbor vertex of $\alpha_\pm(0)$ such that $[\alpha_\pm(0), \alpha_\pm(1)]$ is shared by a white hexagon and a black hexagon. At time $n \in \mathbb{N}$, if $\alpha_\pm(n) \not\in \partial_i D$, then $\alpha_\pm(n)$ is a vertex shared by a black hexagon, a white hexagon, and an uncolored hexagon, denoted by $f^n_\pm$. Let $H_n$ be the function defined on faces, which takes value 1 on the black faces, $-1$ on the white faces, 0 on faces that touch $\partial_i D$, and is discrete harmonic at other faces of $D$. Then $H_n(f^n_-) = -H_n(f^n_+)$. We then color $f_\pm$ black with probability equal to $(1+H_n(f^n_\pm))/2$ and white with probability equal to $(1-H_n(f^n_\pm))/2$ such that $f^n_+$ and $f^n_-$ are colored differently. Let $\alpha_\pm(n+1)$ be the unique neighbor vertex of $\alpha_\pm(n)$ such that $[\alpha_\pm(n), \alpha_\pm(n+1)]$ is shared by a white hexagon and a black hexagon. Increase $n$ by 1, and iterate the above process until $\alpha_+$ and $\alpha_-$ hit $\partial_i D$ at the same time. We always have $\alpha_-(n) = -\alpha_+(n)$, $f^n_- = -f^n_+$, and $H_n(-g) = -H_n(g)$.

From the construction, conditioned on $\alpha_\pm(k)$, $k = 0, 1, \dots, n$, the expected value of $H_{n+1}(f^n_\pm)$ is equal to $(1+H_n(f^n_\pm))/2-(1-H_n(f^n_\pm))/2 = H_n(f^n_\pm)$. And if a face $f$ is colored before time $n$, then its color will not be changed after time $n$, so $H_{n+1}(f) = H_n(f)$. Since $H_{n+1}$ and $H_n$ both vanish on the faces near $\partial_i D$, and are discrete harmonic at all other uncolored faces at time $n+1$ and $n$, resp., so for any face $f$ of $D$, the conditional value of $H_{n+1}(f)$ w.r.t. $\alpha_\pm(k)$, $k = 0, 1, \dots, n$ is equal to $H_n(f)$. Thus for any fixed face $f_0$ of $D$, $H_n(f_0)$ is a martingale.

If $n-1 < t < n$, and $\alpha_\pm(n-1)$ and $\alpha_\pm(n)$ are defined, let $\alpha_\pm(t) = (n-t)\alpha_\pm(n-1) + (t-(n-1))\alpha_\pm(n)$. Then $\alpha_\pm$ becomes a curve in $D$. Let $D_t = D \setminus \alpha_+([0,t]) \setminus \alpha_-([0,t])$. Note that if the side length of the hexagons is very small compared with the size of $D$, then for any face $f$ of $D$, $H_n(f)$ is close to the value of $\widetilde{H}_n$ at the center of $f$, where $\widetilde{H}_n$ is the bounded harmonic function defined on $D_n$, which has a continuation to $\partial D \setminus \{v_+, v_-\}$ and the two sides of $\alpha_\pm([0,t))$ such that $\widetilde{H}_n \equiv 0$ on $\partial_i D$, and $\widetilde{H}_n \equiv \pm 1$ on the curve on $\partial_o D_n$ from $\alpha_\pm(n)$ to $\alpha_\mp(n)$ in the ccw direction. From the last section, we may guess that the distribution of $\alpha_\pm$ tends to that of the square root of an annulus $\mathrm{SLE}_4(P_2(D); P_2(v_\pm) \to \partial_i P_2(D))$ trace when the mesh tends to 0. If at each step of the construction of $\alpha_\pm$, we let $H_n$ be the function which is is equal to 1 on the black faces, $-1$ on the white faces, and is discrete harmonic at all other faces of $D$ including the faces that touch $\partial_i D$, then we get a different pair of curves $\alpha_\pm$. If the mesh is very small compared with the size of $D$, then for any face $f$ of $D$, $H_n(f)$ is close to the value of $\widetilde{H}_n$ at the center of $f$, where $\widetilde{H}_n$ is the bounded harmonic function defined on $D_n$, which has a continuation to $\partial D \setminus \{v_+, v_-\}$ and the two sides of $\alpha_\pm([0,t))$ such that $\partial_{\mathbf{n}} \widetilde{H}_n \equiv 0$ on $\partial_i D$, and $\widetilde{H}_n \equiv \pm 1$ on the curve on $\partial_o D_n$ from $\alpha_\pm(n)$ to $\alpha_\mp(n)$ in the ccw direction. So we also expect the law of $\alpha_\pm$ constructed in this way tends to that of the square root of an annulus $\mathrm{SLE}_4(P_2(D); P_2(v_\pm) \to \partial_i P_2(D))$ trace when the mesh tends to 0.

### 4.3 Annulus $\mathrm{SLE}_8$

Fix $\kappa = 8$. Let $K_t$ and $\varphi_t$, $0 \le t < p$, be the annulus LE hulls and maps, respectively, of modulus $p$, driven by $\xi(t) = \sqrt{\kappa}B(t)$. For $r > 0$, let $\mathbf{T}_r^{(4)}(z) = \frac{1}{4}\mathbf{S}_r(z^4)$ and $\widetilde{\mathbf{T}}_r^{(4)}(z) = \frac{1}{i}\mathbf{T}_r^{(4)}(e^{iz})$. Solve the differential equations:

$$\partial_t \psi_t(z) = \psi_t(z)\mathbf{T}_{p-t}^{(4)}(\psi_t(z)/e^{i\xi(t)/4}), \;\; \psi_0(z) = z;$$

$$\partial_t \widetilde{\psi}_t(z) = \widetilde{\mathbf{T}}_{p-t}^{(4)}(\widetilde{\psi}_t(z) - \xi(t)/4), \;\; \widetilde{\psi}_0(z) = z.$$

Let $P_4$ be the map: $z \mapsto z^4$. Then we have $P_4 \circ \psi_t = \varphi_t \circ P_4$ and $e^i \circ \widetilde{\psi}_t = \psi_t \circ e^i$. Let $L_t := P_4^{-1}(K_t)$ and $\widetilde{L}_t = (e^i)^{-1}(L_t)$. Then $\psi_t$ maps $\mathbb{A}_{p/4} \setminus L_t$ conformally onto $\mathbb{A}_{(p-t)/4}$, and $\widetilde{\psi}_t$ maps $\mathbb{S}_{p/4} \setminus \widetilde{L}_t$ conformally onto $\mathbb{S}_{(p-t)/4}$. Let $G_r$ map $\mathbb{A}_{r/4}$ conformally onto $\{z \in \mathbb{C} : |\operatorname{Re} z| + |\operatorname{Im} z| < 1\} \setminus [-a_r, a_r]$ for some $a_r > 0$ such that $\pm 1$ and $\pm i$ are fixed.

**Proposition 4.2** *For any $z \in \mathbb{A}_{r/4}$, $G_{p-t}(\psi_t(z)/e^{i\xi_t/4})$ is a bounded martingale.*

**Proof.** Let $\widetilde{G}_r := G_r \circ e^i$. For any $z \in \mathbb{A}_{p/4}$, there is $w \in \mathbb{S}_{p/4}$ such that $z = e^i(w)$. Then

$$G_{p-t}(\psi_t(z)/e^{i\xi(t)/4}) = G_{p-t}(\psi_t(e^{iw})/e^{i\xi(t)/4}) = \widetilde{G}_{p-t}(\widetilde{\psi}_t(w) - \xi(t)/4).$$

To prove this proposition, it suffices to show that for any $w \in \mathbb{S}_{p/4}$, $\widetilde{G}_{p-t}(\widetilde{\psi}_t(w) - \xi(t)/4)$ is a local martingale. Let $Z_t = \widetilde{\psi}_t(w) - \xi(t)/4$, then

$$dZ_t = \widetilde{\mathbf{T}}_{p-t}^{(4)}(Z_t)dt - dB(t)/\sqrt{2}.$$

Thus by Ito's formula,

$$d\widetilde{G}_{p-t}(\psi_t(w) - \xi(t)/4) = -\partial_r \widetilde{G}_{p-t}(Z_t)dt + \widetilde{G}'_{p-t}(Z_t)dZ_t + \frac{1}{2}\widetilde{G}''_{p-t}(Z_t)\frac{dt}{2}$$

$$= (-\partial_r \widetilde{G}_{p-t}(Z_t) + \widetilde{G}'_{p-t}(Z_t)\widetilde{\mathbf{T}}_{p-t}^{(4)}(Z_t) + \frac{1}{4}\widetilde{G}''_{p-t}(Z_t))dt - \widetilde{G}'_{p-t}(Z_t)dB(t)/\sqrt{2}.$$

So it suffices to prove the following lemma.

**Lemma 4.2** $-\partial_r \widetilde{G}_r + \widetilde{G}'_r \widetilde{\mathbf{T}}_r^{(4)} + \frac{1}{4}\widetilde{G}''_r \equiv 0$ *in* $\mathbb{S}_{r/4}$.

**Proof.** Let $F_r$ be the left-hand side. Let $Q_r(z) := i(\widetilde{G}_r(z) - 1)^2$. Note that $\widetilde{G}_r$ maps $[0, \pi/2]$ and $[-\pi/2, 0]$ onto the line segments $[1, i]$ and $[-i, 1]$, respectively. Thus $Q_r(z) \to \mathbb{R}$ as $z \in \mathbb{S}_{r/4}$ and $z \to (-\pi/2, \pi/2)$. By reflection principle, $Q_r$ can be extended to an analytic function in a neighborhood of $(-\pi/2, \pi/2)$, and $Q_r(\overline{z}) = \overline{Q_r(z)}$. Since $G_r(\overline{z}) = \overline{G_r(z)}$, so $\widetilde{G}_r(-\overline{z}) = \overline{\widetilde{G}_r(z)}$. It follows that $Q_r(-\overline{z}) = -\overline{Q_r(z)}$. So we have $Q_r(-z) = -Q_r(z)$, and the Taylor expansion of $Q_r$ at 0 is $\sum_{n=0}^{\infty} a_{n,r} z^{2n+1}$. Thus $\widetilde{G}_r(z) = 1 + \sum_{n=0}^{\infty} c_{n,r} z^{2n+1/2}$ for $z$ near 0. So $\partial_r \widetilde{G}_r(z) = O(z^{1/2})$ for $z$ near 0, $\widetilde{G}_r'(z) = 1/2 c_{1,r} z^{-1/2} + O(z^{3/2})$, and $\widetilde{G}_r'''(z) = -1/4 c_{1,r} z^{-3/2} + O(z^{1/2})$. Since $\widetilde{\mathbf{T}}_r^{(4)}(z) = 1/(8z) + O(z)$ near 0, so $\widetilde{G}_r'(z)\widetilde{\mathbf{T}}_r^{(4)}(z) = 1/16 c_{1,r} z^{-3/2} + O(z^{1/2})$. Then we compute $F_r(z) = O(z^{1/2})$ near 0. Similarly, $F_r(z) = O((z - k\pi/2)^{1/2})$ for $z$ near $k\pi/2$, $k \in \mathbb{Z}$.

For $z \in (k\pi, (k+1/2)\pi)$, $k \in \mathbb{Z}$, $\widetilde{G}_r(z) \in (-1)^k + (1-i)\mathbb{R}$. So $F_r(z) \in (1-i)\mathbb{R}$ for $z \in (k\pi, (k+1/2)\pi)$, $k \in \mathbb{Z}$. Similarly, $F_r(z) \in (1+i)\mathbb{R}$ for $z \in ((k-1/2)\pi, k\pi)$, $k \in \mathbb{Z}$. Since $\widetilde{G}_r$ takes real values on $\mathbb{R}_{r/4}$, so $F_r$ also takes real values on $\mathbb{R}_{r/4}$. Let $V_r = \operatorname{Im} F_r$, then $V_r \equiv 0$ on $\mathbb{R}_{r/4}$, and for $k \in \mathbb{Z}$, $\partial_x V + \partial_y V \equiv 0$ on $(k\pi, (k+1/2)\pi)$ and $\partial_x V - \partial_y V \equiv 0$ on $((k-1/2)\pi, k\pi)$, $k \in \mathbb{Z}$. And $V_r(z) \to 0$ as $z \in \mathbb{S}_{r/4}$ and $z \to k\pi/2$, $k \in \mathbb{Z}$. Since $\widetilde{G}_r$ and $\widetilde{\mathbf{T}}_r^{(4)}$ have period $2\pi$, so does $F_r$. Thus $|V_r|$ attains its maximum in $\overline{\mathbb{S}_{r/4}}$ at some $z_0 \in \mathbb{R} \cup \mathbb{R}_{r/4}$. If $z_0 \in \mathbb{R}_{r/4}$ or $z_0 = k\pi/2$ for some $k \in \mathbb{Z}$, then $V_r(z_0) = 0$, and so $V_r$ vanishes in $\mathbb{S}_{r/4}$. Otherwise, either $z_0 \in (k\pi, (k+1/2)\pi)$ or $z_0 \in ((k-1/2)\pi, k\pi)$ for some $k \in \mathbb{Z}$. In either cases, we have $\partial_x V_r(z_0) = 0$, so $\partial_y V_r(z_0) = 0$ too. Thus $F_r'(z_0) = 0$. If $F_r$ is not constant in $\mathbb{S}_{r/4}$, then $F_r(z_0) = 1 + a_m(z - z_0)^m + O((z - z_0)^{m+1}$ for $z$ near $z_0$. Then it is impossible that $\operatorname{Im} F_r(z_0) \ge \operatorname{Im} F_r(z)$ for all $z \in \{|z - z_0| < \varepsilon, \operatorname{Im} z \ge \operatorname{Im} z_0\}$ or $\operatorname{Im} F_r(z_0) \le \operatorname{Im} F_r(z)$ for all $z \in \{|z - z_0| < \varepsilon, \operatorname{Im} z \ge \operatorname{Im} z_0\}$. This contradiction shows that $F_r$ has to be constant in $\mathbb{S}_{r/4}$. Since $F_r(z) \to 0$ as $z \to 0$, so this constant is 0. We again conclude that $V_r$ has to vanish in $\mathbb{S}_{r/4}$. $\square$

# 5 Annulus $\mathrm{SLE}_{8/3}$ and the Restriction Property

In this section, we fix $\kappa = 8/3$ and $\alpha = 5/8$. Let $\varphi_t$ and $K_t$, $0 \le t < p$, be the annulus LE maps and hulls of modulus $p$, driven by $\xi(t) = \sqrt{\kappa} B(t)$, $0 \le t < p$. Let $\widetilde{\varphi}_t$ and $\widetilde{K}_t$, $0 \le t < p$, be the corresponding annulus LE maps and hulls in the covering space. Let $A \ne \emptyset$ be a hull in $\mathbb{A}_p$ w.r.t. $\mathbf{C}_p$ (i.e., $\mathbb{A}_p \setminus A$ is a doubly connected domain whose one boundary component is $\mathbf{C}_p$) such that $1 \notin \overline{A}$. So there is $t > 0$ such that $K_t \cap A = \emptyset$. Let $T_A$ be the biggest $T \in (0, p]$ such that for $t \in [0, T)$, $K_t \cap A = \emptyset$. Let $\varphi_A$ be the conformal map from $\mathbb{A}_p \setminus A$ onto $\mathbb{A}_{p_0}$ such that $\varphi_A(1) = 1$, where $p_0$ is equal to the modulus of $\mathbb{A}_p \setminus A$. Let $K_t' = \varphi_A(K_t)$, $0 \le t < T_A$. Let $h(t)$ equal $p_0$ minus the modulus of $\mathbb{A}_{p_0} \setminus K_t'$. Then $h$ is a continuous increasing function with $h(0) = 0$. So $h$ maps $[0, T_A)$ onto $[0, S_A)$ for some $S_A \in (0, p_0]$. From Proposition 2.1 in [17], $L_s = K_{h^{-1}(s)}$, $0 \le s < S_A$, are the annulus LE hulls of modulus $p_0$, driven by some real continuous function, say $\eta(s)$. Let $\psi_s$, $0 \le s < S_A$, be the corresponding annulus LE maps. Let $\widetilde{\psi}_s$ and $\widetilde{L}_s$, $0 \le s < S_A$, be the annulus LE maps and hulls, respectively, in the covering space.

Let $f_t = \psi_{h(t)} \circ \varphi_A \circ \varphi_t^{-1}$ and $A_t = \varphi_t(A)$. Then for $0 \le t < T_A$, $e^i(\xi(t)) \not\in \overline{A_t}$, and $f_t$ maps $(\mathbb{A}_{p-t} \setminus A_t, \mathbf{C}_{p-t})$ conformally onto $(\mathbb{A}_{p_0-h(t)}, \mathbf{C}_{p_0-h(t)})$. And for any $z_0 \in \mathbf{C}_0 \setminus \overline{A_t}$, if $z \in \mathbb{A}_{p-t} \setminus A_t$ and $z \to z_0$, then $f(z) \to \mathbf{C}_0$. Thus $f_t$ can be extended analytically across $\mathbf{C}_0$ near $e^i(\xi(t))$. A proof similar to those of Lemma 2.1 and 2.2 in [17] shows that $f_t(e^i(\xi(t))) = e^i(\eta(h(t)))$, and $h'(t) = |f_t'(e^i(\xi(t)))|^2$.

Let $\widetilde{\varphi}_A$ be such that $e^i \circ \widetilde{\varphi}_A = \varphi_A \circ e^i$ and $\widetilde{\varphi}_A(0) = 0$. Let $\widetilde{f}_t = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A \circ \widetilde{\varphi}_t^{-1}$. Then $e^i \circ \widetilde{f}_t = f_t \circ e^i$, and so $e^i \circ \widetilde{f}_t(\xi(t)) = e^i(\eta(h(t)))$. Thus $\widetilde{f}_t(\xi(t)) = \eta(h(t)) + 2k\pi$ for some $k \in \mathbb{Z}$. Now we replace $\eta(s)$ by $\eta(s) + 2k\pi$. Then $\eta(s)$, $0 \le s < S_A$, is still a driving function of $L_s$, $0 \le s < S_A$. And we have $\widetilde{f}_t(\xi(t)) = \eta(h(t))$. Moreover, we have $h'(t) = \widetilde{f}_t'(\xi(t))^2$.

Let $\widetilde{A} = (e^i)^{-1}(A)$ and $\widetilde{A}_t = (e^i)^{-1}(A_t)$. For any $t \in [0, T_A)$, and $z \in \mathbb{S}_p \setminus \widetilde{A} \setminus \widetilde{K}_t$, we have $\widetilde{f}_t \circ \widetilde{\varphi}_t(z) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A(z)$. Taking the derivative w.r.t. $t$, we compute

$$\partial_t \widetilde{f}_t(\widetilde{\varphi}_t(z)) + \widetilde{f}_t'(\widetilde{\varphi}_t(z))\mathbf{H}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)) = \widetilde{f}_t'(\xi(t))^2 \mathbf{H}_{p_0-h(t)}(\widetilde{f}_t(\varphi_t(z)) - \widetilde{f}_t(\xi(t))).$$

Since $\widetilde{A}_t = \widetilde{\varphi}_t(A)$ for $0 \le t < T_A$, so for any $t \in [0, T_A)$, and $w \in \mathbb{S}_{p-t} \setminus \widetilde{A}_t$, we have $\widetilde{\varphi}_t^{-1}(w) \in \mathbb{S}_p \setminus \widetilde{A} \setminus \widetilde{K}_t$. Thus

$$\partial_t \widetilde{f}_t(w) = \widetilde{f}_t'(\xi(t))^2 \mathbf{H}_{p_0-h(t)}(\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t))) - \widetilde{f}_t'(w)\mathbf{H}_{p-t}(w - \xi(t)). \tag{10}$$

Recall that

$$\mathbf{H}_r(z) = -i \lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{2kr} + e^{iz}}{e^{2kr} - e^{iz}} = \cot(z/2) + \sum_{k=1}^{\infty} -i\left(\frac{e^{2kr} + e^{iz}}{e^{2kr} - e^{iz}} + \frac{e^{-2kr} + e^{iz}}{e^{-2kr} - e^{iz}}\right)$$

$$= \cot(z/2) + \sum_{k=1}^{\infty} \frac{2\sin(z)}{\cosh(2kr) - \cos(z)}.$$

Let

$$S_r = \sum_{k=1}^{\infty} \frac{2}{\cosh(2kr) - 1} = \sum_{k=1}^{\infty} \frac{1}{\cosh^2(kr)}.$$

Then the Laurent sires expansion of $\mathbf{H}_r$ at 0 is $\mathbf{H}_r(z) = 2/z + (S_r - 1/6)z + O(z^2)$.

Apply the following power series expansions:

$$\mathbf{H}_r(z) = 2/z + O(z);$$

$$\widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t)) + \widetilde{f}_t''(\xi(t))(w - \xi(t)) + O((w - \xi(t))^2);$$

$$\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)) = \widetilde{f}_t'(\xi(t))(w - \xi(t)) + \frac{\widetilde{f}_t''(\xi(t))}{2}(w - \xi(t))^2 + O((w - \xi(t))^3).$$

After some straightforward computation and letting $w \to \xi(t)$, we get

**Lemma 5.1** $\partial_t \widetilde{f}_t(\xi(t)) = -3\widetilde{f}_t''(\xi(t))$.

Now differentiate equation (10) with respect to $w$. We get

$$\partial_t \widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t))^2 \widetilde{f}_t'(w)\mathbf{H}'_{p_0-h(t)}(\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)))$$

$$-\widetilde{f}_t''(w)\mathbf{H}_{p-t}(w-\xi(t)) - \widetilde{f}_t'(w)\mathbf{H}'_{p-t}(w-\xi(t)).$$

Apply the previous power series expansions and the following expansions:

$$\mathbf{H}_r(z) = 2/z + (S_r - 1/6)z + O(z^2);$$

$$\mathbf{H}'_r(z) = -2/z^2 + (S_r - 1/6) + O(z);$$

$$\widetilde{f}_t''(w) = \widetilde{f}_t''(\xi(t)) + \widetilde{f}_t'''(\xi(t))(w-\xi(t)) + O((w-\xi(t))^2);$$

$$\widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t)) + \widetilde{f}_t''(\xi(t))(w-\xi(t)) + \frac{\widetilde{f}_t'''(\xi(t))}{2}(w-\xi(t))^2 + O((w-\xi(t))^3);$$

$$\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)) = \widetilde{f}_t'(\xi(t))(w-\xi(t)) + \frac{\widetilde{f}_t''(\xi(t))}{2}(w-\xi(t))^2$$

$$+\frac{\widetilde{f}_t'''(\xi(t))}{6}(w-\xi(t))^3 + O((w-\xi(t))^4).$$

After some long but straightforward computation and letting $w \to \xi(t)$, we get

**Lemma 5.2**

$$\frac{\partial_t \widetilde{f}_t'(\xi(t))}{\widetilde{f}_t'(\xi(t))} = \frac{1}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 - \frac{4}{3}\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))}$$

$$+\widetilde{f}_t'(\xi(t))^2(S_{p_0-h(t)} - 1/6) - (S_{p-t} - 1/6).$$

From Ito's formula and the above lemma, we have

$$d\widetilde{f}_t'(\xi(t)) = \partial_t \widetilde{f}_t'(\xi(t))dt + \widetilde{f}_t''(\xi(t))d\xi(t) + \frac{\kappa}{2}\widetilde{f}_t'''(\xi(t))dt$$

$$= \widetilde{f}_t''(\xi(t))d\xi(t) + \widetilde{f}_t'(\xi(t))\left(\frac{1}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right.$$

$$\left. + \widetilde{f}_t'(\xi(t))^2(S_{p_0-h(t)} - 1/6) - (S_{p-t} - 1/6)\right) dt.$$

Thus

$$d\widetilde{f}_t'(\xi(t))^\alpha = \alpha\widetilde{f}_t'(\xi(t))^{\alpha-1}d\widetilde{f}_t'(\xi(t)) + \alpha(\alpha-1)\widetilde{f}_t'(\xi(t))^{\alpha-2}\frac{\kappa}{2}\widetilde{f}_t''(\xi(t))^2dt$$

$$= \alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{d\widetilde{f}_t'(\xi(t))}{\widetilde{f}_t'(\xi(t))} + (\alpha-1)\frac{\kappa}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 dt\right)$$

$$= \alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}d\xi(t) + \left(\left(\frac{1}{2} + (\alpha-1)\frac{\kappa}{2}\right)\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2\right.\right.$$

$$\left.\left. + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))} + \widetilde{f}_t'(\xi(t))^2\left(S_{p_0-h(t)} - \frac{1}{6}\right) - \left(S_{p-t} - \frac{1}{6}\right)\right)dt\right)$$

$$= \alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}d\xi(t) + \left(h'(t)\left(S_{p_0-h(t)} - \frac{1}{6}\right) - \left(S_{p-t} - \frac{1}{6}\right)\right)dt\right).$$

The last equality uses $\kappa = 8/3$, $\alpha = 5/8$, and $h'(t) = \widetilde{f}_t'(\xi(t))^2$.

Now we have the following theorem.

**Theorem 5.1**

$$M_t = \widetilde{f}_t'(\xi(t))^{5/8}\exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r - \frac{1}{6}\right)dr\right),$$

*$0 \le t < T_A$, is a bounded martingale.*

**Proof.** From the above computation and Ito's formula, we see tht $M_t$, $0 \le t < T_A$, is a local martingale.

Since $f_t$ maps $\mathbb{A}_{p-t} \setminus A_t$ conformally onto $\mathbb{A}_{p_0-h(t)}$, so by the comparison principle of extremal length, the modulus of $\mathbb{A}_{p_0-h(t)}$ is not bigger than that of $\mathbb{A}_{p-t}$. Thus $p_0 - h(t) \le p - t$. Since $S_r > 0$ for any $r > 0$, so

$$\exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r - \frac{1}{6}\right)dr\right) \le \exp\left(\frac{5}{48}((p-t) - (p_0 - h(t))\right) \le \exp\left(\frac{5p}{48}\right). \quad (11)$$

Let $g_t = f_t^{-1}$, $\widetilde{g}_t = \widetilde{f}_t^{-1}$. Then $g_t \circ e^i = e^i \circ \widetilde{g}_t$. And $g_t$ maps $\mathbb{A}_{p_0-h(t)}$ conformally onto $\mathbb{A}_{p-t} \setminus A_t$. Now $-\ln(g_t(z)/z)$ is a bounded analytic function defined in $\mathbb{A}_{p_0-h(t)}$, $\mathrm{Re}\,(-\ln(g_t(z)/z)) \to (p-t) - (p_0 - h(t))$ as $z \to \mathbf{C}_{p_0-h(t)}$, and any subsequential limit of $\mathrm{Re}\,(-\ln(g_t(z)/z))$ as $z \to \mathbf{C}_0$ is nonnegative. Thus there are some $C \in \mathbb{R}$ and a positive measure $\mu_t$ supported by $\mathbf{C}_0$ of total mass $(p-t)-(p_0-h(t))$ such that for any $z \in \mathbb{A}_{p_0-h(t)}$,

$$-\ln(g_t(z)/z) = \int_{\mathbf{C}_0} \mathbf{S}_{p_0-h(t)}(z/\theta)d\mu_t(\theta) + iC. \tag{12}$$

For any $w \in \mathbb{S}_{p_0-h(t)}$, we have $e^i(w) \in \mathbb{A}_{p_0-h(t)}$, $\ln(g_t(e^i(w))) = i\widetilde{g}_t(w)$, and $\ln(z) = iw$, so

$$-i(\widetilde{g}_t(w) - w) = \int_{\mathbf{C}_0} \mathbf{S}_{p_0-h(t)}(e^i(w)/\theta)d\mu_t(\theta) + iC.$$

If $\widetilde{\mu}_t$ is a measure on $\mathbb{R}$ that satisfies $\mu_t = \widetilde{\mu}_t \circ (e^i)^{-1}$, then for any $w \in \mathbb{S}_{p_0-h(t)}$,

$$\widetilde{g}_t(w) - w = \int_{\mathbb{R}} i\mathbf{S}_{p_0-h(t)} \circ e^i(w-x)d\widetilde{\mu}_t(x) - C = \int_{\mathbb{R}} -\mathbf{H}_{p_0-h(t)}(w-x)d\widetilde{\mu}_t(x) - C.$$

Taking derivative w.r.t. $w$, we have

$$\widetilde{g}_t'(w) - 1 = \int_{\mathbb{R}} -\mathbf{H}'_{p_0-h(t)}(w-x)d\widetilde{\mu}_t(x). \tag{13}$$

From equation (3) and the definition of $\mathbf{H}_r$, we have

$$\mathbf{H}_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - \frac{z}{r} = \frac{\pi}{r}\lim_{M\to\infty}\sum_{k=-M}^{M}\frac{e^{2k\pi^2/r}+e^{-\pi z/r}}{e^{2k\pi^2/r}-e^{-\pi z/r}} - \frac{z}{r}.$$

Thus

$$\mathbf{H}'_r(z) = \frac{\pi^2}{r^2}\sum_{k=-\infty}^{\infty}\frac{-2e^{\pi z/r}e^{2k\pi^2/r}}{(e^{\pi z/r}-e^{2k\pi^2/r})^2} - \frac{1}{r}. \tag{14}$$

So for $z \in \mathbb{R}$, we have $\mathbf{H}'_r(z) < 0$. Apply this to equation (13). We get $\widetilde{g}_t'(\eta(h(t))) > 1$. Thus $\widetilde{f}_t'(\xi(t)) \in (0,1)$. Then from equation (11), we have

$$0 \le M_t \le \exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r - \frac{1}{6}\right)dr\right) \le \exp\left(\frac{5p}{48}\right).$$

Since $M_t$, $0 \le t < T_A$, is uniformly bounded, so it is a bounded martingale. $\square$

Now suppose that $A$ is a smooth hull, i.e., there is a smooth simple closed curve $\gamma : [0,1] \to \mathbb{A}_p \cup \mathbf{C}_0$ with $\gamma((0,1)) \subset \mathbb{A}_p$ and $\gamma(0) \ne \gamma(1) \in \mathbf{C}_0$, and $A$ is bounded by $\gamma$ and an arc on $\mathbf{C}_0$ between $\gamma(0)$ and $\gamma(1)$.

If $T_A < p$, a proof similar to Lemma 6.3 in [8] shows that $\widetilde{f}_t'(\xi(t)) \to 0$ as $t \to T_A$. Thus $M_t \to 0$ as $t \to T_A$ on the event that $T_A < p$. From now on, we suppose $T_A = p$.

Then $K_t$ approaches $\mathbf{C}_p$ as $t \to p$ and is uniformly bounded away from $A$. Then the modulus of $\mathbb{A}_p \setminus K_t \setminus A$ tends to 0 as $t \to p$. Thus $p_0 - h(t) \to 0$ as $t \to p$. So $S_A = p_0$. Now $A_t = \varphi_t(A)$ is bounded by $\gamma_t = \varphi_t(\gamma)$ and an arc on $\mathbf{C}_0$ between $\gamma_t(0)$ and $\gamma_t(1)$. So $A_t$ is also a smooth hull. Thus $f_t$ and $g_t$ both extend continuously to the boundary of the definition domain. And $f_t$ maps $\gamma_t$ to an arc on $\mathbf{C}_0$. Let $I_t$ denote this arc. Since $-\ln(g_t(z)/z)$ also extends continuously to $\mathbf{C}_0$, so the measure $\mu_t$ in equation (12) satisfies

$$d\mu_t(z) = -\mathrm{Re}\,\ln(g_t(z)/z)/(2\pi)d\mathbf{m}(z) = -\ln|g_t(z)|/(2\pi)d\mathbf{m},$$

where $\mathbf{m}$ is the Lebesgue measure on $\mathbf{C}_0$ (of total mass $2\pi$). Since $\ln|g_t(z)| = 0$ for $z \in \mathbf{C}_0 \setminus I_t$, so $\mu_t$ is supported by $I_t$. Let $\widetilde{\gamma}$ be a continuous curve such that $\gamma = e^i \circ \widetilde{\gamma}$. Let $\widetilde{\gamma}_t = \widetilde{\varphi}_t(\widetilde{\gamma})$ and $\widetilde{I}_t = \widetilde{f}_t(\gamma_t)$. Then $e^i(\widetilde{\gamma}_t) = \gamma_t$, $e^i(\widetilde{I}_t) = I_t$, and $\widetilde{I}_t$ is a real interval. Let $\widetilde{\mu}_t$ be a measure supported by $\widetilde{I}_t$ that satisfies $d\widetilde{\mu}_t(z) = \mathrm{Im}\,\widetilde{g}_t(z)/(2\pi)d\mathbf{m}_{\mathbb{R}}$ for $z \in \widetilde{I}_t$, where $\mathbf{m}_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$. Since $-\ln|g_t(e^i(z))| = \mathrm{Im}\,\widetilde{g}_t(z)$, so $\mu_t = \widetilde{\mu}_t \circ (e^i)^{-1}$. Thus equation (13) holds for this $\widetilde{\mu}_t$.

Now $\widetilde{\varphi}_t$ maps $\mathbb{S}_p \setminus \widetilde{K}_t$ conformally onto $\mathbb{S}_{p-t}$. Let $\Sigma_t$ be the union of $\mathbb{S}_p \setminus \widetilde{K}_t$, its reflection w.r.t. $\mathbb{R}$, and $\mathbb{R} \setminus \overline{\widetilde{K}_t}$. By Schwarz reflection principle, $\varphi_t$ extends analytically to $\Sigma_t$, and maps $\Sigma_t$ conformally into $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < p-t\}$. For every $z \in A$, the distance from $z$ to the boundary of $\Sigma_t$ is at least $d_0 = \min\{p, dist(A, K_p)\} > 0$, and the distance from $\varphi_t(z)$ to the boundary of $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < p-t\}$ equals to $p-t$. By Koebe's 1/4 theorem, $|\varphi_t'(z)| \le 4(p-t)/d_0$. Let $H = \max\{\mathrm{Im}\,\widetilde{\gamma}(u) : u \in [0,1]\}$. Since $\widetilde{\gamma}_t = \widetilde{\varphi}_t \circ \widetilde{\gamma}$, so $H_t := \max\{\mathrm{Im}\,\widetilde{\gamma}_t(u) : u \in [0,1]\} \le 4(p-t)H/d_0$. A proof similar as above shows that for any $z \in \widetilde{I}_0$, $|\widetilde{\psi}_{h(t)}(z)| \le 4(p_0 - h(t))/d_1$ for some $d_1 > 0$. Since

$$\widetilde{I}_t = \widetilde{f}_t(\widetilde{\gamma}_t) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A(\widetilde{\gamma}) = \widetilde{\psi}_{h(t)}(\widetilde{I}_0), \tag{15}$$

so $|\widetilde{I}_t| \le 4(p-t)/d_1|\widetilde{I}_0|$. Thus $|\mu_t| = |\widetilde{\mu}_t| \le H_t|I_t| \le 16(p-t)(p_0-h(t))H|\widetilde{I}_0|/(d_0d_1)$. Let $C_0 = 16H|\widetilde{I}_0|/(d_0d_1)$, then

$$(p-t) - (p_0 - h(t)) = |\mu_t| = |\widetilde{\mu}_t| \le C_0(p-t)(p_0 - h(t)). \tag{16}$$

Thus

$$(p_0 - h(t))/(p-t) \ge 1 - C_0(p_0 - h(t)). \tag{17}$$

Since $\widetilde{\mu}_t$ is supported by $\widetilde{I}_t$, so from equation (13) we have

$$\widetilde{g}_t'(\eta(h(t))) - 1 = \int_{\widetilde{I}_t} -\mathbf{H}'_{p_0-h(t)}(\eta(h(t)) - x)d\widetilde{\mu}_t(x).$$

Let $\widetilde{\alpha}(t) = \widetilde{\varphi}_t^{-1}(\xi(t))$. Then $\widetilde{\alpha}(t)$ is a simple curve, and $\alpha(t) = e^i(\widetilde{\alpha}(t)) = \varphi_t^{-1}(e^i(\xi(t)))$ is an annulus $\mathrm{SLE}_{8/3}$ trace. So $K_t = \alpha((0,t])$ for any $t \ge 0$. Thus $\widetilde{K}_t = \cup_{k\in\mathbb{Z}}(\widetilde{\alpha}((0,t])+2k\pi)$.

Let $\widetilde{\beta}(s) = \widetilde{\varphi}_A(\widetilde{\alpha}(h^{-1}(s)))$ for $0 \le s < p_0$. Since $\widetilde{L}_{h(t)} = \widetilde{\varphi}_A(K_t)$ and $\widetilde{\varphi}_A(z + 2k\pi) = \widetilde{\varphi}_A(z) + 2k\pi$, so $\widetilde{L}_{h(t)} = \cup_{k\in\mathbb{Z}}(\widetilde{\beta}((0, h(t)]) + 2k\pi)$. Now we compute

$$\widetilde{\psi}_{h(t)}(\widetilde{\beta}(h(t))) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A \circ \widetilde{\varphi}_t^{-1}(\xi(t)) = \widetilde{f}_t(\xi(t)) = \eta(h(t)).$$

Thus $\widetilde{\psi}_s$ maps the left and right side of $\beta((0, h(t)))$ to intervals $(b_-(t), \eta(h(t)))$ and $(\eta(h(t)), b_+(t))$, respectively, for some $b_-(t) < \eta(h(t)) < b_+(t)$. Therefore $\widetilde{\psi}_{h(t)}$ maps the $\widetilde{L}_{h(t)}$ to $\cup_{k\in\mathbb{Z}}(b_-(t) + 2k\pi, b_+(t) + 2k\pi)$. From equation (15), we have $\cup_{k\in\mathbb{Z}}(l(t) + 2k\pi, r(t) + 2k\pi) \cap \widetilde{I}_t = \emptyset$. So for any $x \in \widetilde{I}_t$ and $k \in \mathbb{Z}$, $|x - (\eta(s) + 2k\pi)| \ge \min\{\eta(h(t)) - b_-(t), b_+(t) - \eta(h(t))\}$.

As $t \to p$, $\widetilde{\beta}(h(t))$ approaches to a point on $\mathbb{R}_{p_0}$, so the extremal distance between the left side of $\widetilde{\beta}((0, h(t))$ and $\mathbb{R}_{p_0}$ in $\mathbb{S}_{p_0} \setminus \widetilde{L}_{h(t)}$ tends to 0. Since $\widetilde{\varphi}_t$ maps $(\mathbb{S}_{p_0} \setminus \widetilde{L}_{h(t)}, \mathbb{R}_{p_0})$ conformally onto $(\mathbb{S}_{p_0-h(t)}, \mathbb{R}_{p_0-h(t)})$, and the left side of $\widetilde{\beta}((0, h(t))$ is mapped to $(b_-(t), \eta(h(t)))$, so the extremal distance between $(b_-(t), \eta(h(t)))$ and $\mathbb{R}_{p_0-h(t)}$ in $\mathbb{S}_{p_0-h(t)}$ tends to 0 as $t \to p$ by the conformal invariance property of extremal length. Thus $(\eta(h(t)) - b_-(t))/(p_0 - h(t)) \to +\infty$ as $t \to p$. Similarly, $(b_+(t) - \eta(h(t)))/(p_0 - h(t)) \to +\infty$ as $t \to p$.

Suppose $R \ge \ln(2)/\pi$. Then $e^{\pi R} \ge 2$, and so $e^{\pi R} - 1 \ge e^{\pi R}/2$. Suppose $r > 0$ and the distance from $x \in \mathbb{R}$ to $\{2k\pi : k \in \mathbb{Z}\}$ is at least $rR$, then there is $k_0 \in \mathbb{Z}$ such that $2(k_0 + 1)\pi - rR \ge x \ge 2k_0\pi + rR$. Thus $2rR \le 2\pi$, and so $r \le \pi/R$. From equation (14), we have

$$-\mathbf{H}_r'(x) = \frac{\pi^2}{r^2}\sum_{k=-\infty}^{k_0}\frac{2e^{\pi(x-2k\pi)/r}}{(e^{\pi(x-2k\pi)/r}-1)^2} + \frac{\pi^2}{r^2}\sum_{k=k_0+1}^{+\infty}\frac{2e^{\pi(2k\pi-x)/r}}{(e^{\pi(2k\pi-x)/r}-1)^2} + \frac{1}{r}$$

$$\le \frac{\pi^2}{r^2}\sum_{k=-\infty}^{k_0}\frac{2e^{\pi(2k_0\pi+rR-2k\pi)/r}}{(e^{\pi(2k_0\pi+rR-2k\pi)/r}-1)^2} + \frac{\pi^2}{r^2}\sum_{k=k_0+1}^{+\infty}\frac{2e^{\pi(2k\pi-(2(k_0+1)\pi-rR))/r}}{(e^{\pi(2k\pi-(2(k_0+1)\pi-rR))/r}-1)^2} + \frac{1}{r}$$

$$= 2\frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{2e^{\pi(2m\pi+rR)/r}}{(e^{\pi(2m\pi+rR)/r}-1)^2} + \frac{1}{r} = \frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{4e^{\pi R+2m\pi^2/r}}{(e^{\pi R+2m\pi^2/r}-1)^2} + \frac{1}{r}$$

$$\le \frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{4e^{\pi R+2m\pi^2/r}}{(e^{\pi R+2m\pi^2/r}/2)^2} + \frac{1}{r} = 16\sum_{m=0}^{\infty}e^{-\pi R-2m\pi^2/r} + \frac{1}{r}$$

$$= \frac{\pi^2}{r^2}\frac{16e^{-\pi R}}{1-e^{-2\pi^2/r}} + \frac{1}{r} \le \frac{\pi^2}{r^2}\frac{16e^{-\pi R}}{1-e^{-2\pi^2/(\pi/R)}} + \frac{1}{r} \le 32\frac{\pi^2}{r^2}e^{-\pi R} + \frac{1}{r}.$$

Let $r(t) = p_0 - h(t)$ and $R(t) = \min\{\eta(h(t)) - b_-(t), b_+(t) - \eta(h(t))\}/r(t)$. Then $r(t) \to 0$ and $R(t) \to +\infty$ uniformly in $\omega$ as $t \to p$, and for any $x \in \widetilde{I}_t$, the distance from $\eta(h(t)) - x$ and $\{2k\pi : k \in \mathbb{Z}\}$ is at least $r(t)R(t)$. There is $t_0 \in (0,p)$ such that for $t \in [t_0, p)$, $R(t) \geq \ln(2)/\pi$ and $r(t) \leq 1/(2C_0)$, where $C_0$ is as in equation (17). From the above displayed formula, we have $-H'_{r(t)}(x) \leq 32\pi^2/r(t)^2 e^{-\pi R(t)} + 1/r$. When $t \in [t_0, p)$, $r(t)/(p-t) \geq 1/2$ by equation (17), and so from the estimation of $-\mathbf{H}'_{r(t)}(x)$ and equation (16), we have

$$\widetilde{g}'_t(\eta(h(t))) - 1 \leq |\widetilde{\mu}_t|(32\frac{\pi^2}{r(t)^2}e^{-\pi R(t)} + \frac{1}{r(t)})$$

$$\leq C_0(p-t)r(t)(32\frac{\pi^2}{r(t)^2}e^{-\pi R(t)} + \frac{1}{r(t)}) \leq 64C_0\pi^2 e^{-\pi R(t)} + C_0(p-t) \to 0,$$

as $t \to p$. Thus $\widetilde{f}'_t(\xi(t)) = 1/\widetilde{g}'_t(\eta(h(t))) \to 1$ as $t \to p$. Recall that the above argument is based on the assumption that $T_A = p$.

Suppose

$$\int_{p_0-h(t)}^{p-t} S_r dr \to 0, \text{ as } t \to p \text{ on the event that } T_A = p. \tag{18}$$

Then $M_t \to 1$ as $t \to T_A$ on the event that $T_A = p$. From the Markov property, we have

$$\widetilde{\varphi}'_A(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_0}^{p}\left(S_r - \frac{1}{6}\right)dr\right) = M_0 = \mathbf{E}\,[\lim_{t\to T_A} M_t] = \mathbf{P}\left(\{T_A = p\}\right).$$

Recall that $p_0$ is the modulus of $\mathbb{A}_p \setminus A$. Let $K_p = \cup_{0\leq t<p} K_t$. Then

$$\mathbf{P}\left(\{K_p \cap A = \emptyset\}\right) = \widetilde{\varphi}'_A(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_0}^{p}\left(S_r - \frac{1}{6}\right)dr\right). \tag{19}$$

If $A$ is not a smooth hull, we may find a sequence of smooth hulls $A_n$ that approaches $A$. Then $\widetilde{\varphi}'_{A_n}(0) \to \widetilde{\varphi}'_A(0)$ and the modulus of $\mathbb{A}_p \setminus A_n$ tends to the modulus of $\mathbb{A}_p \setminus A$, so equation (19) still holds.

Now suppose $B$ is a hull in $\mathbb{A}_{p_0}$ w.r.t. $\mathbf{C}_{p_0}$. Let $D = A \cup \varphi_A^{-1}(B)$. Then $D$ is a hull in $\mathbb{A}_p$ w.r.t. $\mathbf{C}_p$. Let $p_1$ be the modulus of $\mathbb{A}_p \setminus D$, which is also the modulus of $\mathbb{A}_{p_0} \setminus B$. Then $\varphi_D = \varphi_B \circ \varphi_A$, so $\widetilde{\varphi}_D = \widetilde{\varphi}_B \circ \widetilde{\varphi}_A$ and $\widetilde{\varphi}'_D(0) = \widetilde{\varphi}'_B(\widetilde{\varphi}_A(0))\widetilde{\varphi}'_A(0) = \widetilde{\varphi}'_B(0)\widetilde{\varphi}'_A(0)$. From the last paragraph,

$$\mathbf{P}\left(K_p \cap D = \emptyset\}\right) = \widetilde{\varphi}'_D(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p}\left(S_r - \frac{1}{6}\right)dr\right).$$

Thus

$$\mathbf{P}\left(\{K_p \cap D = \emptyset\}|\{K_p \cap A = \emptyset\}\right) = \widetilde{\varphi}'_B(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p_0}\left(S_r - \frac{1}{6}\right)dr\right).$$

If $L_s$, $0 \le s < p_0$, are standard annulus $\mathrm{SLE}_{8/3}$ hulls of modulus $p_0$, and $L_{p_0} = \cup_{0 \le s < p_0} L_s$, then

$$\mathbf{P}\left(\{L_{p_0} \cap B = \emptyset\}\right) = \widetilde{\varphi}_B'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p_0}\left(S_r - \frac{1}{6}\right)dr\right).$$

$$= \mathbf{P}\left(\{K_p \cap D = \emptyset\}|\{K_p \cap A = \emptyset\}\right) = \mathbf{P}\left(\{\varphi_A(K_p) \cap B = \emptyset\}|\{K_p \cap A = \emptyset\}\right).$$

Thus conditioned on the event that $K_p \cap A = \emptyset$, $\varphi_A(K_p)$ has the same distribution as $L_{p_0}$. Then we proved the restriction property of annulus $\mathrm{SLE}_{8/3}$ under the assumption (18).

Unfortunately, the assumption (18) is actually always false. From equation (3) one may compute that $S_r$ is of order $\Theta(1/r^2)$ as $r \to 0$. From (17), $(p-t)-(p_0-h(t)) = |\mu_t|$ is of order $O((p-t)^2)$. In fact, one could prove that it is of order $\Theta((p-t)^2)$. So $\int_{p_0-h(t)}^{p-t} S_r dr$ is uniformly bounded away from 0. Thus it does not tend to 0 as $t \to p$. Therefore we guess that annulus $\mathrm{SLE}_{8/3}$ does not satisfy the restriction property.

Recently, Robert O. Bauer studied in [2] a process defined in a doubly connected domain obtained by conditioning a chordal $\mathrm{SLE}_{8/3}$ in a simply connected domain to avoid an interior contractible compact subset. The process describes a random simple curve connecting two prime ends of a doubly connected domain that lie on the same side, so it is different from the process we study here. That process automatically satisfies the restriction property from the restriction property of chordal $\mathrm{SLE}_{8/3}$. And it satisfies conformal invariance because the set of boundary hulls generates the same $\sigma$-algebra as the Hausdorff metric on the space of simple curves.

**Acknowledgement:** I would like to thank my PhD advisor Nikolai Makarov for his instruction on this work.

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