Random Loewner Chains in Riemann Surfaces

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Abstract

The thesis describes an extension of O. Schramm's SLE processes to complicated plane domains and Riemann surfaces. First, three kinds of new SLEs are defined for simple conformal types. They have properties similar to traditional SLEs. Then harmonic random Loewner chains (HRLC) are defined in finite Riemann surfaces. They are measures on the space of Loewner chains, which are increasing families of closed subsets satisfying certain properties. An HRLC is first defined on local charts using Loewner's equation. Since the definitions in different charts agree with each other, these local HRLCs can be put together to construct a global HRLC. An HRLC in a plane domain can be described by differential equations involving canonical plane domains. Those old and new SLEs are special cases of HRLCs. An HRLC is determined by a parameter $\kappa \geq 0$, a starting point and a target set. When $\kappa = 6$, the HRLC satisfies the locality property. When $\kappa = 2$, the HRLC preserves some observable that resembles the observable for the corresponding loop-erased random walk (LERW). So HRLC_2 should be the scaling limit of LERW. With reasonable assumptions, $\mathrm{HRLC}_{8/3}$ differs from a restriction measure by a conformally invariant density; for $\kappa \in (0, 8/3)$, HRLC_κ differs from a pre-restriction measure by a conformally invariant density. A restriction measure could be constructed from a pre-restriction measure by adding Brownian bubbles.

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Chapter 1

Introduction

1.1 Background

Stochastic Loewner evolution or Schramm-Loewner evolution (SLE), introduced by O. Schramm in [15], is about random growth processes of closed fractal subsets in simply connected (plane) domains (other than $\mathbb{C}$) and in Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The evolution is described by the classical Loewner differential equation with the driving term being a Brownian motion. SLE depends on a parameter $\kappa > 0$, the speed of the Brownian motion, and behaves differently for different value of κ. See [14] by S. Rohde and O. Schramm for the basic fundamental properties of SLE.

Schramm's processes turned out to be very useful. On the one hand, they are amenable to computations, on the other hand, they are related with some statistical physics models. In a series of papers [4]-[8], G. F. Lawler, O. Schramm and W. Werner used SLE to determine the Brownian motion intersection exponents in the plane, identified SLE_2 and SLE_8 with the scaling limits of loop-erased random walk and uniform spanning tree Peano curve, respectively, and conjectured that $\mathrm{SLE}_{8/3}$ is the scaling limit of self avoiding walk. S. Smirnov proved in [17] that SLE_6 is the scaling limit of critical site percolation on the triangular lattice. And O. Schramm and S. Sheffield proved in [16] that the harmonic explorer converges to SLE_4.

For various reasons, a similar theory should also exist for multiply connected domains and even for general Riemann surfaces. We expect that the definition and some study of general SLE will give us better understanding of SLE itself and its physics background.

There are three kinds of SLEs in the literature: radial SLE, chordal SLE, and full plane SLE. They are all conformally invariant. Full plane SLE grows in $\widehat{\mathbb{C}}$, and is the limit case of radial SLE. Radial and chordal SLE grow in simply connected domains. A basic property of these two kinds of SLEs is that if (K_t) has the law of a radial or chordal SLE in a domain D, then for any fixed time b, the increment of (K_t) after b has the same law as (K_t) in the sense of conformal equivalence. This means that there is a conformal map W from D onto $D \setminus K_b$ fixing certain marked point which is the target of (K_t) such that $(W(K_t))$ has the same law as $(K_{b+t} \setminus K_b)$. For radial SLE, the target is an interior point. And for chordal SLE, the target is a prime end. One may see [1] for the definition of a prime end, and we will give an equivalent definition in Chapter 3.

The definitions of SLEs use the fact that all simply connected domains with a marked interior point or prime end are conformally equivalent. This property does not hold for general plane domains or Riemann surfaces. That is the main difficulty in our extension of SLEs.

1.2 Main results

In this thesis, we first define three new kinds of SLEs: strip SLE, annulus SLE, and disc SLE. Strip SLE grows in a simply connected domain whose target is a "boundary arc" which is the set of prime ends between two fixed prime ends. Annulus SLE grows in a doubly connected domain (with finite modulus) whose target is a whole boundary component. Disc SLE is the limit case of annulus SLE, and grows in a simply connected domain. We call them SLEs because they satisfy most properties that the traditional SLEs have.

For further extension of SLEs, we first define Loewner chains in finite Riemann surfaces. Simply or doubly connected domains are simple types of finite Riemann surfaces, and the old and new SLEs all describe measures on Loewner chains. We then provide a method to define conformally invariant measures on Loewner chains in complicated types of Riemann surfaces. We call them the harmonic random Loewner chains, or HRLCs.

An HRLC in a finite Riemann surface depends on a parameter $\kappa \ge 0$, a starting point, and a target set. It is first defined in local charts. In each local chart, after a time-change, the loewner chain becomes a radial or chordal Loewner chain, whose driving function is a Brownian motion plus some drift term, where the drift term carries the information of the total surface and the target set. Ito's formula is used to show that the definition in all charts agree with each other. So they can be pasted together to get a global HRLC. Those six kinds of SLEs are special kinds of HRLCs. For HRLC in plane domains, the growth of the chain can be described by a differential equation with finitely many variables. The canonical domains are used here.

The definition of HRLC is most successful when $\kappa = 6$ and $\kappa = 2$. If $\kappa = 6$, the random Loewner chain has the locality property, which means that the chain does not feel the boundary before hitting it. For $\kappa > 0$ and $\kappa \ne 6$, the chain satisfies the "weak locality". If $\kappa = 2$, the growing chain preserves some observables which resemble the observables for LERW (loop-erased random walk). If we consider plane domain, then this property suggests that HRLC_2 is the scaling limit of a corresponding LERW.

We then show that a Brownian excursion can be constructed by adding Brownian bubbles to an HRLC_2 trace. With reasonable assumptions, we find that $\text{HRLC}_{8/3}$ differs from a restriction measure by a conformally invariant density; and for $0 < \kappa < 8/3$, HRLC_κ differs from a pre-restriction measure also by a conformally invariant density. And a restriction measure can be constructed by adding Brownian bubbles to a pre-restriction measure.

For the extension of SLE_4 and SLE_8, it seems that HRLC_4 and HRLC_8 do not have properties similar to the corresponding SLEs. Some drift terms other than those that are used to define HRLC in local charts are needed to define the random Loewner chain. We expect that the random Loewner chain should preserve observables similar to those for SLE_4 and SLE_8. This work is still in progress now, and is not included in this thesis.

Chapter 2

Various kinds of SLEs

2.1 Hulls and Loewner chains in simple domains

There are three kinds of SLEs in the literature: radial SLE, chordal SLE, and full plane SLE. In this chapter, we define another three kinds of SLEs: strip SLE, annulus SLE, and disc SLE. The content about annulus SLE and disc SLE are chosen from the paper [20].

Those SLEs are measures on the space of (interior) Loewner chains in Riemann sphere $\widehat{\mathbb{C}}$, simply or doubly connected domains. By a simply connected domain we mean a simply connected plane domain that is not $\mathbb{C}$. By a doubly connected domain we mean a doubly connected plane domain whose two boundary components in $\widehat{\mathbb{C}}$ both contain more than one point.

We say K is a hull in a simply connected domain D if $D \setminus K$ is still a simply connected domain. Suppose D is a doubly connected domain with two boundary components S_1 and S_2 in $\widehat{\mathbb{C}}$. We say K is a hull in D on S_j if $D \setminus K$ is a doubly connected domain that has S_{3-j} as one boundary component in $\widehat{\mathbb{C}}$.

Let D be a simply or doubly connected domain and S is a boundary component of D in $\widehat{\mathbb{C}}$. Suppose L maps $[0,T)$, $T \in (0,\infty]$, to the space of hulls in D (on S), such that $L(0) = \emptyset$, $L(t_1) \subsetneqq L(t_2)$ when $t_1 < t_2$, and for any $a \in [0,T)$ and a compact $F \subset D \setminus L(a)$ that has more than one point, the extremal length of the family of path that separates F from $L(t+\varepsilon) \setminus L(t)$ in $D \setminus L(t+\varepsilon)$ tends to 0 as $\varepsilon \to 0^+$, uniformly in $t \in [0,a]$. Then we call L a Loewner chain in D (on S). For such L, there is a unique prime end w of D (on S) such that $L_\varepsilon \to w$ as $\varepsilon \to 0^+$. We say L is started from w.

Suppose D is the Riemann sphere or a simply connected domain. A compact contractible subset K of D that contains more than one point is called an interior hull in D. Then $D \setminus K$ is a simply or doubly connected domain, and the boundary of K is a boundary component of $D \setminus K$. Suppose $p \in D$ and L maps $(-\infty, T)$, $T \in (-\infty, \infty]$, to the space of interior hulls in D such that $\{p\} = \cap L(t)$, $L(t_1) \subsetneqq L(t_2)$ when $t_1 < t_2$, and for each $a \in (-\infty, T)$, $t \mapsto L(a+t)$ is a Loewner chain in $D \setminus L(a)$ (on $\partial L(a)$). Then we say L is an interior Loewner chain in D started from p. Suppose L is a Loewner chain or interior Loewner chain defined on $[0, T)$ or $(-\infty, T)$. Let u be a continuous (strictly) increasing function on $[0, T)$ or $(-\infty, T)$ such that $u(0) = 0$ or $u(-\infty) = -\infty$. Then $t \mapsto L(u^{-1}(t))$ is still a Loewner chain or interior Loewner chain, and is called a time-change of L through u. From the conformal equivalence of extremal length, the conformal image of a (interior) hull or a (interior) Loewner chain is still a (interior) hull or a (interior) Loewner chain.

2.2 Review of SLEs in the literature

2.2.1 Radial SLE

Radial SLE is first defined in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |D| < 1\}$. Given a real continuous function $\xi(t)$ on $[0, a)$, consider the following *radial Loewner equation*:

$$\partial_t \varphi_t(z) = \varphi_t(z) \frac{e^{i\xi(t)} + \varphi_t(z)}{e^{i\xi(t)} - \varphi_t(z)}, \quad \varphi_0(z) = z. \tag{2.2.1}$$

For $t \in [0, a)$, let K_t be the set of points z in $\mathbb{D}$ such that the solution $\varphi_s(z)$ blows up before or at time t. φ_t and K_t, $0 \le t < a$, are called the standard radial LE maps and hulls driven by ξ, respectively. We have $0 \notin K_t$, and φ_t maps $(\mathbb{D} \setminus K_t; 0)$ conformally onto $(\mathbb{D}; 0)$ with $\varphi_t'(0) = e^t$. So K_t is a hull in $\mathbb{D}$. If K is a hull in a simply connected domain D and $p \in D \setminus K$, there is a unique function φ that maps $(D \setminus K; p)$ conformally onto $(D; p)$ such that $\varphi'(p) \ge 1$. Then we let $C_{D;p}(K) := \ln \varphi'(w)$ be the capacity of K in D w.r.t. w. The capacity is 0 iff $K=\emptyset$. For standard radial LE hulls (K_t), we have $C_{\mathbb{D};0}(K_t)=t$ for all t.

C. Pommerenke proved in [10] that if (K_t) is a family of standard radial LE hulls, then $t\mapsto K_t$ is a Loewner chain in $\mathbb{D}$. On the other hand, if $t\mapsto K_t$ is a Loewner chain in $\mathbb{D}$ such that $0\not\in K_t$ and $C_{\mathbb{D};0}(K_t)=t$ for all t, then (K_t) is a family of radial LE hulls. Moreover, if $\xi(t)$ is the driving function and φ_t is the corresponding map, then

$$\{e^{i\xi(t)}\}=\cap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon}\setminus K_t)}. \tag{2.2.2}$$

For $t=0$, this formula means that $t\mapsto K_t$ is started from $e^{i\xi(t)}$ In fact, if we don't assume that $C_{\mathbb{D};0}(K_t)=t$, but let $u(t):=C_{\mathbb{D};0}(K_t)$, then $u(0)=0$, and u is continuous and increasing, and the capacity of $K_{u^{-1}(s)}$ in $\mathbb{D}$ w.r.t. 0 is s for all s. Since the property of Loewner chain is preserved under a time-change, $(K_{u^{-1}(s)})$ is a family of standard radial LE hulls.

Suppose $B(t)$ is a standard Brownian motion, i.e., $B(0)=0$ and $\mathbf{E}\left(B(t)^2\right)=t$, and $\kappa\ge 0$ is a fixed number. The law of a family of standard radial LE hulls (K_t) driven by $\sqrt{\kappa}B(t)$, $0\le t<\infty$, is called the standard radial SLE_κ. It is a measure on the space of Loewner chains in $\mathbb{D}$ started from 1. From Koebe's 1/4 theorem, the distance from 0 to K_t tends to 0 as $t\to\infty$. So we say that the standard radial SLE_κ grows in $\mathbb{D}$ from 1 to 0. Suppose D is a simply connected domain, $z_2\in D$, and z_1 is a prime end, then there is a unique conformal map W that maps $(\mathbb{D};1,0)$ onto $(D;z_1,z_2)$. The radial $\mathrm{SLE}_\kappa(D;z_1\to z_2)$, or radial SLE_κ in D from z_1 to z_2, is defined as the image of the standard radial SLE_κ under the map W.

The radial SLE has the property of symmetry and conformally equivalent time-homogeneity. The symmetry property means that the radial $\mathrm{SLE}_\kappa(D;z_1\to z_2)$ is preserved under the self anti-conformal map of $(D;z_1,z_2)$. The property of conformally equivalent time-homogeneity means the following. Suppose (K_t^1) and (K_t^2) are independent and both have the law of $\mathrm{SLE}_\kappa(D;z_1\to z_2)$. Fix $b\ge 0$. Then there is a random conformal map g from $(D;z_2)$ onto $(D\setminus K_b^1;z_2)$ determined by $(K_t^1,0\le t\le b)$ such that (K_t^3) defined by $K_t^3=K_t^1$ for $0\le t\le b$ and $K_{b+t}^3=K_b^1\cup g(K_{t-b}^2)$ for $t\ge b$

also has the law of $\mathrm{SLE}_\kappa(D; z_1 \to z_2)$. A constant speed time-change of a radial SLE also satisfies these two properties. On the other hand, the above two properties determine a radial SLE up to the parameter κ and a constant speed time-change.

2.2.2 Chordal SLE

Chordal SLE is first defined in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Given a real continuous function $\xi(t)$ on $[0, a)$, consider the following *chordal Loewner equation*:

$$\partial_t \varphi_t(z) = \frac{2}{\varphi_t(z) - \xi(t)}, \quad \varphi_0(z) = z. \tag{2.2.3}$$

For $t \in [0, \infty)$, let K_t be the set of points z in $\mathbb{H}$ such that the solution $\varphi_s(z)$ blows up before or at time t. We call φ_t and K_t, $0 \le t < \infty$, the standard chordal LE maps and hulls driven by ξ, respectively. For each $t \in [0, \infty)$, K_t is a hull in $\mathbb{H}$ bounded from ∞, and φ_t is the conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ that satisfies the *Hydrodynamic normalization*,

$$\varphi_t(z) = z + \frac{t}{z} + O(\frac{1}{|z|^2}), \;\; z \to \infty, \tag{2.2.4}$$

In fact for any bounded hull K in $\mathbb{H}$, there is a unique conformal map φ from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ that satisfies (2.2.4) with φ_t replaced by φ and t by some $c \ge 0$. The constant c is called the capacity of K in $\mathbb{H}$ w.r.t. ∞, denoted by $C_{\mathbb{H};\infty}(K)$. The capacity is 0 iff $K = \emptyset$. So for the standard chordal LE hulls (K_t), $C_{\mathbb{H};\infty}(K_t) = t$ for all t.

If $(K_t, 0 \le t < \infty)$ is a family of standard chordal LE hulls, then $t \mapsto K_t$ is a Loewner chain in $\mathbb{H}$. On the other hand, if $t \mapsto K_t$ is a Loewner chain in $\mathbb{H}$ such that every K_t is bounded and $C_{\mathbb{H};\infty}(K_t) = t$ for all t, then (K_t) is a family of chordal LE hulls. Moreover, if $\xi(t)$ is the driving function and φ_t is the corresponding map, then

$$\{\xi(t)\} = \cap_{\varepsilon > 0} \overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}. \tag{2.2.5}$$

So $t \mapsto K_t$ is started from $\xi(0)$. In fact, if we don't assume that $C_{\mathbb{H};\infty}(K_t) = t$ for all t, then after a time-change similar as that in the radial case, we can make (K_t) to be a family of standard chordal LE hulls.

The law of a family of standard chordal LE hulls driven by $\sqrt{\kappa}B(t)$ is called the standard chordal SLE$_\kappa$. It is a measure on the space of Loewner chains in $\mathbb{H}$ started from 0. From Koebe's 1/4 theorem, the spherical distance from ∞ to K_t tends to 0 as $t \to \infty$. So we say that the standard chordal SLE$_\kappa$ grows in $\mathbb{H}$ from 0 to ∞. If D is a simply connected domain, and $z_1 \neq z_2$ are two prime ends of D, then there is at least one conformal map W from $(\mathbb{H}; 0, \infty)$ onto $(D; z_1, z_2)$. Then we call the image of the standard chordal SLE$_\kappa$ under the map W a chordal SLE$_\kappa(D; z_1 \to z_2)$, or a chordal SLE$_\kappa$ in D from z_1 to z_2. Note that W is not unique. However, if W_1 and W_2 both map $(\mathbb{H}; 0, \infty)$ conformally onto $(D; z_1, z_2)$, then for some $c > 0$, $W_1(z) = W_2(cz)$. On the other hand, the standard chordal SLE$_\kappa$ satisfies the scaling property. That means (cK_t) has the same law as (K_{c^2t}) if (K_t) has the law of the standard chordal SLE$_\kappa$. So $(W_1(K_t))$ has the same law as $W_2(K_{c^2t})$. That means two chordal SLE$_\kappa(D; z_1 \to z_2)$ differ by a constant speed time-change.

Similarly as the radial case, a standard chordal SLE$_\kappa(D; z_1 \to z_2)$ is preserved under the self anti-conformal of $(D; z_1, z_2)$ whose derivatives at z_1 and z_2 are both equal to 1, and has the property of symmetry and conformally equivalent time-homogeneity. And these two properties determine a chordal SLE up to the parameter κ and a constant speed time-change.

2.2.3 Full plane SLE

Full plane SLE grows in the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Suppose ξ is a real continuous function defined on $(-\infty, a)$. From [11], there are an interior Loewner chain $t \mapsto K_t$, $-\infty < t < a$, in $\widehat{\mathbb{C}}$ started from 0, and a family of conformal maps φ_t from $(\widehat{\mathbb{C}} \setminus K_t; \infty)$ onto $(\mathbb{D}; 0)$, $-\infty < t < a$, that satisfies

$$\begin{cases} \partial_t \varphi_t(z) = \varphi_t(z) \frac{e^{i\xi(t)} + \varphi_t(z)}{e^{i\xi(t)} - \varphi_t(z)}; \\ \lim_{t \to -\infty} e^t / \varphi_t(z) = z, \forall z \in \mathbb{C} \setminus \{0\}. \end{cases} \tag{2.2.6}$$

Such φ_t and K_t, $-\infty < t < a$, are unique, and are called the standard full plane LE maps and interior hulls, respectively, driven by ξ. Moreover, we have

$$\{e^{i\xi(t)}\} = \cap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}. \tag{2.2.7}$$

Let $B(t)$ be as usual and $B_+(t) = B(t)$. Let $B_-(t)$ be a standard Brownian motion independent of $B_+(t)$. Let $\mathbf{x}$ be a random variable uniformly distributed on $[0, 2\pi)$ that is independent of $B_\pm(t)$. For a fixed $\kappa \ge 0$, let $\xi_\kappa(t) := \mathbf{x} + \sqrt{\kappa}\mathbf{B}_{\mathrm{sign}(\mathbf{t})}(|\mathbf{t}|)$. Then for any $b \in \mathbb{R}$, $(e^{i\xi(b+t)}/e^{i\xi(b)})$ has the same law as $(e^{i\sqrt{\kappa}B(t)})$. We call the law of the standard full plane LE interior hulls driven by ξ_κ the standard full plane SLE$_\kappa$. If (K_t) has the law of the standard full plane SLE$_\kappa$, from Koebe's 1/4 theorem, the spherical distance from ∞ to K_t tends to 0 as $t \to \infty$. So we say that the standard full plane SLE$_\kappa$ grows in $\widehat{\mathbb{C}}$ from 0 to ∞. Through a conformal map, we could define full plane SLE$_\kappa$ grows in $\widehat{\mathbb{C}}$ from one point to another.

The property of ξ_κ implies that for any $b \in \mathbb{R}$, $(K_{b+t} \setminus K_b)$ has the same law as the image of the standard radial SLE$_\kappa$ under some conformal map from $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus K_b$. So the increments of the full plane SLE$_\kappa$ is always radial SLE$_\kappa$. One may consider a standard full plane SLE$_\kappa$ as the limit as $\varepsilon \to 0^+$ of the radial SLE$_\kappa$ in $\widehat{\mathbb{C}} \setminus \varepsilon\overline{\mathbb{D}}$ from ε to ∞.

2.2.4 Equivalence relations

Suppose (K_t) has the law of the standard chordal SLE$_\kappa$. Let A be a hull in $\mathbb{H}$ that is bounded from 0 and ∞. Let (L_s) has the law of a chordal SLE$_\kappa(\mathbb{H} \setminus A; 0 \to \infty)$. Let T be the first time (K_t) hits A and S the first time that $(\overline{L_s})$ hits A. If $\kappa = 6$, $(K_t, 0 \le t < T)$ and $(L_s, 0 \le s < S)$ have the same law after a time-change. If $\kappa > 0$ and $\kappa \ne 6$, there exists a family of increasing stopping time $\{T_n\}$ and $\{S_n\}$ such that $T = \vee T_n$, $S = \vee S_n$, and the laws of $(K_t, 0 \le t \le T_n)$ and $(L_s, 0 \le s \le S_n)$ are absolutely continuous w.r.t. each other after a time-change.

Suppose D is a simply connected domain, z_1 and z_2 are two distinct prime ends, and $z_3 \in D$. Let (K_t) have the law of a chordal SLE$_\kappa(\Omega; z_1 \to z_2)$ and (L_s) have the law of the radial $\mathrm{SLE}_\kappa(\Omega; z_1 \to z_2)$. Let T be the first time that K_t swallows z_3, S the first time that L_s swallows z_2. If $\kappa = 6$, $(K_t, 0 \le t < T)$ and $(L_s, 0 \le s < S)$ have the same law after a time-change. If $\kappa > 0$ and $\kappa \ne 6$, there exists a family of increasing stopping time $\{T_n\}$ and $\{S_n\}$ such that $T = \vee T_n$, $S = \vee S_n$, and the laws of $(K_t, 0 \le t \le T_n)$ and $(L_s, 0 \le s \le S_n)$ are absolutely continuous w.r.t. each other after a time-change.

The above two equivalence relations are essentially the same thing. We also have the equivalence relations between two radial SLE_κ. The most general form of the equivalence is that we consider two simply connected domain that have a common prime end, and two chordal or radial SLEs starting from this prime end and growing in these two domains respectively with any possible targets. The strong equivalence relation for $\kappa = 6$ is called the locality property, which means that SLE_6 hulls do not feel the boundary and the target before hitting them.

2.2.5 SLE traces

If (K_t) has the law of the standard chordal SLE_κ, there is a.s. a random curve β from $[0, \infty)$ into $\overline{\mathbb{H}}$ such that $\beta(0) = 0$, $\lim_{t\to\infty} \beta(t) = \infty$, and for each t, K_t is the complement of the unbounded component of $\mathbb{H} \setminus \beta(0, t]$. This β is called a standard chordal SLE_κ trace. If $\kappa \le 4$, then β is a simple curve and intersect $\mathbb{R}$ only at 0. In this case, $K_t = \beta(0, t]$. If $\kappa > 4$, then β is not simple and intersect $\mathbb{R}$ at infinitely many points. A general chordal SLE_κ also corresponds to a trace. If $\kappa \le 4$, the trace lies in the domain; if $\kappa > 4$, the trace lies in the *conformal closure* of that domain.

If (K_t) has the law of the standard radial SLE_κ, there is a.s. a random curve β from $[0, \infty)$ into $\overline{\mathbb{D}}$ such that $\beta(0) = 1$, $\lim_{t\to\infty} \beta(t) = 0$, and for each t, K_t is the complement of the component of $\mathbb{H} \setminus \beta(0, t]$ that contains 0. This β is called a standard radial SLE_κ trace. If $\kappa \le 4$, then β is a simple curve and intersect $\partial\mathbb{D}$ only at 1. In this case, $K_t = \beta(0, t]$. If $\kappa > 4$, then β is not simple and intersect $\partial\mathbb{D}$ at infinitely many points. A general radial SLE_κ also corresponds a trace. If $\kappa \le 4$, the trace lies in the domain; if $\kappa > 4$, the trace lies in the *conformal closure* of that domain.

If (K_t) has the law of the standard full plane SLE$_\kappa$, there is a.s. a random curve β from $[-\infty,\infty)$ into $\mathbb{C}$ such that $\beta(-\infty)=0$, $\lim_{t\to\infty}\beta(t)=\infty$, and for each t, K_t is the complement of the unbounded component of $\mathbb{C}\setminus\beta(0,t]$. This β is called a standard full plane SLE$_\kappa$ trace. If $\kappa\le 4$, then β is a simple curve, and $K_t=\beta(-\infty,t]$. If $\kappa>4$, then β is not simple

2.3 Strip SLE

2.3.1 Definition

For a real valued continuous function ξ on $[0,\infty)$, consider the following *strip Loewner equation*:

$$\partial_t\varphi_t(z)=\coth(\frac{\varphi_t(z)-\xi(t)}{2}),\ \ \varphi_0(z)=z. \tag{2.3.1}$$

If $z-\xi(0)\in 2\pi i\mathbb{Z}$, let $\tau(z)=0$; for other $z\in\mathbb{C}$, let $\tau(z)$ be such that $[0,\tau(z))$ is the maximal definition interval of the solution $\varphi_t(z)$. Since $\coth(z/2)$ is analytic, so for each $t\in[0,\infty)$, φ_t must be analytic in $\{\tau>t\}$. From the uniqueness of the solution of ODE, φ_t must be conformal. Since $\coth(z/2)$ is bounded when z is away from $2\pi i\mathbb{Z}$, so if $\tau(z)\in(0,\infty)$, then $\varphi_t(z)-\xi(t)\to 2\pi i\mathbb{Z}$ as $t\to\tau(z)$. Since $\coth(z/2)$ has a period $2\pi i$ and $\coth(\overline{z}/2)=\overline{\coth(z/2)}$, we have $\varphi_t(z+2\pi i)=\varphi_t(z)+2\pi i$ and $\varphi_t(\overline{z})=\overline{\varphi_t(z)}$, and so $\tau(z+2\pi i)=\tau(z)$ and $\tau(\overline{z})=\tau(z)$. For $t\in[0,\infty)$, let J_t be the set of $z\in\mathbb{C}$ such that $\tau(z)\le t$. Then each J_t must be closed, and is symmetric w.r.t. the lines $k\pi i+\mathbb{R}$, $k\in\mathbb{Z}$. Note that $\mathrm{Im}\coth(z/2)=0$ on $\mathbb{R}$ and $2\pi i+\mathbb{R}$ except at the poles 0 and $2\pi i$. For $a>0$, let $\mathbb{S}_a:=\{z\in\mathbb{C}:0<\mathrm{Im}\,z<a\}$. Then for $z\in\mathbb{S}_{2\pi}$, $\varphi_t(z)$ will never cross the lines $\mathbb{R}$ and $2\pi i+\mathbb{R}$. So φ_t maps $\mathbb{S}_{2\pi}\setminus J_t$ conformally into $\mathbb{S}_{2\pi}$. On the other hand, for a fixed $t_0>0$ and $z_0\in\mathbb{S}_{2\pi}$, we set the initial value of (2.3.1) to be $\varphi_{t_0}(z)=z_0$ and consider the solution for $t\le t_0$. Since $\pm\mathrm{Im}\coth(z/2)\le 0$ when $z\in\mathbb{S}_{2\pi}$ and $\pm(\mathrm{Im}\,z-\pi)\le 0$, so as t decreases, $\varphi_t(z)-\xi(t)$ approaches the line $\pi i+\mathbb{R}$, which does not contain any pole of $\coth(z/2)$. This means that $\varphi_t(z)$ will not blow up in the backward direction. So $z=\varphi_0(z)$ exists in $\mathbb{S}_{2\pi}$. Thus φ_t maps $\mathbb{S}_{2\pi}\setminus J_t$ conformally onto $\mathbb{S}_{2\pi}$. Since $\mathrm{Im}\coth(z/2)=0$ on $\pi i+\mathbb{R}$, so for $z\in\pi i+\mathbb{R}$, $\varphi_t(z)$

never blows up and stays on $\pi i + \mathbb{R}$. This means that φ_t maps $\pi i + \mathbb{R}$ onto itself. Finally, we let $K_t := \mathbb{S}_\pi \cap J_t$. Then K_t is a hull in $\mathbb{S}_\pi$ and φ_t maps $\mathbb{S}_\pi \setminus K_t$ conformally onto $\mathbb{S}_\pi$. We call φ_t and K_t the standard strip LE maps and hulls driven by ξ.

Since $\coth(z/2) \to \pm 1$ as $z \in S$ and $\operatorname{Re} z \to \pm\infty$, K_t is bounded and φ_t satisfies the normalization:

$$\lim_{z \to \pm\infty} (\varphi_t(z) - z) = \pm t. \tag{2.3.2}$$

Here $z \to \pm\infty$ means that $z \in S$ and $\operatorname{Re} z \to \pm\infty$. And the limit is uniform in $\operatorname{Im} z$. In fact, if K is any bounded hull in $\mathbb{S}_\pi$ such that $\overline{K} \cap (\pi i + \mathbb{R}) = \emptyset$, then there is a unique conformal map φ_K that maps $\mathbb{S}_\pi \setminus K$ conformally onto $\mathbb{S}_\pi$, maps $\pi i + \mathbb{R}$ onto itself, and satisfies (2.3.2) with φ_t replaced by φ_K and t by some $c \geq 0$. This c is called the capacity of K in $\mathbb{S}_\pi$ w.r.t. $\pi i + \mathbb{R}$, denoted by $C_{\mathbb{S}_\pi;\pi i+\mathbb{R}}(K)$. Thus for the strip LE hulls K_t, $C_{\mathbb{S}_\pi;\pi i+\mathbb{R}}(K_t) = t$ for all t.

Proposition 2.3.1 *The following two conditions are equivalent:*
(i) (K_t) *is a family of standard strip LE hulls.*
(ii) $t \mapsto K_t$ *is a Loewner chain in* $\mathbb{S}_\pi$*, each* K_t *is bounded,* $\overline{K_t} \cap \pi i + \mathbb{R} = \emptyset$*, and* $C_{\mathbb{S}_\pi;\pi i+\mathbb{R}}(K_t) = t$ *for all* t*.*
Moreover, $\{\xi(t)\} = \cap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}$*, where* φ_t *is the corresponding LE map. And if we don't assume that* $C_{\mathbb{S}_\pi;\pi i+\mathbb{R}}(K_t) = t$ *for all* t *in (ii), then after a time-change,* (K_t) *can be made to be a family of standard strip LE hulls.*

Proof. The method is very similar to the proof of its counterparts in the radial and chordal cases. So we omit the proof. $\square$

If $\xi(t) = \sqrt{\kappa}B(t)$. Then the law of (K_t) is called the standard strip SLE_κ maps. Suppose D is a simply connected domain bounded by a Jordan curve, I is a boundary arc and $x \in D \setminus \overline{I}$. Then there is a conformal map W from $\mathbb{S}_\pi$ onto D which can be extended continuously to $\mathbb{R}$ and $\pi i + \mathbb{R}$ and satisfies $W(0) = x$, $W(\pi i + \mathbb{R}) = I$. Then we call the law of $(W(K_t))$ the strip $\mathrm{SLE}_\kappa(D; x \to I)$, or SLE_κ in D from x to I.

Similarly as the radial and chordal cases, the standard strip $\mathrm{SLE}_\kappa(D; x \to I$ is preserved under the self anti-conformal map of $(D; x, I)$, and has the property of conformally equivalent time-homogeneity. And these two properties determine the strip SLE up to the parameter κ and a constant speed time-change.

2.3.2 Equivalence of strip and chordal SLE

Theorem 2.3.1 *Suppose (L_t) has the law of a chordal $SLE_\kappa(\mathbb{S}_\pi; 0 \to +\infty)$. Let T be the first time that $\overline{L_t} \cap \pi i + \mathbb{R} \neq \emptyset$. If $\kappa = 6$, then up to a time change, $(L_t, 0 \le t < T)$ has the same law as the standard strip SLE_κ. If $\kappa > 0$ and $\kappa \neq 6$, then there is an increasing sequence of stopping time $\{T_n\}$ such that $T = \vee T_n$, and for each n, $(L_t, 0 \le t \le T_n)$ has the same law as the standard strip SLE_κ hull stopped at some stopping time.*

Proof. Note that $W(z) = \ln(z+1)$ maps $(\mathbb{H}; 0, \infty)$ conformally onto $(\mathbb{S}_\pi; 0, +\infty)$. We may assume that $L_t = W(K_t)$ and (K_t) are the standard chordal LE hulls driven by $\xi(t) = \sqrt{\kappa}B(t)$. Let (φ_t) be the corresponding standard chordal LE maps.

For $0 \le t < T$, $-\infty$ is not swallowed by L_t, so -1 is not swallowed by K_t, which means that $\varphi_t(-1)$ is defined. Note $\varphi_t \circ W^{-1}$ maps $(\mathbb{S}_\pi \setminus L_t; -\infty, +\infty)$ conformally onto $(\mathbb{H}; \varphi_t(-1), \infty)$, and $W^{-1}(z) = e^z - 1$. Let

$$\widehat{\psi}_t(z) := \ln(\varphi_t(e^z - 1) - \varphi_t(-1)) + \int_0^t \frac{1}{(\varphi_s(-1) - \xi(s))^2} ds.$$

Then $\widehat{\psi}_t$ maps $(\mathbb{S}_\pi \setminus L_t; -\infty, +\infty)$ conformally onto $(\mathbb{S}_\pi; -\infty, +\infty)$, and $\psi_0(z) = z$. We compute

$$\partial_t \widehat{\psi}_t(z) = (\partial_t \varphi_t(e^z - 1) - \partial_t \varphi_t(-1))/(\varphi_t(e^z - 1) - \varphi_t(-1)) + \frac{1}{(\varphi_t(-1) - \xi(t))^2}$$

$$= \left(\frac{2}{\varphi_t(e^z - 1) - \xi(t)} - \frac{2}{\varphi_t(-1) - \xi(t)}\right)/(\varphi_t(e^z - 1) - \varphi_t(-1)) + \frac{1}{(\varphi_t(-1) - \xi(t))^2}$$

$$= \frac{-2}{(\varphi_t(e^z - 1) - \xi(t))(\varphi_t(-1) - \xi(t))} + \frac{1}{(\varphi_t(-1) - \xi(t))^2}$$

$$= \frac{(\varphi_t(e^z - 1) - \xi(t)) - 2(\varphi_t(-1) - \xi(t))}{(\varphi_t(e^z - 1) - \xi(t))(\varphi_t(-1) - \xi(t))^2}.$$

Let

$$\widehat{\zeta}(t) := \ln(\xi(t) - \varphi_t(-1)) + \int_0^t \frac{1}{(\varphi_s(-1) - \xi(s))^2} ds.$$

Then

$$\coth((\widehat{\psi}_t(z) - \widehat{\zeta}(t))/2) = \frac{(\varphi_t(e^z - 1) - \varphi_t(-1)) + (\xi(t) - \varphi_t(-1))}{(\varphi_t(e^z - 1) - \varphi_t(-1)) - (\xi(t) - \varphi_t(-1))}$$

$$= \frac{\varphi_t(e^z - 1) + \xi(t) - 2\varphi_t(-1)}{\varphi_t(e^z - 1) - \xi(t)} = (\varphi_t(-1) - \xi(t))^2 \partial_t \psi_t(z).$$

Now we make the time-change as follows. Let $u(t) := \int_0^t 1/(\varphi_s(z) - \xi(s))^2 ds$, and v be the inverse function of u. Let $\psi_t(z) := \widehat{\psi}_{v(t)}(z)$ and $\xi(t) := \widehat{\xi}(v(t))$. Then

$$\partial_t \psi_t(z) = v'(t) \partial_t \widehat{\psi}_{v(t)}(z)$$

$$= \partial_t \widehat{\psi}_{v(t)}(z)/u'(v(t)) = (\varphi_{v(t)}(z) - \xi(v(t)))^2 \partial_t \psi_{v(t)}(z)$$

$$= \coth((\widehat{\psi}_{v(t)}(z) - \widehat{\zeta}(v(t)))/2) = \coth((\psi_t(z) - \zeta(t))/2).$$

Thus ψ_t and $K_{v(t)}$ are the standard strip LE maps and hulls driven by ζ.

From Ito's formula,

$$d\widehat{\zeta}(t) = (d\xi(t) - \partial_t \varphi_t(-1) dt)/(\xi(t) - \varphi_t(-1)) + \frac{\kappa}{2} \frac{-dt}{(\xi(t) - \varphi_t(-1))^2}$$

$$+ \frac{dt}{(\xi(t) - \varphi_t(-1))^2} = \frac{\sqrt{\kappa} dB(t)}{\xi(t) - \varphi_t(-1)} + (3 - \frac{\kappa}{2}) \frac{dt}{(\xi(t) - \varphi_t(-1))^2}.$$

After the time-change, we have

$$d\zeta(t) = \sqrt{\kappa} d\widetilde{B}(t) + (3 - \frac{\kappa}{2}) dt,$$

where $\widetilde{B}(t)$ is some standard Brownian motion. If $\kappa = 6$, then $\zeta(t) = \sqrt{\kappa}\widetilde{B}(t)$. So $(K_{v(t)})$ has the law of the standard strip SLE_6. If $\kappa > 0$ and $\kappa \neq 6$, $\zeta(t) = \sqrt{\kappa}\widetilde{B}(t) +$ some drift term. The conclusion follows from Girsanov's Theorem ([13]). $\square$

The equivalence theorem implies the existence of the standard strip SLE_κ trace. That means if (K_t) has the law of the standard strip SLE_κ, then there exists a.s. a random curve $\beta : [0,\infty) \to \mathbb{S}_\pi \cup \mathbb{R}$ such that $\beta(0) = 0$ and for each t, K_t is the complement of the unbounded component of $\mathbb{S}_\pi \setminus \beta[0,t]$ in $\mathbb{S}_\pi$. And for $\kappa \le 4$, β is a simple curve and intersects $\mathbb{R}$ only at 0. For $\kappa > 4$, β is not simple and intersects $\mathbb{R}$ at infinitely many points. We call β a standard strip SLE_κ trace. From the existence of SLE_κ trace, we know that the boundary of $\mathbb{S}_\pi \setminus K_t$ is locally connected. Thus φ_t^{-1} has a continuous extension to $\overline{\mathbb{S}_\pi}$. Especially, $\varphi_t^{-1}(\xi(t)) = \beta(t)$. For strip SLE in general simply connected domains, we define the traces as the image of standard traces under a suitable conformal map.

2.3.3 Transience of the strip SLE_κ trace

The goal of this section is to prove that for all $\kappa > 0$, the standard strip SLE $_\kappa$ trace tends to a limit point on $\{\mathrm{Im}\, z = \pi\}$ almost surely. Let K_t and φ_t be the standard strip LE hulls and maps driven by $\xi(t) = \sqrt{\kappa}B(t)$. We may find $C = C(\omega) > 0$, such that $|\xi(t)| \le 1 + Ct^{2/3}$ for all $t \ge 0$. Let $K_\infty = \cup_t K_t$, then we have

Lemma 2.3.1 *K_∞ is bounded.*

Proof. First we may choose $R > 0$ such that $\mathrm{Re}\,\coth(z/2) \ge 1/2$ if $\mathrm{Re}\, z \ge R$. Since $|\xi(t)| \le 1 + C(\omega)t^{2/3}$ for all $t \ge 0$, there is $a = a(\omega) > 0$ such that $a \ge R + 1 - t/2 + \xi(t)$ for all $t \ge 0$. Now suppose $z \in \mathbb{S}_\pi$ and $\mathrm{Re}\, z \ge a$, then $\mathrm{Re}\, z \ge R + 1 + \xi(0)$. Suppose that there is a first $t_0 < \tau(z)$ such that $\mathrm{Re}\,\varphi_t(z) - \xi(t) = R$. Then on $[0, T_0]$, $\mathrm{Re}\,\varphi_t(z) - \xi(t) > R$, and so $\partial_t \mathrm{Re}\,\varphi_t(z) \ge 1/2$. This implies that

$$\mathrm{Re}\,\varphi_{t_0}(z) \ge \mathrm{Re}\, z + t_0/2 \ge a + t_0/2 \ge R + 1 + \xi(t_0),$$

which is a contradiction. Thus $\mathrm{Re}\,\varphi_t(z) - \xi(t) \ge R$ for all $t < \tau(z)$. So $\varphi_t(z)$ will never blow up, which means that $z \notin K_t$ for all $t \ge 0$. Similarly, $z \in \mathbb{S}_\pi$ and $\mathrm{Re}\, z \le -a$ implies that $z \notin K_t$ for all $t \ge 0$. Thus $K_\infty \subset \{|\mathrm{Re}\, z| \le a, |\mathrm{Im}\, z| \le \pi\}$ is bounded. $\square$

Let $X_t(z) := \operatorname{Re}\varphi_t(z) - \xi(t)$. If $z \in \pi i + \mathbb{R}$, then

$$dX_t(z) = \tanh(X_t(z)/2)dt - d\xi(t).$$

Fix $z_0 = x_0 + \pi i \in \pi i + \mathbb{R}$. Write X_t for $X_t(z_0)$ temporarily. Let

$$c_\kappa = \int_{-\infty}^{+\infty} (\cosh(x/2))^{-\frac{4}{\kappa}} dx.$$

Then $0 < c_\kappa < \infty$. Define

$$f_\kappa(x) := \frac{1}{c_\kappa} \int_{-\infty}^{x} (\cosh(t/2))^{-\frac{4}{\kappa}} dt.$$

for $x \in \mathbb{R}$. Then f_κ is continuous and increasing and maps $\mathbb{R}$ onto $(0,1)$. Let $W_t = f_\kappa(X_t)$, by Ito's formula, we have

$$dW_t = f_\kappa'(X_t)dX_t + \frac{\kappa}{2} f_\kappa''(X_t)dt = -(\cosh(X_t/2))^{-\frac{4}{\kappa}}/c_\kappa d\xi(t).$$

Thus W_t is a local martingale. After a time-change, W_t has the same distribution as the Brownian motion starting from $f(x_0)$, stopped when it hits $\{0,1\}$. This implies that $\lim_{t\to\infty} W_t = 0$ or 1 a.s., and $\Pr\{\lim_{t\to\infty} W_t = 1\} = f_\kappa(x_0)$. Thus $\lim_{t\to\infty} X_t = +\infty$ or $-\infty$ a.s., and $\Pr\{\lim_{t\to\infty} X_t = +\infty\} = f_\kappa(x_0)$. Define

$$m_+ = \inf\{x \in \mathbb{R} : \lim_{t\to\infty} \operatorname{Re}\varphi_t(x+\pi i) - \xi(t) = +\infty\};$$

$$m_- = \sup\{x \in \mathbb{R} : \lim_{t\to\infty} \operatorname{Re}\varphi_t(x+\pi i) - \xi(t) = -\infty\}.$$

Since $x_1 < x_2$ implies that $\operatorname{Re}\varphi_t(x_1+\pi i) < \operatorname{Re}\varphi_t(x_2+\pi i)$ for all $t \geq 0$, we have $m_- \leq m_+$; for $x < m_-$, X_t tends to $-\infty$ as $t \to \infty$; and for $x > m_+$, X_t tends to $+\infty$ as $t \to \infty$. Hence $\Pr\{m_+ < x\} \leq f_\kappa(x) \leq \Pr\{m_- \leq x\}$ for all $x \in \mathbb{R}$. Since f_κ is (strictly) increasing, it follows that $m_- = m_+$ almost surely, and their distributions have the density $(\cosh(x/2))^{-\frac{4}{\kappa}}/c_\kappa$ with respect to the Lebesgue measure. By discarding an event of probability 0, we may assume $m_- = m_+$ and denote it by m.

Lemma 2.3.2 $\overline{K_\infty} \cap (\pi i + \mathbb{R}) = \{m + \pi i\}$.

Proof. First we will prove $m + \pi i \in \overline{K_\infty}$. If this is not true, then there are $b, c > 0$ such that $dist(x + \pi i, K_t) > c$, for all $t \geq 0$ and $x \in [m - b, m + b]$. Since $X_t(m-b+\pi i) \to -\infty$ and $X_t(m+b+\pi i) \to +\infty$ as $t \to \infty$, $\varphi_t(m+b+\pi i) - \varphi_t(m-b+\pi i) \to +\infty$ as $t \to \infty$. By mean value theorem, we may find $x_t \in [m - b, m + b]$ such that $|\varphi_t'(x_t + \pi i)| \to \infty$ as $t \to \infty$. From Koebe's 1/4 theorem, we conclude that $dist(x_t + \pi i, K_t \cup \mathbb{R}) \to 0$ as $t \to \infty$. This contradiction shows $m + \pi i \in \overline{K_\infty}$.

Then we will prove that $\overline{K_\infty}$ contains no other point in $\pi i + \mathbb{R}$. First suppose $x_0 > m$. Then as $t \to \infty$ $X_t(x_0 + \pi i) \to +\infty$, and so $\partial_t \varphi_t(x_0 + \pi i) = \tanh(X_t/2) \to 1$. Since $|\xi(t)| \leq 1 + Ct^{2/3}$ for all $t \geq 0$, there is $M = M(\omega)$, such that when $t > M$, $X_t(x_0+\pi i) = \varphi_t(x_0+\pi i) - \xi(t) > t/2$. Thus for all $x \geq x_0$, $X_t(x+\pi i) \geq X_t(x_0+\pi i) > t/2$ when $t > M$. Taking the derivative w.r.t. z on both sides of equation (2.3.1), we get

$$\partial_t \varphi_t'(z) = \frac{-1}{2\sinh((\varphi_t(z) - \xi(t))/2)^2} \varphi_t'(z). \tag{2.3.3}$$

It follows that for $x \in \mathbb{R}$,

$$|\varphi_t'(x + \pi i)| = \exp\left(\int_0^t (1 + \cosh(X_s(x + \pi i))^{-1} ds \right)$$

$$\leq \exp\left(\int_0^M ds + \int_M^\infty (1 + \cosh(t/2))^{-1} ds \right) < \infty.$$

Then by Koebe's 1/4 theorem, for all $x \geq x_0$, $x + \pi i$ is bounded away from K_t uniformly. Thus $\overline{K_\infty} \cap [x_0, +\infty) + \mathbb{R} = \emptyset$. This then implies that $\overline{K_\infty} \cap (\pi i + (m, +\infty)) = \emptyset$. Similarly, $\overline{K_\infty} \cap (\pi i + (-\infty, m)) = \emptyset$. $\square$

For $x > 0$, we have $X_t(x) > 0$ before $\tau(x)$, and $X_t(x)$ satisfies:

$$dX_t(x) = \coth(X_t(x)/2) dt - d\xi(t).$$

We will use an argument similar to that before Lemma 2.3.2. Define g_κ on $(0,\infty)$ such that

$$g_\kappa(x) = -\int_x^\infty (\sinh(x/2))^{-\frac{4}{\kappa}} ds.$$

Then g_κ is continuous and increasing, and maps $(0,+\infty)$ onto $(a_\kappa, 0)$. If $\kappa \le 4$, then $a_\kappa = -\infty$; otherwise, $a_\kappa > -\infty$. Using Ito's formula, we find that $(g_\kappa(X_t))$ is a local martingale. Thus if $\kappa \le 4$, then almost surely $X_t(x) \to +\infty$ as $t \to \tau(x)$ for all $x > 0$. This also shows that a.s. $\tau(x) = \infty$ for all $x > 0$. If $\kappa > 4$, there exists a.s. $r > 0$, such that for all $x > r$, $X_t(x) \to +\infty$ as $t \to \tau(x)$ (which implies that $\tau(x) = \infty$); and for all $x < r$, $X_t(x) \to 0$ as $t \to \tau(x)$. And the distribution of r has a density $(\sinh(x/2))^{-\frac{4}{\kappa}}/|a_\kappa|$ with respect to the Lebesgue measure. Note that $\partial_t \varphi_t(x) = \coth(X_t/2) > 1$ for all $0 < t < \tau(x)$. If $\tau(x) = \infty$, then $X_t(x) = \varphi_t(x) - \xi(t) - t \ge x + t - Ct^{2/3} - 1$ tends to $+\infty$ as $t \to \infty$. Thus $\tau(x) < \infty$ for all $x < r$. We will see later that $\tau(r) < \infty$.

Suppose now $x_0 > 0$ and there is $a > 0$ such that $X_t(x_0) \ge a$ for $0 \le t < \infty$. Then the extremal distance from $[\xi(t), \varphi_t(x_0)]$ to $\pi i + \mathbb{R}$ in $\mathbb{S}_\pi$ is less than some $b = b(a)$. Suppose there is t such that $|\beta(t) - x_0| = p < \pi$. Here β is a standard SLE$_\kappa$ trace that generates (K_t). We may suppose such t_0 is the first time this holds. Then the line segment $(\beta(t_0), x_0)$ lies inside $\mathbb{S}_\pi \setminus K_t$. The extremal distance between $(\beta(t), x_0)$ and $\pi i + \mathbb{R}$ in $\mathbb{S}_\pi$ is no less than $\ln \pi - \ln p$. Now φ_t maps $(x_0, \beta(t))$ to an open curve in $\mathbb{S}_\pi$ with two end points $\xi(t)$ and $\varphi_t(x_0)$. Thus the extremal distance of this curve from $\{\operatorname{Im} z = \pi\}$ in S is less than b. From conformal invariance of the extremal length, we have $\ln \pi - \ln p \le b$. Thus p is bounded from below by a constant depending on a. As $X_t(x) > X_t(x_0) \ge a$ for all $x > x_0$ and $t \ge 0$, the above result implies that the distance between $[x_0, \infty)$ and $\beta[0,\infty)$ is positive. Thus $[x_0,\infty)$ is bounded away from $\beta[0,\infty)$ and $\overline{K_\infty}$. Hence if $\kappa \le 4$, then for all $x > 0$, $[x,\infty)$ is bounded away from $\beta[0,\infty)$. And if $\kappa > 4$, then for all $x > r$, $[x,\infty)$ is bounded away from $\beta[0,\infty)$. Since for all $x \in (0,r)$, $\tau(x) < \infty$, so there is some $t = t(x)$ such that $x \in K_t$, which implies that r is not bounded away from K_∞. Thus $X_t(r) \not\to \infty$, and so $\tau(r) < \infty$. Now r is disconnected from $\pi i + \mathbb{R}$ by $\beta[0,\tau(r)]$ which does not hit (r,∞), so we must have $\beta(\tau(r)) = r$.

Similarly, almost surely we may conclude the following facts. When $\kappa \le 4$, $(-\infty, x]$ is bounded away from $\beta[0,\infty)$ for all $x < 0$. When $\kappa > 4$, there is $l < 0$ such that for all $x \in [l, 0]$, $\tau(x) < \infty$, and $\beta(\tau(l)) = l$; and for all $x < l$, $(-\infty, x]$ is bounded away from $\beta[0,\infty)$ and K_∞. Moreover, the distribution of l has a density $(\sinh(-x/2))^{-\frac{4}{\kappa}}/a_\kappa$ with respect to the Lebesgue measure. For $\kappa \le 4$, we define $r = l = 0$ for convenience.

Theorem 2.3.2 *Almost surely* $\lim_{t\to\infty} \beta(t) = m + \pi i$.

Proof. Let M be the set of all limit points of $\beta(t)$ as $t \to \infty$. Then $M = \cap_t \overline{\beta[t,\infty)}$. By Lemma 2.3.1 and 2.3.2, M is compact and $M \cap \pi i + \mathbb{R} = \{\mathbf{m} + \pi i\}$. Suppose now $M \not\subset \pi i + \mathbb{R}$ holds with a positive probability. We will use the conformally equivalent time-homogeneity of strip SLE$_\kappa$ to find a contradiction. Suppose $\widetilde{\beta}$ also has the law of the standard strip SLE$_\kappa$ trace, and is independent of (K_t). Let $\widetilde{\xi}(t)$ and $\widetilde{\varphi}_t$ be the corresponding driving function and maps. Let $\{\widetilde{\mathcal{F}}_t\}$ be the filtration generated by $\widetilde{\xi}(t)$. Then for any positive finite stopping time T w.r.t. $\{\widetilde{\mathcal{F}}_t\}$, the curve β_* defined by $\beta_*(t) = \widetilde{\beta}(t)$ for $0 \le t \le T$ and $\beta_*(t) = \widetilde{\varphi}_T^{-1}(\widetilde{\xi}(T) + \beta(t-T))$ for $t > T$ has the same law as β. Let $h_T(z) = \widetilde{\varphi}_T^{-1}(\widetilde{\xi}(T) + z)$. Then $h_T(M)$ has the same law as M. From the strip LE equation, for any $z \in \overline{\mathbb{S}}_\pi$, $\operatorname{Im} h_t(z) \ge \operatorname{Im} z$. The strict inequality holds when $z \in \mathbb{S}_\pi$. For $d \in (0,1)$, let T_d be the first t such that $\operatorname{Im}\widetilde{\beta}(t) = \pi - d$. Then $h_{T_d}(0) = \widetilde{\beta}(T_d)$. Let a_d be the biggest $a < 0$ such that $h_{T_d}(a) \in \mathbb{R}$. Let b_d be the smallest $b > 0$ such that $h_{T_d}(b) \in \mathbb{R}$. Then for $a_d < x < b_d$, $\operatorname{Im} h_{T_d}(x) > 0 = \operatorname{Im} x$. Since an annulus of inner radius d and outer radius π disconnects all curves in $\mathbb{S}_\pi$ from $h_{T_d}(-\infty, a_d)$ to $h_{T_d}(0,+\infty)$, so the extremal distance from $(-\infty, a_d)$ to $(0,\infty)$ in $\mathbb{S}_\pi$ is not less than $(\ln(\pi) - \ln(d))/\pi$, which tends to ∞ as $d \to 0$. Thus $a_d \to -\infty$ uniformly as $d \to 0$. Similarly, $b_d \to +\infty$ uniformly as $d \to 0$. Since M is a bounded set, there is $R > 0$ such that $M \subset \{z : |\operatorname{Re} z| < R\}$ and $M \not\subset \pi i + \mathbb{R}$ with a positive probability. If d is small enough, we have $|a_d|, |b_d| > R$. Then for any $z \in M \setminus (\pi i + \mathbb{R})$, either $z \in \mathbb{S}_\pi$ or $a_d < z < b_d$. In both cases, we have $\operatorname{Im} h_{T_d}(z) > \operatorname{Im} z$. Thus on this event, h_{T_d} strictly increases $\min \operatorname{Im} M$. So $h_{T_d}(M)$ cannot have the same law as M, which is a contradiction. Thus $M \subset \pi i + \mathbb{R}$ almost surely. So M has to be $\{m + \pi i\}$, which means that $\lim_{t\to\infty} \beta(t) = m + \pi i$. □

Now we may define $\beta(\infty) = m + \pi i$. Then β is a continuous path in $\overline{\mathbb{S}_\pi}$ which grows from 0 to $m + \pi i$, and intersects $\pi i + \mathbb{R}$ at only one point $m + \pi i$. And $\overline{\mathbb{S}_\pi} \setminus \overline{K_\infty}$ has two components. We denote the left one by S_-, and the right one by S_+. If $\kappa \le 4$, β is a simple curve and intersects $\mathbb{R}$ only at 0. If $\kappa > 4$, then β is not a simple path. The intersection of β with $\mathbb{R}$ has lower bound $l < 0$ and upper bound $r > 0$.

Lemma 2.3.3 *For all $\kappa > 0$, if $z \in S_\pm$, then $\tau(z) = \infty$, and $X_t(z) \to \pm\infty$ as $t \to \infty$.*

Proof. For $z \in S_+$, we may find a C^1 path α in S_+ from z to $m + 1 + \pi i$ with a finite length. Denote d the distance of α from K_∞, then $d > 0$. By Koebe's 1/4 theorem, $|\varphi_t'(\alpha(s))|$ is bounded by $4\mathrm{Im}\, z/\min\{\mathrm{Im}\, z, d\} < 4(\pi/d + 1)$. Thus the length of $\varphi_t \circ \alpha$ is uniformly bounded in t. Hence $|X_t(z) - X_t(m + 1 + \pi i)|$ is uniformly bounded. Since $X_t(m + 1 + \pi i) \to +\infty$, so does $X_t(z)$. The case that $z \in S_-$ is similar. □

2.3.4 Cardy's formula

Let $h(z) = \coth(z/2)$, $Z_t(z) = \varphi_t(z) - \xi(t)$, $X_t = \mathrm{Re}\, Z_t$ and $Y_t = \mathrm{Im}\, Z_t$. Then $dX_t = \mathrm{Re}\, h(Z_t)dt - d\xi(t)$, and $dY_t = \mathrm{Im}\, h(Z_t)dt$. We have

Lemma 2.3.4 *Suppose f is analytic in $\mathbb{S}_\pi$, and satisfies $f'h + \frac{\kappa}{2}f'' = 0$. Then $f(Z_t)$ is a local martingale.*

Proof. Let $U = \mathrm{Re}\, f$ and $V = \mathrm{Im}\, f$. Then $U_x = V_y = \mathrm{Re}\, f'$, $-U_y = V_x = \mathrm{Im}\, f'$, $U_{xx} = \mathrm{Re}\, f''$ and $V_{xx} = \mathrm{Im}\, f''$. By Ito's formula,

$$dU(Z_t) = U_x dX_t + U_y dY_t + \frac{\kappa}{2} U_{xx} dt$$

$$= \mathrm{Re}\, f' \mathrm{Re}\, h dt - \mathrm{Re}\, f' d\xi(t) - \mathrm{Im}\, f' \mathrm{Im}\, h dt + \frac{\kappa}{2} \mathrm{Re}\, f'' dt$$

$$= \mathrm{Re}\,(f'h + \frac{\kappa}{2} f'')dt - \mathrm{Re}\, f' d\xi(t) = -\mathrm{Re}\, f' d\xi(t);$$

and

$$dV(Z_t) = V_x dX_t + V_y dY_t + \frac{\kappa}{2} V_{xx} dt$$
$$= \mathrm{Im}\, f' \mathrm{Re}\, h dt - \mathrm{Im}\, f' d\xi(t) + \mathrm{Re}\, f' \mathrm{Im}\, h dt + \frac{\kappa}{2} \mathrm{Im}\, f'' dt$$
$$= \mathrm{Im}\,(f'\varphi + \frac{\kappa}{2} f'')dt - \mathrm{Im}\, f' d\xi(t) = -\mathrm{Im}\, f' d\xi(t).$$

Thus $df(Z_t) = -f'(Z_t)d\xi(t)$, and so $f(Z_t)$ is a local martingale. □

Now we suppose $\kappa > 4$ and ABC is a triangle with $\angle B = \angle C = \frac{2}{\kappa}\pi$. Suppose f maps $\mathbb{S}_\pi$ conformally onto ABC, such that $f(0) = A$, $f(+\infty) = B$ and $f(-\infty) = C$. We may check that f satisfies the condition of Lemma 2.3.4. In fact, let $g(z) = f(\ln z)$. Then g maps $\mathbb{H}$ conformally onto ABC and $g(1) = A$, $g(\infty) = B$ and $g(0) = C$. We then have

$$\frac{g''(z)}{g'(z)} = -\frac{4}{\kappa} \cdot \frac{1}{z-1} + \left(\frac{2}{\kappa} - 1\right) \cdot \frac{1}{z}.$$

Thus $g'(z) = C(z-1)^{-\frac{4}{\kappa}} z^{\frac{2}{\kappa}-1}$. Since $f(z) = g(e^z)$, we have

$$f'(z) = C(e^z - 1)^{-\frac{4}{\kappa}} (e^z)^{\frac{2}{\kappa}} = C(\sinh(z/2))^{-\frac{4}{\kappa}}.$$

Hence $f''(z) = -\frac{2}{\kappa}\varphi(z) f'(z)$ as desired. Note that f is bounded, and it extends continuously to the boundary of $\mathbb{S}_\pi$ and is analytic on $[0, +\infty)$, $(-\infty, 0]$ and $\{\mathrm{Im}\, z = \pi\}$. Thus we have

Theorem 2.3.3 *$f(Z_t(z))$, $0 \le t < \tau(z)$, is a martingale for $z \in \overline{\mathbb{S}_\pi} \setminus \{0\}$.*

By Lemma 2.3.3, $f(Z_t(z))$ may only tend to A, B, or C depending on $z \in K_\infty$, S_+ or S_-. Hence we have

Corollary 2.3.1 *Suppose $f(z) = aA + bB + cC$ with $a, b, c \in \mathbb{R}$, and $a + b + c = 1$. Then the probability of $z \in K_\infty$, $z \in S_+$ or $z \in S_-$ is a, b or c, respectively. Thus $f(m + \pi i)$, $f(r)$ and $f(l)$ are uniformly distributed on the sides $[B, C]$, $[A, B]$ and $[A, C]$, respectively. And the expected area of $f(K_\infty)$, $f(S_+)$ and $f(S_-)$ are all equal to area$(ABC)/3$.*

If $\kappa \in [4/3, 4]$, we choose $A = \infty$ and still let $\angle B = \angle C = \frac{2}{\kappa}\pi$. And let f be defined in the same way. Now ABC is not a triangle. Theorem 2.3.3 holds with the term martingale replaced by local martingale. For $\kappa = 4$, ABC is bounded by two half lines and a line segment orthogonal to them. If we project $f(Z_t)$ to that line segment, we also get a bounded martingale. If $\kappa = 2$, ABC is a half plane. In all above cases, from the definition of general strip SLE$_\kappa$, $(f(K_t))$ has the law of strip SLE$_\kappa(ABC; A \to BC)$.

2.4 Annulus SLE

2.4.1 Definition

Annulus SLE grows in a doubly connected domain. For $p \in (0, \infty)$, we denote by $\mathbb{A}_p$ the standard annulus of modulus p:

$$\mathbb{A}_p := \{z \in \mathbb{C} : e^{-p} < |z| < 1\}.$$

Let $\mathbf{C}_p := e^{-p}\partial\mathbb{D}$. So $\mathbb{A}_p$ is bounded by $\mathbf{C}_0$ and $\mathbf{C}_p$. Every doubly connected domain D is conformally equivalent to a unique $\mathbb{A}_p$, where $p = M(D)$ is the modulus of D. We may first define SLE on the standard annuli, and then extend the definition to arbitrary doubly connected domains via conformal maps.

Denote

$$\mathbf{S}_p(z) = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{e^{2kp} + z}{e^{2kp} - z}.$$

Let $\xi : [0, a) \to \mathbb{R}$, $a \in (0, p]$, be a continuous function. Consider the following *annulus Loewner equation*:

$$\partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}_{p-t}(\varphi_t(z)/e^{i\xi(t)}), \quad \varphi_0(z) = z. \tag{2.4.1}$$

For $0 \le t < a$, let K_t be the set of $z \in \mathbb{A}_p$ such that the solution $\varphi_s(z)$ blows up before or at time t. We call K_t and φ_t, $0 \le t < p$, the standard modulus p annulus

LE hulls and maps driven by ξ.

We recall some facts about $\mathbf{S}_r$:

(i) $\mathbf{S}_r$ is analytic in $\mathbb{C}\setminus\{0\}\setminus\{e^{2kr}:k\in\mathbb{Z}\}$;

(ii) $\{e^{2kr}:k\in\mathbb{Z}\}$ are simple poles of S_r;

(iii) $\operatorname{Re}\mathbf{S}_r\equiv 1$ on $\mathbf{C}_r=\{z\in\mathbb{C}:|z|=e^{-r}\}$;

(iv) $\operatorname{Re}\mathbf{S}_r\equiv 0$ on $\mathbf{C}_0\setminus\{1\}$;

(v) $\operatorname{Re}\mathbf{S}_r>0$ in $\mathbb{A}_r$; and

(vi) $\operatorname{Im}\mathbf{S}_r\equiv 0$ on $\mathbb{R}\setminus\{0\}\setminus\{\text{poles}\}$.

Moreover, suppose f is an analytic function in $\mathbb{A}_r$, $\operatorname{Re} f$ is non-negative, and $\operatorname{Re} f(z)$ tends to a constant c as $z\to\mathbf{C}_r$, then there is some positive measure $\mu=\mu(f)$ on $\mathbf{C}_0$ of total mass c such that

$$f(z)=\int_{\mathbf{C}_0}\mathbf{S}_r(z/\chi)d\mu(\chi)+iC, \tag{2.4.2}$$

for some real constant C. If $\operatorname{Re} f(z)$ tends to zero as z approaches the complement of an arc α of $\mathbf{C}_0$, then $\mu(f)$ is supported by $\overline{\alpha}$. If f is bounded, then the radial limit of f on $\mathbf{C}_0$ exists a.e., and $d\mu(f)/d\mathbf{m}=f|_{C_0}$. The proof is similar to that of the Poisson integral formula.

Divide both sides of equation (2.4.1) by $\varphi_t(z)$ and take the real part. We get

$$\partial_t \ln|\varphi_t(z)|=\operatorname{Re}\mathbf{S}_{p-t}(\varphi_t(z)/e^{i\xi(t)}). \tag{2.4.3}$$

From the values of $\operatorname{Re}\mathbf{S}_{p-t}$ on $\mathbf{C}_{p-t}$ and $\mathbf{C}_0$ we see that if $z\in\mathbf{C}_0\setminus\{1\}$, then $\varphi_t(z)\in\mathbf{C}_0\setminus\{1\}$ until it blows up; if $z\in\mathbf{C}_p$, then $\varphi_t(z)\in\mathbf{C}_{p-t}$ for $0\le t<p$. Thus for $z\in\mathbb{A}_p$, $\varphi_t(z)$ stays between $\mathbf{C}_0$ and $\mathbf{C}_{p-t}$ until it blows up. So φ_t maps $\mathbb{A}_p\setminus K_t$ into $\mathbb{A}_{p-t}$. The fact that $\mathbf{S}_{p-t}$ is analytic implies that for every $t\in[0,p)$, φ_t is a conformal map of $\mathbb{A}_p\setminus K_t$. By considering the backward flow, it is easy to see that φ_t maps $\mathbb{A}_p\setminus K_t$ onto $\mathbb{A}_{p-t}$. Thus K_t is a hull in $\mathbb{A}_p$ on $\mathbf{C}_0$. In general, if D is a doubly connected domain with boundary components S and S', and K is a hull in D on S, then the capacity of K in D w.r.t. S', denoted by $C_{D,S'}(K)$, is defined as the value

of $M(D) - M(D \setminus K)$. The capacity is non-negative, and it is 0 iff $K = \emptyset$. Here we have $C_{\mathbb{A}_p;C_p}(K_t) = t$ for all t.

Proposition 2.4.1 *The following two statements are equivalent:*
(i) (K_t) *is a family of standard modulus* p *LE hulls;*
(ii) $t \mapsto K_t$ *is a Loewner chain* $\mathbb{A}_p$ *on* $\mathbf{C}_0$*, and* $C_{\mathbb{A}_p,C_p}(K_t) = t$ *for all* t*.*
Moreover, $\{e^{i\xi(t)}\} = \cap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}$*, where* $\xi(t)$ *is the driving function, and* φ_t *is the corresponding map. And if we don't assume* $C_{A_p,C_p}(K_t) = t$ *for all* t *in (ii), then we can make* (K_t) *to be a family of standard annulus LE hulls through a time-change.*

Proof. The method is very similar to the proof of its counterparts in the radial and chordal cases. So we omit the most part of it. One thing we want to show here is how we derive φ_t from K_t in the proof of (ii) implies (i). We first choose $\widehat{\varphi}_t$ that maps $\mathbb{A}_p \setminus K_t$ conformally onto $\mathbb{A}_{p-t}$ such that $\widehat{\varphi}_t(\mathbf{C}_p) = \mathbf{C}_{p-t}$ and $\widehat{\varphi}_t(e^{-p}) = e^{t-p}$. Then we can prove that $\widehat{\varphi}_t$ satisfies the equation

$$\partial_t\widehat{\varphi}_t(z) = \widehat{\varphi}_t(z)(\mathbf{S}_{p-t}(\widehat{\varphi}_t(z)/e^{i\widehat{\xi}(t)}) - i\mathrm{Im}\,\mathbf{S}_{p-t}(e^{t-p}/\widehat{\chi}_t)), \;\; \widehat{\varphi}_0(z) = z,$$

for some continuous $\widehat{\xi} : [0, p) \to \mathbb{R}$. And $\{e^{i\widehat{\xi}(t)}\} = \cap_{\varepsilon>0}\overline{\widehat{\varphi}_t(K_{t+\varepsilon} \setminus K_t)}$. Define

$$\theta(t) = \int_0^t \mathrm{Im}\,\mathbf{S}_{p-s}(e^{s-p}/e^{i\widehat{\xi}}(s))ds,$$

$\xi(t) = \theta(t) + \widehat{\xi}(t)$ and $\varphi_t(z) = e^{i\theta(t)}\widehat{\varphi}_t(z)$, for $t \in [0, p)$. Then $\varphi_0(z) = \widehat{\varphi}_0(z) = z$, φ_t maps $\mathbb{A}_p \setminus K_t$ conformally onto $\mathbb{A}_{p-t}$, $\{\chi_t\} = \cap_{u>0}\overline{\varphi_t(K_{t+u} \setminus K_t)}$, and

$$\partial_t \ln \varphi_t(z) = \partial_t \ln \widehat{\varphi}_t(z) + i\theta'(t) = \mathbf{S}_{p-t}(\widehat{\varphi}_t(z)/e^{i\widehat{\xi}(t)}) = \mathbf{S}_{p-t}(\varphi_t(z)/e^{i\xi(t)}).$$

Thus $\partial_t\varphi_t(z) = \varphi_t(z)\mathbf{S}_{p-t}(\varphi(z)/e^{i\xi(t)})$. So K_t, $0 \le t < p$, are the standard modulus p annulus LE hulls, driven by ξ. □

If $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < p$, then the law of the standard modulus p annulus LE hulls driven by ξ is called the standard modulus p annulus SLE_κ. It is a measure on the space of Loewner chains in $\mathbb{A}_p$ started from 1. Suppose D is a doubly connected domain with one boundary component S, and w is a prime end of D that does not lie on S. Then there exists a conformal map from $(\mathbb{A}_p; 1, \mathbf{C}_p)$, where $p = M(D)$, onto $(D; w, S)$. the annulus $\mathrm{SLE}_\kappa(D; w \to S)$ is defined as the image of the standard modulus p annulus SLE_κ under W.

The annulus $\mathrm{SLE}_\kappa(D; w \to S)$ is preserved under the self anti-conformal map of $(D; w, S)$, and has the property of conformally equivalent time-homogeneity. The meaning is the following. Suppose $p = M(D)$. Fix $b \in (0, p)$. Let $p_a = p - b$. Suppose (K_t) has the law of the annulus $\mathrm{SLE}_\kappa(D; w \to S)$, (K_t^1) has the law of the standard modulus p_1 annulus SLE_κ, and (K_t) and (K_t^1) are independent. Then there is a conformal map g from $\mathbb{A}_{p_1}$ onto $D \setminus K_b$ that is determined by $(K_t, 0 \le t \le b)$; and the process (K_t^2) defined by $K_t^2 = K_t$ for $0 \le t \le b$ and $K_t^2 = K_b \cup g^{-1}(K_{t-b}^1)$ for $b \le t \le p_1$ also has the law of the annulus $\mathrm{SLE}_\kappa(D; w \to S)$. The above two properties do not determine a measure on standard annulus LE hulls up to a parameter κ. For example, suppose h is any continuous increasing function on $[0, \infty)$ such that $h(0) = 0$. For $p > 0$, let $\xi^p(t) := B(h(p-t)) - B(h(p))$. For each $p > 0$, let μ^p be the law of the standard modulus p annulus LE hulls driven by ξ^p. Then $\{\mu^p\}$ satisfies the above properties, and μ^p is a standard modulus p annulus SLE only if $h(t)/t$ is constant on $(0, p]$.

2.4.2 Equivalence of annulus and radial SLE

Suppose Ω is a simply connected domain, a is a prime end, and b is an interior point. Suppose $F \supsetneqq \{b\}$ is an interior hull in Ω. Then $\Omega \setminus F$ is a doubly connected domain with two boundary components $\partial\Omega$ and ∂F. Let $p := M(\Omega \setminus F)$. For a fixed $\kappa > 0$, let (K_t) has the law of radial $\mathrm{SLE}_\kappa(\Omega; a \to b)$, and (L_s) has the law of annulus $\mathrm{SLE}_\kappa(\Omega \setminus F; a \to \partial F)$. Let T_F be the first time that K_t intersects F. Then we have

Theorem 2.4.1 *(i)If $\kappa = 6$, the law of $(K_t)_{0 \le t < T_F}$, is equal to that of $(L_s)_{0 \le s < p}$, up to a time-change.*

(ii)If $\kappa > 0$ and $\kappa \ne 6$, there exist two sequences of stopping times $\{T_n\}$ and $\{S_n\}$ such

that $T = \vee_n T_n$, $p = \vee_n S_n$, and for each $n \in \mathbb{N}$, the law of $(K_t)_{0\le t\le T_n}$ is equivalent to that of $(L_s)_{0\le s\le S_n}$, up to a time-change.

By conformal invariance, we may assume that $\Omega = \mathbb{D}$, $a = 1$ and $b = 0$. Then $(K_t, 0 \le t < \infty)$ has the law of the standard radial SLE_κ. Suppose φ_t and $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$, are the corresponding standard radial LE maps and driving function, respectively. Suppose W maps $\mathbb{D} \setminus F$ conformally onto $\mathbb{A}_p$ so that $W(1) = 1$. Since $t \mapsto K_t$, $0 \le t < \infty$, is a Loewner chain in $\mathbb{D}$, it is clear that $t \mapsto K_t$, $0 \le t < T_F$, is a Loewner chain in $\mathbb{D} \setminus F$ on $\mathbf{C}_0$. By conformal invariance, $t \mapsto W(K_t)$, $0 \le t < T_F$, is a Loewner chain in $\mathbb{A}_p$ on $\mathbf{C}_0$. Let $u(t) := C_{D,\partial F}(K) = C_{\mathbb{A}_p, C_p}(W(K))$. Then u is a continuous increasing function and maps $[0, T_F)$ onto $[0, p)$. Let v be the inverse of u. By Proposition 2.4.1, $W(K_{v(s)})$, $0 \le s < p$, are the standard annulus modulus p LE hulls driven by some continuous function $\zeta : [0, p) \to \mathbb{R}$. Let ψ_s, $0 \le s < p$, be the corresponding standard annulus LE maps.

Now φ_t maps $\mathbb{D} \setminus F \setminus K_t$ conformally onto $\mathbb{D} \setminus \varphi_t(F)$. Let $f_t := \psi_{u(t)} \circ W \circ \varphi_t^{-1}$. Then f_t maps $\mathbb{D} \setminus \varphi_t(F)$ conformally onto $\mathbb{A}_{p-u(t)}$, and $f_t(\mathbf{C}_0) = \mathbf{C}_0$. By Schwarz reflection, we may extend f_t conformally to Σ_t, which is the union of $\mathbb{D} \setminus \varphi_t(F)$, $\mathbf{C}_0$, and the reflection of $\mathbb{D} \setminus \varphi_t(F)$ w.r.t. $\mathbf{C}_0$. Note that f_t maps $\varphi_t(K_{t+a} \setminus K_t)$ to $\psi_{u(t)}(W(K_{t+a}) \setminus W(K_t))$ for $a > 0$. From Proposition 2.4.1, we see that $\{e^{i\zeta(t)}\} = \cap_{a>0}\overline{\psi_{u(t)}(W(K_{t+a}) \setminus W(K_t))}$. And from formula (2.2.2), we know that $\{e^{i\xi(t)}\} = \cap_{a>0}\overline{\varphi_t(K_{t+a} \setminus K_t)}$. Thus $e^{i\zeta(t)} = f_t(e^{i\xi(t)})$.

Recall that for a hull K in a dimply connected domain D and $z \in D \setminus K$, $C_{D;z}(K)$ is the capacity of K in D w.r.t. z. Similarly as Lemma 2.8 in [4], using the integral formulas for capacities of hulls in $\mathbb{D}$ and $\mathbb{A}_p$, it is not hard to derive the following Lemma:

Lemma 2.4.1 *Suppose $x, y \in \mathbf{C}_0$, and G is a conformal map from a neighborhood U of x onto a neighborhood V of y such that $G(U \cap \mathbb{D}) = V \cap \mathbb{D}$. Fix any $p > 0$. For every $\varepsilon > 0$, there is $r = r(\varepsilon) > 0$ such that if K is a non-empty hull in $\mathbb{D}$ on $\mathbf{C}_0$ and $K \subset \mathbf{B}(x; r)$, which is the open ball of radius r about x, then $G(K)$ is a hull in $\mathbb{A}_p$*

on $\mathbf{C}_0$, *and*

$$\left|\frac{C_{A_p,C_p}(G(K))}{C_{\mathbb{D},0}(K)} - |G'(x)|^2\right| < \varepsilon.$$

Now $\varphi_t(K_{t+a} \setminus K_t)$ is a hull in $\mathbb{D}$, $\varphi_{t+a} \circ \varphi_t^{-1}$ maps $\mathbb{D} \setminus \varphi_t(K_{t+a} \setminus K_t)$ conformally onto $\mathbb{D}$, fixes 0, and $(\varphi_{t+a} \circ \varphi_t^{-1})'(0) = e^a$. So the capacity of $\varphi_t(K_{t+a} \setminus K_t)$ in $\mathbb{D}$ w.r.t. 0 is a. Similarly, $\psi_{u(t)}(W(K_{t+a} \setminus W(K_t))$ is a hull in $\mathbb{A}_{p-u(t)}$ on $\mathbf{C}_0$, and the capacity is $u(t+a) - u(t)$. From Lemma 2.4.1 we conclude that $u'_+(t) = |f'_t(e^{i\xi(t)})|^2$.

Let $H := \{(t,z) : 0 \le t < T_F, z \in \Sigma_t\}$ and $G(\xi) = \{(t, e^{i\xi(t)}) : 0 \le t < T_F\}$. By the definition of f_t, we see that $(t,z) \mapsto f'_t(z)$ is continuous in $H \setminus G(\chi)$. Note that f'_t is analytic in Σ_t for each $t \in [0, T_F)$. The maximum principle implies that $(t,z) \mapsto f'_t(z)$ is continuous in H. In particular, $t \mapsto f'_t(e^{i\xi(t)})$ is continuous. So we have

Lemma 2.4.2 *$u(t)$ is C^1 continuous, and $u'(t) = |f'_t(e^{i\xi(t)})|^2$.*

The fact $W(1) = 1$ implies that $e^{i\zeta(0)} = 1$. We may choose $\zeta(0) = 0$, and lift f_t to the covering space. Let e^i denote the function $z \mapsto e^{iz}$. Let $\widetilde{\Sigma}_t := (e^i)^{-1}(\Sigma_t)$. There is a unique family of conformal maps $\widetilde{f}_t$ on $\widetilde{\Sigma}_t$ such that $e^i \circ \widetilde{f}_t = f_t \circ e^i$, $\widetilde{f}_t(\xi(t)) = \zeta(t)$, and $\widetilde{f}_t$ takes real values on $\mathbb{R}$. Then we have $u'(t) = \widetilde{f}'_t(\xi(t))^2$.

Lemma 2.4.3 *$(t,x) \mapsto \widetilde{f}_t(x)$ is $C^{1,\infty}$ continuous on $[0,T_F) \times \mathbb{R}$. And for all $t \in [0, T_F)$, $\partial_t \widetilde{f}_t(\xi(t)) = -3\widetilde{f}''_t(\xi(t))$.*

Proof. For any $t \in [0, T_F)$, and $z \in \mathbb{D} \setminus F \setminus K_t$, we have $f_t \circ \varphi_t(z) = \psi_{u(t)} \circ W(z)$. Taking the derivative w.r.t. t, we compute

$$\partial_t f_t(\varphi_t(z)) + f'_t(\varphi_t(z))\varphi_t(z)\frac{e^{i\xi(t)} + \varphi_t(z)}{e^{i\xi(t)} - \varphi_t(z)}$$

$$= u'(t)\psi_{u(t)}(W(z))\mathbf{S}_{p-u(t)}(\psi_{u(t)}(W(z))/e^{i\zeta(t)}).$$

By Lemma 2.4.2, $u'(t) = |f'_t(e^{i\xi(t)})|^2$. Thus for any $t \in [0, T_F)$ and $z \in \mathbb{D} \setminus F \setminus K_t$,

$$\partial_t f_t(\varphi_t(z)) = |f'_t(e^{i\xi(t)})|^2 f_t(\varphi_t(z))\mathbf{S}_{p-u(t)}(f_t(\varphi_t(z))/f_t(e^{i\xi(t)}))$$

$$-f_t'(\varphi_t(z))\varphi_t(z)\frac{e^{i\xi(t)}+\varphi_t(z)}{e^{i\xi(t)}-\varphi_t(z)}.$$

For any $t\in[0,T_F)$, and $w\in\mathbb{D}\setminus\varphi_t(F)$, we have $\varphi_t^{-1}(w)\in\mathbb{D}\setminus F\setminus K_t$. Thus

$$\partial_t f_t(w)=|f_t'(e^{i\xi(t)})|^2 f_t(w)\mathbf{S}_{p-u(t)}(f_t(w)/f_t(e^{i\xi(t)}))-f_t'(w)w\frac{e^{i\xi(t)}+w}{e^{i\xi(t)}-w}.$$

Let $g_t(w)$ be the right-hand side of the above formula for $t\in[0,T_F)$ and $w\in\Sigma_t\setminus\{e^{i\xi(t)}\}$. Then for each $t\in[0,T_F)$, $g_t(w)$ is analytic in $\Sigma_t\setminus\{e^{i\xi(t)}\}$. And $(t,w)\mapsto g_t(w)$ is $C^{0,\infty}$ continuous on $H\setminus G(\xi)$.

Now fix $t_0\in[0,T_F)$. Let us compute the limit of $g_{t_0}(w)$ when $w\to e^{i\xi(t_0)}$. Since

$$\mathbf{S}_{p-u(t_0)}(f_{t_0}(w)/f_{t_0}(e^{i\xi(t_0)}))-\frac{f_{t_0}(e^{i\xi(t_0)})+f_{t_0}(w)}{f_{t_0}(e^{i\xi(t_0)})-f_{t_0}(w)}\to 0,\ \text{ as } w\to\chi_{t_0},$$

so the limit of $g_{t_0}(w)$ is equal to the limit of the following function:

$$|f_{t_0}'(e^{i\xi(t_0)})|^2 f_{t_0}(w)\frac{f_{t_0}(e^{i\xi(t_0)})+f_{t_0}(w)}{f_{t_0}(e^{i\xi(t_0)})-f_{t_0}(w)}-f_{t_0}'(w)w\frac{e^{i\xi(t_0)}+w}{e^{i\xi(t_0)}-w}.$$

Let $w=e^{ix}$, we may express the above formula in term of x, $\xi(t_0)$ and $\widetilde{f}_{t_0}$, as follows:

$$\widetilde{f}_{t_0}'(\xi(t_0))^2 e^{i\widetilde{f}_{t_0}(x)}\frac{e^{i\widetilde{f}_{t_0}(\xi(t_0))}+e^{i\widetilde{f}_{t_0}(x)}}{e^{i\widetilde{f}_{t_0}(\xi(t_0))}-e^{i\widetilde{f}_{t_0}(x)}}-\widetilde{f}_{t_0}'(x)e^{i\widetilde{f}_{t_0}(x)}\frac{e^{i\xi(t_0)}+e^{ix}}{e^{i\xi(t_0)}-e^{ix}}$$

$$=-ie^{i\widetilde{f}_{t_0}(x)}[\widetilde{f}_{t_0}'(\xi(t_0))^2\cot(\frac{\widetilde{f}_{t_0}(x)-\widetilde{f}_{t_0}(\xi(t_0))}{2})-\widetilde{f}_{t_0}'(x)\cot(\frac{x-\xi(t_0)}{2})].$$

By expanding the Laurent series of $\cot(z/2)$ near 0, we see that the limit of the above formula is $3ie^{i\widetilde{f}_{t_0}(\xi(t_0))}\widetilde{f}_{t_0}''(\xi(t_0))=3if_{t_0}(e^{i\xi(t_0)})\widetilde{f}_{t_0}''(\xi(t_0))$. Therefore g_t has an analytic extension to Σ_t for each $t\in[0,T_F)$. The maximum principle also implies that $g_t(w)$ is $C^{0,\infty}$ continuous in H, and $\partial_t f_t(w)=g_t(w)$ holds in the whole H. Thus $f_t(w)$ is $C^{1,\infty}$ continuous on $[0,T_F)\times\mathbf{C}_0$, and $\widetilde{f}_t(w)$ is $C^{1,\infty}$ continuous on $[0,T_F)\times\mathbb{R}$. Finally,

$$\partial_t\widetilde{f}_t(\xi(t))=\frac{i\partial_t f_t(e^{i\xi(t)})}{f_t(e^{i\xi(t)})}=\frac{ig_t(e^{i\xi(t)})}{f_t(e^{i\xi(t)})}=\frac{-3f_t(e^{i\xi(t)})\widetilde{f}_t''(\xi(t))}{f_t(e^{i\xi(t)})}=-3\widetilde{f}_t''(\xi(t)).\ \square$$

Proof of Theorem 2.4.1. Note that $\zeta(t) = \widetilde{f}_t(\xi(t))$, $\xi(t) = \sqrt{\kappa}B(t)$, and from the last lemma, $\partial_t \widetilde{f}_t(\xi(t)) = -3\widetilde{f}_t''(\xi(t))$. By Itô's formula, we have

$$d\zeta(u(t)) = \widetilde{f}_t'(\xi(t))d\xi(t) + (\frac{\kappa}{2} - 3)\widetilde{f}_t''(\xi(t))dt.$$

Since $u'(t) = \widetilde{f}_t'(\xi(t))^2$, so

$$d\zeta(t) = d\xi^1(t) + (\frac{\kappa}{2} - 3)\widetilde{f}_{v(s)}''(\xi(t))/\widetilde{f}_{v(s)}'(\xi(t))^2 dt,$$

where $\xi^1(t) = \sqrt{\kappa}B^1(t)$, $0 \le t < p$, and $B^1(t)$ is another standard Brownian motion. Note that $\zeta_0 = 0$. If $\kappa = 6$, then $\zeta(t) = \xi^1(t)$, $0 \le t < p$. Thus $(W(K_{v(s)}))_{0 \le s < p}$ has the law of the standard modulus p annulus $\mathrm{SLE}_{\kappa=6}$. So $(K_{v(s)})_{0\le s<p}$ has the same law as $(L_s)_{0 \le s < p}$.

If $\kappa > 0$ and $\kappa \ne 6$, then $d\zeta(s) = d\xi^1(s)$ + a drift term. The remaining part follows from Girsanov's Theorem. □

From this proof, it is clear that if a family of standard modulus p annulus LE hulls (K_t) is equivalent to radial SLE_κ in the sense of the above theorem, then the driving function is $\sqrt{\kappa}B(t)$ plus some C^1 continuous function. If the law of (K_t) also satisfies the properties of symmetry and the conformally equivalent time-homogeneity, then the driving function must be $\sqrt{\kappa}B(t)$. So the standard annulus SLE_κ is determined by the three properties.

This equivalence theorem implies the a.s. existence of annulus SLE trace. Suppose (K_t) has the law of the standard modulus p annulus SLE_κ. Then there is a.s. a random curve $\beta : [0, p) \to \mathbb{A}_p \cup \mathbf{C}_0$ with $\beta(0) = 1$ such that for each t, K_t is the complement in $\mathbb{A}_p$ of the component of $\mathbb{A}_p \setminus \beta[0, t]$ whose boundary contains $\mathbf{C}_p$. If $\kappa \le 4$, then β is simple and $K_t = \beta(0, t]$. Such β is called a standard modulus p annulus SLE_κ trace. Through a conformal map, we could define annulus SLE_κ trace in an arbitrary doubly connected domain.

From the strong equivalence for $\kappa = 6$, if β is a standard modulus p annulus SLE_6 trace, then $\lim_{t\to p} \beta(t)$ exists on $\mathbf{C}_p$ a.s.. The same is true for $\kappa = 2$ thanks to the convergence of LERW to SLE_2(see Chapter 4). It is not known now whether this is true for other κ.

2.5 Disc SLE

2.5.1 Definition

Proposition 2.5.1 *Suppose $\xi : (-\infty, a) \to \mathbb{R}$, $-\infty < a \le 0$, is continuous. Then there is an interior Loewner chain $t \mapsto K_t$, $-\infty < t < a$, in $\mathbb{D}$ started from 0, and a family of maps φ_t, $-\infty < t < a$, such that each φ_t maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{A}_{|t|}$ with $\varphi_t(\mathbf{C}_0) = \mathbf{C}_{|t|}$, and*

$$\begin{cases} \partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}_{|t|}(\varphi_t(z)/e^{i\xi(t)}), & -\infty < t < a; \\ \lim_{t\to -\infty} e^t/\varphi_t(z) = z, & \forall z \in \mathbb{D} \setminus \{0\}. \end{cases} \tag{2.5.1}$$

Such K_t and φ_t are uniquely determined by ξ, and $\{e^{i\xi(t)}\} = \cap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}$. We call K_t and φ_t, $-\infty < t < a$, the standard disc LE interior hulls and maps driven by ξ.

Before the proof, we need the notation of convergence of plane domain sequences. We say that a sequence of plane domains $\{\Omega_n\}$ converges to a plane domain Ω, or $\Omega_n \to \Omega$, if
(i) every compact subset of Ω lies in Ω_n, for n large enough;
(ii) for every $z \in \partial\Omega$ there exists $z_n \in \partial\Omega_n$ for each n such that $z_n \to z$.
Note that a sequence of domains may have more than one limits. The following lemma is similar to Theorem 1.8, the Carathéodory kernel theorem, in [12].

Lemma 2.5.1 *Suppose $\Omega_n \to \Omega$, f_n maps Ω_n conformally onto G_n, and f_n converges to some function f on Ω uniformly on each compact subset of Ω. Then either f is constant on Ω, or f maps Ω conformally onto some domain G. And in the latter case, $G_n \to G$ and f_n^{-1} converges to f^{-1} uniformly on each compact subset of G.*

Proof of Proposition 2.5.1. For fixed $r \in (-\infty, a)$, let φ_t^r, $r \le t < a$, be the solution of

$$\partial_t \varphi_t^r(z) = \varphi_t^r(z)\mathbf{S}_{|t|}(\varphi_t^r(z)/e^{i\xi(t)}), \ \ \varphi_r^r(z) = z. \tag{2.5.2}$$

For $r \le t < a$, let K_t^r be the set of $z \in \mathbb{A}_{|r|}$ such that $\varphi_s^r(z)$ blows up at some time $s \in [r,t]$. Then $s \mapsto K_{r+s}^r$, $0 \le s < a - r$, is a Loewner chain in $\mathbb{A}_{|r|}$ on $\mathbf{C}_0$, and φ_t^r maps $\mathbb{A}_{|r|} \setminus K_t^r$ conformally onto $\mathbb{A}_{|t|}$ with $\varphi_t^r(\mathbf{C}_{|r|}) = \mathbf{C}_{|t|}$. By the uniqueness of the solution of ODE, if $t_1 \le t_2 \le t_3 < a$, then $\varphi_{t_3}^{t_2} \circ \varphi_{t_2}^{t_1}(z) = \varphi_{t_3}^{t_1}(z)$, for $z \in \mathbb{A}_{|t_1|} \setminus K_{t_3}^{t_1}$. For $t < 0$, define $R_t(z) = e^t/z$. Then R_t maps $\mathbb{A}_{|t|}$ conformally onto itself, and exchanges its two boundary components. Define $\widehat{\varphi}_t^r = R_t \circ \varphi_t^r \circ R_r$, and $\widehat{K}_t^r = R_r(K_t^r)$. Then $\widehat{K}_t^r$ is a hull in $\mathbb{A}_{|r|}$ on $\mathbf{C}_{|r|}$, and $\widehat{\varphi}_t^r$ maps $\mathbb{A}_{|r|} \setminus \widehat{K}_t^r$ conformally onto $\mathbb{A}_{|t|}$ with $\widehat{\varphi}_t^r(\mathbf{C}_0) = \mathbf{C}_0$. We also have $\widehat{\varphi}_{t_3}^{t_2} \circ \widehat{\varphi}_{t_2}^{t_1}(z) = \widehat{\varphi}_{t_3}^{t_1}(z)$, for $z \in \mathbb{A}_{|t_1|} \setminus \widehat{K}_{t_3}^{t_1}$, if $t_1 \le t_2 \le t_3 < a$. And $\widehat{\varphi}_t^r$ satisfies

$$\partial_t \widehat{\varphi}_t^r(z) = \widehat{\varphi}_t^r(z)\widehat{\mathbf{S}}_{|t|}(\widehat{\varphi}_t^r(z)/e^{-i\xi(t)}), \ \ \widehat{\varphi}_r^r(z) = z,$$

where $\widehat{\mathbf{S}}_p(z) := 1 - \mathbf{S}_p(e^{-p}/z)$ for $p > 0$. A simple computation gives:

$$|\widehat{\mathbf{S}}_p(z)| \le 8e^{-p}/|z|, \ \text{ if } 4e^{-p} \le |z| \le 1.$$

We then have

$$|\widehat{\varphi}_t^r(z) - z| \le 8e^t, \ \text{ if } \ r \le t < 0, \ \text{ and } \ 12e^t \le |z| \le 1. \tag{2.5.3}$$

Now let $\widehat{\psi}_t^r$ be the inverse of $\widehat{\varphi}_t^r$. If $t_1 \le t_2 \le t_3 < a$, then $\widehat{\psi}_{t_2}^{t_1} \circ \widehat{\psi}_{t_3}^{t_2}(z) = \widehat{\psi}_{t_3}^{t_1}(z)$, for any $z \in \mathbb{A}_{|t_3|}$. For fixed $t \in (-\infty, a)$, $\{\widehat{\psi}_t^r : r \in (-\infty, t]\}$ is a family of uniformly bounded conformal maps on $\mathbb{A}_{|t|}$, so is a normal family. This implies that we can find a sequence $r_n \to -\infty$ such that for any $m \in \mathbb{N}$, $\{\widehat{\psi}_{-m}^{r_n}\}$ converges to some $\widehat{\psi}_{-m}$, uniformly on each compact subset of $\mathbb{A}_m$. Let $\beta_n = \widehat{\psi}_{-m}^{r_n}(\mathbf{C}_{m/2})$. Then β_n is a Jordan curve in $\mathbb{A}_{|r_n|} \setminus \widehat{K}_{-m}^{r_n}$ that separates the two boundary components. So 0 is contained in the Jordan domain determined by β_n. Note that $\{\widehat{\psi}_{-m}^{r_n}\}$ maps $\mathbb{A}_{m/2}$ onto the domain bounded by β_n and $\mathbf{C}_0$, whose modulus has to be $m/2$. So β_n is not contained in $\mathbf{B}(0;e^{-m/2})$. This implies that the diameter of β_n is not less than $e^{-m/2}$. So $\widehat{\psi}_{-m}$ can't be a constant. By Lemma 2.5.1, $\widehat{\psi}_{-m}$ maps $\mathbb{A}_m$ conformally onto some domain D_{-m}, and $\widehat{\psi}^{r_n}_{-m}(\mathbb{A}_m) \to D_{-m}$. Since $\widehat{\psi}^{r_n}_{-m}(\mathbb{A}_m) = \mathbb{A}_{|r_n|} \setminus \widehat{K}^{r_n}_{-m} \subset \mathbb{D}\setminus\{0\}$, $D_{-m} \subset \mathbb{D}\setminus\{0\}$. Since $M(\mathbb{A}_{|r_n|} \setminus \widehat{K}^{r_n}_{-m}) = m$, there is some $a_m \in (0,1)$ such that $\overline{\mathbf{B}(0;e^{r_n})} \cup \widehat{K}^{r_n}_{-m} \subset \mathbf{B}(0;e^{-a_m})$ for all r_n. So $\mathbb{A}_{a_m}$ contains no boundary points of $\mathbb{A}_{|r_n|} \setminus \widehat{K}^{r_n}_{-m} = \widehat{\psi}^{r_n}_{-m}(\mathbb{A}_m)$. Since these domains converge to D_{-m} as $n \to \infty$, so $\mathbb{A}_{a_m}$ contains no boundary points of D_{-m}, which means that either $\mathbb{A}_{a_m} \subset D_{-m}$ or $\mathbb{A}_{a_m} \cap D_{-m} = \emptyset$. Now let $\gamma_n = \widehat{\psi}^{r_n}_{-m}(\mathbf{C}_{a_m/2})$. For the same reason as β_n, we have $\gamma_n \not\subset \mathbf{B}(0;e^{-a_m/2})$. So there is $z_n \in \mathbf{C}_{a_m/2}$ such that $|\widehat{\psi}^{r_n}_{-m}(z_n)| \ge e^{-a_m/2}$. Let z_0 be any subsequential limit of $\{z_n\}$, then $z_0 \in \mathbf{C}_{a_m/2} \subset \mathbb{A}_m$ and $|\widehat{\psi}_{-m}(z_0)| \ge e^{-a_m/2}$, so $\widehat{\psi}_{-m}(z_0) \in \mathbb{A}_{a_m}$. Thus $D_{-m} \cap \mathbb{A}_{a_m} \ne \emptyset$, and so $\mathbb{A}_{a_m} \subset D_{-m}$. Hence D_{-m} has one boundary component $\mathbf{C}_0$. Using similar arguments, we have $\widehat{\psi}_t(\mathbf{C}_0) = \mathbf{C}_0$.

If $r_n < -m_1 < -m_2$, then $\widehat{\psi}^{r_n}_{-m_1} \circ \widehat{\psi}^{-m_1}_{-m_2} = \widehat{\psi}^{r_n}_{-m_2}$, which implies $\widehat{\psi}_{-m_1} \circ \widehat{\psi}^{-m_1}_{-m_2} = \widehat{\psi}_{-m_2}$. For $t \in (-\infty, 0)$, choose $m \in \mathbb{N}$ with $-m \le t$, define $\widehat{\psi}_t = \widehat{\psi}_{-m} \circ \widehat{\psi}^{-m}_t$ and $D_t = \widehat{\psi}_t(\mathbb{A}_{|t|})$. It is easy to check that the definition of $\widehat{\psi}_t$ is independent of the choice of m, and the following properties hold. For all $t \in (-\infty, 0)$, D_t is a doubly connected subdomain of $\mathbb{D}\setminus\{0\}$ that has one boundary component $\mathbf{C}_0$, and $\widehat{\psi}_t(\mathbf{C}_0) = \mathbf{C}_0$; $\widehat{\psi}^{r_n}_t$ converges to $\widehat{\psi}_t$, uniformly on each compact subset of $\mathbb{A}_{|t|}$. If $r < t < 0$, then $\widehat{\psi}_t = \widehat{\psi}_r \circ \widehat{\psi}^r_t$; $D_t \subsetneqq D_r$, and $D_r \setminus D_t = \widehat{\psi}_r(\widehat{K}^r_t)$.

Let $\widehat{\varphi}_t$ on D_t be the inverse of $\widehat{\psi}_t$. By Lemma 2.5.1, $\widehat{\varphi}^{r_n}_t$ converges to $\widehat{\varphi}_t$ as $n \to \infty$, uniformly on each compact subset of D_t. Thus from formula (2.5.3), we have $|\widehat{\varphi}_t(z) - z| \le 8e^t$, if $12e^t \le |z| < 1$. It follows that $\lim_{t\to-\infty} \widehat{\varphi}_t(z) = z$, for any $z \in \mathbb{D}\setminus\{0\}$. We also have $\widehat{\varphi}_t(z) = \widehat{\varphi}^{-m}_t \circ \widehat{\varphi}_{-m}(z)$, if $-m \le t < 0$ and $z \in D_t$. Let $\varphi_t = R_t \circ \widehat{\varphi}_t$ on D_t. Then φ_t maps D_t conformally onto $\mathbb{A}_{|t|}$, takes $\mathbf{C}_0$ to $\mathbf{C}_{|t|}$, and

$$\lim_{t\to-\infty} e^t/\varphi_t(z) = \lim_{t\to-\infty} \widehat{\varphi}_t(z) = z, \text{ for any } z \in \mathbb{D}\setminus\{0\}.$$

If $-m \le t$, then $\varphi_t(z) = \varphi^{-m}_t \circ R_{-m} \circ \widehat{\varphi}_{-m}(z)$, $\forall z \in D_t$. By formula (2.5.2), we have

$$\partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}_{|t|}(\varphi_t(z)/e^{i\xi(t)}), \quad -m \le t < 0.$$

Since we may choose $m \in \mathbb{N}$ arbitrarily, formula (2.5.1) holds.

Let $K_t = \mathbb{D} \setminus D_t$. Since D_t is a doubly connected subdomain of $\mathbb{D} \setminus \{0\}$ with a boundary component $\mathbf{C}_0$, K_t is an interior hull in $\mathbb{D}$ and $0 \in K_t$. The fact $M(D_t) = |t| \to \infty$ as $t \to -\infty$ implies that the diameter of K_t tends to 0 as $t \to -\infty$. So $\{0\} = \cap K_t$. If $t_1 < t_2 < a$, then $K_{t_1} \subsetneqq K_{t_2}$, as $D_{t_1} \supsetneqq D_{t_2}$. Fix any $r \in (-\infty, a)$. For $t \in [r, a)$, $K_t \setminus K_r = D_r \setminus D_t = \widehat{\psi}_r(\widehat{K}_t^r)$. From conformal invariance, $s \to \widehat{\psi}_r(\widehat{K}_{r+s}^r)$, $0 \le s < a - r$, is a Loewner chain in D_r on ∂K_r. Thus $t \mapsto K_t$ is an interior Loewner chain in $\mathbb{D}$ started from 0.

For any $t \in (-\infty, a)$ and $\varepsilon \in (0, a - t)$, we have

$$\varphi_t(K_{t+\varepsilon} \setminus K_t) = \varphi_t(\widehat{\psi}_t(\widehat{K}_{t+\varepsilon}^t)) = R_t \circ \widehat{\varphi}_t \circ \widehat{\psi}_t \circ R_t(K_{t+\varepsilon}^t) = K_{t+\varepsilon}^t.$$

Since $(K_{t+\varepsilon}^t, 0 \le \varepsilon < a - t)$ is a family of standard modulus p annulus LE hulls driven by $\xi(t + \cdot)$, so we have $\{\xi(t)\} = \cap_{\varepsilon > 0} \overline{K_{t+\varepsilon}^t}$, from which follows that $\{\xi(t)\} = \cap_{\varepsilon > 0} \overline{\varphi_t(K_{t+\varepsilon} \setminus K_t)}$.

Suppose $t \mapsto K_t^*$, $-\infty < t < a$, is an interior Loewner chain in $\mathbb{D}$ started from 0, and φ_t^*, $-\infty < t < a$, is a family of maps such that for each t, φ_t^* maps $\mathbb{D} \setminus K_t^*$ conformally onto $\mathbb{A}_{|t|}$ and formula (2.5.1) holds with φ_t replaced by φ_t^*. By the uniqueness of the solution of ODE, we have $\varphi_t^* = \varphi_t^r \circ \varphi_r^*$, if $r \le t < 0$. So $R_t \circ \varphi_t^* = \widehat{\varphi}_t^r \circ R_r \circ \varphi_r^*$. Now choose $r = r_n$ and let $n \to \infty$. Since $R_{r_n} \circ \varphi_{r_n}^* \to \mathrm{id}$ by formula (2.5.1), and $\widehat{\varphi}_t^{r_n} \to \widehat{\varphi}_t$, so $R_t \circ \varphi_t^* = \widehat{\varphi}_t$, from which follows that $\varphi_t^* = R_t \circ \widehat{\varphi}_t = \varphi_t$ and $K_t^* = K_t$. $\square$

Proposition 2.5.2 *Suppose $t \mapsto K_t$, $-\infty < t < a$, is an interior Loewner chain in $\mathbb{D}$ started from 0 such that $M(\mathbb{D} \setminus K_t) = |t|$ for each t. Then $(K_t$, $-\infty < t < a$, is a family of standard disc LE interior hulls. And if we don't assume that $M(\mathbb{D} \setminus K_t) = |t|$ for each t, then after a time-change, we can make (K_t) to be a family of standard disc interior LE hulls.*

Proof. We only need to consider the case that $M(\mathbb{D} \setminus K_t) = |t|$, for all $-\infty < t < a$. For each $t \in (-\infty, a)$, choose g_t^* which maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{A}_{|t|}$ so that

$g_t^*(1) = 1$. Let $\varphi_t^* = R_t \circ g_t^*$, where $R_t(z) = e^t/z$. Then φ_t^* maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{A}_{|t|}$ with $\varphi_t^*(\mathbf{C}_0) = \mathbf{C}_{|t|}$ and $\varphi_t^*(1) = e^t$. For any $r \le t < 0$, let $K_{r,t}^* = \varphi_r^*(K_t \setminus K_r)$. Then for fixed $r < a$, $s \mapsto K_{r,r+s}^*$, $0 \le s < a - r$), is a Loewner chain in $\mathbb{A}_{|r|}$ on $\mathbf{C}_0$. Now $\varphi_t^* \circ (\varphi_r^*)^{-1}$ maps $\mathbb{A}_{|r|} \setminus K_{r,t}^*$ conformally onto $\mathbb{A}_{|t|}$, and satisfies $\varphi_t^* \circ (\varphi_r^*)^{-1}(e^r) = e^t$. From the proof of Proposition 2.4.1, there exists some continuous $\xi_r^* : [r, 0) \to \mathbb{R}$ such that for $r \le t < 0$,

$$\partial_t \varphi_t^* \circ (\varphi_r^*)^{-1}(w) = \varphi_t^* \circ (\varphi_r^*)^{-1}(w)[\mathbf{S}_{|t|}(\varphi_t^* \circ (\varphi_r^*)^{-1}(w)/e^{i\xi_r^*(t)}) - i\mathrm{Im}\, \mathbf{S}_{|t|}(e^t/e^{i\xi_r^*(t)})].$$

It then follows that

$$\partial_t \varphi_t^*(z) = \varphi_t^*(z)[\mathbf{S}_{|t|}(\varphi_t^*(z)/e^{i\xi_r^*(t)}) - i\mathrm{Im}\, \mathbf{S}_{|t|}(e^t/e^{i\xi_r^*(t)})], \ \ r \le t < 0.$$

So $e^{i\xi_{r_1}^*(t)} = e^{i\xi_{r_2}^*(t)}$ if $r_1, r_2 \le t$. We then can construct a continuous $\xi^* : (-\infty, a) \to \mathbb{R}$, such that

$$\partial_t \varphi_t^*(z) = \varphi_t^*(z)[\mathbf{S}_{|t|}(\varphi_t^*(z)/e^{i\xi^*(t)}) - i\mathrm{Im}\, \mathbf{S}_{|t|}(e^t/e^{i\xi^*(t)})], \quad -\infty \le t < a.$$

Consequently,

$$\partial_t g_t^*(z) = g_t^*(z)[\widehat{\mathbf{S}}_{|t|}(\varphi_t^*(z)/e^{-i\xi^*(t)}) - i\mathrm{Im}\, \widehat{\mathbf{S}}_{|t|}(e^{i\xi^*(t)})], \quad -\infty \le t < a.$$

Since $|\widehat{\mathbf{S}}_{|t|}(z)| \le 8e^t$ when $4e^t \le |z| \le 1$, $|\mathrm{Im}\, \widehat{\mathbf{S}}_{|t|}(e^{i\xi^*(t)})|$ decays exponentially as $t \to -\infty$. Let $\theta(t) = \int_{-\infty}^t \mathrm{Im}\, \widehat{\mathbf{S}}_{|s|}(e^{i\xi^*(s)})ds$, $g_t(z) = e^{i\theta(t)} g_t^*(z)$, and $\xi(t) = \xi^*(t) - \theta(t)$. Then g_t maps $\mathbb{D} \setminus K_t$ conformally onto $\mathbb{A}_{|t|}$ with $g_t(\mathbf{C}_0) = \mathbf{C}_0$, and

$$\partial_t \ln g_t(z) = \partial_t \ln g_t^*(z) + i\theta'(t) = \widehat{\mathbf{S}}_{|t|}(g_t^*/e^{-i\xi^*(t)}) = \widehat{\mathbf{S}}_{|t|}(\varphi_t/e^{-i\xi(t)}).$$

Thus $\partial_t g_t(z) = g_t(z)\widehat{\mathbf{S}}_{|t|}(g_t(z)/e^{-i\xi(t)})$. From the estimation of $\widehat{\mathbf{S}}_{|t|}$, we have

$$|g_t(z) - g_r(z)| \le 8e^t, \ \text{ if } \ 12e^t \le |g_r(z)| \le 1, \ \text{ and } \ r \le t < 0.$$

Since K_t contains 0 and $M(\mathbb{D}\setminus K_t)=|t|$, the diameter of K_t tends to zero as $t\to-\infty$. Let $D_t=\mathbb{D}\setminus K_t$. Then for any sequence $t_n\to-\infty$, we have $D_{t_n}\to\mathbb{D}\setminus\{0\}$. Since g_{t_n} is uniformly bounded, there is a subsequence that converges to some function g on $\mathbb{D}\setminus\{0\}$ uniformly on each compact subset of $\mathbb{D}\setminus\{0\}$. By checking the image of $\mathbf{C}_1$ under g_{t_n} similarly as in the proof of Proposition 2.5.1, we see that g cannot be constant. So by Lemma 2.5.1, g maps $\mathbb{D}\setminus\{0\}$ conformally onto some domain D_0 which is a subsequential limit of $\mathbb{A}_{|t_n|}=g_{t_n}(D_{t_n})$. Since $t_n\to-\infty$, D_0 has to be $\mathbb{D}\setminus\{0\}$ and so $g(z)=\chi z$ for some $\chi\in\mathbf{C}_0$. Now this χ may depend on the subsequence of $\{t_n\}$. But we always have $\lim_{t\to-\infty}|g_t(z)|=|z|$ for any $z\in\mathbb{D}\setminus\{0\}$. Now fix $z\in\mathbb{D}\setminus\{0\}$, there is $s(z)<0$ such that when $r\le t<s(z)$, we have $12e^t\le|g_r(z)|\le 1$. Therefore $|g_t(z)-g_r(z)|\le 8e^t$ for $r\le t<s(z)$. Thus $\lim_{t\to-\infty}g_t(z)$ exists for every $z\in\mathbb{D}\setminus\{0\}$. Since we have a sequence $t_n\to-\infty$ such that $\{g_{t_n}\}$ converges pointwise to $z\mapsto\chi z$ on $\mathbb{D}\setminus\{0\}$ for some $\chi\in\mathbf{C}_0$, so $\lim_{t\to-\infty}\varphi_t(z)=\chi z$, for all $z\in\mathbb{D}\setminus\{0\}$. Finally, let $\varphi_t(z)=R_t\circ g_t(z/\chi)$. Then φ_t maps $\mathbb{D}\setminus K_t$ conformally onto $\mathbb{A}_{|t|}$, takes $\mathbf{C}_0$ to $\mathbf{C}_{|t|}$, and satisfies (disc LC1). $\square$

We still use $B(t)$ to denote a standard Brownian motion. Let $\mathbf{x}$ be some uniform random point on $[0,2\pi)$, independent of $B(t)$. For $\kappa>0$ and $-\infty<t<0$, write $\xi_\kappa(t)=\mathbf{x}+\sqrt{\kappa}B(|t|)$. The process $(e^{i\xi(t)})$ is determined by the following properties: for any fixed $r<0$, $(e^{i\xi(t)}/e^{i\xi(r)},r\le t<0)$ has the same law as $(e^{iB(\kappa(t-r))},r\le t<0)$ and is independent from $e^{i\xi(r)}$. If K_t and φ_t, $-\infty<t<0$, are the standard disc interior LE hulls and maps, respectively, driven by ξ_κ, then we call the law of (K_t) the standard disc SLE$_\kappa$. Suppose D is a simply connected domain and $p\in D$. Let W map $(\mathbb{D};0)$ conformally onto $(D;p)$ and $W'(0)>0$. Then the disc SLE$_\kappa(D;p\to\partial D)$ is defined as the image of the standard disc SLE$_\kappa$ under the map W. The existence of standard annulus SLE$_\kappa$ trace then implies the a.s. existence of standard disc SLE$_\kappa$ trace, which is a curve $\beta:[-\infty,0)\to\mathbb{D}$ such that $\gamma(-\infty)=0$, and for each $t\in(-\infty,0)$, K_t is the complement of the unbounded component of $\mathbb{C}\setminus\gamma[-\infty,t]$. If $\kappa\le 4$, the trace is a simple curve; otherwise, it is not simple.

2.5.2 Equivalence of disc and full plane SLE

Theorem 2.5.1 *Let (K_t) has the law of the standard full plane SLE_6. Suppose D is a simply connected domain that contains 0. Let τ be the first t such that $K_t \not\subset \Omega$. Then $(K_t, 0 \le t < \tau)$ has the law of the disc $SLE_6(D; 0 \to \partial D)$ after a time-change.*

Proof. Let φ_t and ξ be the full plane LE maps and driving function corresponding to (K_t). Let $\mathcal{F}_t$ be the σ-algebra determined by $\{e^{i\xi(s)} : s \le t\}$. Since $t \mapsto K_t$, $-\infty < t < \infty$ is an interior Loewner chain in $\widehat{\mathbb{C}}$ started from 0, it is clear that $t \mapsto K_t$, $-\infty < t < \tau$, is an interior Loewner chain in D started from 0. Let $u(t) = -M(\Omega \setminus K_t)$, for $-\infty < t < \tau$. Then u is continuous and increasing, and maps $(-\infty, \tau)$ onto $(-\infty, 0)$. Let v be the inverse of u. Then for any $t \in (-\infty, 0)$, $v(t)$ is a $(\mathcal{F}_t)$ stopping time. Define $\widehat{\mathcal{F}}_t := \mathcal{F}_{v(t)}$. Let W map $(D; 0)$ conformally onto $(\mathbb{D}; 0)$. By Proposition 2.5.2 $(W(K_{v(s)}))$ is a family of standard disc LE hulls. Let ψ_s and ζ be the standard disc LE maps and driving function corresponding to $(W(K_{v(s)}))$. Define $f_t := \psi_t \circ W \circ (\varphi_{v(t)})^{-1}$. Similarly as before, we have $f_t(e^i(\xi(v(t)))) = e^i(\zeta(t))$. Then $(e^{i\zeta(t)})$, (ψ_t), and (f_t) are $(\widehat{\mathcal{F}}_t)$ adapted. We have $\widetilde{f}_t$ that satisfies $e^i \circ \widetilde{f}_t = f_t \circ e^i$ and $\widetilde{f}_t(\xi(v(t)) = \zeta(t)$. Then one can check that $v'(t) = (\partial_x \widetilde{f}_t(\xi(v(t))))^{-2}$ and $\partial_t \widetilde{f}_t(\xi(v(t)) = -3\partial_x^2 \widetilde{f}_t(\xi(v(t))/\partial_x \widetilde{f}_t(\xi(v(t))$. Now fix $r < 0$. Let $\xi^r(t) := \xi(v(r+t)) - \xi(v(r))$. There is a $(\widehat{\mathcal{F}}_{r+t}, t \ge 0)$ standard Brownian motion $B^r(t)$ such that $d\xi^r(t) = \sqrt{6}v'(r+t)dB^r(t)$. From Ito's formula, we see that $\zeta(r+t) - \zeta(r) = \sqrt{6}B^r(t)$. This also implies that $\zeta(r+t) - \zeta(r)$ is independent of $e^{i\zeta(s)}$, $s \le r$. Thus $(e^{i\zeta(r)})$ has the same law as $(e^{i\xi_6(t)})$. So $(W(K_{v(t)}))$ has the law of the standard disc SLE_6, from which follows that $(K_{v(t)})$ has the law of the disc $\mathrm{SLE}_6(D; 0 \to \partial D)$. $\square$

Corollary 2.5.1 *The distribution of the hitting point of full plane SLE_6 trace at $\partial\Omega$ is the harmonic measure valued at 0.*

An immediate consequence of this corollary is that the plane SLE_6 hull stopped at the hitting time of $\partial\Omega$ has the same law as the hull generated by a plane Brownian motion started from 0 and stopped on exiting Ω. This result has been announced and proved in [19] and [8]. See them for details.

Chapter 3

Harmonic random Loewner chain

3.1 Some notations

3.1.1 Finite Riemann surfaces and conformal structure

Suppose R is a compact Riemann surface. A subdomain D of R is called a finite Riemann surface if $R \setminus D$ is a union of finitely many mutually disjoint compact contractible subsets of R, each of which contains more than one point. We call R the underlying surface of D, and each component of $R \setminus D$ an island of D. A compact Riemann surface is also considered as a finite Riemann surface. The simply and doubly connected domains are two special kinds of finitely Riemann surfaces.

Suppose f maps a finite Riemann surface D_1 conformally onto a finite Riemann surface D_2. Then f induces a one-to-one correspondence $\widehat{f}$ from the set of islands of D_1 to the set of islands of D_2 such that for any island A of D_1, $z \in D$ and $z \to A$ iff $f(z) \to \widehat{f}(A)$. So D_1 and D_2 have the same number of islands. If K is a hull in D_1 on A, then $f(K)$ is a hull in D_2 on $\widehat{f}(A)$.

For any finite Riemann surface D, there exists f that maps D conformally onto a finite Riemann surface E whose islands are all surrounded by analytic Jordan curves. We call such f a boundary smoothing map of D. Suppose $f_j : D \to E_j$, $j = 1, 2$, are two boundary smoothing maps. Then $f_2 \circ f_1^{-1}$ maps E_1 conformally onto E_2. Since E_1 and E_2 are all bounded by analytic Jordan curves, each of which bounds an island, so $f_2 \circ f_1^{-1}$ induces a one-to-one correspondence J from the set of Jordan curves that bound E_1 to the set of Jordan curves that bound E_2 such that for any one of these analytic Jordan curves γ, $z \in E_1$ and $z \to \gamma$ iff $f_2 \circ f_1^{-1}(z) \to J(\gamma)$. From Schwarz's reflection, $f_2 \circ f_1^{-1}$ can be extended conformally across γ, and the extension maps γ onto $J(\gamma)$.

Now consider the set of all pairs (f, z) such that f is a boundary smoothing map of D, and $z \in \overline{f(D)}$. Two pairs (f_1, z_1) and (f_2, z_2) are equivalent if the conformal extension of $f_2 \circ f_1^{-1}$ maps z_1 to z_2. Let $\widehat{D}$ be the set of all equivalent classes. There is a unique conformal structure on $\widehat{D}$ such that for any boundary smoothing map f of D, $z \mapsto [(f, z)]$ is a conformal map from $\overline{f(D)}$ onto $\widehat{D}$. Then $z \mapsto [(f, f(z))]$ is a conformal map from D into $\widehat{D}$ independent of the choice of f. So we may view D as a subset of $\widehat{D}$, and call $\widehat{D}$ the conformal closure of D. From the construction, a conformal map between finite Riemann surfaces can be extended to a conformal map between their conformal closures.

We call $\widehat{\partial} D := \partial D \setminus D$ the conformal boundary of D. It is clear that $\widehat{\partial} D$ is a union of finitely many mutually disjoint analytic Jordan curves, each of which is called a side of D, and corresponds to an island A of D such that $z \in D$ tends to a side in $\widehat{D}$ iff $z \in D$ tends to the corresponding island. We call a point on $\widehat{\partial} D$ a prime end of D. This is equivalent to the prime ends defined in [1] and [12]. In fact, the definition in [1] describes the property of a sequence of points in D that converges to a point on $\widehat{\partial} D$, and the definition in [12] describes a neighborhood basis bounded by crosscuts of a point on $\widehat{\partial} D$.

If $z_0 \in \partial D$, the boundary of D in its underlying surface R, corresponds to a prime end w of D such that $z \in D$ and $z \to z_0$ in R iff $z \to w$ in $\widehat{D}$, then we may also view z_0 as the prime end w. This may happen if there is a neighborhood V of z_0 in R such that $\partial D \cap V$ is a simple curve, and V is divided by that curve into two parts, one in D, the other outside D; or there is a neighborhood V of z_0 in R such that $\partial D \cap V$ is a simple curve started from z_0.

3.1.2 Hulls and Loewner chains

Suppose F is an island of a finite Riemann surface D, and corresponds to the side α of D. A closed subset K of D is called a hull in D on ∂F or on α if $F \cup K$ is a compact contractible subset of R. Then $D \setminus K$ is still a finite Riemann surface with the underlying surface R, $F \cup K$ is an island of $D \setminus K$, and other islands of $D \setminus K$ are the islands of D other than F. A compact contractible subset of D that contains more than one point is called an interior hull.

A Loewner chain in D on an island F is a function L from $[0, T)$ for some $T \in (0, \infty]$ into the space of hulls in D on F such that $L(0) = \emptyset$, $L(t_1) \subsetneqq L(t_2)$ when $t_1 < t_2$, and for any fixed $b \in (0, T)$, and any compact subset F of $D \setminus L(b)$, the extremal length of the family of curves in $D \setminus L(t + \varepsilon)$ that separate F from $L(t + \varepsilon) \setminus L(t)$ tends to 0 as $\varepsilon \to 0$, uniformly in $t \in [0, b]$. An interior Loewner chain started from $z_0 \in D$ is a function L from $(-\infty, T)$ for some $T \in (-\infty, \infty]$ into the space of interior hulls in D such that $\{z_0\} = \cap L(t)$, $L(t_1) \subsetneqq L(t_2)$ when $t_1 < t_2$, and for any fixed $b \in (-\infty, T)$, $L(b + t)$, $0 \le t < a - b$, is a Loewner chain in $D \setminus L(b)$ on $L(b)$. Let Δ be a function on the sets of Loewner chains or interior Loewner chains such that the definition interval of L is $[0, \Delta(L))$ or $(-\infty, \Delta(L))$. The definitions of (interior) hull and Loewner chain clearly extend those defined in Section 2.1.

Suppose L is a Loewner chain in D on a side α. Fix $t \in [0, \Delta(L))$. Let d_t be a metric on $\widehat{D \setminus L(t)}$. From the definition of a Loewner chain, we see that the d_t-diameter of $L(t + \varepsilon) \setminus L(t)$ tends to 0 as $\varepsilon \to 0^+$. So there is a unique prime end $w(t)$ of $D \setminus L(t)$ that lies in the intersection of the closure of $L(t + \varepsilon) \setminus L(t)$ in $\widehat{D \setminus L(t)}$. We call $w(t)$ the prime end determined by L at time t. Especially, $w(0) \in \alpha$, and we say that L is started from $w(0)$. For example, a radial or chordal $\mathrm{SLE}_\kappa(D; a \to b)$ is supported by the set of Loewner chains in D started from a.

If f maps D_1 conformally onto D_2, and K is a hull in D_1 on a side α, then $f(K)$ is a hull in D_2 on the side $f(\alpha)$. From conformal invariance of extremal length, if L is a Loewner chain in D_1 on α, then $f \circ L$ is a Loewner chain in D_2 on $f(\alpha)$. This property holds even locally. That means if L is started from the prime end w on the side α_1 of D_1, U is a neighborhood in D of w such that $L(t) \subset U$ for all $0 \le t < \Delta(L)$, and f maps U conformally into a finite Riemann surface D_2 such that $z \in U$ and $z \to \alpha_1$ iff $f(z) \to \alpha_2$ for some side α_2 of D_2, then $f \circ L$ is a Loewner chain in D_2 started from $f(w_1) \in \alpha_2$.

3.1.3 Topology and measure structure

Suppose D is a finite Riemann surface with the underlying surface R. Let d be any metric on R. Then d induces a Hausdorff metric $d^{\mathcal{H}}$ on $\mathrm{Cld}^*(R)$, the space of nonempty closed subsets of R. Consider $\mathrm{Cld}(D)$, the space of closed subsets of D, as a subspace of $\mathrm{Cld}^*(R)$ through the inclusion map $K \mapsto K \cup (R \setminus D)$. Then $d^{\mathcal{H}}$ restricted to $\mathrm{Cld}(D)$ induces a topology $\mathcal{T}_D^{\mathcal{H}}$ on $\mathrm{Cld}(D)$. It is easy to see that $\mathcal{T}_D^{\mathcal{H}}$ is independent of the choice of d, and is conformally invariant, which means that for any conformal map f between two finite Riemann surfaces D_1 and D_2, $K \mapsto f(K)$ is a homeomorphism from $\mathrm{Cld}(D_1)$ onto $\mathrm{Cld}(D_2)$. The topology $\mathcal{T}_D^{\mathcal{H}}$ then induces a σ-algebra $\mathcal{F}_D^{\mathcal{H}}$.

Let the σ-algebra on the space of (interior) Loewner chains be generated by the sets $\{L : t < \Delta(L), L(t) \in A\}$, where $t \in \mathbb{R}$ or $\mathbb{R}_+$ and $A \in \mathcal{F}_D^{\mathcal{H}}$. The range of A could be replaced by $A = \{\cap F = \emptyset\}$, where F is a compact subset of D.

Now we consider the radial Loewner equation. The map from ξ to the family of standard radial LE hulls (K_t) driven by ξ is a map dr from $C[0,a)$ to the space of Loewner chains in $\mathbb{D}$. Now fixed a t and a compact subset F of $\mathbb{D}$. From the equation, if $dr(\xi_0)(t) \cap F = \emptyset$, then there is $\varepsilon > 0$ such that $dr(\xi)(t) \cap F = \emptyset$ if $\|\xi - \xi_0\|_t := \max\{|\xi(s) - \xi_0(s)| : 0 \le s \le t\}$ is less than ε. So dr is a measurable function. Since $\xi(t) = \sqrt{\kappa}B(t)$ gives a measure on $C[0,\infty)$, the standard radial SLE_κ is a measure on the space of Loewner chains in $\mathbb{D}$. Similarly, the standard chordal SLE_κ is a measure on the space of Loewner chains in $\mathbb{H}$. By conformal invariance, any radial or chordal SLE in a simply connected domain D is a measure on the space of Loewner chains in D.

3.1.4 Positive harmonic functions

Let D and R be as before. Suppose I is a side or side arc of D. Then there is a unique bounded continuous function H defined on $\widehat{D}$ taking away the extreme points of I such that H is harmonic and positive in D, $H \equiv 1$ on I, and $H \equiv 0$ on $\hat{\partial} D \setminus \overline{I}$. This H is called the harmonic measure function in D of I. For any $z \in D$, $H(z)$ is equal to the probability that a two-dimensional Brownian motion in $\widehat{D}$ started from z hits I before $\hat{\partial} D \setminus I$.

Suppose $p \in D$. There is a unique continuous function G defined on $\widehat{D} \setminus \{p\}$ such that G is harmonic and positive in $D \setminus \{p\}$, $G \equiv 0$ on $\hat{\partial} D$, and the limit as $z \to p$ of $G(z) + \ln|z-p|/(2\pi)$ exists. This G is called the Green function in D with the pole at p.

Suppose $w \in \hat{\partial} D$ is a prime end. There is a continuous function M defined on $\widehat{D} \setminus \{w\}$ such that M is harmonic and positive in D and $M \equiv 0$ on $\widehat{D} \setminus \{w\}$. This M is called a minimal function in D with the pole at p. The name comes from the fact that if any other positive harmonic function f in D is bounded by M then $f = cM$ for some $c \in (0, 1]$. If M is a minimal function in D with the pole at w, then any minimal function in D with the pole at w is equal to cM for some $c \in (0, \infty)$.

There are various ways to normalize a minimal function in D with the pole at $w \in \hat{\partial} D$. Suppose W is a conformal map from a neighborhood U of w in $\widehat{D}$ into $\overline{\mathbb{H}}$ such that $W(U \cap \hat{\partial} D) \subset \mathbb{R}$. Then there is a unique minimal function M in D with the pole at p such that $M \circ W^{-1}(z) + \operatorname{Im} 1/(z - W(p)) \to 0$ as $z \in W(U)$ and $z \to \mathbb{R}$. We say this M is normalized by W.

Suppose I is an arc of $\hat{\partial} D$, and W maps a neighborhood U of I into $\overline{\mathbb{H}}$ such that $W(U \cap \hat{\partial} D) \subset \mathbb{R}$. For each $w \in I$, let M_w be the minimal function in D with the pole at w normalized by W. Let H_I be the harmonic measure function in D of I. Then we have $\pi H_I(z) = \int_\alpha M_w(z) ds \circ W(w)$ for any $z \in D$. Thus if $p_0 \in \alpha$, and I_n is a descending sequence of subarcs of I such that $x_0 = \cap I_n$, then $M_{p_0}(z) = \lim_{n \to \infty} \pi H_{I_n}(z)/|W(I_n)|$, where H_{I_n} is the harmonic measure function in D of I_n.

We may also relate minimal functions with Green functions. For $q \in D$, let G_q be the Green function in D with the pole at q. The limit of $G_q(z)/\mathrm{Im}\, W(q)$ as $q \in U$ and $q \to p_0$ is the minimal function in D with the pole at p_0 normalized by W/π.

3.2 Definition

3.2.1 Conformally invariant SDE

Suppose $\{\mathcal{F}_t : t \geq 0\}$ is a filtration, $B_1(t)$ is a $\{\mathcal{F}_t\}$ standard Brownian motion. Suppose T_1 is a $\{\mathcal{F}_t\}$ stopping time, $\xi_1(t)$ is an $\{\mathcal{F}_t : t \geq 0\}$ adapted continuous function defined on $[0, T_1)$, $R_1(t, x)$ is an $\{\mathcal{F}_t : t \geq 0\}$ adapted $\mathrm{C}^{1,2}$ continuous function defined on a neighborhood of $\{(t, \xi(t)) : t \in [0, T_1)\}$, and they satisfy

$$d\xi_1(t) = \sqrt{\kappa} dB_1(t) + (3 - \kappa/2) \frac{\partial_x R_1(t, \xi_1(t))}{R_1(t, \xi_1(t))} dt, \quad \xi_1(0) = 0. \tag{3.2.1}$$

Let K_t^1 and φ_t^1, $0 \leq t < T_1$, be the standard radial LE hulls and maps driven by ξ_1. Let

$$u_1(t) = \int_0^t R_1(s, \xi_1(s))^{-2} ds, \tag{3.2.2}$$

$0 \leq t < T_1$. Let v_1 be the inverse of u_1. Let $M(t) = K^1_{v_1(t)}$, then M is a Loewner chain in $\mathbb{D}$ started from 1.

Now suppose W maps a neighborhood U of 1 in $\overline{\mathbb{D}}$ that contains all $M(t)$ conformally into $\overline{\mathbb{D}} \setminus \{0\}$ such that $W(U \cap \partial\mathbb{D}) \subset \partial\mathbb{D}$ and $W(1) = 1$. Let $v_2(t) := C_{\mathbb{D};0}(W(M(t)))$, and $u_2(t)$ be the inverse of $v_2(t)$. Let $K_t^2 = W(M(u_2(t)))$. Then K_t^2 is a family of standard radial LE hulls. Let φ_t^2 and ξ_2 be the corresponding radial LE maps and driving function with $\xi_2(0) = 0$. Note that $W_t := \varphi^2_{v_2(t)} \circ W \circ \varphi^1_{v_1(t)}$ maps a neighborhood of $e^{i\xi_1(v_1(t))}$ in $\overline{\mathbb{D}}$ conformally onto a neighborhood of $e^{i\xi_2(v_2(t))}$ in $\overline{\mathbb{D}}$ such that $W_t(e^{i\xi_1(v_1(t))}) = e^{i\xi_2(v_2(t))}$. We may choose f_t such that $W_t \circ e^i = e^i \circ f_t$ and $f_t(\xi_1(v_1(t))) = \xi_2(v_2(t))$.

Theorem 3.2.1 *We have* $u_2(t) = \int_0^t R_2(s, \xi_2(s))^{-2} ds$, *and*

$$d\xi_2(t) = \sqrt{\kappa} dB_2(t) + (3 - \kappa/2) \frac{\partial_x R_2(t, \xi_2(t))}{R_2(t, \xi_2(t))} dt, \tag{3.2.3}$$

where $B_2(t)$ *is a* $\{\mathcal{F}_{u_2 \circ v_1(t)}\}$ *standard Brownian motion, and* R_2 *is defined by*

$$R_2(v_2(t), f_t(x)) = R_1(v_1(t), x)/\partial_x f_t(x). \tag{3.2.4}$$

Proof. Note that $v_1'(t) = R_1(v_1(t), \xi_1(v_1(t)))^{-2}$, we have

$$d\xi_1(v_1(t)) = \frac{\sqrt{\kappa} d\widetilde{B}_1(t)}{R_1(v_1(t), \xi_1(v_1(t)))} + (3 - \frac{\kappa}{2}) \frac{\partial_x R_1(v_1(t), \xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^3} dt,$$

where $\widetilde{B}_1(t)$ is a $\{\mathcal{F}_{v_1(t)}\}$ standard Brownian motion. Similarly as before, we can prove that $v_2'(t)/v_1'(t) = \partial_x f_t(\xi_1(v_1(t)))^2$, and

$$\partial_t f_t(\xi_1(v_1(t))) = \frac{-3\partial_x^2 f_t(\xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^2}.$$

Since $f_t(\xi_1(v_1(t))) = \xi_2(v_2(t))$, from Ito's formula, we have

$$d\xi_2(v_2(t)) = \frac{\partial_x f_t(\xi_1(v_1(t)))\sqrt{\kappa} d\widetilde{B}_1(t)}{R_1(v_1(t), \xi_1(v_1(t)))} + (3 - \frac{\kappa}{2}) \cdot$$

$$[\partial_x f_t(\xi_1(v_1(t))) \frac{\partial_x R_1(v_1(t), \xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^3} - \frac{\partial_x^2 f_t(\xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^2}] dt.$$

From (3.2.4), we have

$$\partial_x R_2(v_2(t), f_t(x)) \partial_x f_t(x) = \partial_x R_1(v_1(t), x)/\partial_x f_t(x) - R_1(v_1(t), x) \partial_x^2 f_t(x)/\partial_x f_t(x).$$

Let $x = \xi_1(v_1(t))$, then $f_t(x) = \xi_2(v_2(t))$. Then we compute

$$\frac{\partial_x R_2(v_2(t), \xi_2(v_2(t)))}{R_2(v_2(t), \xi_2(v_2(t)))^3} = \partial_x f_t(\xi_1(v_1(t))) \frac{\partial_x R_1(v_1(t), \xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^3} - \frac{\partial_x^2 f_t(\xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^2}.$$

Thus

$$d\xi_2(v_2(t)) = \frac{\sqrt{\kappa}d\widetilde{B}_1(t)}{R_2(v_2(t),\xi_2(v_2(t)))} + (3-\frac{\kappa}{2})\frac{\partial_x R_2(v_2(t),\xi_2(v_2(t)))}{R_2(v_2(t),\xi_2(v_2(t)))^3}dt.$$

Since

$$v_2'(t) = v_1'(t)\partial_x f_t(\xi_1(v_1(t)))^2 = \frac{\partial_x f_t(\xi_1(v_1(t)))^2}{R_1(v_1(t),\xi_1(v_1(t)))^2} = R_2(v_2(t),\xi_2(v_2(t)))^{-2},$$

we have

$$d\xi_2(t) = \sqrt{\kappa}dB_2(t) + (3-\frac{\kappa}{2})\frac{\partial_x R_2(t,\xi_2(t))}{R_2(t,\xi_2(t))}dt$$

for some $\{\mathcal{F}_{u_2\circ v_1(t)}\}$ standard Brownian motion $B_2(t)$. $\square$

Now suppose D is a finite Riemann surface, α is a side of D. Fix $p_0\in D$. Let Ω be a neighborhood of $\partial\mathbb{D}$ in $\mathbb{D}\setminus\{0\}$, Σ a neighborhood of α in $D\setminus\{p_0\}$, and W map Ω conformally onto Σ such that $W(z)\to\alpha$ iff $z\in\Omega$ and $z\to\partial\mathbb{D}$. Given $\xi\in C[0,a)$, let φ_t^ξ and K_t^ξ, $0\le t<a$, denote the standard radial LE maps and hulls driven by ξ. If $K_t^\xi\in\Omega$, then $W(K_t^\xi)$ is a hull in D on α. So $D\setminus W(K_t^\xi)$ is still a finite Riemann surface which contains p_0. Let $P_{t,W}^\xi$ denote the Green function in $D\setminus W(K_t^\xi)$ with the pole at p_0. As $z\in\varphi_t^\xi(\Omega\setminus K_t^\xi)$ and $z\to\partial\mathbb{D}$, we have $P_{t,W}^\xi\circ W\circ(\varphi_t^\xi)^{-1}(z)\to 0$. By reflection principle, $P_{t,W}^\xi\circ W\circ(\varphi_t^\xi)^{-1}$ extends harmonicly across $\partial\mathbb{D}$. Let $\lambda\in\mathbb{R}$ and $A\in C[0,\infty)$, we have the following theorem. The proof will be postponed to the next section.

Theorem 3.2.2 *The solution of the equation*

$$\xi(t) = A(t) + \lambda\int_0^t(\partial_x\partial_y/\partial_y)(P_{s,W}^\xi\circ W\circ(\varphi_s^\xi)^{-1}\circ e^i)(\xi(s))ds. \tag{3.2.5}$$

exists uniquely. Suppose $[0,T(\xi))$ *is the maximum definition interval of* ξ. *Then* $\cup_{0\le t<T}K_t^\xi$ *intersects every Jordan curve* J *in* Ω *that together with* $\partial\mathbb{D}$ *bounds a doubly connected domain contained in* Ω. *For any* $a\ge 0$, *let* S_a *be the set of* $A\in C[0,\infty)$ *such that* $T(\xi)>a$. *Then* S_a *is an open subset in the semi-norm* $\|\cdot\|_a$. *And the maps* $S_a\ni A\mapsto\xi|_{[0,a]}$ *and* $S_a\ni A\mapsto\partial_y(P_{t,W}^\xi\circ W\circ(\varphi_t^\xi)^{-1}\circ e^i)(\xi(t))|_{t\in[0,a]}$ *are* $(\|\cdot\|_a,\|\cdot\|_a)$

continuous.

For a fixed $\kappa \geq 0$, let $\lambda = 3 - \kappa/2$ and $A(t) = \sqrt{\kappa}B(t)$, where $B(t)$ is as usual the standard Brownian motion. Suppose $w_0 \in \alpha$ is a prime end. We choose W such that $W(1) = w_0$. Let

$$u(t) := \int_0^t (\partial_y(P^{\xi}_{s,W} \circ W \circ (\varphi^{\xi}_s)^{-1} \circ e^i)(\xi(s)))^{-2} ds.$$

Let v be the inverse of u. Define $L(t) := W(K^{\xi}_{v(t)})$. From Theorem 3.2.2, the map $A \mapsto L$ is measurable. So the law of $A(t)$ which is a Brownian motion gives a low of L. The law of L is called the local HRLC$_\kappa$ in D from w_0 to p_0, or HRLC$_\kappa(D; w_0 \to p_0)$ in the chart (Σ, W). In fact, From the next Corollary we see that the law does not depend on W, so we may omit W.

Corollary 3.2.1 *Suppose L_1 and L_2 have the laws of local HRLC$_\kappa(D; w_0 \to p_0)$ in the (Σ_1, W_1) and (Σ_2, W_2), respectively. For $j = 1, 2$, let S_j be the first t such that $L_j(t) \notin \Omega_{3-j}$; or $\Delta(L_j)$ if such t does not exist. Then L_1 restricted to $[0, S_1)$ has the same law as L_2 restricted to $[0, S_2)$.*

Proof. Let $\Sigma = \Sigma_1 \cap \Sigma_2$, then for $j = 1, 2$, L_j restricted to $[0, S_j)$ has the law of local HRLC$_\kappa(D; w_0 \to p_0)$ in the chart (Σ, W_j). So it suffices to show that if $\Sigma_1 = \Sigma_2$, then the law of L_1 is the same as that of L_2. Let $\Sigma := \Sigma_1$, $\Omega_j = W_j^{-1}(\Sigma_j)$, $j = 1, 2$, and $U := W_2^{-1} \circ W_1$. Then U maps $(\Omega_1; 1)$ conformally onto $(\Omega_2; 1)$ and $U(\Omega_1 \cap \partial\mathbb{D}) \subset \partial\mathbb{D}$.

From the definition, there is a standard Brownian motion $B_1(t)$, a random continuous function ξ_1 that satisfies

$$\xi_1(t) = \sqrt{\kappa}B_1(t) + (3 - \kappa/2) \int_0^t (\partial_x \partial_y / \partial_y)(P^{\xi_1}_{s,W_1} \circ W_1 \circ (\varphi^{\xi_1}_s)^{-1} \circ e^i)(\xi_1(s)) ds. \quad (3.2.6)$$

And $L_1(t) = W_1(K^{\xi_1}_{v_1(t)})$, where v_1 is the inverse of u_1, and

$$u_1(t) = \int_0^t \partial_y(P^{\xi_1}_{s,W_1} \circ W_1 \circ (\varphi^{\xi_1}_s)^{-1} \circ e^i)(\xi_1(s))^{-2} ds.$$

Let $\{\mathcal{F}_t\}$ be the filtration generated by $B_1(t)$, $M(t) := K^{\xi_1}_{v_1(t)} = W_1^{-1}(L_1(t))$, and

$$R_1(t,x) := \partial_y(P^{\xi_1}_{t,W_1} \circ W_1 \circ (\varphi^{\xi_1}_t)^{-1} \circ e^i)(x).$$

Then B_1 is an $\{\mathcal{F}_t\}$ standard Brownian motion. From Theorem 3.2.2, ξ_1 and R_1 are all $\{\mathcal{F}_t\}$ adapted, and formulas (3.2.2) and (3.2.1) are satisfied. Let $v_2(t) := C_{\mathbb{D};0}(U(M(t)))$, and $u_2(t)$ be the inverse of $v_2(t)$. Let $K_t := U(M(u_2(t)))$. Then K_t is a family of standard radial LE hulls driven by some ξ_2. Let $U_t := \varphi^{\xi_2}_{v_2(t)} \circ W \circ \varphi^{\xi_1}_{v_1(t)}$. Then U_t maps a neighborhood of $e^{i\xi_1(v_1(t))}$ in $\overline{\mathbb{D}}$ conformally onto a neighborhood of $e^{i\xi_2(v_2(t))}$ in $\overline{\mathbb{D}}$ such that $U_t(e^{i\xi_1(v_1(t))}) = e^{i\xi_2(v_2(t))}$. We may choose f_t such that $U_t \circ e^i = e^i \circ f_t$ and $f_t(\xi_1(v_1(t))) = \xi_2(v_2(t))$.

From Theorem 3.2.1, We have $u_2(t) = \int_0^t R_2(s,\xi_2(s))^{-2} ds$, and

$$d\xi_2(t) = \sqrt{\kappa} dB_2(t) + (3-\kappa/2)\frac{\partial_x R_2(t,\xi_2(t))}{R_2(t,\xi_2(t))} dt,$$

where B_2 is a standard Brownian motion $B_2(t)$, and R_2 is defined by

$$R_2(v_2(t), f_t(x)) = R_1(v_1(t),x)/\partial_x f_t(x).$$

After a simple computation, we have

$$R_2(t,x) = \partial_y(P^{\xi_2}_{t,W_2} \circ W_2 \circ (\varphi^{\xi_2}_t)^{-1} \circ e^i)(x).$$

Thus equation (3.2.6) holds with the subscript 1 replaced by 2. This implies that $L_1(t) = W_2(K^{\xi_2}_{v_2(t)})$ has the same law as L_2. $\square$

There exists a sequence of neighborhoods $\{\Sigma_n\}$ of w_0 in $\widehat{\mathbb{D}} \setminus \{p_0\}$ that satisfy (i) for each n, there is a conformal map W_n from a neighborhood Ω_n of 1 in $\overline{\mathbb{D}} \setminus \{0\}$ onto Σ_n such that $W_n(1) = w_0$ and $W_n(\Omega_n \cap \partial\mathbb{D}) \subset \hat{\partial} D$; and (ii) if K is any hull in D on the side that contains w_0 and $p_0 \notin K$, then there is at least one Σ_n that contains K. From Corollary 3.2.1, there is a unique measure μ on the space of Loewner chains L in D started from w_0 avoiding p_0, i.e., $p_0 \notin L(t)$ for all $0 \le t < \Delta(L)$, such that if L

has the law μ then L restricted to any chart Σ is a local $\mathrm{HRLC}_\kappa(D;w_0\to p_0)$ in Σ. This μ is called the (global) $\mathrm{HRLC}_\kappa(D;w_0\to p_0)$.

There are some other possible definitions of $P^\xi_{t,W}$ in (3.2.5) such that Theorem 3.2.2 and Corollary 3.2.1 hold with no changes or minor changes. For example, Let I be a side arc of D such that $w_0\not\in\overline{I}$. We may define $P^\xi_{t,W}$ to be the harmonic measure function of I in $D\backslash W(K^\xi_t)$ as long as there is a neighborhood of $\overline{I}$ in ∂D that does not intersect $W(K^\xi_t)$. Then the law of L defined before Corollary 3.2.1 is called the local $\mathrm{HRLC}_\kappa(D;w_0\to I)$ in Σ. Let w_1 be a prime end of D other than w_0, and J maps a neighborhood of w_1 conformally onto a neighborhood of 0 in $\mathbb{H}$ such that $J(w_1)=0$. If there is a neighborhood of w_1 in $\widehat{D}$ that does not intersect $W(K^\xi_t)$, then We may define $P^\xi_{t,W}$ to be the minimal function in $D\setminus W(K^\xi_t)$ with the pole w_1, normalized by J. Then the law of L is called the local $\mathrm{HRLC}_\kappa(D;w_0\to w_1)$ in Σ, normalized by J. In fact, two local $\mathrm{HRLC}_\kappa(D;w_0\to w_1)$ in the same chart differ only by a constant speed time-change. We may also define the (global) $\mathrm{HRLC}_\kappa(D;w_0\to I)$ or $\mathrm{HRLC}_\kappa(D;w_0\to w_1)$ normalized by J, similarly as the case that the target is a prime end.

Proposition 3.2.1 *Suppose (K^r_t) has the law of the radial $SLE_\kappa(D;w_0\to p_0)$. Then $t\mapsto K^r_{t/(4\pi^2)}$ has the law of the $HRLC_\kappa(D;w_0\to p_0)$. Suppose (K^c_t) has the law of a chordal $SLE_\kappa(D;w_1\to w_2)$. Then $t\mapsto K^c_t$ has the law of an $HRLC_\kappa(D;w_1\to w_2)$. Suppose (K^s_t) has the law of the strip $SLE_\kappa(D;w_0\to I)$. Then $t\mapsto K^s_{t/\pi^2}$ has the law of the $HRLC_\kappa(D;w_0\to I)$. Suppose (K^a_t) has the law of the annulus $SLE_\kappa(D;w_0\to S)$. Then $t\mapsto K_{p-\sqrt[3]{p^3-3t}}$ has the law of the $HRLC_\kappa(D;w_0\to S)$.*

Proof. The proof is similar as that of Theorem 3.2.1. □

3.2.2 Existence and uniqueness

The goal of this subsection is to prove Theorem 3.2.2. We need the following lemmas.

Lemma 3.2.1 *For any $a>0$ and compact subset F of $\mathbb{D}$, there are $\delta, C>0$ depending on a and F such that if $t\in[0,a]$ and $\zeta,\eta\in C[0,t]$ satisfy $\|\zeta-\eta\|_t:=$*

$\sup_{0\le s\le t}|\zeta(s)-\eta(s)|<\delta$, *then for any* $z\in F$, $\varphi_t^\zeta\circ(\varphi_t^\eta)^{-1}(z)$ *is well defined, and*

$$|z-\varphi_t^\zeta\circ(\varphi_t^\eta)^{-1}(z)|\le Ct\|\zeta-\eta\|_t,\ \ \forall z\in F,$$

where φ_t^η *and* φ_t^ζ *are standard radial LE maps driven by* η *and* ζ, *respectively.*

Proof. Choose $\varepsilon\in(0,1)$ such that $F\subset(1-\varepsilon)\mathbb{D}$. Let $d>0$ be the distance between F and $(1-\varepsilon)\partial\mathbb{D}$. There is $C_\varepsilon>0$ such that for $\chi_1,\chi_2\in\partial\mathbb{D}$ and $x_1,x_2\in(1-\varepsilon)\overline{\mathbb{D}}$,

$$\left|x_1\frac{\chi_1+x_1}{\chi_1-x_1}-x_2\frac{\chi_2+x_2}{\chi_2-x_2}\right|\le C_\varepsilon(|\chi_1-\chi_2|+|x_1-x_2|). \tag{3.2.7}$$

Fix $z\in F$ and $t\in[0,a]$. Let s_0 be the first $s>0$ that is equal to t or

$$f(s):=|\varphi_s^\eta\circ(\varphi_t^\eta)^{-1}(z)-\varphi_s^\zeta\circ(\varphi_t^\eta)^{-1}(z)|$$

is equal to d. Then for $0\le s\le s_0$,

$$|\varphi_s^\eta\circ(\varphi_t^\eta)^{-1}(z)|\le|\varphi_t^\eta\circ(\varphi_t^\eta)^{-1}(z)|=|z|.$$

From the definition of d, we see that for $0\le s\le s_0$, $\varphi_s^\eta\circ(\varphi_t^\eta)^{-1}(z),\varphi_s^\zeta\circ(\varphi_t^\eta)^{-1}(z)\in(1-\varepsilon)\overline{\mathbb{D}}$. So $\varphi_s^\zeta\circ(\varphi_t^\eta)^{-1}(z)$ is well defined. Since $\varphi_0^\eta=\varphi_0^\zeta=\mathrm{id}$, $f(0)=0$. From (2.2.1) and (3.2.7), we conclude that

$$f(s)\le C_\varepsilon(\|\zeta-\eta\|_t s+\int_0^s f(r)dr),\ \ 0\le s\le s_0.$$

From a standard argument of differential equations, we get

$$f(s_0)\le 3(e^{s_0C\varepsilon}-1)\|\zeta-\eta\|_t\le Cs_0\|\zeta-\eta\|_t, \tag{3.2.8}$$

where $C:=3(e^{C\varepsilon a}-1)/a$. Let $\delta=d/(Ca)$. If $\|\zeta-\eta\|_t<\delta$, then $f(s_0)<d$, so $s_0=t$. This ends the proof. □

Lemma 3.2.2 *Suppose* $a>0$ *and* β *is a Jordan curve in* Ω *such that the doubly* *connected domain bounded by β and $\partial\mathbb{D}$ is contained in Ω. There are $\delta, \varepsilon, C > 0$ depending on D, p_0, W, Ω, β and a, such that if $t \in (0,a]$ and $\zeta, \eta \in C[0,t]$ satisfy $\|\zeta - \eta\|_t < \delta$ and $K_a^\zeta \cap \beta = \emptyset$, then for any $z \in \mathbb{D}$ with $|z| \geq 1 - \varepsilon$, we have $(\varphi_t^\zeta)^{-1}(z), (\varphi_t^\eta)^{-1}(z) \in \Omega$, and*

$$|P_{t,W}^\zeta \circ W \circ (\varphi_t^\zeta)^{-1}(z) - P_{t,W}^\eta \circ W \circ (\varphi_t^\eta)^{-1}(z)| \leq Ct\|\zeta - \eta\|_t.$$

Proof. We can find another Jordan curve γ in Ω disjoint from β such that the doubly connected domain bounded by γ and $\partial\mathbb{D}$ is contained in Ω and contains β. Let m be the modulus of the doubly connected domain bounded by β and γ. By conformal invariance and the comparison principle of moduli, the modulus of the doubly connected domain bounded by $\varphi_t^\zeta(\gamma)$ and $\partial\mathbb{D}$ is at least m. So there is $\varepsilon_0 = \varepsilon_0(m) > 0$ such that $\varphi_t^\zeta(\gamma) \subset (1-\varepsilon_0)\mathbb{D}$. Let $J_k = (1 - \varepsilon_0/k)\partial\mathbb{D}$, $k = 1, 2, 3$, and $J_4 = \sqrt{1 - \varepsilon_0/3}\partial\mathbb{D}$.

Let $F = \sqrt{1 - \varepsilon_0/3}\overline{\mathbb{D}}$. From Lemma 3.2.1, we have $C_1, \delta_1 > 0$ depending on a and F such that if $\|\zeta - \eta\|_t < \delta_1$, then for all $z \in F$, $\varphi_t^\zeta \circ (\varphi_t^\eta)^{-1}(z)$, $\varphi_t^\eta \circ (\varphi_t^\zeta)^{-1}(z)$ exist, and

$$|z - \varphi_t^\zeta \circ (\varphi_t^\eta)^{-1}(z)|, |z - \varphi_t^\eta \circ (\varphi_t^\zeta)^{-1}(z)| \leq C_1 t\|\zeta - \eta\|_t. \tag{3.2.9}$$

Since J_1 lies between $\varphi_t^\zeta(\gamma)$ and $\partial\mathbb{D}$, so does J_3. Thus $(\varphi_t^\zeta)^{-1}(J_3)$ lies between γ and $\partial\mathbb{D}$. So for any $z \in \mathbb{D}$ such that $|z| \geq 1 - \varepsilon_0/3$, $(\varphi_t^\zeta)^{-1}(z)$ lies in the domain bounded by γ and $\partial\mathbb{D}$, which is contained in Ω. If $\|\zeta - \eta\|_t < \delta = \min\{\delta_1, \varepsilon_0/(6C_1 a)\}$, from $\varphi_t^\zeta(\gamma) \subset F$ and (3.2.9) we have $|\varphi_t^\eta(z) - \varphi_t^\zeta(z)| \leq C_1 t\|\zeta - \eta\|_t < \varepsilon_0/6$ for all $z \in \gamma$. Note $\varphi_t^\zeta(\gamma) \subset (1-\varepsilon_0)\mathbb{D}$. Thus $\varphi_t^\eta(\gamma) \subset (1 - \varepsilon_0/2)\mathbb{D}$. This means that J_2 lies between $\varphi_t^\eta(\gamma)$ and $\partial\mathbb{D}$. Similarly, we have $(\varphi_t^\eta)^{-1}(z)$ lies in Ω, for any $z \in \mathbb{D}$ such that $|z| \geq 1 - \varepsilon_0/3$. Thus $P_{t,W}^\zeta \circ W \circ (\varphi_t^\zeta)^{-1}$ and $P_{t,W}^\eta \circ W \circ (\varphi_t^\eta)^{-1}$ are well defined on a domain that contains $\{z \in \mathbb{D} : |z| \geq 1 - \varepsilon_0/3\}$ when $\|\zeta - \eta\|_t < \delta$. It is clear that they are harmonic, and they have vanished continuation at $\partial\mathbb{D}$. Now suppose

$\|\zeta-\eta\|_t<\delta$, define

$$M=\sup_{z\in J_3}|P^\eta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)-P^\zeta_{t,W}\circ W\circ(\varphi^\zeta_t)^{-1}(z)|;$$

$$N=\sup_{z\in J_4}|P^\eta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)-P^\zeta_{t,W}\circ W\circ(\varphi^\zeta_t)^{-1}(z)|.$$

Since a plane Brownian motion started from $z\in J_4$ has probability $1/2$ to hit J_3 before $\partial\mathbb{D}$, so $N_t\le M_t/2$.

Note that $P^\eta_{t,W}-P^\zeta_{t,W}$ is harmonic in $D\setminus W(K^\zeta_t\cup K^\eta_t)$ including the pole p_0, and has vanished continuation on the sides of D other than α which contains w_0. It is clear that $(\varphi^\eta_t)^{-1}(J_3)$ and $(\varphi^\eta_t)^{-1}(J_4)$ are disjoint from K^η_t. Since $J_3,J_4\subset F$, and $\|\zeta-\eta\|_t<\delta\le\delta_1$, so $\varphi^\zeta\circ(\varphi^\eta_t)^{-1}(J_3)$ and $\varphi^\zeta\circ(\varphi^\eta_t)^{-1}(J_4)$ are defined. Thus $(\varphi^\eta_t)^{-1}(J_3)$ and $(\varphi^\eta_t)^{-1}(J_4)$ are disjoint from K^ζ_t. Since $J_3,J_4\subset\{z\in\mathbb{D}:|z|\ge 1-\varepsilon_0/3\}$, so from the last paragraph, $(\varphi^\eta_t)^{-1}(J_3)$ and $(\varphi^\eta_t)^{-1}(J_4)$ lie in Ω. Thus $W\circ(\varphi^\eta_t)^{-1}(J_3)$ and $W\circ(\varphi^\eta_t)^{-1}(J_4)$ are two Jordan curves in D, and the latter disconnects the former from $W(K^\zeta_t))\cup W(K^\eta_t)$ and the side that w_0 lies on. Now define

$$M'=\sup_{z\in J_3}|P^\eta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)-P^\zeta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)|;$$

$$N'=\sup_{z\in J_4}|P^\eta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)-P^\zeta_{t,W}\circ W\circ(\varphi^\eta_t)^{-1}(z)|.$$

From the maximal principle, we have $M'\le N'$.

Let A_1 be the closure of the domain bounded by γ and J_1. Since $|(\varphi^\zeta_t)^{-1}(z)|<|z|$, and J_1 lies between $\varphi^\zeta_t(\gamma)$ and $\partial\mathbb{D}$, so $(\varphi^\zeta_t)^{-1}(J_1)\subset A_1$. It is clear that $P^\zeta_{t,W}\le P_0$, the Green function in D with the pole at p_0. So $P^\zeta_{t,W}\circ W\circ(\varphi^\xi_t)^{-1}$ on J_1 is bounded by the maximum of $P_0\circ W$ on A_1. Let A_2 be the domain bounded by J_2 and $\partial\mathbb{D}$. Since $P^\zeta_{t,W}\circ W\circ(\varphi^\xi_t)^{-1}$ vanishes on $\partial\mathbb{D}$, by reflection principle and Harnack principle, the gradient of $P^\zeta_{t,W}\circ W\circ(\varphi^\zeta_t)^{-1}$ on A_2 is bounded by some $C_2>0$ depending on D, p, W, A_1 and ε_0. Since $J_3,J_4\subset F$, so by (3.2.9), if $\|\zeta-\eta\|_t<\delta$, then

$$|z-\varphi^\zeta_t\circ(\varphi^\eta_t)^{-1}(z)|<C_1t\delta<\varepsilon_0/6,\ \ \forall z\in J_3\cup J_4.$$

It follows that the line segment $[z, \varphi_t^\zeta \circ (\varphi_t^\eta)^{-1}(z)]$ is contained in A_2, for $z \in J_3 \cup J_4$.

We now have

$$|M - M'| \le \sup_{z \in J_3} |P_{t,W}^\zeta \circ W \circ (\varphi_t^\zeta)^{-1}(z) - P_{t,W}^\zeta \circ W \circ (\varphi_t^\eta)^{-1}(z)|.$$

$$\le \sup_{w \in A_2} |\nabla(P_{t,W}^\zeta \circ W \circ (\varphi_t^\zeta)^{-1})(w)| \cdot \sup_{z \in J_3} |z - \varphi_t^\zeta \circ (\varphi_t^\eta)^{-1}(z)| \le C_1 C_2 t \|\zeta - \eta\|_t.$$

Similarly, $|N - N'| \le C_1 C_2 t \|\zeta - \eta\|_t$. Thus

$$M \le M' + C_1 C_1 t \|\zeta - \eta\|_t \le N' + C_1 C_2 t \|\zeta - \eta\|_t$$

$$\le N + 2C_1 C_2 t \|\zeta - \eta\|_t \le M/2 + 2C_1 C_2 t \|\zeta - \eta\|_t,$$

which implies that $M \le 4C_1 C_2 t \|\zeta - \eta\|_t$. Let $C = 4C_1 C_2$ and $\varepsilon = \varepsilon_0/3$. The proof of this lemma is completed by the maximal principle. □

Lemma 3.2.3 *Let a and β be as in Lemma 3.2.1. Then there are $\delta, C > 0$ depending on D, p_0, W, Ω, β and a, such that if $t \in (0, a]$ and $\zeta, \eta \in C[0,t]$ satisfy $\|\zeta - \eta\|_t < \delta$ and $K_a^\zeta \cap \beta = \emptyset$, then*

$$\begin{aligned} &|(\partial_x \partial_y/\partial_y)(P_{t,W}^\zeta \circ W \circ (\varphi_t^\zeta)^{-1} \circ e^i)(\zeta(t)) \\ -&(\partial_x \partial_y/\partial_y)(P_{t,W}^\eta \circ W \circ (\varphi_t^\eta)^{-1} \circ e^i)(\eta(t))| \quad \le \quad C(1+t)\|\zeta - \eta\|_t. \end{aligned} \tag{3.2.10}$$

Proof. Let ε_0, J_k, $k = 1, \ldots, 4$, and A_l, $l = 1, 2$, be defined as in the proof of Lemma 3.2.2. In that proof we see that for all $\xi \in C[0,t]$, $0 < t \le a$, $P_{t,W}^\xi \circ W \circ (\varphi_t^\xi)^{-1}$ on J_1 is uniformly bounded. By reflection principle and Harnack principle, this implies that $\partial_x^k \partial_y (P_{t,W}^\xi \circ W \circ (\varphi_t^\xi)^{-1} \circ e^i)$, $k = 0, 1, 2$, are uniformly bounded on $\mathbb{R}$. Let $P(z)$ denote the Green function with the pole at p_0, in the subdomain of D that is bounded by $W(J_2)$ and the sides of D that does not contains w_0. Then $P_{t,W}^\xi(z) \ge P(z) \ge C_0$ on $W(J_1)$, for some $C_0 > 0$. This implies that $|\partial_y(P_{t,W}^\xi \circ W \circ (\varphi_t^\xi)^{-1} \circ e^i)|$ on $\mathbb{R}$ is uniformly bounded from below by $C_0/(-\ln(1-\varepsilon_0))$. Since $\partial_x(\partial_x \partial_y/\partial_y) = \partial_x^2 \partial_y/\partial_y - (\partial_x \partial_y/\partial_y)^2$, we proved that for some $C > 0$,

$$|\partial_x(\partial_x\partial_y/\partial_y)(P^{\zeta}_{t,W} \circ W \circ (\varphi^{\zeta}_t)^{-1} \circ e^i)(z)| \le C. \tag{3.2.11}$$

By Lemma 3.2.2 there are $\delta, C_1 > 0$ such that

$$|P^{\zeta}_{t,W} \circ W \circ (\varphi^{\zeta}_t)^{-1}(z) - P^{\eta}_{t,W} \circ W \circ (\varphi^{\eta}_t)^{-1}(z)| \le C_1 t\|\zeta - \eta\|_a,$$

for $t \in (0, a]$ and $z \in J_3$. By Harnack principle, there is $C_2 > 0$ such that for all $z \in \mathbb{R}$,

$$|\partial_y(P^{\zeta}_{t,W} \circ W \circ (\varphi^{\zeta}_t)^{-1} \circ e^i)(z) - \partial_y(P^{\eta}_{t,W} \circ W \circ (\varphi^{\eta}_t)^{-1} \circ e^i)(z)| \le C_2 t\|\zeta - \eta\|_t,$$

$$|\partial_x\partial_y(P^{\zeta}_{t,W} \circ W \circ (\varphi^{\zeta}_t)^{-1} \circ e^i)(z) - \partial_x\partial_y(P^{\eta}_{t,W} \circ W \circ (\varphi^{\eta}_t)^{-1} \circ e^i)(z)| \le C_2 t\|\zeta - \eta\|_t.$$

From the first part of the proof, ∂_y and $\partial_x\partial_y$ are uniformly bounded, and ∂_y is uniformly bounded from zero, so we have

$$\begin{aligned} &|(\partial_x\partial_y/\partial_y)(P^{\zeta}_{t,W} \circ W \circ (\varphi^{\zeta}_t)^{-1} \circ e^i)(z) \\ &-(\partial_x\partial_y/\partial_y)(P^{\eta}_{t,W} \circ W \circ (\varphi^{\eta}_t)^{-1} \circ e^i)(z)| \le Ct\|\zeta - \eta\|_t. \end{aligned} \tag{3.2.12}$$

The proof is completed by combining (3.2.11) and (3.2.12). □

Lemma 3.2.4 *There are $a, C > 0$ such that for $z \in \mathbb{R}$, $t \in (0, a]$, and $\zeta, \eta \in C[0, t]$, the inequality (3.2.10) holds.*

Proof. There is a Jordan curve β in Ω such that the doubly connected domain bounded by β and $\partial\mathbb{D}$ is contained in Ω. We can find $a > 0$ such that any hull in $\mathbb{D}$ of capacity w.r.t. 0 less a is disjoint from β. Thus for any $t \in (0, a]$ and $\xi \in C[0, t]$, $K^{\xi}_t \cap \beta = \emptyset$. From Lemma 3.2.3, the inequality (3.2.10) holds when $\zeta, \eta \in C[0, t]$ with $\|\zeta - \eta\|_t < \delta$ for some $\delta > 0$. The condition $\|\zeta - \eta\|_t < \delta$ can be dropped because we can choose $\xi_0 = \zeta, \xi_1, \ldots, \xi_n = \eta$ in $C[0, t]$ such that $\|\xi_{j-1} - \xi_j\|_t < \delta$ and $\sum_1^n \|\xi_{j-1} - \xi_j\|_t = \|\zeta - \eta\|_t$. □

Proof of Theorem 3.2.2. Let $\xi_0 = A$. Define ξ_n inductively:

$$\xi_{n+1}(t) = A(t) + \lambda \int_0^t (\partial_x \partial_y / \partial_y)(P_{s,W}^{\xi_n} \circ W \circ (\varphi_s^{\xi_n})^{-1} \circ e^i)(\xi_n(s))ds. \tag{3.2.13}$$

Compare $P_{t+\varepsilon,W}^{\xi} \circ W \circ (\varphi_{t+\varepsilon}^{\xi})^{-1}$ with $P_{t,W}^{\xi} \circ W \circ (\varphi_t^{\xi})^{-1}$ in the similar way as we compare $P_{t,W}^{\zeta} \circ W \circ (\varphi_t^{\zeta})^{-1}$ with $P_{t,W}^{\eta} \circ W \circ (\varphi_t^{\eta})^{-1}$, and we find that $(\partial_x \partial_y / \partial_y)(P_{s,W}^{\xi} \circ W \circ (\varphi_s^{\xi})^{-1} \circ e^i)(\xi_n(t))$ is continuous in t. Thus ξ_n are well defined and continuous.

Choose $\varepsilon_0 > 0$ such that $(1-\varepsilon)\partial\mathbb{D} \subset \Omega$ for $\varepsilon \in (0, \varepsilon_0]$. Let a and C be in Lemma 3.2.4 depending on ε_0. We may chose a small enough such that Lemma 3.2.4 implies $\|\xi_{n+2} - \xi_{n+1}\|_a \le \|\xi_{n+1} - \xi_n\|_a/2$. Thus $\{\xi_n\}$ is a Cauchy sequence in $C[0,a]$. Let $\xi \in C[0,a]$ be the limit of $\{\xi_n\}$. Then ξ solves (3.2.5) for $t \in [0,a]$. If there is another solution ξ' on $[0,a]$ of (3.2.5), then a similar argument gives $\|\xi - \xi'\|_a \le \|\xi - \xi'\|_a/2$, which forces that $\xi' = \xi$. Thus the solution of (3.2.5) exists uniquely on $[0,a]$.

Now suppose (3.2.5) has two different solutions ξ_1 and ξ_2. Let $b > 0$ be the biggest t such that $\xi_1(t) = \xi_2(t)$. Let w_b be the prime end of $D_b := D \setminus W(K_b^{\xi_1})$ determined by the Loewner chain $t \mapsto W(K_t^{\xi_1})$ at time b. Then $\Sigma_b := \Sigma \setminus W(K_b^{\xi_1})$ is a neighborhood of w_b in D_b. Moreover, $\Omega_b := \varphi_b^{\xi_1}(\Omega \setminus K_b^{\xi_1})$ is a neighborhood of $e^{i\xi_1(b)}$ in $\mathbb{D}$, and $W_b := W \circ (\varphi_b^{\xi})^{-1}$ maps Ω_b conformally onto Σ_b and can be extended conformally to the boundary such that $W_b(e^{i\xi_1(b)}) = w_b$ and $W_b(\Sigma_b \cap \partial\mathbb{D}) \subset \hat{\partial} D_b$. Let $\widetilde{P}_{t,W_b}^{\zeta}$ be the Green function in $D_b \setminus W_b(K_t^{\zeta})$ with the pole at p_0. For $j = 1, 2$, let $\zeta_j(t) := \xi_j(b+t)$. Then $\varphi_{b+t}^{\xi_j} = \varphi_t^{\zeta_j} \circ \varphi_b^{\xi_j}$, and $K_{b+t}^{\xi_j} = K_b^{\xi_j} \cup (\varphi_b^{\xi})^{-1}(K_t^{\zeta_j})$. Thus $\widetilde{P}_{t,W_b}^{\zeta_j} = P_{b+t,W}^{\xi_j}$. So we have

$$\widetilde{P}_{t,W_b}^{\zeta_j} \circ W_b \circ (\varphi_t^{\zeta_j})^{-1} = P_{b+t,W}^{\xi_j} \circ W \circ (\varphi_{b+t}^{\xi_j})^{-1}.$$

Let $\widetilde{A}(t) = A(b+t) - A(b) + \xi_1(b)$. Then for $j = 1, 2$, ζ_j solves the equation

$$\zeta_j(t) = \widetilde{A}(t) + \lambda \int_0^t (\partial_x \partial_y / \partial_y)(\widetilde{P}_s^{\zeta_j} \circ W_b \circ (\varphi_s^{\zeta_j})^{-1} \circ e^i)(\zeta_j(s))ds. \tag{3.2.14}$$

From the first part of this proof, we see that $\zeta_1(t) = \zeta_2(t)$ for $t \in (0,c)$, for some $c > 0$. Thus $\xi_1(t) = \xi_2(t)$ for $0 \le t \le b+c$, which contradicts the choice of b. Thus the solution of (3.2.5) is unique.

Now let $[0, T)$ be the maximal definition interval of the solution ξ of (3.2.5). Suppose that $\cup_{0 \le t < T} K_t^\xi$ does not intersect some Jordan curve J_1 in Ω such that the doubly connected domain bounded by J_1 and $\partial\mathbb{D}$ is contained in Ω. Then we can find another Jordan curve J_2 that has the above properties that J_1 has, and is disconnected by J_1 from $\partial\mathbb{D}$. Let $m > 0$ be the modulus of the doubly connected domain bounded by J_1 and J_2. Then for any $t \in [0, T)$, the modulus of the doubly connected domain bounded by $\partial\mathbb{D} \cup K_t^\xi$ and J_2 is greater than m. There is $\varepsilon_0 = \varepsilon_0(m) > 0$ such that $\varphi_t^\xi(J_2) \subset (1 - \varepsilon_0)\mathbb{D}$ for all $0 \le t < T$. Let a depending on ε_0 be as in the first part of this proof. Choose $b \in [0, T)$ so that $T < b + a$. Let $\Omega_b := \varphi_t^\xi(\Omega \setminus K_b^\xi)$. Then $(1-\varepsilon)\partial\mathbb{D} \subset \Omega_b$ for $0 < \varepsilon \le \varepsilon_0$. Thus (3.2.14) has a solution ζ on $[0, a]$. Let $\xi'(t) = \xi(t)$ for $0 \le t \le b$, and $\xi'(t) = \zeta(t - b)$ for $b \le t \le b + a$. Then ξ' solves (3.2.5). Note $T < b + a$. By the uniqueness of the solution, $\xi'(t) = \xi(t)$ for $0 \le t < T$. So the solution ξ can be extended to $[0, b + a]$. This contradicts the assumption that $[0, T)$ is the maximal definition interval of the solution.

Suppose ξ_0 is the solution for A_0 and is defined on $[0, a]$. To prove that S_a is open in $\|\cdot\|_a$ and $S_a \ni A \mapsto \xi|_{[0,a]}$ is $(\|\cdot\|_a, \|\cdot\|_a)$ continuous, it suffices to show that if $A_n \to A_0$ in $\|\cdot\|_a$, and ξ_n are solutions for A_n, then ξ_n is defined on $[0, a]$ for n big enough and $\|\xi_n - \xi_0\|_a \to 0$. Let β be a Jordan curve in Ω disjoint from $K_a^{\xi_0}$ such that the doubly connected domain bounded by β and $\partial\mathbb{D}$ is contained in Ω. Let $\delta > 0$ be as in Lemma 3.2.3. If $\|\xi_n - \xi_0\|_t < \delta$ for $t \in (0, a]$, then from Lemma 3.2.3,

$$|\xi_n(s) - \xi_0(s)| \le \|A_n - A_0\|_a + C \int_0^s |\xi_n(r) - \xi_0(r)| dr,$$

for some $C > 0$ and $s \in [0, t]$. This implies that $|\xi_n(s) - \xi_0(s)| \le e^{Cs}\|A_n - A_0\|_a$. If n is big enough, then $\|A_n - A_0\|_a < e^{-Ca}\delta$. Suppose ξ_n is not defined on $[0, a]$, then there is some $t_0 \in [0, a)$ such that $K_{t_0}^{\xi_n} \cap \beta \ne \emptyset$. However, since

$$\|\xi_n - \xi_0\|_{t_0} \le e^{Ct_0}\|A_n - A_0\|_a < \delta,$$

and $K_{t_0}^{\xi_0} \cap \beta = \emptyset$, from Lemma 3.2.1, $K_{t_0}^{\xi_n}$ does not intersect β. The contradiction shows that ξ_n is defined on $[0, a]$. From a similar argument, we conclude that there is no $t \in [0, a]$ such that $|\xi_n(t) - \xi_0(t)| \geq \delta$. It follows that $\|\xi_n - \xi_0\|_a \leq e^{Ca}\|A_n - A_0\|_a$. Thus $\xi_n \to \xi_0$ in $\|\cdot\|_a$. From the proofs of the Lemmas it is clear that $\xi \mapsto \partial_y(P_{t,W}^{\xi} \circ W \circ (\varphi_t^{\xi})^{-1} \circ e^i)(\xi(t))|_{t\in[0,a]}$ is $(\|\cdot\|_a, \|\cdot\|_a)$ continuous on the set of $\xi \in [0, a]$ such that $K_a^{\xi} \subset \Omega$. So it is also true if the first ξ in ghe above sentence is replaced by $A \in S_a$. $\square$

Suppose $P_{t,W}^{\xi}$ is defined to be the harmonic function of some side arc I of D in $D \setminus W(L_t^{\xi})$ such that $w_0 \notin \overline{I}$, or the minimal function in $D \setminus W(L_t^{\xi})$ with the pole at some prime end $w_1 \neq w_0$, normalized by $(J, q, 1)$. If I or w_1 does not lie on α, then Theorem 3.2.2 holds without any change, and the proof is also the same. If I or w_1 lies on α, then Theorem 3.2.2 holds with the change of the long-range behavior of L_t^{ξ}, which is $\cup_{0\leq t<T} L_t^{\xi}$ *either* intersects every Jordan curve β in Ω such that the doubly connected domain bounded by β and $\partial\mathbb{D}$ is contained in Ω, *or* intersects every neighborhood of I or w_1. The proof is a bit more complicated, but the basic idea is the same. And we choose to omit the proof.

3.2.3 Interior HRLC

A harmonic random interior Loewner chain in a finite Riemann surface extends the notations of full plane SLE and disc SLE. It is a law of random interior Loewner chains module time-changes. In another word, it is a measure on the equivalence classes of interior Loewner chains, where two interior Loewner chains are equivalent iff one can be converted into the other through a time-change.

Suppose D is a finite Riemann surface, and $p_0 \neq p_1 \in D$. Let W map a neighborhood of Ω of 0 in $\mathbb{D}$ conformally onto a neighborhood Σ of p_0 in $D \setminus \{p_1\}$ such that $W(0) = p_0$. For any $\xi \in C(-\infty, a)$, let L_t^{ξ} and ψ_t^{ξ}, $0 \leq t < a$, be the standard full plane LE hulls and maps driven by ξ, respectively. Let $Q_{t,W}^{\xi}$ be the Green function in $D \setminus W(L_t^{\xi})$ with the pole at p_1. We then have

Theorem 3.2.3 *For any $A \in C(-\infty,\infty)$ and $\lambda \in \mathbb{R}$, the solution of*

$$\xi(t) = A(t) + \lambda \int_{-\infty}^{t} (\partial_x \partial_y/\partial_y)(Q^{\xi}_{s,W} \circ W \circ (\psi^{\xi}_s)^{-1} \circ e^i)(\xi(s))ds \tag{3.2.15}$$

exists uniquely. Suppose $(-\infty, T(\xi))$ is the maximal definition interval of ξ, then $\cup_{t<T} L^{\xi}_t$ is not contained in any closed Jordan domain in Ω. For any $a \in \mathbb{R}$, let U_a be the set of A such that $T(\xi) > a$. Then $U_a \in \mathcal{F}_a$, which is in the σ-algebra determined by $A(t)$, $t \le a$. Moreover, $U_a \ni A \mapsto \xi|_{(-\infty,a]}$ is $(\mathcal{F}_a, \mathcal{F}_a)$ measurable.

Sketch of the proof. Let $\xi_0 = A$ and define ξ_n inductively:

$$\xi_{n+1}(t) = A(t) + \lambda \int_{-\infty}^{t} (\partial_x \partial_y/\partial_y)(Q^{\xi_n}_{s,W} \circ W \circ (\psi^{\xi_n}_s)^{-1} \circ e^i)(\xi_n(s))ds.$$

Note that $Q^{\xi}_{s,W} \circ W \circ (\psi^{\xi}_s)^{-1}$ is positive and harmonic in $\mathbb{D} \setminus C_1 e^s \overline{\mathbb{D}}$ for some $C_1 > 0$. It follows that

$$|(\partial_x \partial_y/\partial_y)(Q^{\xi}_{s,W} \circ W \circ (\psi^{\xi}_s)^{-1} \circ e^i)(z)| \le C_2 |s| e^s$$

for some $C_2 > 0$ and all $z \in \mathbb{R}$, when $s < 0$ and $|s|$ is big enough. So all ξ_n are well defined on $(-\infty, a)$ for some $a \in \mathbb{R}$, and are continuous. Especially, $\|\xi_1 - \xi_0\|_t \le C_3 |t| e^t$ for $t \in (-\infty, a]$. We may choose $b \le a$ such that for $t \le b$,

$$\begin{aligned} &\lambda|(\partial_x \partial_y/\partial_y)(Q^{\zeta}_{t,W} \circ W \circ (\psi^{\zeta}_t)^{-1} \circ e^i)(\zeta(t)) \\ &-(\partial_x \partial_y/\partial_y)(Q^{\eta}_{t,W} \circ W \circ (\psi^{\eta}_t)^{-1} \circ e^i)(\eta(t))| \quad \le \quad \|\zeta - \eta\|_t/4. \end{aligned} \tag{3.2.16}$$

From $\|\xi_1 - \xi_0\|_t \le C_3|t|e^t$ for $t \in (-\infty, b]$, we have $\|\xi_{n+1} - \xi_n\|_t \le C_3 |t| e^t / 2^n$ for all $n \in \mathbb{N}$ and $t \in (-\infty, b]$. Thus $\{\xi_n\}$ is a Cauchy sequence in $C(-\infty, b]$. The limit ξ solves (3.2.15) on $(-\infty, b]$. If another function ξ' also solves (3.2.15), then ξ' must agree with ξ on $(-\infty, b]$.

To find the value of ξ after b, we define $D_b := D \setminus W(L^{\xi}_b)$ and $W_b := W \circ (\psi^{\xi}_b)^{-1}$. Let $A_b(t) = A(b+t) - A(b) + \xi(b)$. Let φ^{ζ}_t and K^{ζ}_t be the radial LE maps and hulls driven by ξ. Let $\widetilde{P}^{\zeta}_{t,W_b}$ be the Green function in $D_b \setminus W_b(K^{\zeta}_t)$ with the pole at p_1.

From Theorem 3.2.2, the solution of the equation

$$\zeta(t) = A_b(t) + \lambda \int_0^t (\partial_x \partial_y / \partial_y)(\widetilde{P}^{\zeta}_{t,W_b} \circ W_b \circ (\varphi^{\zeta}_t)^{-1} \circ e^i)(\zeta(s))ds$$

exists uniquely. We define $\xi(t) = \zeta(t-b)$ for $t \geq b$. Then this ξ solves (3.2.15). The other parts of this theorem can be proved similarly. $\square$

Now fix $\kappa \geq 0$. Let $A(t) = \xi_\kappa(t)$ defined in Section 2.2.3, and $\lambda = 3 - \kappa/2$. Let ξ be the solution of (3.2.15) and $[0, T)$ the maximal definition interval. Then the law of $[0, T) \ni t \mapsto W(L^{\xi}_t)$ is called the local interior HRLC$_\kappa$ in D from p_0 to p_1, or HRLC$_\kappa(D; p_0 \to p_1)$, in the chart (Σ, W).

Theorem 3.2.4 *Suppose L_1 and L_2 have the laws of local HRLC$_\kappa(D; p_0 \to p_1)$ in (Σ_1, W_1) and (Σ_2, W_2), respectively. For $j = 1, 2$, let S_j be the first t such that $L_j(t) \notin \Omega_{3-j}$; or $\Delta(L_j)$ if such t does not exist. Then L_1 restricted to $[0, S_1)$ has the same law as L_2 restricted to $[0, S_2)$, after a time-change*

Proof. The idea of the proof is a combination of the proof of Theorem 3.2.1 and the proof of Theorem 2.5.1. $\square$

The (global) HRLC$_\kappa(D; p_0 \to p_1)$ is the measure on the space of interior Loewner chains module time-change that are started from p_0 and disjoint from p_1 such that when restricted to any Ω it has the law of local HRLC$_\kappa(D; p_0 \to p_1)$ in (Ω, W) module the time-change. From the last theorem, such a measure exists uniquely.

If we define $Q^{\xi}_{t,W}$ to be the harmonic measure function in $D \setminus W(L^{\xi}_t)$ of a fixed side arc I of D, then we obtain HRLC$_\kappa(p_0 \to I)$. If we define $Q^{\xi}_{t,W}$ be the minimal function in $D \setminus W(L^{\xi}_t)$ with the pole at a fixed prime end w_1 of D, normalized by $(J, w_1, 1)$, then we obtain HRLC$_\kappa(p_0 \to w_1)$.

3.3 HRLC in canonical plane domains

If the underlying surface of a finite Riemann surface D is the Riemann sphere $\widehat{\mathbb{C}}$ and $\infty \notin D$, then D is called a multiply connected (plane) domain, or n-connected domain, if D has n sides. There are some special types of multiply connected domains, which are determined by finitely many real parameters. A circular canonical domain is obtained by removing from a open disc or annulus finitely many mutually disjoint arcs on the circles centered at the center of the disc or annulus. If the disc is the unit disc $\mathbb{D}$, it is called a type D domain. If the annulus is the standard annulus $\mathbb{A}_p$ for some $p > 0$, it is called a type A domain. A flat canonical domain is obtained by removing from an open half plane or open strip (a domain bounded by two parallel lines) finitely many mutually disjoint line segments that are parallel to the boundary line(s) of the half plane or strip. If the half plane is the upper half plane $\mathbb{H}$ (right half plane $-i\mathbb{H}$, or lower half plane $-\mathbb{H}$, resp.), then it is called a type H (RH, or LH, resp.) domain. If the strip is a standard strip $\mathbb{S}_p$ for some $p > 0$, then it is called a type S domain.

Suppose α is a side of D, and $p \in D$. Then there is g that maps D conformally onto a type D domain such that $g(\alpha) = \partial\mathbb{D}$ and $g(p) = 0$. Such g is unique up to a rotation. Suppose $\alpha_1 \neq \alpha_2$ are two sides of D. Then there is g that maps D conformally onto a type A domain such that $g(\alpha_1) = \partial\mathbb{D}$ and $g(\alpha_2) = \mathbf{C}_p$, the inner boundary component of $\overline{g(D)}$. Such g is also unique up to a rotation. Suppose w is a prime end of D. Then there is g that maps D conformally onto a type H, RH, or LH domain such that $g(w) = \infty$. Such g is unique up to a translation and a dilation. Suppose $w_- \neq w_+$ are two prime ends of D on one side α, then there is g that maps D conformally onto a type S domain such that $g(w_\pm) = \pm\infty$. Such g is unique up to a translation and a dilation.

Now we focus on type D domains. For $m \in \mathbb{N}$, let $\mathcal{T}_m$ denote the set of $(p_1, \dots, p_m; \theta_1^-, \dots, \theta_m^-; \theta_1^+, \dots, \theta_m^+) \in \mathbb{R}^{3m}$ such that for each $1 \le j \le m$, $p_j < 0$, $\theta_j^- < \theta_j^+ < \theta_j^- + 2\pi$, and

$$F_j := \{\exp(p + i\theta) : \theta_j^- \le \theta \le \theta_j^+\}, \;\; 1 \le j \le m, \tag{3.3.1}$$

are mutually disjoint. Let

$$\Omega(\omega) := \mathbb{D} \setminus \cup_{j=1}^m F_j(\omega).$$

Then each $(m+1)$-connected type D domain is $\Omega(\omega)$ for some $\omega \in \mathcal{T}_m$.

For each $\omega \in \mathcal{T}_m$ and $\chi \in \partial\mathbb{D}$ we can find $\mathbf{S}(w, \chi, \cdot)$ that maps $\Omega(\omega)$ conformally onto a type RH domain so that χ is mapped to ∞ and

$$\lim_{z \to \chi} \mathbf{S}(w, \chi, z) - \frac{\chi + z}{\chi - z} = 0. \tag{3.3.2}$$

Under this normalization, $\mathbf{S}(w, \chi, \cdot)$ is uniquely determined. And $\mathbf{S}(w, \chi, \cdot)$ corresponds each $F_j(\omega)$ to one vertical line segments.

$\mathbf{S}(w, \chi, \cdot)$ has continuation at the two ends of $F_j(\omega)$, and the values have the same real part. For $\omega \in \mathcal{T}_m$ and $\xi \in \mathbb{R}$, We may denote

$$\begin{cases} q_j(\omega, \xi) + i\sigma_j^{\pm}(\omega, \xi) &= \mathbf{S}(\omega, e^{i\xi}, \exp(p_j(\omega) + i\theta_j^{\pm}(\omega))); \\ V(\omega, \xi) &= (q_1, \ldots, q_m; \sigma_1^-, \ldots, \sigma_m^-; \sigma_1^+, \ldots, \sigma_m^+)(\omega, \xi). \end{cases} \tag{3.3.3}$$

So V is a $\mathbb{R}^{3m}$ valued function on $\mathcal{T}_m \times \mathbb{R}$.

Lemma 3.3.1 *Suppose h is a real harmonic function in $M(\omega)$ for some $\omega \in \mathcal{T}_m$ such that the harmonic conjugates of h exist. And h is continuous on $\overline{\Omega(\omega)} = \overline{\mathbb{D}}$ so that the value of h on each $F_j(\omega)$ is constant C_j. Then we have*

$$h(z) = \int_{\partial\mathbb{D}} h(\chi) Re\, \mathbf{S}(w, \chi, z) d\mathbf{m}(\chi), \tag{3.3.4}$$

where $\mathbf{m}$ is the uniform probability measure on $\partial\mathbb{D}$.

Proof. Let $I(z)$ be equal to the right hand side of (3.3.4). Then $I(z) = I_1(z) + I_2(z)$, where

$$I_1(z) = \int_{\partial\mathbb{D}} h(\chi) \mathrm{Re}\, \frac{\chi + z}{\chi - z} d\mathbf{m}(\chi), \ \text{ and}$$

$$I_2(z) = \int_{\partial\mathbb{D}} h(\chi) \mathrm{Re} \left(\mathbf{S}(w, \chi, z) - \frac{\chi + z}{\chi - z} \right) d\mathbf{m}(\chi).$$

From the property of Poisson kernel we know that $I(z)$ is harmonic in $\mathbb{D}$, and the continuation of I to $\partial\mathbb{D}$ coincides with h. From the definition of $\mathbf{S}(w, \chi, z)$, we may check that for fixed $\omega \in \mathcal{T}_m$, $\mathrm{Re}\,(\mathbf{S}(w, \chi, z) - \frac{\chi+z}{\chi-z})$ tends to 0 as $z \in \Omega(\omega)$ tends to $\partial\mathbb{D}$, uniformly in χ. Thus the continuation of I_2 to $\partial\mathbb{D}$ is constant 0. Therefore the continuation of I to $\partial\mathbb{D}$ coincides with h. Since $\operatorname{Re}\mathbf{S}(w,\chi,z)$ has constant continuation to each $F_j(w)$ by definition, so does $I(z)$ by formula (3.3.4). And the same is true for $h(z)$ by assumption. Thus $I-h$ has constant continuation to each $F_j(\omega)$. We have proved that $I-h$ has constant 0 continuation to $\partial\mathbb{D}$. Now both I and h has harmonic conjugate, so does $I-h$. Thus the constant continuation of $I-h$ to each boundary component of $\Omega(\omega)$ has to be equal. So they are all equal to 0, which implies $h\equiv I$, as desired. $\square$

Theorem 3.3.1 *Suppose $\omega_0\in\mathcal{T}_m$ and L is a Loewner chain in $\Omega(\omega_0)$ started from $e^{i\xi_0}\in\partial\mathbb{D}$ that avoids 0. Then after a time-change of L, there are a real valued continuous function ξ and a differentiable $\mathcal{T}_m$ valued function ω defined on $[0,\Delta(L))$ with $\xi(0)=\xi_0$ and $\omega(0)=\omega_0$, a family of φ_t that maps $\Omega(\omega_0)\setminus L(t)$ conformally onto $\Omega(\omega(t))$ such that $\varphi_0(z)=z$, $\varphi_t(0)=0$, $\varphi_t(F_j(\omega_0))=F_j(\omega(t))$, $1\le j\le n$, $0\le t<\Delta(L)$, and*

$$\partial_t\varphi_t(z)=\varphi_t(z)\mathbf{S}(\omega(t),e^{i\xi(t)},\varphi_t(z)). \tag{3.3.5}$$

The time-change, ξ, ω, and φ_t are uniquely determined, and $\omega'(t)=V(\omega(t),\xi(t))$.

Proof. For each $t\in[0,\Delta(L))$ we can find $\rho(t)\in\mathcal{T}_m$ and a function ψ_t that maps $\Omega(\omega_0)\setminus L(t)$ conformally onto $\Omega(\rho(t))$ such that $\psi_t(0)=0$ and $\psi_t'(0)>0$. Then $\Omega(\rho(t))$ and ψ_t are uniquely determined. For $t=0$, $\Omega(\rho(0))=\Omega(\omega_0)$ and ψ_0 is the identity.

Now fix $a\in[0,\Delta(L))$. Define

$$L_a(t)=\psi_t(L(a+t)\setminus L(a)) \ \text{ and } \ \psi_{a,t}=\psi_{a+t}\circ\psi_a^{-1}.$$

Then L_a is a Loewner chain in $\Omega(\rho(a))$ started from $\partial\mathbb{D}$ and $\psi_{a,t}$ maps $\Omega(\rho(a))\setminus L_a(t)$ conformally onto $\Omega(\rho(a+t))$. If $r>0$ is smaller than the absolute value of the first m coordinates of $\rho(a+t)$ then we define $\rho^r(a+t)=\rho(a+t)+(r,\dots,r;0,\dots,0;0\dots,0)\in\mathcal{T}_m$. The function $z\mapsto e^{-r}z$ maps $\Omega(\rho^r(a+t))$ conformally onto a subdomain of $\Omega(\rho(a+t))$, and it corresponds $F_j(\rho^r(a+t))$ with $F_j(\rho(a+t))$ for $1\le j\le m$. Now define

$$Q^r_{a,t}(z) = -\ln|\psi^{-1}_{a,t}(e^{-r}z)/z| = -\text{Re}\ \ln(\psi^{-1}_{a,t}(e^{-r}z)/z)$$

on $\Omega(\rho^r(a+t))$. Then $Q^r_{a,t}$ is a real harmonic function in $\Omega(\rho^r(a+t))$ with harmonic conjugates, and $Q^r_{a,t}$ has a constant continuation to each $F_j(\rho^r(a+t))$. And for $\chi \in \partial\mathbb{D}$, $Q^r_{a,t}(\chi) = -\ln|\psi_{a,t}(e^{-r}\chi)| > 0$. By Lemma 3.3.1, we have

$$Q^r_{a,t}(z) = \int_{\partial\mathbb{D}} Q^r_{a,t}(\chi)\text{Re}\,\mathbf{S}(\rho^r(a+t),\chi,z)dm(\chi) = \int_{\partial\mathbb{D}} \text{Re}\,\mathbf{S}(\rho^r(a+t),\chi,z)d\mu^r_{a,t}(\chi), \tag{3.3.6}$$

where $\mu^r_{a,t}$ is a measure on $\partial\mathbb{D}$ so that $d\mu^r_{a,t}/d\mathbf{m} = Q^r_{a,t}$. So $\mu^r_{a,t}$ is a positive measure. Note that $Q^r_{a,t}(0) = r + \ln\psi'_{a,t}(0)$. As $r \to 0^+$, $\mathbf{S}(\rho^r(a+t),\chi,0) \to \mathbf{S}(\rho(a+t),\chi,0)$, uniformly in $\chi \in \partial\mathbb{D}$. And $\text{Re}\,\mathbf{S}(\rho(a+t),\chi,0)$ is positive, uniformly in $\chi \in \partial\mathbb{D}$. So we may find $b, \varepsilon > 0$ such that for $r \in (0,b)$, $\text{Re}\,\mathbf{S}(\rho^r(a+t),\chi,0) > \varepsilon$. We may choose $C > 0$ so that $|Q^r_{a,t}(0)| \le C$ for $r \in (0,b)$. Then formula (3.3.6) implies that $|\mu^r_{a,t}| \le C/\varepsilon$ for $r \in (0,b)$. So there is a sequence $r_n \to 0$ such that $\mu^{r_n}_{a,t}$ converges to some positive measure $\mu_{a,t}$ on $\partial\mathbb{D}$ in the weak* topology. Since for each fixed $z \in \Omega(\rho(t))$, $Q^{r_n}_{a,t}(z) \to -\ln|\varphi^{-1}_{a,t}(z)/z|$ and $\mathbf{S}(\rho^{r_n}(a+t),\chi,z) \to \mathbf{S}(\rho(a+t),\chi,z)$, uniformly in $\chi \in \partial\mathbb{D}$, so we have

$$-\ln|\psi^{-1}_{a,t}(z)/z| = \int_{\partial\mathbb{D}} \text{Re}\,\mathbf{S}(\rho(t),\chi,z)d\mu_{a,t}. \tag{3.3.7}$$

From this we see that $\mu_{a,t}$ is independent of the sequence $\{r_n\}$, so it is the weak* limit of $\mu^r_{a,t}$ as $r \to 0^+$. If $\mu_{a,t} = 0$ then $\psi^{-1}_{a,t}(z) = \chi z$ for some $\chi \in \partial\mathbb{D}$, from which follow that $L_a(t) = \emptyset$, which is impossible. Thus $\mu_{a,t}$ is strictly positive. This implies $\psi'_{a+t}(0) > \psi'_a(0)$, and so $t \mapsto \psi'_t(0)$ is increasing on $[0,\Delta(L))$.

Consider $t_\infty \in [0,\Delta)$ and a sequence t_n in $[0,\Delta(L))$ that converges to t_∞. Since $\{\psi_{t_n}\}$ is uniformly bounded, so is a normal family. We may find a subsequence of ψ_{t_n} that converges to some ψ uniformly on each compact subset of $\Omega(\omega_0) \setminus L(t_\infty)$. Since $\psi'_{t_n}(0) \ge 1$ for all n, so ψ can't be a constant map. By Lemma 2.5.1, ψ is a conformal map, and a subsequence of $\Omega(\rho(t_n))$ converges to $\psi(\Omega(\omega_0) \setminus L(t_\infty))$, which must be a $(m+1)-$connected domain since $\Omega(\omega_0) \setminus L(t_\infty)$ is. Thus $\psi(\Omega(\omega_0) \setminus L(t_\infty)) = \Omega(\rho)$ for some $\rho \in \mathcal{T}_m$. Since all $\psi_{t_n}(0)=0$ and $\psi_{t_n}'(0)>0$, we have $\psi(0)=0$ and $\psi'(0)>0$. Thus $\psi=\psi_{t_\infty}$ and $\Omega(\rho)=\Omega(\rho(t_\infty))$. Since all subsequential limits of ψ_{t_n} is ψ_{t_∞}, we conclude that ψ_{t_n} converges to ψ_{t_∞} uniformly on each compact subset of $\Omega(\omega_0)\backslash L(t_\infty)$ and $\Omega(\rho(t_n))\to\Omega(\rho(t_\infty))$. We may change $\rho(t)$ without changing $\Omega(\rho(t))$ such that $\rho(0)=\omega_0$ and $\rho(t)$ is continuous on $[0,\Delta(L))$. Then ψ_t maps $F_j(w_0)$ onto $F_j(\rho(t))$ for each j. Thus $\mathbf{S}(\rho(t),\chi,z)$ is continuous in both t and χ. The continuity of ψ_t in t implies that $t\mapsto\psi_t'(0)$ is continuous. We have proved that it is increasing. After a time-change of L, we may assume that $\psi_t'(0)=e^t$ for all $t\in[0,\Delta(L))$.

Fix $T\in(0,\Delta(L))$. Choose $T'\in(T,\Delta(L))$. Let J be a Jordan curve in $\Omega(\omega_0)\setminus L(T')$ that surrounds $\cup_1^m F_j(\omega_0)\cup\{0\}$. For $a\in[0,T]$ and $t\in(0,T'-T]$, let $\Gamma_{a,t}^J$ be the family of curves in $\Omega(\omega_0)\setminus J\setminus L(a+t)$ that disconnect J from $L(a+t)\setminus L(a)$ and touch $\partial\mathbb{D}\cup L(a+t)$. From the definition of Loewner chains and conformal invariance of extremal length, the extremal length of $\psi_a(\Gamma_{a,t}^J)$ and $\psi_{a+t}(\Gamma_{a,t}^J)$ tend to 0 as $t\to 0^+$, uniformly in $a\in[0,T]$. These two curve families both lie in $\mathbb{D}$, whose area is finite, so the minimum length of $\psi_a(\Gamma_{a,t}^J)$ and $\psi_{a+t}(\Gamma_{a,t}^J)$ tends to 0 as $t\to0^+$, uniformly in $a\in[0,T]$. So there is a positive function l on $(0,T'-T]$ with $\lim_{t\to0^+}l(t)=0$ such that for all $a\in[0,T]$ and $t\in(0,T'-T]$, there is $\alpha_{a,t}\in\psi_a(\Gamma_{a,t}^J)$ and $\beta_{a,t}\in\psi_{a+t}(\Gamma_{a,t}^J)$ with lengths less than $l(t)$. Since $\alpha_{a,t}$ disconnects $\psi_a(L(a+t)\setminus L(t))=L_a(t)$ from $\psi_a(J)$, so the diameter of $L_a(t)$ is less than $l(t)$ when $l(t)$ is small enough. Thus for fixed $a\in[0,T]$, $\cap_{t>0}\overline{L_a(t)}$ is a single point on $\partial\mathbb{D}$, denoted by $\chi(t)$. Especially, $\chi(0)=e^{i\xi_0}$. Now we consider $\beta_{a,t}$. The set of points on $\partial\mathbb{D}$ that are not disconnected by $\beta_{a,t}$ from $\psi_{a+t}(J)$ is an open arc, denoted by $I_{a,t}^2$. Let $I_{a,t}^1=\partial\mathbb{D}\backslash I_{a,t}^2$. Since $\psi_{a,t}^{-1}$ maps $\Omega(\rho(a+t))$ conformally onto $\Omega(\rho(a))\backslash L_a(t)$ and $\psi_{a,t}^{-1}(\beta_{a,t})$ disconnects $L_a(t)$ from $\psi_a(J)$, so $\psi_{a,t}(z)$ approaches $\partial\mathbb{D}$ as z approaches $I_{a,t}^2$. Thus $Q_{a,t}^r(\chi)=-\ln|\psi_{a,t}^{-1}(e^{-r}\chi)|\to0$ as $r\to0^+$ for all $\chi\in I_{a,t}^2$. Since $\mu_{a,t}$ is the weak* limit of $\mu_{a,t}^r$ with $d\mu_{a,t}^r/dm=Q_{a,t}^r$, so $\mu_{a,t}$ is supported by $I_{a,t}^1$. We may choose $t_1>0$ so that $l(t)<1/5$ if $t<t_1$. Now suppose $t\in(0,t_1)$. That the diameter of $L_a(t)$ is less than $l(t)$ implies that for $\chi\in I_{a,t}^1$, the upper limit of $Q_{a,t}^r(\chi)$ when $r\to0^+$ is less than $-\ln(1-l(t))$. Note that the length

of $I^1_{a,t}$ is less than $2l(t)$, so $m(I^1_{a,t}) < l(t)/\pi$. Thus

$$|\mu_{a,t}| = \mu_{a,t}(I^1_{a,t}) < -\ln(1-l(t))l(t)/\pi < l(t)^2.$$

Formula (3.3.7) implies that

$$-\ln(\psi^{-1}_{a,t}(z)/z) = \int_{\partial\mathbb{D}} \mathbf{S}(\rho(a+t),\chi,z)d\mu_{a,t}(\chi) + iC_{a,t},$$

for some $C_{a,t} \in \mathbb{R}$. Since $(\psi^{-1}_{a,t})'(0) = -t$, we may set $-\ln(\psi^{-1}_{a,t}(z)/z) = t$ for $z = 0$. Thus

$$C_{a,t} = -\int_{\partial\mathbb{D}} \operatorname{Im}\mathbf{S}(\rho(a+t),\chi,0)d\mu_{a,t}(\chi).$$

From the definition of $\mathbf{S}(\omega,\chi,z)$, we see that

$$\mathbf{S}(\rho(a+t),\chi,z) - i\operatorname{Im}\mathbf{S}(\rho(a+t),\chi,0) - \frac{\chi+z}{\chi-z}$$

is uniformly bounded in $a \in [0,T]$, $t \in (0,t_1)$, $\chi \in \partial\mathbb{D}$, and $z \in \Omega(\rho(a+t))$. Thus there is $C > 0$ such that

$$|\mathbf{S}(\rho(a+t),\chi,z) - i\operatorname{Im}\mathbf{S}(\rho(a+t),\chi,0)| \le C/|z-\chi|.$$

We may choose $d > 0$ such that the distance of $\psi_u(J)$ from $\partial\mathbb{D}$ is greater than d for all $u \in [0,T']$. Then there is $t_2 \in (0,t_1)$ such that $l(t) < \min\{d/(5C), 1/10\}$ for $t \in (0,t_2)$. From now on, we always suppose $a \in [0,T]$ and $t \in (0,t_2)$. Since $\beta_{a,t}$ has length less than $l(t)$ and touches $\partial\mathbb{D}$, we may choose a crosscut $\gamma_{a,t}$ in $\mathbb{D}$ with two ends on $\partial\mathbb{D}$ separating $\psi_{a+t}(J)$ from $\beta_{a,t}$ so that the distance from any point of $\gamma_{a,t}$ to $\beta_{a,t}$ is between $Cl(t)$ and $3Cl(t)$. Now $\gamma_{a,t}$ divides $\Omega(\rho(a+t))$ into two parts. Let $\Omega_{a,t}$ denote the component that contains $\beta_{a,t}$. Then the diameter of $\Omega_{a,t}$ is not bigger than that of $\gamma_{a,t}$, which is less than $7l(t)$. For $z \in \Omega(\rho(a+t)) \setminus \Omega_{a,t}$, the distance between z and $I^1_{a,t}$ is at least $Cl(t)$. Thus for all $z \in \Omega(\rho(a+t)) \setminus \Omega_{a,t}$,

$$|\ln(\psi^{-1}_{a,t}(z)/z)| \le \int_{I^1_{a,t}} |\mathbf{S}(\rho(a+t), \chi, z) - i\mathrm{Im}\,\mathbf{S}(\rho(a+t), \chi, 0)| d\mu_{a,t}(\chi)$$

$$\le C/(Cl(t))|\mu_{a,t}| \le l(t),$$

which implies that $|\psi^{-1}_{a,t}(z) - z| \le e^{l(t)} - 1 \le 2l(t)$ as $l(t) \le 1/10$. Since this is true for $z \in \gamma_{a,t}$, so the diameter of $\psi^{-1}_{a,t}(\gamma_{a,t})$ is less than $11l(t)$. Note that the image of $\Omega_{a,t}$ under $\psi^{-1}_{a,t}$ is the domain bounded by $\psi^{-1}_{a,t}(\gamma_{a,t})$ and $L_a(t) \cup \partial\mathbb{D}$, whose diameter is not bigger than that of $\psi^{-1}_{a,t}(\gamma_{a,t})$. It follows that for all $z \in \Omega_{a,t}$,

$$|\psi^{-1}_{a,t}(z) - z| < (7 + 2 + 11)l(t) = 20l(t). \tag{3.3.8}$$

The above formula is in fact ture for $z \in \Omega(\rho(a+t))$. Suppose $0 \le a_1 < a_2 \le T$ and $d_0 = a_2 - a_1 < t_2/2$. Note that

$$L_{a_2}(t_2/2) = \psi_{a_2}(L(a_2 + t_2/2) \setminus L(a_2)), \text{ and}$$

$$L_{a_1}(t_2/2 + d_0) = \psi_{a_1}(L(a_2 + t_2/2) \setminus L(a_1)).$$

Choose $z_0 \in L(a_2 + t_2/2) \setminus L(a_2) \subset L(a_2 + t_2/2) \setminus L(a_1)$, then $\psi_{t_2}(z_0) \in L_{a_2}(t_2/2)$, $\psi_{t_1}(z_0) \in L_{a_1}(t_2/2 + d_0)$, and $|\psi_{t_2}(z_0) - \psi_{t_1}(z_0)| \le 20l(t)$ by formula (3.3.8). Since $\chi(a_2) \in \overline{L_{a_2}(t_2/2)}$, $\chi(a_1) \in \overline{L_{a_1}(t_2/2 + d_0)}$, and the diameters of $L_{a_1}(t_2/2 + d_0)$ and $L_{a_2}(t_2/2)$ are both less than $l(t)$, so $|\chi(t_1) - \chi(t_2)| \le 22l(t)$. Consequently χ is continuous.

Similarly, the distance between $I^1_{a,t}$ and $L_a(t)$ is less than $20l(t)$. Since $\chi(t) \in \overline{L_a(t)}$ and the diameters of $I^1_{a,t}$ and $L_a(t)$ are both less than $l(t)$, so $I^1_{a,t}$ lies in the ball of radius $22l(t)$ about $\chi(t)$. Now we return to the formula

$$-\ln(\psi^{-1}_{a,t}(z)/z) = \int_{I^1_{a,t}} (\mathbf{S}(\rho(a+t), \chi, z) - i\mathrm{Im}\,\mathbf{S}(\rho(a+t), \chi, 0) d\mu_{a,t}(\chi). \tag{3.3.9}$$

Set $z = 0$ and let both sides be divided by t, we then have

$$1 = \int_{I^1_{a,t}} \operatorname{Re}\mathbf{S}(\rho(a+t), \chi, 0)\frac{d\mu_{a,t}}{t}(\chi).$$

Let $t \to 0^+$, then the support of $\mu_{a,t}$ tends to $\chi(a)$, so $|\mu_{a,t}|/t \to 1/\operatorname{Re}\mathbf{S}(\rho(a), \chi(a), 0)$. This then implies that

$$\text{weak*} - \lim_{t\to 0^+} \mu_{a,t}/t = \delta_{\chi(a)}/\operatorname{Re}\mathbf{S}(\rho(a), \chi(a), 0).$$

Thus

$$\partial_t^+(-\ln(\psi_{a,t}^{-1}(z)/z))|_{t=0} = \frac{\mathbf{S}(\rho(a), \xi(a), z) - i\operatorname{Im}\mathbf{S}(\rho(a), \chi(a), 0)}{\operatorname{Re}\mathbf{S}(\rho(a), \xi(a), 0)}.$$

From the definition of $\psi_{a,t}$ and the fact that χ is continuous, we have

$$\partial_t \ln(\psi_t(z)) = \frac{\mathbf{S}(\rho(t), \chi(t), \psi_t(z)) - i\operatorname{Im}\mathbf{S}(\rho(t), \chi(t), 0)}{\operatorname{Re}\mathbf{S}(\rho(t), \chi(t), 0)}. \tag{3.3.10}$$

This formula holds for $t \in [0, T]$. However, since we may choose T to be arbitrarily close to $\Delta(L)$ and define $\xi(t)$ accordingly, formula (3.3.10) then holds for all $t \in [0, \Delta(L))$. Now for $t \in [0, \Delta(L)]$, define

$$h(t) = \int_0^t \operatorname{Re}\mathbf{S}(\rho(s), \chi_s, 0)ds.$$

After a time-change through h, formula (3.3.10) becomes

$$\partial_t \ln(\psi_t(z)) = \mathbf{S}(\rho(t), \chi(t), \psi_t(z)) - i\operatorname{Im}\mathbf{S}(\rho(t), \chi(t), 0). \tag{3.3.11}$$

Finally, we define $\beta(t) = \int_0^t \operatorname{Im}\mathbf{S}(\rho(s), \chi_s, 0)ds$, let $\varphi_t = e^{i\beta(t)}\psi_t$, $\iota(t) = e^{i\beta(t)}\chi(t)$, and $\omega(t) = \rho(t) + (0, \dots, 0; \beta(t), \dots, \beta(t); \beta(t), \dots, \beta(t))$. Since $\beta(0) = 0$, so $\varphi_0 = \psi_0$ is the identity, $\iota(0) = \chi(0) = e^{i\xi_0}$ and $\omega(0) = \omega_0$. We may choose a continuous real valued function ξ such that $\iota = e^{i\xi}$ and $\xi(0) = \xi_0$. Since $F_j(\omega(t))$ and $\Omega(\omega(t))$ are rotations of $F_j(\rho(t))$ and $\Omega(\rho(t))$, respectively, by $e^{i\beta(t)}$, so φ_t maps $\Omega(\omega_0) \setminus L(t)$ conformally onto $\Omega(\omega(t))$ and corresponds $F_j(\omega_0)$ with $F_j(\omega(t))$ for each j. Moreover,

$\mathbf{S}(\omega(t), \iota(t), \varphi_t(z)) = \mathbf{S}(\rho(t), \chi(t), \psi_t(z))$. Thus from (3.3.11),

$$\partial_t \ln \varphi_t(z) = \partial_t \ln \psi_t(z) + i\beta'(t) = \mathbf{S}(\omega(t), \chi(t), \varphi_t(z)),$$

from which follows that $\partial_t \varphi_t(z) = \varphi_t(z)\mathbf{S}(\omega(t), e^{i\xi(t)}, \varphi_t(z))$.

Suppose for $k = 1, 2$, there are a continuous increasing function h_k on $[0, \Delta(L))$ with $h_k(0) = 0$, a continuous real valued function ξ_k and a $\mathcal{T}_m$ valued function ω_k on $[0, h_k(\Delta(L))$ with $\xi_k(0) = \xi_0$, $\omega_k(0) = \omega_0$, and φ_t^k that maps $\Omega(\omega_0) \setminus L(h_k(t))$ conformally onto $\Omega(\omega_k(t))$ and fixes 0 so that φ_0^k is identity and

$$\partial_t \varphi_t^k(z) = \varphi_t^k(z)\mathbf{S}(\omega_k(t), e^{i\xi_k(t)}, \varphi_t^k(z)). \tag{3.3.12}$$

Note that $\varphi^k_{h_k^{-1}(t)}$ maps $\Omega(\omega_0) \setminus L(t)$ conformally onto a type D domain and fixes 0. Therefore $\varphi^2_{h_2^{-1}(t)}(z) = Q(t)\varphi^1_{h_1^{-1}(t)}(z)$ for some $Q(t) \in \partial\mathbb{D}$. Let $h = h_1^{-1} \circ h_2$. Then $\varphi_t^2(z) = Q(h_2(t))\varphi^1_{h(t)}(z)$. It is clear that Q is continuous and $Q(0) = 1$. We may write $Q(t) = e^{i\eta(t)}$ so that η is continuous and $\eta(0) = 0$. Then

$$\ln \varphi_t^2(z) = i\eta(h_2(t)) + \ln \varphi^1_{h(t)}(z).$$

Formula (3.3.12) implies $\partial_t \ln \varphi_t^k(z) = \mathbf{S}(\omega_k(t), e^{i\xi_k(t)}, \varphi_t^k(z))$. This then implies that $\tau := \eta \circ h_2$ and h are differentiable, and

$$\partial_t \ln \varphi_t^2(z) = i\tau'(t) + h'(t)\partial_t \ln \varphi^1_{h(t)}(z), \tag{3.3.13}$$

from which follows that

$$\mathbf{S}(\omega_2(t), e^{i\xi_2(t)}, \varphi_t^2(z)) = i\tau'(t) + h'(t)\mathbf{S}(\omega_1(h(t)), e^{i\xi_1(h(t))}, \varphi^1_{h(t)}(z)). \tag{3.3.14}$$

Now use $\varphi_t^2(z) = e^{i\tau(t)}\varphi^1_{h(t)}(z)$. Let

$$\omega_3(t) = \omega_2(t) - (0, \ldots, 0; \tau(t), \ldots, \tau(t); \tau(t), \ldots, \tau(t))$$

and $\xi_3(t) = \xi_2(t) - \tau(t)$, then

$$\mathbf{S}(\omega_2(t), e^{i\xi_2(t)}, \varphi_t^2(z)) = \mathbf{S}(\omega_3(t), e^{i\xi_3(t)}, \varphi_{h(t)}^1(z)).$$

This together with (3.3.14) implies that

$$\mathbf{S}(\omega_3(t), e^{i\xi_3(t)}, z) = i\tau'(t) + h'(t)\mathbf{S}(\omega_1(h(t)), e^{i\xi_1(t)}, z),$$

for any $z \in \Omega(\omega_1(h(t)))$. Let $z \longrightarrow e^{i\xi_1(t)}$, then the right hand side tends to ∞, so does the left hand side. Thus $e^{i\xi_3(t)} = e^{i\xi_1(t)}$. Recall that $\mathbf{S}(\omega_3(t), e^{i\xi_3(t)}, z) - \frac{e^{i\xi_3(t)}+z}{e^{i\xi_3(t)}-z}$ tends to 0 as $z \longrightarrow e^{i\xi_3(t)}$. Thus

$$\lim_{z \to \chi^1(h(t))} \left(i\tau'(t) + h'(t)\mathbf{S}(\omega_1(h(t)), e^{i\xi_1(h(t))}, z) - \frac{e^{i\xi_1(h(t))} + z}{e^{i\xi_1(h(t))} - z} \right) = 0.$$

On the other hand, $\mathbf{S}(\omega_1(h(t)), e^{i\xi_1(h(t))}, z) - \frac{e^{i\xi_1(h(t))}+z}{e^{i\xi_1(h(t))}-z}$ tends to 0 as $z \longrightarrow \chi^1(h(t))$. We must have $\tau'(t) \equiv 0$ and $h'(t) \equiv 1$. Thus $Q(t) \equiv 1$, $h_1 = h_2$, $e^{i\xi_2} = e^{i\xi_1}$, $\varphi_t^1 = \varphi_t^2$ and $\Omega(\omega_1(t)) = \Omega(\omega_2(t))$. Since ω_k and ξ_k are continuous and $\omega_k(0) = \omega_0$, $\xi_k(0) = \xi_0$, so $\omega_1 = \omega_2$ and $\xi_1 = \xi_2$.

Finally, we need to prove that $\omega'(t) = V(\omega(t), \xi(t))$. Suppose α_j is the side of $\Omega(\omega_0)$ that corresponds to $F_j(\omega_0)$, $1 \le j \le m$. Since φ_t corresponds $F_j(\omega_0)$ with $F_j(\omega(t))$, and $F_j(\omega(t))$ is locally connected (see [12]), φ_t can be extended continuously to α_j, $1 \le j \le m$. Under the map φ_t, each end point of some $F_j(\omega_0)$ has exactly one pre-image, and other points of $F_j(\omega_0)$ all have two pre-images. Suppose $w_j^{\pm}(t) = \varphi_t^{-1}(\exp(p_j(t) + i\theta_j^{\pm}(t)))$. Note that $G_t := \mathbf{S}(e^{i\xi(t)}, \cdot) \circ \varphi_t$ maps $\Omega(\omega_0)$ conformally onto a type RH domain, and each α_j corresponds with a vertical line segment. So G_t can be extended continuously to all α_j. And we have $\partial_t \varphi_t(z) = \varphi_t(z) G_t(z)$ for all $z \in \cup \alpha_j$. We may find ψ_t defined on $\cup \alpha_j$ such that $\varphi_t = \exp \circ \psi_t$, $\partial_t \psi_t(z) = G_t(z)$, and

$$\psi_t(\alpha_j) = [p_j(\omega(t)) + i\theta_j^-(\omega(t)), p_j(\omega(t)) + i\theta_j^+(\omega(t))].$$

Thus $p_j(\omega(t)) = \operatorname{Re} \psi_t(\alpha_j)$, $\theta_j^-(\omega(t)) = \min \operatorname{Im} \rho_t(\alpha_j)$, and $\theta_j^+(\omega(t)) = \max \operatorname{Im} \rho_t(\alpha_j)$.

Let $z_j^{\pm}(t) \in \alpha_j$ be the unique prime ends such that $\psi_t(z_j^{\pm}(t)) = p_j(w(t)) + i\theta_j^{\pm}(w(t))$. From the definition of G_t, q_j and $\sigma_j^{\pm}$, we see that

$$G_t(z_j^{\pm}(t)) = q_j(\omega(t), \xi(t)) + i\sigma_j^{\pm}(\omega(t), \xi(t)). \tag{3.3.15}$$

Now fix $t_0 \in [0, \Delta(L))$. We have

$$d_t p_j(\omega(t))|_{t=t_0} = \partial_t \mathrm{Re}\, \psi_t(z_j^-(t_0))|_{t=t_0} = \mathrm{Re}\, G_{t_0}(z_j^-(t_0)) = q_j(\omega(t_0), \xi(t_0)). \tag{3.3.16}$$

Suppose f_j maps a neighborhood U_j of $z_j^-(t)$ in $\widehat{\Omega(\omega_0)}$ conformally onto a neighborhood of 0 in $\overline{\mathbb{H}}$ such that $f_j(z_j^-) = 0$ and $f_j(U_j \cap \alpha_j) \subset \mathbb{R}$. For $z \in \alpha_j$ near $z_j^-(t_0)$, $\psi_{t_0}(z)$ behaves like $\psi_{t_0}(z_j^-(t_0)) + iC_j f_j(z)^2$ for some $C_j \in \mathbb{R}$. This implies that for $z \in U_j \cap \alpha_j$,

$$\mathrm{Im}\, \psi_{t_0}(z) - \mathrm{Im}\, \psi_{t_0}(z_j^-(t_0)) \geq C_j f_j(z)^2.$$

On the other hand, G_{t_0} is Lipschitz on α_j, so for $z \in \alpha_j$,

$$|G_{t_0}(z) - G_{t_0}(z_j^-(t_0))| \leq C_j' |f_j(z)|$$

for some $C_j' > 0$. Since $\partial_t \psi_t(z) = G_t(z)$, so

$$\mathrm{Im}\, \psi_{t_0+\varepsilon}(z) - \mathrm{Im}\, \psi_{t_0}(z) = \varepsilon \mathrm{Im}\, G_{t_0}(z) + o(\varepsilon).$$

Thus for all $z \in B_j$,

$$\begin{aligned} \mathrm{Im}\, \psi_{t_0+\varepsilon}(z) &\geq \mathrm{Im}\, \psi_{t_0}(z_j^-(t_0)) + \varepsilon \mathrm{Im}\, G_{t_0}(z_j^-(t_0)) + o(\varepsilon) + C_j f_j(z)^2 - \varepsilon C_j' |f_j(z)| \\ &\geq \mathrm{Im}\, \psi_{t_0}(z_j^-(t_0)) + \varepsilon \mathrm{Im}\, G_{t_0}(z_j^-(t_0)) + o(\varepsilon) - \frac{(C_j')^2}{4C_j} \varepsilon^2. \end{aligned}$$

This implies that

$$\theta_j^-(\omega(t_0+\varepsilon)) = \min \mathrm{Im}\, \psi_{t_0+\varepsilon}(\alpha_j) \geq \mathrm{Im}\, \psi_{t_0}(z_j^-(t_0)) + \varepsilon \mathrm{Im}\, G_{t_0}(z_j^-(t_0)) + o(\varepsilon).$$

On the other hand

$$\theta_j^-(\omega(t_0+\varepsilon)) \le \operatorname{Im}\psi_{t_0+\varepsilon}(z_j^-(t_0)) = \operatorname{Im}\psi_{t_0}(z_j^-(t_0)) + \varepsilon \operatorname{Im} G_{t_0}(z_j^-(t_0)) + o(\varepsilon).$$

Thus by the definition of $z_j^-(t_0)$ and formula (3.3.15), we have

$$d_t\theta_j^-(\omega(t))|_{t=t_0} = \operatorname{Im} G_{t_0}(z_j^-(t_0)) = \sigma_j^-(\omega(t_0), \chi(t_0)).$$

Similarly, we can prove the above formula when the superscript $-$ is replaced by $+$. These together with formula (3.3.16) finish the proof. □

Now suppose L has the law of the HRLC$_\kappa(\Omega(\omega_0; 1 \to 0)$ for some $\kappa \ge 0$. From Theorem 3.3.1, there are a continuous increasing function v on $[0, \Delta(L))$, a real valued function ξ and a $\mathcal{T}_m$ valued function ω on $[0, v(\Delta(L))$ with $\xi(0) = 0$ and $\omega(0) = \omega_0$, and a family of maps φ_t that maps $\Omega(\omega_0) \setminus L(u(t))$ conformally onto $\Omega(\omega(t))$, where u is the inverse function of v, such that φ_0 is an identity and

$$\partial_t\varphi_t(z) = \varphi_t(z)\mathbf{S}(\omega(t), e^{i\xi(t)}, \varphi_t(z)).$$

And they are all uniquely determined by L. For $\omega \in \mathcal{T}_m$, let G_ω be the Green function in $\Omega(\omega)$ with the pole at 0.

Theorem 3.3.2 *There is a standard Brownian motion $B(t)$ such that*

$$\xi(t) = \sqrt{\kappa}B(t) + (3 - \kappa/2)\int_0^t (\partial_x\partial_y/\partial_y)(G_{\omega(s)} \circ e^i)(\xi(s))ds.$$

And $u'(t) = 1/\partial_y(G_{\omega(t)} \circ e^i)(\xi(t))^2$.

Proof. The idea of the proof is similar as those of all equivalence theorems that we have encountered. So we omit the proof. □

Now write $A(\omega, \xi) := (\partial_x\partial_y/\partial_y)(G_\omega \circ e^i)(\xi)$ for $\omega \in \mathcal{T}_m$ and $\xi \in \mathbb{R}$. Let $d(t) :=$

$\xi(t)-\sqrt{\kappa}B(t)$. Then ω and d satisfy the differential equations:

$$\begin{cases} \omega'(t) &= V(\omega(t), d(t)+\sqrt{\kappa}B_c(t)); \\ d'(t) &= (3-\kappa/2)A(\omega(t), d(t)+\sqrt{\kappa}B(t)), \end{cases} \tag{3.3.17}$$

with the initial value $\omega(0)=\omega_0$ and $d(0)=0$. Now it is interesting to ask whether we can start from a standard Brownian motion B and get ω and ξ by solving the system (3.3.17). The answer is not known now. The difficulty is that it is not obvious that V and A are Lipschitz. So we may not be able to apply the existence and uniqueness theorem for the solution of an ordinary differential equation.

We may also consider other types of canonical domains. Let $\mathcal{T}^A_m$ be the set of $(p_0,p_1,\dots,p_m;\theta^-_1,\dots,\theta^-_m;\theta^+_1,\dots,\theta^+_m)\in\mathbb{R}^{3m+1}$ such that for each $1\le j\le m$, $p_0<p_j<0$, $\theta^-_j<\theta^+_j<\theta^-_j+2\pi$, and

$$F_j:=\{\exp(p+i\theta):\theta^-_j\le\theta\le\theta^+_j\},\ \ 1\le j\le m,$$

are mutually disjoint. Let

$$\Omega(\omega):=\mathbb{A}_{|p_0(\omega)|}\setminus\cup_{j=1}^m F_j(\omega).$$

Then each $(m+2)$-connected type A domain is $\Omega(\omega)$ for some $\omega\in\mathcal{T}_m$.

For $\omega\in\mathcal{T}^A_m$ and $\chi\in\partial\mathbb{D}$, let $\mathbf{S}(\omega,\chi,\cdot)$, $q_j(\omega,\chi)$ and $\sigma^{\pm}(\omega,\chi)$, $1\le j\le m$, be defined in exactly the same way as in the case of type D domains. We let $q_0(\omega,\chi)$ be the real part of $\mathbf{S}(\omega,\chi,\mathbf{C}_{|p_0(\omega)|})$, and

$$V(\omega,\chi):=(q_0,q_1,\dots,q_m;\sigma^-_1,\dots,\sigma^-_m;\sigma^+_1,\dots,\sigma^+_m)(\omega,\chi).$$

Then Theorem 3.3.1 still holds with $\mathcal{T}_m$ replaced by $\mathcal{T}^A_m$. If $m=0$, then the equation becomes the annulus Loewner equation.

Suppose L has the law of $\mathrm{HRLC}_\kappa(\Omega(\omega_0);1\to\mathbf{C}_{|p_0(\omega_0)|})$ for some $\kappa\ge0$. Then from Theorem 3.3.1 for $\mathcal{T}^A_m$, there are a continuous increasing function v on $[0,\Delta(L))$, a real valued function ξ and a $\mathcal{T}_m^A$ valued function ω on $[0, v(\Delta(L))$ with $\xi(0) = 0$ and $\omega(0) = \omega_0$, and a family of maps φ_t that maps $\Omega(\omega_0) \setminus L(u(t))$ conformally onto $\Omega(\omega(t))$, where u is the inverse function of v, such that φ_0 is an identity and

$$\partial_t \varphi_t(z) = \varphi_t(z) \mathbf{S}(\omega(t), e^{i\xi(t)}, \varphi_t(z)).$$

For any $\omega \in \mathcal{T}_m^A$, let H_ω be the harmonic measure function of $\mathbf{C}_{|p_0(\omega)|}$ in $\Omega(\omega)$. Then Theorem 3.3.2 holds for type A domains with G replaced by H.

For the type H or S domains, we need differential equations similar to the chordal and strip Loewner equations. Let $\mathcal{T}_m^H$ be the set of $(p_1, \dots, p_m; \theta_1^-, \dots, \theta_m^-; \theta_1^+, \dots, \theta_m^+)$ $\in \mathbb{R}^{3m}$ such that for each $1 \le j \le m$, $p_j > 0$, $\theta_j^- < \theta_j^+$, and

$$F_j := [\theta_j^- + ip_j, \theta_j^+ + ip_j], \ \ 1 \le j \le m,$$

are mutually disjoint. Let

$$\Omega(\omega) := \mathbb{H} \setminus \cup_{j=1}^m F_j(\omega).$$

Then each $(m+1)$-connected type H domain is $\Omega(\omega)$ for some $\omega \in \mathcal{T}_m^H$. We then have the following theorem.

For $\omega \in \mathcal{T}_m^H$ and $\xi \in \mathbb{R}$, there is a unique $\mathbf{R}(\omega, \xi, \cdot)$ that maps $\Omega(\omega)$ conformally onto a type LH domain such that ξ is mapped to ∞, and

$$\mathbf{R}(\omega, \xi, z) - \frac{2}{z - \xi} \to 0, \ \text{ as } \ z \to \xi.$$

Then we may denote

$$\begin{cases} \sigma_j^\pm(\omega, \xi) + iq_j(\omega, \xi) &= \mathbf{R}(\omega, \xi, \theta_j^\pm(\omega) + ip_j(\omega)); \\ V(\omega, \xi) &= (q_1, \dots, q_m; \sigma_1^-, \dots, \sigma_m^-; \sigma_1^+, \dots, \sigma_m^+)(\omega, \xi). \end{cases} \tag{3.3.18}$$

So V is a $\mathbb{R}^{3m}$ valued function on $\mathcal{T}_m^H \times \mathbb{R}$.

Theorem 3.3.3 *Suppose $\omega_0 \in \mathcal{T}_m^H$ and L is a Loewner chain in $\Omega(\omega_0)$ started from $\xi_0 \in \mathbb{R}$. Then after a time-change of L, there are a real valued continuous function*

ξ and a differentiable $\mathcal{T}_m^H$ valued function ω defined on $[0, \Delta(L))$ with $\xi(0) = \xi_0$ and $\omega(0) = \omega_0$, a family of φ_t that maps $\Omega(\omega_0) \setminus L(t)$ conformally onto $\Omega(\omega(t))$ such that $\varphi_0(z) = z$, $\varphi_t(\infty) = \infty$, $\varphi_t(F_j(\omega_0) = F_j(\omega(t))$, $1 \le j \le n$, $0 \le t < \Delta(L)$, and

$$\partial_t \varphi_t(z) = \mathbf{R}(\omega(t), \xi(t), \varphi_t(z)). \tag{3.3.19}$$

The time-change, $\omega(t)$, $\xi(t)$ and φ_t are uniquely determined by L. And $\omega'(t) = V(\omega(t), \xi(t))$.

Suppose L has the law of an $\mathrm{HRLC}_\kappa(\Omega(\omega_0); 0 \to \infty)$ for some $\kappa \ge 0$. From Theorem 3.3.3 there are a continuous increasing function v on $[0, \Delta(L))$, a real valued function ξ and a $\mathcal{T}_m^H$ valued function ω on $[0, v(\Delta(L))$ with $\xi(0) = 0$ and $\omega(0) = \omega_0$, and a family of maps φ_t that maps $\Omega(\omega_0) \setminus L(u(t))$ conformally onto $\Omega(\omega(t))$, where u is the inverse function of v, such that φ_0 is an identity and

$$\partial_t \varphi_t(z) = \mathbf{R}(\omega(t), \xi(t), \varphi_t(z)).$$

For $\omega \in \mathcal{T}_m$, let M_ω be the minimal function in $\Omega(\omega)$ with the pole at ∞, normalized by $z \mapsto -1/z$.

Theorem 3.3.4 *There is a standard Brownian motion $B(t)$ such that*

$$\xi(t) = \sqrt{\kappa} B(t) + (3 - \kappa/2) \int_0^t (\partial_x \partial_y / \partial_y) M_{\omega(s)}(\xi(s)) ds.$$

And $u'(t) = a^2 / \partial_y M_{\omega(t)}(\xi(t))^2$ for some $a > 0$.

The case of type S domain is similar as that of type H domains. Let $\mathcal{T}_m^S$ be the set of $(p_0, p_1, \dots, p_m; \theta_1^-, \dots, \theta_m^-; \theta_1^+, \dots, \theta_m^+) \in \mathbb{R}^{3m+1}$ such that for each $1 \le j \le m$, $p_0 > p_j > 0$, $\theta_j^- < \theta_j^+$, and

$$F_j := [\theta_j^- + i p_j, \theta_j^+ + i p_j], \;\; 1 \le j \le m,$$

are mutually disjoint. Let

$$\Omega(\omega) := \mathbb{S}_{p_0} \setminus \cup_{j=1}^{m} F_j(\omega).$$

Then each $(m+1)$-connected type S domain is $\Omega(\omega)$ for some $\omega \in \mathcal{T}_m^S$. For $\omega \in \mathcal{T}_m^S$ and $\xi \in \mathbb{R}$, let $\mathbf{R}(\omega, \xi, \cdot)$, q_j and $\sigma_j^{\pm}$ be defined in exactly the same way as that in the case of type H domains. Let

$$V(\omega, \xi) := (0, q_1, \dots, q_m; \sigma_1^-, \dots, \sigma_m^-; \sigma_1^+, \dots, \sigma_m^+)(\omega, \xi).$$

Then Theorem 3.3.3 still holds with $\mathcal{T}_m^H$ replaced by $\mathcal{T}_m^S$. Since the first coordinate of V is 0, so $p_0(\omega(t)) \equiv p_0(\omega_0)$. If $m = 0$ and $p_0 = \pi$, then this becomes the strip Loewner equation.

Suppose L has the law of the $\mathrm{HRLC}_\kappa(\Omega(\omega_0); 0 \to ip_0(\omega_0) + \mathbb{R})$ for some $\kappa \geq 0$. From Theorem 3.3.3 for $\mathcal{T}_m^S$ domains there are a continuous increasing function v on $[0, \Delta(L))$, a real valued function ξ and a $\mathcal{T}_m^S$ valued function ω on $[0, v(\Delta(L))$ with $\omega(0) = \omega_0$ and $\xi(0) = 0$, and a family of maps φ_t that maps $\Omega(\omega_0) \setminus L(u(t))$ conformally onto $\Omega(\omega(t))$, where u is the inverse function of v, such that φ_0 is an identity and

$$\partial_t \varphi_t(z) = \mathbf{R}(\omega(t), \xi(t), \varphi_t(z)).$$

For $\omega \in \mathcal{T}_m$, let H_ω be the harmonic measure function of $ip_0(\omega) + \mathbb{R}$ in $\Omega(\omega)$. Then Theorem 3.3.4 holds for type S domains with M replaced by H and the arbitrary $a > 0$ be replaced by 1.

Chapter 4

Loop-erased random walk and HRLC$_2$

4.1 Loop-erased random walk

The graphs that we will consider in this chapter are connected simple nondirected graphs such that the degree of each vertex is finite. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph G. If two vertices v_1 and v_2 of G are adjacent, we write $v_1 \sim v_2$.

A path X on G is a map from $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}$ or $\overline{\mathbb{N}}_m := \{n \in \overline{\mathbb{N}} : n \leq m\}$ for some $m \in \overline{\mathbb{N}}$ to $V(G)$ such that $X(n) \sim X(n-1)$. X is called a simple path if the map is an injection. The edges of X are those $\{X(n), X(n+1)\}$. We say the length of X is ∞ if X is defined on $\overline{\mathbb{N}}$; or m if X is defined on $\overline{\mathbb{N}}_m$.

The loop-erasure of a path X of length m is defined as follows. Let $\sigma(0)$ be the biggest n such that $X(n) = X(0)$. When $\sigma(k)$ is defined, if $\sigma(k) = m$ then let $\tau = \sigma(k)$ and we stop here; otherwise, let $\sigma(k+1)$ be the biggest n such that $X(n) = X(\sigma(k)+1)$. The loop-erasure $LE(X)$ of X is defined on $\overline{\mathbb{N}}_\tau$ such that $LE(X)(j) = X(\sigma(j))$. It is clear that $LE(X)$ is a simple path, and starts and ends at the same points as X. See [3] for details.

A (simple) random walk on a graph G started from $v_0 \in V(G)$ is a random infinite path X such that $X(0) = v_0$, and

$$\Pr[X(n+1) = v|X(0), \ldots, X(n)] = 1/\deg(X(n), \quad \text{if } \; v \sim X(n).$$

A subset A of $V(G)$ is reachable if for a random walk X on G started from $v_0 \in V(G)$ will hit A almost surely. This property in fact does not depend on the choice of v_0. If A is reachable, then A could be set as the boundary of G, and the pair (G, A) is called a graph with boundary, where the points of A are called boundary vertices, and the points of $V(G) \setminus A$ are called interior vertices. A random walk in G started from v_0 stopped on hitting A is called a random walk in (G, A) started from v_0. This random walk only hits one point of A, and has finite length almost surely. The loop-erasure of this stopped random walk is called the LERW in (G, A) started from v_0.

Now suppose D is a multiply connected (plane) domain, $0 \in \partial D$ and there is $a > 0$ such that the line segment $(0, a] \subset D$. Then as $z \to \partial D$ along $(0, a]$, z converges to a prime end of D. Let 0_+ denote that prime end. For $\delta > 0$, we consider $\delta\mathbb{Z}^2$ be as a graph whose edges are the pairs of nearest points of $\delta\mathbb{Z}^2$. Each edge of $\delta\mathbb{Z}^2$ could also be considered as a closed line segment. If $0 < \delta < a$, then $\delta = (\delta, 0) \in \delta\mathbb{Z}^2 \cap D$. Let $\widetilde{D}^\delta$ be the biggest connected subgraph of $\delta\mathbb{Z}^2$ containing δ whose vertices and edges are all contained in D. Let $\partial_E D^\delta$ be the set of pairs $\{p, v\}$ where $p \in \partial D$ and $v \in V(D^\delta)$ such that there is an edge e of $\delta\mathbb{Z}^2$ satisfying $[v, p) \subset e \cap D$. Let $\partial_V D^\delta$ be the set of $p \in \partial D$ that is contained in any edge of $\partial_E D^\delta$. We let D^δ be the union of $\widetilde{D}^\delta$ with the edge set $\partial_E D^\delta$ and the vertex set $\partial_V D^\delta$. Then D^δ is called the grid approximation of D in $\delta\mathbb{Z}^2$. It is a connected graph, and the degree of each vertex is at most 4.

The vertices and edges of $\widetilde{D}^\delta$ are called interior vertices and edges of D^δ; and $\partial_V D^\delta$ and $\partial_E D^\delta$ are called the sets of boundary vertices and edges of D^δ. From the recurrence property of a random walk on $\delta\mathbb{Z}^2$, we see that $\partial_V D^\delta$ is reachable, so it could be set as a boundary set of D^δ. And as $z \to \partial D$ along any edge $e \in \partial_E D^\delta$, z converges to a prime end of D. We say that e intersects or hits $\hat{\partial} D$ at that prime end. From the construction of D^δ, for any $v \in V(D^\delta)$ there is a path on D^δ connecting δ and v whose edges are all contained in D with the only possible exception at the last edge when $v \in \partial D$.

Now suppose $p_0 \in D \cap c\mathbb{Z}^2$ for some $c > 0$. Then if $n \in \mathbb{N}$ is big enough, we have $p_0 \in V(D^{\delta_n})$, where $\delta_n = c/n$. Let X be an LERW on $(D^{\delta_n}, \partial_V D^{\delta_n} \cup \{p_0\})$

started from δ_n conditioned to hit p_0, i.e., conditioned on the event that X stopes at p_0, which should have a positive probability because there is a path on D^{δ_n} from δ_n to p_0 without passing $\partial_V D^{\delta_n}$. It is also the loop-erasure of the random walk on $(D^{\delta_n}, \partial_V D^{\delta_n} \cup \{p_0\})$ started from δ_n conditioned to hit p_0. We construct a curve from X which is the union of all edges of X and the line segment $[0, \delta_n]$. This curve is a simple path from 0_+ to p_0. We expect that as $n \to \infty$, this curve converges to $\mathrm{HRLC}_2(D; 0_+ \to p_0)$ trace in a suitable sense.

Suppose I is a side arc of D. Then if $\delta > 0$ is small enough, there exist edges of $\partial_E D^\delta$ that intersect $\hat{\partial} D$ at I. Let X be an LERW on $(D^\delta, \partial_V D^\delta)$ started from δ conditioned to hit $\hat{\partial} D$ at I. We construct a curve from X which is the union of all edges of X and the line segment $[0, \delta]$. This curve is a simple path from 0 to a prime end on I. We expect that as $\varepsilon \to 0^+$, this curve converges to $\mathrm{HRLC}_2(D; 0_+ \to I)$ trace in a suitable sense.

Suppose $p_1 \in \partial D \cap c\mathbb{Z}^2$ for some $c > 0$, and ∂D is flat near p_1, which means that there is some $\varepsilon > 0$ such that $(p_1 + \varepsilon\mathbb{D}) \cap D = p_1 + (\varepsilon\mathbb{D} \cap u\mathbb{H})$, where $u = \pm 1$ or $\pm i$. Then p_1 represents a prime end of D. And if $n \in \mathbb{N}$ is big enough, we have $p_1 \in V(D^{\delta_n})$, where $\delta_n = c/n$. Let X be an LERW on $(D^{\delta_n}, \partial_V D^{\delta_n})$ started from δ_n conditioned to hit p_1. We construct a curve from X which is the union of all edges of X and the line segment $[0, \delta_n]$. This curve is a simple path from 0_+ to p_1. We expect that as $n \to \infty$, this curve converges to $\mathrm{HRLC}_2(D; 0_+ \to p_1)$ trace in a suitable sense.

Now we suppose $0 \in D$ instead of $0 \in \partial D$ and let $\widetilde{D}^\delta$ be the biggest connected subgraph of $\delta\mathbb{Z}^2$ containing 0 whose vertices and edges are all contained in D. Construct $\mathbb{D}^\delta$ from $\widetilde{D}^\delta$ in the same way as before. Suppose I is a side arc of D. Let X be an LERW on $(D^\delta, \partial_V D^\delta)$ started from 0 conditioned to hit $\hat{\partial} D$ at I. We construct a curve from X which is the union of all edges of X. We expect that as $\varepsilon \to 0^+$, this curve converges to the interior $\mathrm{HRLC}_2(D; 0 \to I)$ trace in a suitable sense. Similarly, we may construct LERW that is supposed to converge to the interior $\mathrm{HRLC}_2(D; 0 \to p)$ trace for an interior point p or a prime end p.

4.2 Observables

4.2.1 Observables for LERW

Suppose f is defined on $V(G)$, the vertex set of G. Then $\Delta_G f$ is defined by

$$\Delta_G(f)(v) = \sum_{w\sim v}(f(w) - f(v)).$$

Suppose S_1, S_2, S_3 are subsets of $V(G)$, let $\Gamma^{S_3}_{S_1,S_2}$ denote the path X of finite length m for some $m \in \overline{\mathbb{N}}$ such that $X(0) \in S_1$, $X(m) \in S_2$, and $X(n) \in S_3$ for $1 \le n \le m-1$. Suppose X is a path of finite length m. The reversal of X is a path $R(X)$ defined on $\overline{\mathbb{N}}_m$ such that $R(X)(n) = X(m-n)$. Let P_0, P_1 and P_2 be defined by

$$P_0(X) = \prod_{j=1}^{m-1} \frac{1}{\deg(X(j))},\ P_1(X) = \prod_{j=0}^{m-1} \frac{1}{\deg(X(j))},\ P_2(X) = \prod_{j=0}^{m} \frac{1}{\deg(X(j))}.$$

Then $P_0(X) = P_0(R(X))$ and $P_2(X) = P_2(R(X))$.

Lemma 4.2.1 *Suppose A and B are disjoint subsets of V, and $A\cup B$ is reachable. Let $f(v)$ be the probability that the random walk on $(G, A\cup B)$ started from v hits A. Then f is the unique bounded function on V that satisfies $f \equiv 1$ on A, $f \equiv 0$ on B, and $\Delta_G f \equiv 0$ on $C := V(G) \setminus (A \cup B)$. Moreover $\sum_{v\in B} \Delta_G f(v) = -\sum_{v\in A} \Delta_G f(v) > 0$.*

Proof. The proof is elementary. For the last statement, note that $\sum_{v\in B} \Delta_G f(v) = \sum P_0(X)$ where X runs over the non-empty set $\Gamma^C_{B,A}$; and $-\sum_{v\in A} \Delta_G f(v) = \sum P_0(X)$ where X runs over $\Gamma^C_{A,B}$. The values of the two summations are equal because the reverse map R is a one-to-one correspondence between $\Gamma^C_{B,A}$ and $\Gamma^C_{A,B}$, and $P_0(X) = P_0(R(X))$. □

Lemma 4.2.2 *Let A, B, C and f be as in Lemma 4.2.1. Fix $x \in C$. Let $h(v)$ be equal to the probability that a random walk on $(G, A\cup B)$ started from v hits x. Then*

$$\sum_{v\in A} \Delta_G h(v) = f(x)(-\Delta_G h(x)).$$

Proof. From the proof of Lemma 4.2.1, we have

$$f(x) = \sum_{X \in \Gamma^C_{x,A}} P_1(X) = \sum_{Y \in \Gamma^C_{x,x}} P_2(Y) \sum_{Z \in \Gamma^{C\setminus\{x\}}_{x,A}} P_0(Z) = \sum_{Y \in \Gamma^C_{x,x}} P_2(Y) \sum_{v \in A} \Delta_G h(v),$$

and

$$1 = \sum_{X \in \Gamma^C_{x,A\cup B}} P_1(X) = \sum_{Y \in \Gamma^C_{x,x}} P_2(Y) \sum_{Z \in \Gamma^{C\setminus\{x\}}_{x,A\cup B}} P_0(Z) = \sum_{Y \in \Gamma^C_{x,x}} P_2(Y)(-\Delta_G h(x)).$$

So we proved this lemma. □

Let $L(A,B) = \sum_{v\in B} \Delta_G f(v)$ for the f in Lemma 4.2.1. Then $L(A,B) = L(B,A)$ is non-decreasing in both A and B. If A or B is finite, then $L(A,B) < \infty$. Suppose $L(A,B) < \infty$. If $x \in C = V(G)\setminus A\setminus B$, then $L(A\cup\{x\},B) \le L(A,B)+L(x,B) < \infty$. Thus if $A' \setminus A$ and $B' \setminus B$ are both finite, then $L(A',B') < \infty$.

Lemma 4.2.3 *Let A, B, C and f be as in Lemma 4.2.1. Suppose $L(A,B) < \infty$. Fix $x \in C$ such that $f(x) > 0$. Then there is a unique bounded function g on V such that $g \equiv 1$ on A; $g \equiv 0$ on B; $\Delta_G g \equiv 0$ on $C\setminus\{x\}$; and $\sum_{v\in A} \Delta_G g(v) = 0$. Moreover, such g is non-negative and satisfies $\sum_{v\in B\cup\{x\}} \Delta_G g(v) = 0$ and $\Delta_G g(x) = -L(A,B)/f(x)$.*

Proof. Suppose g satisfies the first group of properties. Let $I = g - f$. Then I is bounded, $I \equiv 0$ on $A \cup B$ and $\Delta_G I \equiv 0$ on $C \setminus \{x\}$. Thus $I(v) = I(x)h(v)$, where h is as in Lemma 4.2.2. Then by Lemma 4.2.1 and 4.2.2,

$$0 = \sum_{v\in A} \Delta_G g(v) = \sum_{v\in A} \Delta_G (I+f)(v) = -I(x)f(x)\Delta_G h(x) - L(A,B).$$

Thus $I(x) = L(A,B)/(-f(x)\Delta_G h(x))$ is uniquely determined. Therefore g is unique.

On the other hand, if we define $g = f + hL(A,B)/(-f(x)\Delta_G h(x))$, then from the last paragraph, we see that g satisfies the first group of properties. Since f and

h are non-negative, and $-\Delta_G h(x) = L(x, A \cup B) > 0$ by Lemma 4.2.1, so g is also non-negative. By Lemma 4.2.1 and 4.2.2,

$$\sum_{v \in B \cup \{x\}} \Delta_G g(v) = L(A,B) + \Delta_G f(x) + \sum_{v \in B \cup \{x\}} \Delta_G h(v) L(A,B)/(-f(x)\Delta_G h(x))$$

$$= L(A,B) - \sum_{v \in A} \Delta_G h(v) L(A,B)/(-f(x)\Delta_G h(x)) = L(A,B) - L(A,B) = 0.$$

Finally, $\Delta_G g(x) = \Delta_G h(x) \cdot L(A,B)/(-f(x)\Delta_G h(x)) = -L(A,B)/f(x)$. $\square$

Let A, B and f be as in Lemma 4.2.1. Suppose $v_0 \in V(G)$ is such that $f(v_0) > 0$. Then a random walk on $(G, A \cup B)$ started from v_0 hits A with a positive probability, and so does the LERW on $(G, A \cup B)$ started from v_0. Let X be a LERW on $(G, A \cup B)$ started from v_0 conditioned to hit A. Suppose $X(n)$ is defined and does not lie on A. Then from [3] we know that

$$\Pr[X(n+1) = v | X(0), \ldots, X(n)] = \frac{f_n(v)}{\sum_{w \sim X(n)} f_n(w)}, \text{ if } v \sim X(n);$$

$= 0$ if $v \not\sim X(n)$, where f_n is the f in Lemma 4.2.1 with B replaced by $B_n := B \cup \{X(j), 0 \le j \le n\}$.

Now we assume that $L(A,B) < \infty$. Let f_n and B_n be as above. Let $C_n := V(G) \backslash A \backslash B_n$. Let g_n be the g in Lemma 4.2.3 with $x = X(n)$ and B replaced by B_{n-1}. One should note that $L(A, B_n) < \infty$ because $B_n \setminus B$ is finite, and $f_{n-1}(X(n)) > 0$ because $X(n + \cdot)$ is a path from $X(n)$ to A without passing through B_{n-1}.

Theorem 4.2.1 *Let $\overline{A}$ be the union of A with all vertices of G that are adjacent to A. Fix any $w_0 \in V(G) \setminus B \setminus \{v_0\}$. Conditioned on the event that $X(j) = v_j$, $0 \le j \le k$, $v_k \notin \overline{A}$, and $f_k(w_0) > 0$, the expectation of $g_{k+1}(w_0)$ is equal to $g_k(w_0)$, which is determined by v_j, $0 \le j \le k$. Thus $g_k(w_0)$ is a discrete martingale up to the first time X hits $\overline{A}$, or B_k disconnects w_0 from A.*

Proof. Let S be the set of v such that $v \sim v_k$ and $f_k(v) > 0$. Then the conditional probability that $X(k+1) = u$ is $f_k(u)/\sum_{v \in S} f_k(v)$ for $u \in S$. For $v \in S$, let g_{k+1}^v

be the g in Lemma 4.2.3 with $x = v$ and B replaced by B_k. Then with probability $f_k(u)/\sum_{v\in S} f_k(v)$, $g_{k+1} = g^u_{k+1}$. Thus the conditional expectation of $g_{k+1}(w_0)$ is equal to $\widetilde{g}_k(w_0)$, where

$$\widetilde{g}_k(v) := \sum_{u\in S} f_k(u) g^u_{k+1}(v) / \sum_{u\in S} f_k(u).$$

Then $\widetilde{g}_k \equiv 0$ on B_k, $\equiv 1$ on A; $\Delta_G \widetilde{g}_k \equiv 0$ on $C_k \setminus S$, and $\sum_{v\in A} \Delta_G \widetilde{g}_k(v) = 0$. Moreover,

$$\Delta_G \widetilde{g}_k(v) = \frac{f_k(v)\Delta_G g^v_{k+1}(v)}{\sum_{u\in S} f_k(u)} = -\frac{L(B_k, A)}{\sum_{u\in S} f_k(u)}, \quad \forall v \in S,$$

by Lemma 4.2.3. Now define $\widehat{g}_k$ on $V(G)$ such that $\widehat{g}_k(w_k) = L(B_k, A)/\sum_{u\in S} f_k(u)$; $\widehat{g}_k(v)$ equals to $\widehat{g}_k(w_k)$ times the probability that a simple random walk on G started from v hits w_k before B_{k-1} for those $v \in C_k$ such that $f_k(v) = 0$; and $\widehat{g}_k(v) = \widetilde{g}_k(v)$ for all other $v \in V(G)$. Then $\Delta_G \widehat{g}_k \equiv 0$ on C_k, $\widehat{g}_k \equiv 0$ on $B_k \setminus \{w_k\}$, and $\widehat{g}_k \equiv 1$ on A. Since $w_k \notin \overline{A}$, and for $v \in C_k$ such that $f_k(v) = 0$ we have $v \notin \overline{A}$, so $\sum_{v\in A} \Delta_G \widehat{g}_k(v) = \sum_{v\in A} \Delta \widetilde{g}_k(v) = 0$. Now $\widehat{g}_k$ satisfies all properties of g_k. The uniqueness of g_k implies that $\widehat{g}_k \equiv g_k$. Since $f_k(w_0) > 0$, we have $g_k(w_0) = \widehat{g}_k(w_0) = \widetilde{g}_k(w_0)$. $\square$

Remark. The functions g_k are called observables for this LERW. We may have different kinds of observables for an LERW. Let $h_k(v)$ be the probability that a random walk on $(G, A \cup B_k)$ started from v hits $X(k)$. Let $g'_k(v) = c_k h_k(v)$, where $c_k > 0$ is chosen so that $\sum_{v\in A} \Delta_G g'_k(v) = 1$. Then $g'_k \equiv 0$ on $A \cup B_k \setminus \{X(k)\}$ and $\Delta_G g'_k \equiv 0$ on C_k. The definition of g'_k does not require that $L(A, B) < \infty$. And Theorem 4.2.1 still holds if g_k is replaced by g'_k.

4.2.2 Observables for HRLC$_2$

Suppose D is a finite Riemann surface, $p_0 \in D$ and w_0 is a prime end of D. Let L has the law of HRLC$_2(D; w_0 \to p_0)$. For each $t \in [0, \Delta(L))$, let $w(t)$ be the prime end (of $D \setminus L(t)$) determined by L at time t. Let M_t be the minimal function in $D \setminus L(t)$ with the pole at $w(t)$, normalized by $M_t(p_0) = 1$.

Theorem 4.2.2 *For any fixed $z \in D$, $(M_t(z), 0 \le t < T_z)$ is a local martingale,* *where T_z is the fist t such that $z \in L(t)$ or $t = \Delta(L)$.*

Proof. Suppose α is the side of D that contains w_0. Let Ω be a neighborhood of $\partial\mathbb{D}$ in $\overline{\mathbb{D}} \setminus \{0\}$, Σ a neighborhood of α in $\widehat{D} \setminus \{p_0\}$, and W map Ω conformally onto Σ such that $W(\partial\mathbb{D}) = \alpha$ and $W(1) = w_0$. Let L_Σ be the part of L that is contained in Σ. Let L' be a time change of L_Σ such that the capacity of $K_t := W^{-1}(L'(t))$ in $\mathbb{D}$ w.r.t. 0 is t. Let $w'(t)$ be the prime end determined by L' at time t. Let M'_t be the minimal function in $D \setminus L'(t)$ with the pole at $w'(t)$ normalized by $M'_t(p_0) = 1$. Since the property of local martingale does not change after a time-change. It suffices to show that $M'_t(z)$ is a local martingale for any $z \in D$.

Now (K_t) is a family of standard radial LE hulls driven by some function ξ with $\xi(0) = 0$. Let φ_t be the corresponding LE maps. From the definition of HRLC, there is a standard Brownian motion $B(t)$ such that

$$d\xi(t) = \sqrt{2}dB(t) + 2(\partial_x\partial_y/\partial_y)(G_t \circ W \circ \varphi_t^{-1} \circ e^i)(\xi(t))dt,$$

where G_t is the Green function in $D \setminus L'(t)$ with the pole at p_0.

Note that $W \circ \varphi_t^{-1}$ maps $\partial\mathbb{D}$ to the side α_t of $D \setminus L'(t)$ that contains $w'(t)$, and $w'(t) = W \circ \varphi_t^{-1} \circ e^i(\xi(t))$. Let $S(t, w; \cdot)$ be the minimal function in $D \setminus L'(t)$ with the pole at $W \circ \varphi_t^{-1} \circ e^i(w)$, normalized by $S(t, w; p_0) = 1$. Then $M'_t(z) = S(t, \xi(t); z)$. It is standard to check that S is $C^{1,2,\infty}$ continuous. Thus $M'_t(z)$ is a semi-martingale. We may write $dM'_t(z) = I^1_t(z)dB(t) + I^2_t(z)dt$. Since all M'_t are harmonic and vanish on the sides of D other than α, so I^1_t and I^2_t should also have these properties.

Define $f(t, w; z) = S(t, w; W \circ \varphi_t^{-1} \circ e^i(z))$. Then $M'_t(W(e^{iz})) = f(t, \xi(t); \widetilde{\varphi}_t(z))$, where $\widetilde{\varphi}_t$ is a conformal map that satisfies $e^i \circ \widetilde{\varphi}_t = \varphi_t \circ e^i$. From radial Loewner equation, we have that $\widetilde{\varphi}_t$ satisfies $\partial_t\widetilde{\varphi}_t(z) = \cot((\widetilde{\varphi}_t(z) - \xi(t))/2)$. From Ito's formula, we have

$$dM'_t(W(e^{iz})) = \partial_t f dt + \partial_w f \cdot 2(\partial_x\partial_y/\partial_y)(G_t \circ W \circ \varphi_t^{-1} \circ e^i)(\xi(t))dt + \partial_w^2 f dt$$

$$+\partial_x f \mathrm{Re}\cot(\frac{\widetilde{\varphi}_t(z) - \xi(t)}{2})dt + \partial_y f \mathrm{Im}\cot(\frac{\widetilde{\varphi}_t(z) - \xi(t)}{2})dt + \partial_w f \cdot \sqrt{2}dB(t),$$

where all derivatives of f are valued at $(t, \xi(t); \widetilde{\varphi}_t(z))$ and ∂_x and ∂_y are derivatives with respect to the last parameter $z \in \mathbb{C}$. Define

$$\mathbf{D}_t f(z) = \partial_t f + \partial_w^2 f + \partial_w f \cdot 2(\partial_z^2/\partial_z)(G_t \circ W \circ \varphi_t^{-1} \circ e^i)(\xi(t))$$

$$+\partial_x f \mathrm{Re}\, \cot(\frac{z-\xi(t)}{2})dt + \partial_y f \mathrm{Im}\, \cot(\frac{z-\xi(t)}{2}),$$

where all derivatives of f are valued at $(t, \xi(t); z)$. Then

$$dM_t'(W(e^{iz})) = \mathbf{D}_t f(\widetilde{\varphi}_t(z))dt + \partial_w f(t, \xi(t); \widetilde{\varphi}_t(z)) \cdot \sqrt{2}dB(t). \tag{4.2.1}$$

Note that $f(t, w; \cdot)$ has simple poles at $w + 2k\pi$, vanishes otherwhere on $\mathbb{R}$, and has period 2π. So we may write $f(t, w; z) = p(t, w; z) + r(t, w; z)$ such that $p(t, w; z) = c(t, w)\mathrm{Im}\, \cot((z - w)/2)$ for some $c(t, w) \in \mathbb{R}$ and $r(t, w; \cdot)$ vanishes everywhere on $\mathbb{R}$. It is clear that $\mathbf{D}_t f$ vanishes on $\mathbb{R} \setminus \{\xi(t) + 2k\pi : k \in \mathbb{Z}\}$. Now $\mathbf{D}_t f = \mathbf{D}_t p + \mathbf{D}_t r$. $\mathbf{D}_t r$ contributes at most a simple pole at $\xi(t)$, which comes from the terms

$$\partial_x r(t, \xi(t); z)\mathrm{Re}\, \cot(\frac{z-\xi(t)}{2})dt + \partial_y r(t, \xi(t); z)\mathrm{Im}\, \cot(\frac{z-\xi(t)}{2}).$$

$\mathbf{D}_t p$ contributes poles of order at most 3 at $\xi(t)$. The pole of order 3 comes from the terms

$$\partial_w^2 p(t, \xi(t); z) + \partial_x p(t, \xi(t); z)\mathrm{Re}\, \cot(\frac{z-\xi(t)}{2})dt + \partial_y p(t, \xi(t); z)\mathrm{Im}\, \cot(\frac{z-\xi(t)}{2}).$$

Plug in $p(t, w; z) = c(t, w)\mathrm{Im}\, \cot((z - w)/2)$. We find that the same pole comes from

$$c(t, \xi(t))\mathrm{Im}\, (\partial_w^2 \cot(\frac{z-w}{2})|_{w=\xi(t)} + \partial_z \cot(\frac{z-\xi(t)}{2}) \cot(\frac{z-\xi(t)}{2})),$$

which is in fact constant 0. Thus $\mathbf{D}_t p$ contributes no pole of order 3 at $\xi(t)$. Now the pole of order 2 comes from the terms

$$\partial_w^2 p(t, \xi(t); z) + \partial_w p(t, \xi(t); z)2(\partial_z^2/\partial_z)(G_t \circ W \circ \varphi_t^{-1} \circ e^i)(\xi(t)),$$

which has the same coefficient in $1/(z-w)^2$ as

$$2\partial_w c(t,\xi(t))\partial_w \cot(\frac{z-w}{2})|_{w=\xi(t)}$$

$$+c(t,\xi(t))\partial_w \cot(\frac{z-w}{2})|_{w=\xi(t)}2(\partial_z^2/\partial_z)(G_t\circ W\circ\varphi_t^{-1}\circ e^i)(\xi(t)).$$

We now want to prove that the pole of order 2 vanishes, we need that

$$\partial_w \ln c(t,\xi(t)) = -\partial_x \ln(\partial_y(G_t\circ W\circ\varphi_t^{-1}\circ e^i))(\xi(t)).$$

It suffices to show that $c(t,w)\partial_y(G_t\circ W\circ\varphi_t^{-1}\circ e^i)(w)$ is constant in w.

Choose an analytic Jordan curve β_0 in Σ such that the domain bounded by α and β_0 is contained in Σ. Choose J that maps a neighborhood of 0 in $\mathbb{C}$ conformally onto some neighborhood of p_0 such that $J(0)=p_0$. For $\varepsilon>0$ small enough, $\beta_\varepsilon = J(\varepsilon\partial\mathbb{D})$ is well defined. Let V be the subdomain of D bounded by β_0, β_ε and the sides of D other than α. Apply Green's formula to the region V. We compute

$$\int_{\partial V} S(t,w;z)\partial_{\mathbf{n}}G_t(z)ds(z) = \int_{\partial V} G_t(z)\partial_{\mathbf{n}}S(t,w;z)ds(z),$$

$\mathbf{n}$ denoting the outward unit normal vector on ∂V. Since G_t and $S(t,w;\cdot)$ vanish on sides of D other than α, so the integrations above can be restricted to $\beta_0\cup\beta_\varepsilon$. Since on β_ε, $G_t(z)=O(-\ln\varepsilon)$, $\partial_{\mathbf{n}}S_{t,w}(z)=O(1)$, and the length of β_ε is $O(\varepsilon)$, so as $\varepsilon\to 0^+$, we have $\int_{\beta_\varepsilon} G_t(z)\partial_{\mathbf{n}}S_{t,w}(z)ds(z)\to 0$. On the other hand, we find that $\int_{\beta_\varepsilon}|\partial_{\mathbf{n}}G_t(z)|ds(z)$ is uniformly bounded in ε, and $\int_{\beta_\varepsilon}\partial_{\mathbf{n}}G_t(z)ds(z)=1$. Since the value of $S_{t,w}$ at β_ε tends to $S_{t,w}(a)=1$, so $\int_{\beta_\varepsilon} S_{t,w}(z)\partial_{\mathbf{n}}G_t(z)ds(z)\to 1$ as $\varepsilon\to 0^+$. Thus

$$1=\int_{\beta_0} G_t(z)\partial_{\mathbf{n}}S_{t,w}(z)ds(z) - \int_{\beta_0} S_{t,w}(z)\partial_{\mathbf{n}}G_t(z)ds(z)$$

$$=\int_{\varphi_t\circ W^{-1}(\beta_0)} G_t\circ W\circ\varphi_t^{-1}(z)\partial_{\mathbf{n}}S(t,w;\cdot)\circ W\circ\varphi_t^{-1}(z)ds(z)$$

$$-\int_{\varphi_t\circ W^{-1}(\beta_0)} S(t,w;\cdot)\circ W\circ\varphi_t^{-1}(z)\partial_{\mathbf{n}}G_t\circ W\circ\varphi_t^{-1}(z)ds(z).$$

For simplicity of notations, we write $\widetilde{h}_t = G_t \circ W \circ \varphi_t^{-1}$, $h_t = \widetilde{h}_t \circ e^i$, and $\widetilde{f}(t, w; \cdot) = S(t, w; \cdot) \circ W \circ \varphi_t^{-1}$. Then we see that $f = \widetilde{f} \circ e^i$.

For $\varepsilon > 0$ small, let γ_ε be the image of $\{w + \varepsilon e^{i\theta} : 0 \le \theta \le \pi\} \cup [w + \varepsilon, w + 2\pi - \varepsilon]$ under the map e^i. Let U denote the domain bounded by $\varphi_t \circ W^{-1}(\beta_0)$ and γ_ε. Apply Green's formula to U. Note that the outer unit normal vectors for V and for U at each point of $\varphi_t \circ W^{-1}(\beta_0)$ are opposite to each other. Thus

$$1 = \int_{\gamma_\varepsilon} \widetilde{h}_t(z)\partial_{\mathbf{n}}\widetilde{f}(t, w; z) - \widetilde{f}(t, w; z)\partial_{\mathbf{n}}\widetilde{h}_t(z)ds(z)$$

$$= \int_{e^i(\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\})} \widetilde{h}_t(z)\partial_{\mathbf{n}}\widetilde{f}(t, w; z) - \widetilde{f}(t, w; z)\partial_{\mathbf{n}}\widetilde{h}_t(z)ds(z)$$

$$= \int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}} h_t(z)\partial_{\mathbf{n}}f(t, w; z) - f(t, w; z)\partial_{\mathbf{n}}h_t(z)ds(z),$$

where $\mathbf{n}$ is the unit normal vector pointed towards w. Remember that $f(t, w; z) = p(t, w; z) + r(t, w; z)$, $p(t, w; z) = c(t, w)\text{Im}\cot((z - w)/2)$, and $r(t, w; \cdot)$ has no pole and vanishes every where on $\mathbb{R}$. Let $\widehat{p}(t, w; z) = c(t, w)\text{Im}\,(2/(z-w))$, and $\widehat{r}(t, w; z) = f(t, w; z) - \widehat{p}(t, w; z)$. Then $\widehat{r}$ has the similar property as r. And we have

$$1 = \int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}} h_t(z)\partial_{\mathbf{n}}\widehat{p}(t, w; z) - \widehat{p}(t, w; z)\partial_{\mathbf{n}}h_t(z)ds(z).$$

There is an analytic function F_t in $w + \varepsilon\mathbb{D}$ such that $h_t = \text{Im}\, F_t$ and $F_t(w) = 0$. Then $F_t'(w) = \partial_y h_t(w)$. We may write $F_t(z) = \partial_y h_t(w)(z - w) + \widehat{F}_t(z)$. On the circle $w + \varepsilon\partial\mathbb{D}$, $|\widehat{F}_t(z)| = O(\varepsilon^2)$, $|\partial_{\mathbf{n}}\widehat{F}_t(z)| \le |\widehat{F}_t'(z)| = O(\varepsilon)$, $\widehat{p}(t, w; z) = O(\varepsilon^{-1})$, $\partial_{\mathbf{n}}\widehat{p}(t, w; z) = O(\varepsilon^{-2})$, and $\int_{w+\varepsilon\partial\mathbb{D}} ds = O(\varepsilon)$. Thus

$$\int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}} \text{Im}\,\widehat{F}_t(z)\partial_{\mathbf{n}}\widehat{p}(t, w; z) - \widehat{p}(t, w; z)\partial_{\mathbf{n}}\text{Im}\,\widehat{F}_t(z)ds(z) = O(\varepsilon),$$

from which follows that

$$1 = 2\partial_y h_t(w)c(t, w) \lim_{\varepsilon\to 0^+} \int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}} \text{Im}\,(z - w)\partial_{\mathbf{n}}\text{Im}\,\frac{1}{z - w}ds(z)$$

$$-2\partial_y h_t(w)c(t,w)\lim_{\varepsilon\to 0^+}\int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}}\mathrm{Im}\,\frac{1}{z-w}\partial_{\mathbf{n}}\mathrm{Im}\,(z-w)ds(z)$$

Now

$$\int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}}\mathrm{Im}\,(z-w)\partial_{\mathbf{n}}\mathrm{Im}\,\frac{1}{z-w}ds(z)=-\pi/2;$$

and

$$\int_{\{w+\varepsilon e^{i\theta}:0\le\theta\le\pi\}}\mathrm{Im}\,\frac{1}{z-w}\partial_{\mathbf{n}}\mathrm{Im}\,(z-w)ds(z)=-\pi/2.$$

Thus

$$c(t,w)\partial_y(G_t\circ W\circ\varphi_t^{-1}\circ e^i)(w)=c(t,w)\partial_y h_t(w)=-1/(2\pi)$$

is constant in w.

So we proved that the pole of $\mathbf{D}_t f$ at $\xi(t)$ of order 2 vanishes. From formula (4.2.1), $I_t^2(W(e^{iz}))=\mathbf{D}_t f(\widetilde{\varphi}_t(z))$. Thus $I_t^2\circ W\circ\varphi_t^{-1}\circ e^i=\mathbf{D}_t f$. This means that $I_t^2\circ W\circ\varphi_t^{-1}\circ e^i$ vanishes on $\mathbb{R}\setminus\{\xi(t)+2k\pi : k\in\mathbb{Z}\}$, and has at most simples at $\{\xi(t)+2k\pi : k\in\mathbb{Z}\}$. So I_t^2 is equal to some $c_t\in\mathbb{R}$ times M_t'. Since $M_t'(p_0)\equiv 1$, we have $I_t^2(p_0)\equiv 0$, so $c_t=0$ and $I_t^2\equiv 0$. Thus $dM_t'(z)=I_t^1 dB(t)$, which means that $M_t'(z)$ is a local martingale. After a time-change, we have proved that $M_t(z)$ is a local martingale up to the time that $L(t)$ is not contained in Σ. Since we can find a sequence of Σ_n such that $\Delta(L)=\vee T_n$, where T_n is the first time L leaves Σ_n, so $M_t(z)$ is a local martingale for $t\in[0,\Delta(L))$. □

The functions M_t are called observables for L. Now suppose L has the law of $\mathrm{HRLC}_2(D;w_0\to w_1)$, where w_1 is a prime end of D other than w_0. Let Q maps a neighborhood U of w_1 conformally onto $\mathbb{H}$ such that $Q(U\cap\hat\partial D)\subset\mathbb{R}$. Let M_t be the minimal function in $D\setminus L(t)$ with the pole at $w(t)$, the prime end determined by L at time t, normalized by $\partial_y M_t\circ Q^{-1}(Q(p_1))=1$. Then Theorem 4.2.2 also holds.

Suppose I is an side arc of D such that $w_0\notin\overline{I}$. Let L have the law of $\mathrm{HRLC}_2(D;w_0\to I)$. Let M_t be the minimal function in $D\setminus L(t)$ with the pole at $w(t)$, the prime end determined by L at time t, normalized by $\int_I\partial_{\mathbf{n}}M_t(w)ds(w)=1$, where $\mathbf{n}$ is the inward unit normal vector. Then Theorem 4.2.2 still holds. Suppose I is a whole side of D. Let h_t be the harmonic measure function of I in $D\setminus L(t)$. There is $c_t>0$ such that $M_t^1 := h_t + c_t M_t$ satisfies $\int_I \partial_{\mathbf{n}} M_t^1(w) ds(w) = 0$. Then Theorem 4.2.2 holds with M_t replaced by M_t^1.

If L has the law of an interior HRLC_2 aiming at an interior point, a prime end, or a side arc, then we could define functions M_t or M_t' in the same way as we define them for HRLC_2 started from a prime end. Then for any fixed $z \in D$, $(M_t(z), -\infty < t < T_z)$ is a local martingale, and the same is true for M_t'.

4.2.3 Resemblance

We now consider the LERWs on D^δ defined in Section 4.1. Suppose X is an LERW on $(D^{\delta_m}, \partial_V D^{\delta_m} \cup \{p_0\})$ started from δ_m conditioned to hit $p_0 \in D \cap c\mathbb{Z}^2$, where $\delta_m = c/m$. Let $A = \{p_0\}$, $B = \partial_V D^{\delta_m}$, $B_n = B \cup \{X(j) : 0 \le j \le n\}$ and $C_n = V(D^{\delta_m}) \cap D \setminus B_n$. Let g_n be as in Theorem 4.2.1. Then g_n is discrete harmonic on C_n, vanishes on B_{n-1}, and $g_n(p_0) = 1$. So when m is big, g_n is closed to the minimal function M in $D \setminus \cup_{j=1}^n [X(j-1), X(j)]$ with the pole at $X(n)$, normalized by $M(p_0) = 1$. And M is similar to the observable M_t for $\mathrm{HRLC}_2(D; 0_+ \to p_0)$.

Suppose X is an LERW on $(D^{\delta_m}, \partial_V D^{\delta_m})$ started from δ_m conditioned to hit $p_1 \in \partial D \cap c\mathbb{Z}^2$, where ∂D is flat near p_1, and $\delta_m = c/m$. Let $A = \{p_1\}$, $B = \partial_V D^{\delta_m} \setminus A$, and B_n and C_n be defined similarly as above. Let g_n' be as in the remark after Theorem 4.2.1. Then g_n is discrete harmonic on C_n, vanishes on $A \cup B_{n-1}$, and $\Delta g_n(p_1) = 1$. So when m is big, $\delta_m g_n$ is closed to the minimal function M in $D \setminus \cup_{j=1}^n [X(j-1), X(j)]$ with the pole at $X(n)$, normalized by $\partial_{\mathbf{n}} M(p_1) = 1$. And M is similar to the observable M_t for $\mathrm{HRLC}_2(D; 0_+ \to p_1)$.

Suppose X is an LERW on $(D^\delta, \partial_V D^\delta)$ started from δ_m conditioned to hit a side arc I of D. Let A be the set of vertices of $\partial_V D^\delta$ that lie on I, $B = \partial_V D^\delta \setminus A$, and B_n and C_n be defined similarly as above. Let g_n' be as in the remark after Theorem 4.2.1. Then g_n is discrete harmonic on C_n, vanishes on $A \cup B_{n-1}$, and $\sum_{v \in A} \Delta g(v) = 1$. So when δ is small, g_n is closed to the minimal function M in $D \setminus \cup_{j=1}^n [X(j-1), X(j)]$ with the pole at $X(n)$, normalized by $\int_I \partial_{\mathbf{n}} M ds = 1$. And M is similar to the observable M_t for $\mathrm{HRLC}_2(D; 0_+ \to I)$. If I is a whole side of D, we could let g_n be as in Theorem

4.2.1. Then when δ is small, g_n is closed to M', which is linear combination of M and the harmonic measure function of I in $D \setminus \cup_{j=1}^{n}[X(j-1), X(j)]$ such that $M' \equiv 1$ on I and $\int_I \partial_{\mathbf{n}} M ds = 0$. This M' is similar to the observable M'_t for $\mathrm{HRLC}_2(D; 0_+ \to I)$.

Similarly, in the case that $0 \in D$, the observables for an LERW on D^δ started from 0 conditioned to hit certain vertex set resemble the observables for the corresponding interior HRLC_2 in D started from 0.

4.3 Convergence of LERW to HRLC_2

It is proved in [7] that a LERW on a discrete approximation of a simply connected domain D started from $0 \in D$ stopped on hitting the boundary converges to radial $\mathrm{SLE}_2(D; \mathbf{x} \to 0)$ trace, where $\mathbf{x}$ is a random point on ∂D that has harmonic measure in D valued at 0. In that proof, first the observables for LERW are given; then they are proved to converge to some continuous harmonic functions; these facts are then used to show that the driving function of the LERW converges to a Brownian motion with speed 2; finally some nice behaviors of LERW paths are used to show that the LERW curve converges to the radial SLE_2 trace uniformly in probability.

In that paper, some subgraph of $\mathbb{Z}^2$ is used to approximate the simply connected domain, and the inner radius with respect to the target point (which is 0 there) is used to describe the extent that the graph approximates the domain. After a rescaling, the inner radius means the distance from 0 to the boundary of the domain divided by the length of the mesh. However, it seems not easy to find counterparts of the inner radius for other types of HRLC_2.

In this section we will show that when a doubly connected domain D has the property as in Section 4.1 with $0 \in \partial D$, I_1 is the side that contains the prime end 0_+, and I_2 is the other side of D, the LERW on $(D^\delta, \partial_V D^\delta)$ started from δ conditioned to hit I_2 converges to the annulus $\mathrm{SLE}_2(D; 0_+ \to I_2)$ trace as $\delta \to 0$. The content is chosen from the paper [20]. We will follow the order of the proof in [7]. To prove the convergence of observables for LERW to a continuous harmonic function, we use the method of domain convergence. Although it is still about a special case, the proof seems more likely to be extended to general cases.

For $\delta > 0$ small enough, let X^δ be the LERW on $(D^\delta, \partial_V D^\delta)$ started from δ conditioned to hit I_2. Let υ be the length of X^δ, i.e., $X^\delta(\upsilon) \in I_2$. Let $X^\delta(-1) = 0$. We may extend X^δ to be a continuous function on $[-1, \upsilon]$ such that X^δ is linear on $[n-1, n]$, $0 \le n \le \upsilon$. For $-1 \le s < \upsilon$, let $T(s)$ be the capacity of $X^\delta(-1, s]$ in D w.r.t. I_2, then T is a continuous increasing function and maps $[-1, \upsilon)$ onto $[0, p)$, where p is the modulus of D. Let S be the inverse function of T. Let $\beta^\delta(t) = X^\delta(S(t))$. Let β^0 be an annulus $\mathrm{SLE}_2(D; 0_+ \to I_2)$ trace.

Theorem 4.3.1 *For every $q \in (0, p)$ and $\varepsilon > 0$, there is a $\delta_0 > 0$ depending on q and ε such that for $\delta \in (0, \delta_0)$ there is a coupling of the processes β^δ and β^0 such that*

$$\Pr[\sup\{|\beta^\delta(t) - \beta^0(t)| : t \in [q, p)\} > \varepsilon] < \varepsilon.$$

Moreover, if the impression of the prime end 0_+ is a single point, then the theorem holds with $q = 0$.

Here a coupling of two random processes A and B is a probability space with two random processes A' and B', where A' and B' have the same law as A and B, respectively. In the above statement (as is customary) we don't distinguish between A and A' and between B and B'. The impression (see [12]) of a prime end is the intersection of the closures in $\mathbb{C}$ of all neighborhoods of that prime end.

4.3.1 Convergence of the driving functions

For $a < b$, let $\mathbb{A}_{a,b}$ be the annulus bounded by $\mathbf{C}_a$ and $\mathbf{C}_b$. For any $0 < q < p$, there is a smallest $l(p,q) \in (0, p)$ such that if K is a hull in $\mathbb{A}_p$ on $\mathbf{C}_0$ with the capacity (w.r.t. $\mathbf{C}_p$) less than q, then K does not intersect $\mathbf{C}_{l(p,q)}$. Using the fact that for any $0 < s \le r$, $\mathrm{Re}\,\mathbf{S}_r$ attains its unique maximum and minimum on $\overline{\mathbb{A}_{s,r}}$ at e^{-s} and $-e^{-s}$, respectively, it is not hard to derive the following Lemma.

Lemma 4.3.1 *Fix $0 < q < p$, let $r \in (l(p,q), p)$. There are $\iota \in (0, 1/2)$ and $M > 0$ depending on p, q and r, which satisfy the following properties. Suppose φ_t, $0 \le t < p$,* *are some modulus p standard annulus LE maps driven by ξ on $[0,p)$. Then we have $|\partial_z \mathbf{S}_{p-t}(\varphi_t(z)/e^{i\xi(t)})| \le M$, for all $t \in [0,q]$ and $z \in \mathbb{A}_{r,p}$. Moreover,*

$$\mathbb{A}_{\iota(p-t),p-t} \supset \varphi_t(\mathbb{A}_{r,p}) \supset \mathbb{A}_{(1-\iota)(p-t),p-t}, \quad \forall t \in [0,q].$$

We may lift the φ_t to the covering space, and find a conformal map $\widetilde{\varphi}_t$ such that $e^i \circ \widetilde{\varphi}_t = \varphi_t \circ e^i$, $\widetilde{\varphi}_0$ is an identity map, and $\widetilde{\varphi}_t$ is continuous in t. Then we have

$$\partial_t \widetilde{\varphi}_t(z) = \widetilde{\mathbf{S}}_{p-t}(\varphi_t(z) - \xi(t)),$$

where $\widetilde{\mathbf{S}}_r(z) = \frac{1}{i}\mathbf{S}_r(e^{iz})$. If we let $\widetilde{\mathbb{A}}_{a,b} := (e^i)^{-1}(\mathbb{A}_{a,b})$, then with the assumption of the above lemma, we have

$$\widetilde{\mathbb{A}}_{\iota(p-t),p-t} \supset \widetilde{\varphi}_t(\widetilde{\mathbb{A}}_{r,p}) \supset \widetilde{\mathbb{A}}_{(1-\iota)(p-t),p-t}, \quad \forall t \in [0,q].$$

It is clear that $\widetilde{\mathbf{S}}_r$ has period 2π, is meromorphic in $\mathbb{C}$ with poles $\{2k\pi + i2mr : k, m \in \mathbb{Z}\}$, $\operatorname{Im} \widetilde{\mathbf{S}}_r \equiv 0$ on $\mathbb{R} \setminus \{\text{poles}\}$, and $\operatorname{Im} \widetilde{\mathbf{S}}_r \equiv -1$ on $\widetilde{\mathbf{C}}_r := ri + \mathbb{R}$. It is also easy to check that $\widetilde{\mathbf{S}}_r$ is an odd function, and the principal part of $\widetilde{\mathbf{S}}_r$ at 0 is $2/z$. So $\widetilde{\mathbf{S}}_r(z) = 2/z + az + O(z^3)$ near 0, for some $a \in \mathbb{R}$. It is possible to explicit this kernel using classical functions in [2]:

$$\widetilde{\mathbf{S}}_r(z) = 2\zeta(z) - \frac{2}{\pi}\zeta(\pi)z = \frac{1}{\pi}\frac{\partial_v \theta}{\theta}(\frac{z}{2\pi}, \frac{ir}{\pi}),$$

where ζ is the Weierstrass zeta function with basic periods $(2\pi, i2r)$, and $\theta = \theta(v,\tau)$ is Jacobi's theta function. The following lemma is a direct consequence of the heat-type differential equation satisfied by θ: $(\partial_v^2 - 4i\pi\partial_\tau)\theta = 0$. The symbols $'$ and $''$ in the lemma denote the first and second derivatives w.r.t. z.

Lemma 4.3.2 $\partial_r \widetilde{\mathbf{S}}_r - \widetilde{\mathbf{S}}_r \widetilde{\mathbf{S}}_r' - \widetilde{\mathbf{S}}_r'' \equiv 0.$

Let $K_t^\delta = \beta^\delta(0,t]$, for $0 \le t < p$. Suppose W maps D conformally onto $\mathbb{A}_p$ so that $W(0_+) = 1$. Then $t \mapsto W(K_t^\delta)$ is a Loewner chain in $\mathbb{A}_p$ on $\mathbf{C}_0$ such that $C_{A_p,C_p}(W(K_t^\delta)) = t$. By Proposition 2.4.1, $(W(K_t^\delta), 0 \le t < p)$ is a family of modulus p standard annulus LE hulls driven by some real continuous function ξ_t^δ on $[0,p)$ with $\xi^\delta(0) = 0$. Let φ_t^δ be the corresponding LE maps. We want to prove that as $\delta \to 0$, the law of ξ^δ converges to the law of $\sqrt{2}B$, where $B(t)$ is a standard Brownian motion.

Define

$$E_{-1}^\delta = \partial_V D^\delta \cap I_1, \;\; F^\delta = \partial_V D^\delta \cap I_2, \;\; N_{-1}^\delta = V(D^\delta) \cap D,$$

and

$$E_k^\delta = E_{-1}^\delta \cup \{X^\delta(0), \dots, X^\delta(k)\}, \;\; N_k^\delta = N_{-1}^\delta \setminus \{X^\delta(0), \dots, X^\delta(k)\},$$

for $0 \le k < v$. Let f_k be the f in Lemma 4.2.1 with $G = D^\delta$, $A = F^\delta$ and $B = E_k^\delta$; let g_k be the g in Lemma 4.2.3 with $G = D^\delta$, $A = F^\delta$, $B = E_{k-1}^\delta$, and $x = X^\delta(k)$, for $0 \le k < v$. Note that one of I_1 and I_2 must be bounded, so one of F^δ and E_{k-1}^δ must be finite, which implies $L(E_{k-1}^\delta, F^\delta) < \infty$. And since $X^\delta(k+\cdot)$ is a path on D^δ from $X^\delta(k)$ to F^δ without passing through E_{k-1}^δ, we have $f_{k-1}(X^\delta(k)) > 0$, so g_k is well defined. From Theorem 4.2.1, (g_k) is a $\{\mathcal{F}_k\}$ martingale, where $\mathcal{F}_k$ denotes the σ-algebra generated by $X^\delta(0)$, $X^\delta(1)$, $\dots$, $X^\delta(k \wedge v)$.

Now fix $q_0 \in (0,p)$. Let $q_1 = (q_0+p)/2$. Choose $p_1 \in (l(p,q_1), p)$, and let $p_2 = (p_1+p)/2$. Denote $\alpha_j = W^{-1}(\mathbf{C}_{p_j})$, $j = 1,2$. Then α_1 and α_2 are disjoint Jordan curves in D such that α_j disconnects α_{3-j} from I_j, $j = 1,2$. For $j = 1,2$, let U_j be the subdomain of D bounded by α_j and I_j, and $V_j^\delta = V(D^\delta) \cap U_j$. Let L^δ be the set of simple lattice paths w on D^δ of finite length such that $w(0) \in I_1$, $w(n) \in V_1^\delta$ for all $n > 0$, and there is a path on D^δ from the last vertex $P(w)$ of w to I_2 without passing through I_1 or other vertices of w. For $w \in L^\delta$ of length k, denote

$$E_w^\delta = E_{-1}^\delta \cup \{w(0), \dots, w(k)\}, \;\text{ and }\; N_w^\delta = N_{-1}^\delta \setminus \{w(0), \dots, w(k)\}.$$

Let g_w be the g in Lemma 4.2.3 with $G = D^\delta$, $A = F^\delta$, $B = E^\delta_w \setminus \{P(w)\}$, and $x = P(w) = w(k)$. Now define $D_w = D \setminus \cup_{j=1}^k [w(j-1), w(j)]$. Let u_w be the non-negative harmonic function in D_w whose continuation is constant 1 on I_2, constant 0 on $\cup_{j=0}^k [w(j-1), w(j)] \cup I_1$ except at $P(w)$, and $\int_{I_2} \partial_{\mathbf{n}} u_w(z) ds(z) = 0$. It is intuitive to guess that g_w should be close to u_w. In fact, we have the following proposition. The proof is postponed to the next section.

Proposition 4.3.1 *Given any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if $0 < \delta < \delta(\varepsilon)$ and $w \in L^\delta$, then $|g_w(v) - u_w(v)| < \varepsilon$, for any $v \in V_2^\delta$.*

Let $n_\infty = \lceil S(q_0) \rceil$, where $\lceil x \rceil$ is the smallest integer that is not less than x. Then n_∞ is a $\{\mathcal{F}_k\}$ stopping time. For $0 \le k \le n_\infty - 1$, $T(k) \le q_0 < q_1$, so from the choice of p_1, we see that $W(X^\delta(k))$ lies in the domain bounded by $\mathbf{C}_{p_1}$ and $\mathbf{C}_0$, so $X^\delta(k)$ lies in the domain bounded by I_1 and α_1. Note that $X^\delta(-1) = 0 \in I_1$. So for $-1 \le k \le n_\infty - 1$, if δ is small, then $[X^\delta(k), X^\delta(k+1)]$ can be disconnected from I_2 by an annulus centered at $X^\delta(k)$ with inner radius δ and outer radius $dist(\alpha_1, I_2)$. So as $\delta \to 0$, the conjugate extremal distance between I_2 and $[X^\delta(k), X^\delta(k+1)]$ in $D \setminus \cup_{0 \le j \le k} [X^\delta(j-1), X^\delta(j)]$ tends to 0, uniformly in $-1 \le k \le n_\infty - 1$. It follows that $T(k+1) - T(k)$ and $\max\{|\xi^\delta(t) - \xi^\delta(T(k))| : T(k) \le t \le T(k+1)\}$ tend to 0 as $\delta \to 0$, uniformly in $-1 \le k \le n_\infty - 1$. Since $T(n_\infty - 1) \le q_0$, we may choose δ small enough such that $T(n_\infty) < q_1$. Since $p_1 \in (l(p, q_1), p)$, and $\alpha_1 = W^{-1}(\mathbf{C}_{p_1})$, so $X^\delta[-1, k] \cap \alpha_1 = \emptyset$ for $0 \le k \le n_\infty$. So for $0 \le k \le n_\infty$, $w_k := X^\delta(\cdot - 1)|_{\overline{\mathbb{N}}_{k+1}}$ is contained in L^δ, and $g_{w_k} = g_k$. Since $\varphi^\delta_{T(k)} \circ W$ maps $(D_{w_k}, P(w_k))$ conformally onto $(\mathbb{A}_{p-T(k)}, e^{i\xi^\delta(T(k))})$, so

$$u_{w_k}(z) = \mathbf{S}_{p-T(k)}(\varphi^\delta_{T(k)} \circ W(z)/e^{i\xi^\delta(T(k))}).$$

Now fix $d > 0$. Define a non-decreasing sequence $(n_j)_{j \ge 0}$ inductively. Let $n_0 = 0$. Let n_{j+1} be the first integer $n \ge n_j$ such that $T(n) - T(n_j) \ge d^2$, or $|\xi^\delta(T(n)) - \xi^\delta(T(n_j))| \ge d$, or $n = n_\infty$, whichever comes first. Then n_j's are stopping times w.r.t. $\{\mathcal{F}_k\}$, and they are bounded above by n_∞. If we let δ be smaller than some constant depending on d, then $T(n_{j+1}) - T(n_j) \le 2d^2$ and $|\xi^\delta(T(s)) - \xi^\delta(T(n_j))| \le 2d$ for all $s \in [n_j, n_{j+1}]$ and $j \ge 0$. Let $\mathcal{F}'_j = \mathcal{F}_{n_j}$. Then for any $v \in V_2^\delta$, $(g_{n_j}(v), 0 \le j < \infty)$ is an $\{\mathcal{F}'_j\}$ martingale. By Proposition 4.3.1 for any $z \in W(V_2^\delta)$ and $0 \le j \le k$,

$$\mathbf{E}\,[\mathrm{Re}\,\mathbf{S}_{p-T(n_k)}(\varphi^\delta_{T(n_k)}(z)/e^{i\xi^\delta(T(n_k))})|\mathcal{F}'_j] = \mathrm{Re}\,\mathbf{S}_{p-T(n_j)}(\varphi^\delta_{T(n_j)}(z)/e^{i\xi(T(n_j))}) + o_\delta(1).$$

As δ tends to 0, the set $W(V_2^\delta)$ tends to be dense in $\mathbb{A}_{p_2,p}$. So for any $z \in \mathbb{A}_{p_2,p}$, there is some $z_0 \in W(V_2^\delta)$ such that $|z - z_0| = o_\delta(1)$. Note that $T(n_j) \le T(n_k) \le T(n_\infty) \le q_1$ for $0 \le j \le k$. Using the boundedness of the derivative in Lemma 4.3.1 with $q = q_1$ and $r = p_2$, we then have that for all $z \in \mathbb{A}_{p_2,p}$,

$$\mathbf{E}\,[\mathrm{Re}\,\mathbf{S}_{p-T(n_k)}(\varphi^\delta_{T(n_k)}(z)/e^{i\xi^\delta(T(n_k))})|\mathcal{F}'_j] = \mathrm{Re}\,\mathbf{S}_{p-T(n_j)}(\varphi^\delta_{T(n_j)}(z)/e^{i\xi^\delta(T(n_j))}) + o_\delta(1).$$

Then we have for all $z \in \widetilde{\mathbb{A}}_{p_2,p}$,

$$\mathbf{E}\,[\mathrm{Im}\,\widetilde{\mathbf{S}}_{p-T(n_k)}(\widetilde{\varphi}^\delta_{T(n_k)}(z) - \xi^\delta(T(n_k))|\mathcal{F}'_j] = \mathrm{Im}\,\widetilde{\mathbf{S}}_{p-T(n_j)}(\widetilde{\varphi}^\delta_{T(n_j)}(z) - \xi^\delta(T(n_j))) + o_\delta(1). \tag{4.3.1}$$

In Lemma 4.3.1, let $q = q_1$ and $r = p_2$, then we have some $\iota \in (0, 1/2)$ such that

$$\widetilde{\mathbb{A}}_{\iota(p-t),p-t} \supset \widetilde{\varphi}^\delta_t(\widetilde{\mathbb{A}}_{p_2,p}) \supset \widetilde{\mathbb{A}}_{(1-\iota)(p-t),p-t}, \tag{4.3.2}$$

for $0 \le t \le q_1$.

Proposition 4.3.2 *There are an absolute constant $C > 0$ and a constant $\delta(d) > 0$ such that if $\delta < \delta(d)$, then for all $j \ge 0$,*

$$|\mathbf{E}\,[\xi^\delta(T(n_{j+1})) - \xi^\delta(T(n_j))|\mathcal{F}'_j]| \le Cd^3, \textit{ and}$$

$$|\mathbf{E}\,[(\xi^\delta(T(n_{j+1})) - \xi^\delta(T(n_j)))^2/2 - (T(n_{j+1}) - T(n_j))|\mathcal{F}'_j]| \le Cd^3.$$

Proof. Fix some $j \geq 0$. Let $a = T(n_j)$ and $b = T(n_{j+1})$. Then $0 \leq a \leq b \leq q_1$. And if δ is less than some $\delta_1(d)$, we have $|b-a| \leq 2d^2$ and $|\xi^\delta(c) - \xi^\delta(a)| \leq 2d$, for any $c \in [a,b]$. Now suppose $z \in \widetilde{\mathbb{A}}_{p_2,p}$, and consider

$$F := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_b(z) - \xi^\delta(b)) - \widetilde{\mathbf{S}}_{p-a}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a)).$$

Then $F = F_1 + F_2$, where

$$F_1 := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_b(z) - \xi^\delta(b)) - \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a)),$$

$$F_2 := \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a)) - \widetilde{\mathbf{S}}_{p-a}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a)).$$

Then for some $c_1 \in [a,b]$, $F_1 = F_3 + F_4 + F_5$, where

$$F_3 := \widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a))[(\widetilde{\varphi}^\delta_b(z) - \widetilde{\varphi}^\delta_a(z)) - (\xi^\delta(b) - \xi^\delta(a))],$$

$$F_4 := \widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a))[(\widetilde{\varphi}^\delta_b(z) - \widetilde{\varphi}^\delta_a(z)) - (\xi^\delta(b) - \xi^\delta(a))]^2/2,$$

$$F_5 := \widetilde{\mathbf{S}}'''_{p-b}(\widetilde{\varphi}^\delta_{c_1}(z) - \xi^\delta(c_1))[(\widetilde{\varphi}^\delta_b(z) - \widetilde{\varphi}^\delta_a(z)) - (\xi^\delta(b) - \xi^\delta(a))]^3/6.$$

And for some $c_2 \in [a,b]$, we have

$$F_2 = -\partial_r \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a))(b-a) + \partial_r^2 \widetilde{\mathbf{S}}_{p-c_2}(\widetilde{\varphi}^\delta_a(z) - \xi^\delta(a))(b-a)^2/2. \quad (4.3.3)$$

Now for some $c_3 \in [a,b]$, we have

$$\widetilde{\varphi}^\delta_b(z) - \widetilde{\varphi}^\delta_a(z) = \partial_r \widetilde{\varphi}^\delta_{c_3}(z)(b-a) = \widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}^\delta_{c_3}(z) - \xi^\delta(c_3))(b-a). \quad (4.3.4)$$

For some $c_4 \in [c_3, b]$, we have

$$\widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}^\delta_{c_3}(z) - \xi^\delta(c_3)) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^\delta_{c_3}(z) - \xi^\delta(c_3)) + \partial_r \widetilde{\mathbf{S}}_{p-c_4}(\widetilde{\varphi}^\delta_{c_3}(z) - \xi^\delta(c_3))(b-c_3). \quad (4.3.5)$$

For some $c_5 \in [a, c_3]$, we have

$$\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^{\delta}_{c_3}(z) - \xi^{\delta}(c_3)) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^{\delta}_{a}(z) - \xi^{\delta}(a))$$

$$+\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}^{\delta}_{c_5}(z) - \xi^{\delta}(c_5))[(\widetilde{\varphi}^{\delta}_{c_3}(z) - \widetilde{\varphi}^{\delta}_{a}(z)) - (\xi^{\delta}(c_3) - \xi^{\delta}(a))]. \tag{4.3.6}$$

Once again, there is $c_6 \in [a, c_3]$ such that

$$\widetilde{\varphi}^{\delta}_{c_3}(z) - \widetilde{\varphi}^{\delta}_{a}(z) = \partial_r \widetilde{\varphi}^{\delta}_{c_6}(z)(c_3 - a) = \widetilde{\mathbf{S}}_{p-c_6}(\widetilde{\varphi}^{\delta}_{c_6}(z) - \xi^{\delta}(c_6))(c_3 - a). \tag{4.3.7}$$

We have the freedom to choose d arbitrarily small. Now suppose $d < (1-\iota)(p-q_1)/2$. Then

$$p - a \leq p - b + 2d \leq (p-b) + (1-\iota)(p-q_1) \leq (2-\iota)(p-b).$$

Thus for any $m \leq M \in [a, b]$, $p - m \leq (2-\iota)(p-M)$. By formula (4.3.2),

$$\widetilde{\varphi}^{\delta}_{m}(z) - \xi^{\delta}(m) \in \widetilde{\mathbb{A}}_{\iota(p-m), p-m} \subset \widetilde{\mathbb{A}}_{\iota(p-M),(2-\iota)(p-M)}.$$

So the values of $\widetilde{\mathbf{S}}_{p-M}$, $\partial_r \widetilde{\mathbf{S}}_{p-M}$, $\partial_r^2 \widetilde{\mathbf{S}}_{p-M}$, $\widetilde{\mathbf{S}}'_{p-M}$, $\widetilde{\mathbf{S}}''_{p-M}$ and $\widetilde{\mathbf{S}}'''_{p-M}$ at $\widetilde{\varphi}^{\delta}_{m}(z) - \xi^{\delta}(m)$ are uniformly bounded. In formula (4.3.3), consider $m = a$ and $M = c_2$. Since $|b - a| \leq 2d^2$, we have

$$F_2 = -\partial_r \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^{\delta}_{a}(z) - \xi^{\delta}(a))(b-a) + O(d^4).$$

Similarly, formula (4.3.7) implies

$$\widetilde{\varphi}^{\delta}_{c_3}(z) - \widetilde{\varphi}^{\delta}_{a}(z) = O(c_3 - a) = O(d^2).$$

This together with formulae (4.3.5),(4.3.6) and $\xi^{\delta}(c_3) - \xi^{\delta}(a) = O(d)$ implies that

$$\widetilde{\mathbf{S}}_{p-c_3}(\widetilde{\varphi}^{\delta}_{c_3}(z) - \xi^{\delta}(c_3)) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}^{\delta}_{a}(z) - \xi^{\delta}(a)) + O(d).$$

By formula (4.3.4), we have

$$\widetilde{\varphi}_b^\delta(z) - \widetilde{\varphi}_a^\delta(z) = \widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))(b-a) + O(d^3) = O(d^2).$$

Thus $F_5 = O(d^3)$,

$$F_4 = \widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))(\xi^\delta(b) - \xi^\delta(a))^2/2 + O(d^3), \text{ and}$$

$$F_3 = \widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))[\widetilde{\mathbf{S}}_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))(b-a) - (\xi^\delta(b) - \xi^\delta(a))] + O(d^3).$$

Note that $F = F_2 + F_3 + F_4 + F_5$. Using Lemma 4.3.2, we get

$$F = \widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))[(\xi^\delta(b) - \xi^\delta(a))^2/2 - (b-a)]$$

$$-\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a^\delta(z) - \xi^\delta(a))(\xi^\delta(b) - \xi^\delta(a)) + O(d^3).$$

By formula (4.3.1), if δ is smaller than some $\delta_2(d)$, then the conditional expectation of

$$\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(\widetilde{\varphi}_a^\delta(z)-\xi^\delta(a))[(\xi^\delta(b)-\xi^\delta(a))^2/2-(b-a)]-\operatorname{Im}\widetilde{\mathbf{S}}'_{p-b}(\widetilde{\varphi}_a^\delta(z)-\xi^\delta(a))[\xi^\delta(b)-\xi^\delta(a)]$$

w.r.t. $\mathcal{F}'_j$ is bounded by $C_1 d^3$.

By formula (4.3.2), for any $w \in \widetilde{\mathbb{A}}_{(1-\iota)(p-a),p-a}$, the conditional expectation of

$$\operatorname{Im}\widetilde{\mathbf{S}}''_{p-b}(w)[(\xi^\delta(b) - \xi^\delta(a))^2/2 - (b-a)] - \operatorname{Im}\widetilde{\mathbf{S}}'_{p-b}(w)[\xi^\delta(b) - \xi^\delta(a)] \tag{4.3.8}$$

w.r.t $\mathcal{F}'_j$ is bounded by $C_1 d^3$, if δ is small enough (depending on d).

Now suppose $d < (p-q_1)\iota/(4-4\iota)$. Then

$$(1-\iota)(p-a) < (1-\iota/2)(p-b) < p-a.$$

Thus $i(1-\iota/2)(p-b) \in \widetilde{\mathbb{A}}_{(1-\iota)(p-a),p-a}$. We may check

$$\operatorname{Im} \widetilde{\mathbf{S}}''_{p-b}(i(1-\iota/2)(p-b)) > 0, \text{ and } \operatorname{Im} \widetilde{\mathbf{S}}'_{p-b}(i(1-\iota/2)(p-b)) = 0.$$

So we can find $C_2 > 0$ such that for all $b \in [0, q_1]$, $\operatorname{Im} \widetilde{\mathbf{S}}''_{p-b}(i(1-\iota/2)(p-b)) > C_2$. Let $w = i(1-\iota/2)(p-b)$ in formula (4.3.8), then we get

$$|\mathbf{E}\,[(\xi^\delta(b) - \xi^\delta(a))^2/2 - (b-a)|\mathcal{F}'_j]| \le C_3 d^3.$$

Since $\operatorname{Im} \widetilde{\mathbf{S}}''_{p-b}(w)$ is uniformly bounded on $\widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$, so for all $w \in \widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$,

$$\operatorname{Im} \widetilde{\mathbf{S}}'_{p-b}(w)|\mathbf{E}\,[\xi^\delta(b) - \xi^\delta(a)|\mathcal{F}'_j]| \le C_4 d^3. \tag{4.3.9}$$

We may check that

$$x_b := \operatorname{Im} \widetilde{\mathbf{S}}_{p-b}(\pi + i(1-\iota/2)(p-b)) - \operatorname{Im} \widetilde{\mathbf{S}}_{p-b}(i(1-\iota/2)(p-b)) > 0.$$

So x_b is greater than some absolute constant $C_5 > 0$ for $b \in [0, q_1]$. Then there exists $w_b \in \widetilde{\mathbf{C}}_{(1-\iota/2)(p-b)}$ such that

$$|\operatorname{Im} \widetilde{\mathbf{S}}'_{p-b}(w_b)| = |\partial_x \operatorname{Im} \widetilde{\mathbf{S}}_{p-b}(w_b)| = x_b/\pi \ge C_5/\pi.$$

Plugging $w = w_b$ in formula (4.3.9), we then have $|\mathbf{E}\,[\xi^\delta(b) - \xi^\delta(a)|\mathcal{F}'_j]| \le C_6 d^3$. $\square$

The following Theorem about the convergence of the driving process can be deduced from Proposition 4.3.2 by using the Skorokhod Embedding Theorem. It is very similar to Theorem 3.6 in [7]. So we omit the proof.

Theorem 4.3.2 *For every $q_0 \in (0, p)$ and $\varepsilon > 0$ there is a $\delta_0 > 0$ depending on q_0 and ε such that for $\delta < \delta_0$ there is a coupling of the processes ξ^δ and $\sqrt{2}B$ such that*

$$\Pr[\sup\{|\xi^\delta(t) - \sqrt{2}B(t)\}| : t \in [0, q_0]\} > \varepsilon] < \varepsilon.$$

4.3.2 Convergence of the curves

In this subsection, we will prove Theorem 4.3.1. We need two well-known lemmas about random walks on $\delta\mathbb{Z}^2$. The metric by default is the Euclidean metric. The superscript # is used to denote the spherical metric.

Lemma 4.3.3 *Suppose $v \in \delta\mathbb{Z}^2$ and K is a connected set on the plane that has Euclidean (spherical, resp.) diameter at least R. Then the probability that a random walk on $\delta\mathbb{Z}^2$ started from v will exit $\mathbf{B}(v; R)$ ($\mathbf{B}^\#(v; R)$, resp.) before using an edge of $\delta\mathbb{Z}^2$ that intersects K is at most $C_0((\delta + dist(v, K))/R)^{C_1}$ ($C_0((\delta + dist^\#(v, K))/R)^{C_1}$, resp.) for some absolute constants $C_0, C_1 > 0$.*

Lemma 4.3.4 *Suppose U is a plane domain, and has a compact subset K and a non-empty open subset V. Then there are positive constants δ_0 and C depending on U, V and K, such that when $\delta < \delta_0$, the probability that a random walk on $\delta\mathbb{Z}^2$ started from some $v \in \delta\mathbb{Z}^2 \cap K$ will hit V before exiting U is greater than C.*

The following lemma about random walks on D^δ is an easy consequence of the above two lemmas and the Markov property of random walks.

Lemma 4.3.5 *For every $d > 0$, there are $\delta_0, C > 0$ depending on d such that if $\delta < \delta_0$ and $v \in \delta\mathbb{Z}^2 \cap D$ is such that $dist^\#(v, I_1) > d$, then the probability that a random walk on D^δ started from v hits I_2 before I_1 is at least C.*

Lemma 4.3.6 *For every $q \in (0, p)$ and $\varepsilon > 0$, there are $d, \delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, the probability that $dist^\#(\beta^\delta[q, p), I_1) \geq d$ is at least $1 - \varepsilon$.*

Proof. For $k = 1, 2, 3$, let $J_k = W^{-1}(\mathbf{C}_{q/k})$. Then J_1, J_2, J_3 are disjoint Jordan curves in D that separate I_2 from I_1. And J_2 lies in the domain, denoted by Λ, bounded by J_1 and J_3. Moreover, the modulus of the domain bounded by J_k and I_2 is $p - q/k$. Let τ^δ be the first n such that the edge $[X^\delta(n-1), X^\delta(n)]$ intersects J_2. Then τ^δ

is a stopping time. If δ is smaller than the distance between $J_1 \cup J_3$ and J_2, then $X^\delta(\tau^\delta) \in \Lambda$ and $X^\delta[-1, \tau^\delta]$ does not intersect J_1. Thus $M(D \setminus X^\delta(-1, \tau^\delta]) \geq p - q$, and so $T(\tau^\delta) \leq q$. So it suffices to prove that when δ and d are small enough, the probability that X^δ will get within spherical distance d from I_1 after time τ^δ is less than ε. Let RW_v^δ denote a random walk on $(D^\delta, \partial_V D^\delta)$ started from v, and CRW_v^δ denote that RW_v^δ conditioned to hit I_2. Since X^δ is obtained by erasing loops of $\mathrm{CRW}_\delta^\delta$, it suffices to show that the probability that $\mathrm{CRW}_\delta^\delta$ will get within spherical distance d from I_1 after it hits Λ, tends to zero as $d, \delta \to 0$. Since CRW^δ is a Markov chain, it suffices to prove that the probability that CRW_v^δ will get within spherical distance d from I_1 tends to zero as $d, \delta \to 0$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. By Lemma 4.3.5, there is $a > 0$ such that for δ small enough, the probability that RW_v^δ hits I_2 before I_1 is greater than a, for all $v \in \delta\mathbb{Z}^2 \cap \Lambda$. By Markov property, for every $v \in \delta\mathbb{Z}^2 \cap \Lambda$, the probability that CRW_v^δ will get within spherical distance d from I_1 is less than

$$\frac{1}{a} \cdot \sup\{\Pr[\mathrm{RW}_w^\delta \text{ hits } I_2 \text{ before } I_1] : w \in V(D^\delta) \cap D \text{ and } dist^\#(w, I_1) < d\},$$

which tends to 0 as $d, \delta \to 0$ by Lemma 4.3.3. So the proof is finished. □

Lemma 4.3.7 *For every $q \in (0, p)$ and $\varepsilon > 0$, there are $M, \delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, the probability that $\beta^\delta[q, p) \subset \mathbf{B}(0; M)$ is at least $1 - \varepsilon$.*

Proof. We use the notations of the last lemma. It suffices to prove that the probability that $\mathrm{RW}_v^\delta \not\subset \mathbf{B}(0; M)$ tends to zero as $\delta \to 0$ and $M \to \infty$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. Let $K = \mathbb{C} \setminus D$, then K is unbounded, and the distance between $v \in \Lambda$ and K is uniformly bounded from below by some $d > 0$. Let $r > 0$ be such that $\Lambda \subset \mathbf{B}(0; r)$. For $M > r$, let $R = M - r$, then for $v \in \delta\mathbb{Z}^2 \cap \Lambda$, RW_v^δ should exit $\mathbf{B}(v; R)$ before $\mathbf{B}(0; M)$. By Lemma 4.3.3, the probability that $\mathrm{RW}_v^\delta \not\subset \mathbf{B}(0; M)$ is less than $C_0((\delta + d)/(M - r))^{C_1}$, which tends to 0 as $\delta \to 0$ and $M \to \infty$, uniformly in $v \in \delta\mathbb{Z}^2 \cap \Lambda$. □

Lemma 4.3.8 *For every $\varepsilon > 0$, there are $q \in (0, p)$ and $\delta_0 > 0$ depending on ε such* *that when $\delta < \delta_0$, with probability greater than $1-\varepsilon$, the diameter of $\beta^\delta[q,p)$ is less than ε.*

Proof. The idea is as follows. Note that as $q \to p$, the modulus of $D \setminus \beta^\delta(0,q]$ tends to zero. So for any fixed $a \in (0,p)$, the spherical distance between $\beta^\delta[a,q]$ and I_2 tends to zero as $q \to p$. By Lemma 4.3.7, if M is big and δ is small, the event that $\beta^\delta[a,q]$ does not lie in $\mathbf{B}(0;M)$ is of small probability. Thus on the complement of this event, the Euclidean distance between $\beta^\delta[a,q]$ and I_2 tends to zero, which means that β^δ gets to some point near I_2 in the Euclidean metric before time q. By Lemma 4.3.3, RW_v^δ does not go far before hitting ∂D if v is near I_2. The same is true for CRW_v^δ because by Lemma 4.3.5, RW_v^δ hits I_2 before I_1 with a probability bigger than some positive constant when v is near I_2. Since X^δ is the loop-erasure of $\mathrm{CRW}_\delta^\delta$, X^δ does not go far after it gets near I_2, nor does β^δ. So the diameter of $\beta^\delta[q,p)$ is small. $\square$

Definition 4.3.1 *Let $z \in \mathbb{C}$, $r, \varepsilon > 0$. A (z,r,ε)-quasi-loop in a path ω is a pair $a, b \in \omega$ such that $a, b \in \mathbf{B}(z;r)$, $|a-b| \le \varepsilon$, and the subarc of ω with endpoints a and b is not contained in $\mathbf{B}(z;2r)$. Let $\mathcal{L}^\delta(z,r,\varepsilon)$ denote the event that $\beta^\delta[0,p)$ has a (z,r,ε)-quasi-loop.*

Lemma 4.3.9 *If $\overline{\mathbf{B}(z;2r)} \cap I_1 = \emptyset$, then $\lim_{\varepsilon\to 0} \Pr[\mathcal{L}^\delta(z,r,\varepsilon)] = 0$, uniformly in δ.*

Proof. This lemma is very similar to Lemma 3.4 in [15]. There are two points of difference between them. First, here we are dealing with the loop-erased *conditional* random walk. With Lemma 4.3.5, the hypothesis $\overline{\mathbf{B}(z;2r)} \cap I_1 = \emptyset$ guarantees that for some v near $\partial\mathbf{B}(z;2r)$, the probability that RW_v^δ hits I_2 before I_1 is bounded away from zero uniformly. Second, our LERW is stopped when it hits I_2, while in Lemma 3.4 in [15], the LERW is stopped when it hits some single point. It turns out that the current setting is easier to deal with. See [15] for more details. $\square$

Proposition 4.3.3 *For every $q \in (0,p)$ and $\varepsilon > 0$, there are $\delta_0, a_0 > 0$ depending on q and ε such that for $\delta < \delta_0$, with probability at least $1-\varepsilon$, β^δ satisfies the following* *property. If $q \le t_1 < t_2 < p$, and $|\beta^\delta(t_1) - \beta^\delta(t_2)| < a_0$, then the diameter of $\beta^\delta[t_1, t_2]$ is less than ε.*

Proof. For $d, M > 0$, let $\Lambda_{d,M}$ denote the set of $z \in \mathbf{B}(0; M)$ such that $dist^{\#}(z, I_1) \ge d$, and $\mathcal{A}^\delta_{d,M}$ denote the event that $\beta^\delta[q,p) \subset \Lambda_{d,M}$. By Lemma 4.3.6 and 4.3.7, there are $d_0, M_0, \delta_0 > 0$ such that for $\delta < \delta_0$, $\Pr[\mathcal{A}^\delta_{d_0,M_0}] > 1 - \varepsilon/2$. Note that the Euclidean distance between Λ_{d_0,M_0} and I_1 is greater than $d_0/2$. Choose $0 < r < \min\{\varepsilon/4, d_0/4\}$. There are finitely many points $z_1, \ldots, z_n \in \Lambda_{d_0,M_0}$ such that $\Lambda_{d_0,M_0} \subset \cup_1^n \mathbf{B}(z_j; r/2)$. For $a > 0$, $1 \le j \le n$, let $\mathcal{B}^\delta_{j,a}$ denote the event that $\beta^\delta[0,p)$ does not have a (z_j, r, a)-quasi-loop. Since $r < d_0/4$, we have $\overline{\mathbf{B}(z_j; 2r)} \cap I_1 = \emptyset$. By Lemma 4.3.9, there is $a_0 \in (0, r/2)$ such that $\Pr[\mathcal{B}^\delta_{j,a_0}] \ge 1 - \varepsilon/(2n)$ for $1 \le j \le n$. Let $\mathcal{C}^\delta = \cap_1^n \mathcal{B}^\delta_{j,a_0} \cap \mathcal{A}^\delta_{d_0,M_0}$. Then $\Pr[\mathcal{C}^\delta] > 1 - \varepsilon$ if $\delta < \delta_0$. And on the event $\mathcal{C}^\delta$, if there are $t_1 < t_2 \in [q,p)$ satisfying $|\beta^\delta(t_1) - \beta^\delta(t_2)| < a_0$, then $\beta^\delta(t_1)$ lies in some ball $\mathbf{B}(z_j; r/2)$, so $\beta^\delta(t_2) \in \mathbf{B}(z_j; r)$ as $a_0 < r/2$. Since β^δ does not have a (z_j, r, a_0)-quasi-loop, $\beta^\delta[t_1, t_2] \subset \mathbf{B}(z_j; 2r)$. This then implies that the diameter of $\beta^\delta[t_1, t_2]$ is not bigger than $4r$, which is less than ε. $\square$

Proof of Theorem 4.3.1. Let $K^0_t = \beta^0(0,t]$, $0 \le t < p$. Then $(W(K^0_t))$ has the law of modulus p standard annulus SLE_2. Let ξ^0 be the driving function. Then $\xi^0 = \sqrt{2}B$ for some standard Brownian motion B. By Theorem 4.3.2, we may assume that all ξ^δ and ξ^0 are in the same probability space, and for every $q \in (0,p)$ and $\varepsilon > 0$ there is an $\delta_0 > 0$ depending on q and ε such that for $\delta < \delta_0$,

$$\Pr[\sup\{|\xi^\delta(t) - \xi^0(t)| : t \in [0,q]\} > \varepsilon] < \varepsilon.$$

Since β^δ and β^0 are determined by ξ^δ and ξ^0, respectively, all β^δ and β^0 are also in the same probability space. For the first part of this theorem, it suffices to prove that for every $q \in (0,p)$ and $\varepsilon > 0$ there is $\delta_0 = \delta_0(q,\varepsilon) > 0$ such that for $\delta < \delta_0$,

$$\Pr[\sup\{|\beta^\delta(t) - \beta^0(t)| : t \in [q,p)\} > \varepsilon] < \varepsilon. \tag{4.3.10}$$

Now choose any sequence $\delta_n \to 0$. Then it contains a subsequence δ_{n_k} such that for each $q \in (0,p)$, $\xi^{\delta_{n_k}}$ converges to ξ^0 uniformly on $[0,q]$ almost surely. Here we use the fact that a sequence converging in probability contains an a.s. converging subsequence. For simplicity, we write δ_n instead of δ_{n_k}. Let $\varphi_t^{\delta_n}$ (φ_t^0, resp.), $0 \le t < p$, be the modulus p standard annulus LE maps driven by ξ^{δ_n} (ξ^0, resp.). Let $\Omega_t^{\delta_n} := \mathbb{A}_p \setminus W(\beta^{\delta_n}(0,t])$, and $\Omega_t^0 := \mathbb{A}_p \setminus W(\beta^0(0,t])$. Fix $q \in (0,p)$. Suppose K is a compact subset of Ω_q^0. Then for every $z \in K$, $\varphi_t^0(z)$ does not blow up on $[0,q]$. Since the driving function ξ^{δ_n} converges to ξ^0 uniformly on $[0,q]$, so if n is big enough, then for every $z \in K$, $\varphi_t^{\delta_n}(z)$ does not blow up on $[0,q]$, which means that $K \subset \Omega_q^{\delta_n}$. Moreover, $\varphi_q^{\delta_n}$ converges to φ_q^0 uniformly on K. It follows that $\Omega_q^{\delta_n} \cap \Omega_q^0 \to \Omega_q^0$ as $n \to \infty$. By Lemma 2.5.1, $(\varphi_q^{\delta_n})^{-1}$ converges to $(\varphi_q^0)^{-1}$ uniformly on each compact subset of $\mathbb{A}_{p-q}$, and so $\Omega_q^{\delta_n} = (\varphi_q^{\delta_n})^{-1}(\mathbb{A}_{p-q}) \to (\varphi_q^0)^{-1}(\mathbb{A}_{p-q}) = \Omega_q^0$. Now we denote $D_t^{\delta_n} := D \setminus \beta^{\delta_n}(0,t] = W^{-1}(\Omega_t^{\delta_n})$, and $D_t^0 := D \setminus \beta^0(0,t] = W^{-1}(\Omega_t^0)$. Then we have $D_q^{\delta_n} \to D_q^0$ for every $q \in (0,p)$.

Fix $\varepsilon > 0$ and $q_1 < q_2 \in (0,p)$. Let $q_0 = q_1/2$ and $q_3 = (q_2+p)/2$. By Proposition 4.3.3, there are $n_1 \in \mathbb{N}$ and $a \in (0, \varepsilon/2)$ such that for $n \ge n_1$, with probability at least $1-\varepsilon/3$, β^{δ_n} satisfies: if $q_0 \le t_1 < t_2 < p$, and $|\beta^{\delta_n}(t_1) - \beta^{\delta_n}(t_2)| < a$, then the diameter of $\beta^{\delta_n}[t_1,t_2]$ is less than $\varepsilon/3$. Let $\mathcal{A}_n$ denote the corresponding event. Since β^0 is continuous, there is $b > 0$ such that with probability $1-\varepsilon/3$, we have $|\beta^0(t_1) - \beta^0(t_2)| < a/2$ if $t_1, t_2 \in [q_0, q_3]$ and $|t_1 - t_2| \le b$. Let $\mathcal{B}$ denote the corresponding event. We may choose $q_0 < t_0 < t_1 = q_1 < \cdots < t_{m-1} = q_2 < t_m < q_3$ such that $t_j - t_{j-1} < b$ for $1 \le j \le m$. Since $\beta^0(t_j) \notin \beta^0(0, t_{j-1}]$ for $1 \le j \le m$, there is $r \in (0, a/4)$ such that with probability at least $1-\varepsilon/3$, $\overline{\mathbf{B}(\beta^0(t_j); r)} \subset D_{t_{j-1}}^0$ for all $0 \le j \le m$. We now use the convergence of $D_t^{\delta_n}$ to D_t^0 for $t = t_0, \dots, t_m$. There exists $n_2 \in \mathbb{N}$ such that for $n \ge n_2$, with probability at least $1-\varepsilon/3$, $\overline{\mathbf{B}(\beta^0(t_j); r)} \subset D_{t_{j-1}}^{\delta_n}$, and there is some $z_j^n \in \partial D_{t_j}^{\delta_n} \cap \mathbf{B}(\beta^0(t_j); r)$, for all $1 \le j \le m$. Let $\mathcal{C}_n$ denote the corresponding event. Then on the event $\mathcal{C}_n$, $z_j^n \in \partial D_j^{\delta_n} \setminus \partial D_{j-1}^{\delta_n}$, so $z_j^n = \beta^{\delta_n}(s_j^n)$ for some $s_j^n \in (t_{j-1}, t_j]$. Let $\mathcal{D}_n = \mathcal{A}_n \cap \mathcal{B} \cap \mathcal{C}_n$. Then $\Pr[\mathcal{D}_n] \ge 1 - \varepsilon$, for $n \ge n_1 + n_2$.

And on the event $\mathcal{D}_n$,

$$|z_j^n - z_{j+1}^n| \le 2r + |\beta^0(t_j) - \beta^0(t_{j+1})| \le 2r + a/2 < a, \ \forall 1 \le j \le m-1,$$

as $|t_j - t_{j+1}| \le b$. Thus the diameter of $\beta^{\delta_n}[s_j^n, s_{j+1}^n]$ is less than $\varepsilon/3$. It follows that for any $t \in [s_j^n, s_{j+1}^n] \subset [t_{j-1}, t_{j+1}]$,

$$|\beta^0(t) - \beta^{\delta_n}(t)| \le |\beta^0(t) - \beta^0(t_j)| + |\beta^0(t_j) - z_j^n| + |z_j^n - \beta^{\delta_n}(t)| \le a/2 + r + \varepsilon/3 < \varepsilon.$$

Since $[q_1, q_2] = [t_1, t_{m-1}] \subset \cup_{j=1}^{m-1}[s_j^n, s_{j+1}^n]$, we have now proved that for n big enough, with probability at least $1-\varepsilon$, $|\beta^{\delta_n}(t) - \beta^0(t)| < \varepsilon$ for all $t \in [q_1, q_2]$. By Lemma 4.3.8, for any $\varepsilon > 0$, there is $q(\varepsilon) \in (0, p)$ such that if n is big enough, with probability at least $1 - \varepsilon$, the diameter of $\beta^{\delta_n}[q(\varepsilon), p)$ is less than ε. For any $S \in [q(\varepsilon), p)$, by the uniform convergence of β^{δ_n} to β^0 on the interval $[q(\varepsilon), S]$, it follows that with probability at least $1 - \varepsilon$, the diameter of $\beta^0[q(\varepsilon), S)$ is no more than ε, nor is the diameter of $\beta^0[q(\varepsilon), p)$. Now for fixed $q \in (0, p)$ and $\varepsilon > 0$, choose $q_1 \in (q, p) \cap (q(\varepsilon/3), p)$. Then with probability at least $1 - \varepsilon/3$, the diameter of $\beta^0[q_1, p)$ is less than $\varepsilon/3$. And if n is big enough, then with probability at least $1 - \varepsilon/3$, the diameter of $\beta^{\delta_n}[q_1, p)$ is less than $\varepsilon/3$. Moreover, if n is big enough, we may require that with probability at least $1 - \varepsilon/3$, $|\beta^{\delta_n}(t) - \beta^0(t)| \le \varepsilon/3$ for all $t \in [q, q_1]$. Thus $|\beta^{\delta_n}(t) - \beta^0(t)| \le \varepsilon$ for all $t \in [q, p)$ with probability at least $1 - \varepsilon$, if n is big enough. Since $\{\delta_n\}$ is chosen arbitrarily, we proved formula (4.3.10).

Now suppose that the impression of 0_+ is the a single point, which must be 0. From [12], we see that $W^{-1}(z) \to 0$ as $z \in \mathbb{A}_p$ and $z \to 1$. From above, it suffices to prove that for any $\varepsilon > 0$, we can choose $q \in (0, p)$ and $\delta_0 > 0$ such that for $\delta < \delta_0$, with probability at least $1 - \varepsilon$, the diameters of $\beta^\delta(0, q]$ and $\beta^0(0, q]$ are less than ε. Since W^{-1} is continuous at 1, we need only to prove the same is true for the diameters of $W(\beta^\delta(0, q])$ and $W(\beta^0(0, q])$. Note that they are the modulus p standard annulus LE hulls at time q, driven by χ_t^δ and χ_t^0, respectively. By Theorem 4.3.2, if δ and q are small, then the diameters of $\chi^\delta[0, q]$ and $\chi^0[0, q]$ are uniformly small with probability near 1, so are the diameters of $W(\beta^\delta(0,q])$ and $W(\beta^0(0,q])$. $\square$

4.4 Convergence of observables

The goal of this section is to prove Proposition 4.3.1. The proof is sort of long. The main difficulty is that we need the approximation to be uniform in the domains. The tool we can use is Lemma 2.5.1. However, the limit of a domain sequence in general does not have good boundary conditions, even if every domain in the sequence has. Prime ends and crosscuts are used to describe the boundary correspondence under conformal maps. Some ideas of the proof come from [7].

We will often deal with a function defined on a subset of $\delta\mathbb{Z}^2$. Suppose f is such a function. For $v \in \delta\mathbb{Z}^2$ and $z \in \mathbb{Z}^2$, if $f(v)$ and $f(v+\delta z)$ are defined, then define

$$\nabla_z^\delta f(v) = (f(v+\delta z) - f(v))/\delta,$$

We say that f is δ-harmonic in $\Omega \subset \mathbb{C}$ if f is defined on $\delta\mathbb{Z}^2 \cap \Omega$ and all $v \in \delta\mathbb{Z}^2$ that are adjacent to vertices of $\delta\mathbb{Z}^2 \cap \Omega$ so that for all $v \in \delta\mathbb{Z}^2 \cap \Omega$,

$$f(v+\delta) + f(v-\delta) + f(v+i\delta) + f(v-i\delta) = 4f(v).$$

The following lemma is well known.

Lemma 4.4.1 *Suppose Ω is a plane domain that has a compact subset K. For $l \in \mathbb{N}$, let $z_1, \dots, z_l \in \mathbb{Z}^2$. Then there are positive constants δ_0 and C depending on Ω, K, and $z_1, \dots, z_l$, such that for $\delta < \delta_0$, if f is non-negative and δ-harmonic in Ω, then for all $v_1, v_2 \in \delta\mathbb{Z}^2 \cap K$,*

$$\nabla_{z_1}^\delta \cdots \nabla_{z_l}^\delta f(v_1) \le C f(v_2).$$

This is also true for $l = 0$, which means that $f(v_1) \le C f(v_2)$.

For $a, b \in \delta\mathbb{Z}$, denote

$$S_{a,b}^\delta := \{(x,y) : a \le x \le a+\delta, b \le y \le b+\delta\}.$$

Suppose A is a subset of $\delta\mathbb{Z}^2$, let S_A^δ be the union of all $S_{a,b}^\delta$ whose four vertices are in A. If f is defined on A, we may define a continuous function $\mathrm{CE}^\delta f$ on S_A^δ, as follows. For $(x,y) \in S_{a,b}^\delta \subset S_A^\delta$, define

$$\begin{aligned}\mathrm{CE}^\delta f(x,y) = \quad & (1-s)(1-t)f(a,b) + (1-s)tf(a,b+\delta) \\ & +s(1-t)f(a+\delta,b) + stf(a+\delta,b+\delta),\end{aligned}$$

where $s = (x-a)/\delta$ and $t = (y-b)/\delta$. Then $\mathrm{CE}^\delta f$ is well defined on S_A^δ, and agrees with f on $S_A^\delta \cap A$. Moreover, on $S_{a,b}^\delta$, $\mathrm{CE}^\delta f$ has a Lipschitz constant not bigger than two times the maximum of $|\nabla_{(1,0)}^\delta f(a,b)|$, $|\nabla_{(0,1)}^\delta f(a,b)|$, $|\nabla_{(1,0)}^\delta f(a,b+\delta)|$, $|\nabla_{(0,1)}^\delta f(a+\delta,b)|$. And for any $u \in \mathbb{Z}^2$,

$$\mathrm{CE}^\delta \nabla_u^\delta f(z) = (\mathrm{CE}^\delta f(z+\delta u) - \mathrm{CE}^\delta f(z))/\delta,$$

when both sides are defined.

Proof of Proposition 4.3.1. Suppose the proposition is not true. Then we can find $\varepsilon_0 > 0$, a sequence of lattice paths $w_n \in L^{\delta_n}$ with $\delta_n \to 0$, and a sequence of points $v_n \in V_2^{\delta_n}$, such that $|g_{w_n}(v_n) - u_{w_n}(v_n)| > \varepsilon_0$ for all $n \in \mathbb{N}$. For simplicity of notations, we write g_n for g_{w_n}, u_n for u_{w_n}, and D_n for D_{w_n}. Let p_n be the modulus of D_n. The remaining of the proof is composed of four steps.

4.4.1 The limits of domains and functions

By comparison principle of extremal length, we have $p \geq p_n \geq M(U_2) > 0$. By passing to a subsequence, we may assume that $p_n \to p_0 \in (0,p]$. Then $\mathbb{A}_{p_n} \to \mathbb{A}_{p_0}$. Let Q_n map D_n conformally onto $\mathbb{A}_{p_n}$ so that $Q_n(z) \to 1$ as $z \in D_n$ and $z \to P(w_n)$. Then $u_n = \mathrm{Re}\,\mathbf{S}_{p_n} \circ Q_n$. Now Q_n^{-1} maps $\mathbb{A}_{p_n}$ conformally onto $D_n \subset D$. Thus $\{Q_n^{-1}\}$ is a normal family. By passing to a subsequence, we may assume that Q_n^{-1} converges to some function J uniformly on each compact subset of $\mathbb{A}_{p_0}$. Using some argument similar to that in the proof of Theorem 4.3.1, we conclude that J maps $\mathbb{A}_{p_0}$ conformally onto some domain D_0, and $D_n \to D_0$. Let $Q_0 = J^{-1}$ and $u_0 = \mathrm{Re}\,\mathbf{S}_{p_0} \circ Q_0$. Then Q_n and u_n converge to Q_0 and u_0, respectively, uniformly on each compact subset of D_0. Moreover, we have $U_2 \cup \alpha_2 \subset D_0 \subset D$. Thus I_2 is a side of D_0. Let I_1^n and I_1^0 denote the other side of D_n and D_0, respectively.

Let $\{K_m\}$ be a sequence of compact subsets of D_0 such that $D_0 = \cup_m K_m$, and for each m, K_m disconnects I_1^0 from I_2 and $K_m \subset \mathrm{int}\, K_{m+1}$. Let $K_m^n = K_m \cap \delta_n \mathbb{Z}^2$. Now fix m. If n is big enough depending on m, we can have the following properties. First, $K_m \subset D_n$ and $K_m^n \subset V(D^{\delta_n})$, so g_n is δ_n-harmonic on K_m. Second, K_m^n disconnects all lattice paths on D^{δ_n} from I_2 to I_1^n. Now let RW_v^n be a random walk on D^{δ_n} started from $v \in V(D^{\delta_n})$, and τ_m^n the hitting time of RW_v^n at $I_2 \cup K_m^n$. By the properties of g_n, if v is in D and between K_m and I_2, then $(g_n(\mathrm{RW}_v^n(j)), 0 \le j \le \tau_n^m)$ is a martingale, so $g_n(v) = \mathbf{E}\,[g_n(\mathrm{RW}_n^v(\tau_n^m)]$. Now suppose $g_n(v) > 1$ for all $v \in K_n^m$. Choose $v_0 \in V(D^{\delta_n}) \cap D$ that is adjacent to some vertex of $F^{\delta_n} = V(D^{\delta_n}) \cap I_2$. Then $g_n(v_0) = \mathbf{E}\,[g_n(\mathrm{RW}_n^{v_0}(\tau_n^m))] \ge 1$. The equality holds iff there is no lattice path on D^{δ_n} from v_0 to K_m^n. By the definition of D^{δ_n}, we know that the equality can not always hold. It follows that $\sum_{u \in F^{\delta_n}} \Delta_{D^{\delta_n}} g_n(u) > 0$, which contradicts the definition of g_n. Thus there is $v \in K_m^n$ such that $g_n(v) \le 1$. Note that g_n is non-negative. By Lemma 4.4.1, if n is big enough depending on m, then g_n on K_m^n is uniformly bounded in n. Similarly for any $z_1, \dots, z_l \in \mathbb{Z}^2$, $\nabla_{z_1}^{\delta_n} \cdots \nabla_{z_l}^{\delta_n} g_n$ on K_m^n is uniformly bounded in n, if n is big enough depending on m, and $z_1, \dots, z_l \in \mathbb{Z}^2$.

We just proved that for a fixed m, if n is big enough depending on m, then g_n on K_{m+1}^n is δ_n-harmonic and uniformly bounded in n. We may also choose n big such that every lattice square of $\delta_n \mathbb{Z}^2$ that intersects K_m is contained in K_{m+1}, and so $\mathrm{CE}^{\delta_n} g_n$ on K_m is well defined, and is uniformly bounded in n. Using the boundedness of $\nabla_u^{\delta_n} g_n$ on K_{m+1}^n for $u \in \{1, i\}$, we conclude that $\{\mathrm{CE}^{\delta_n} g_n\}$ on K_m is uniformly continuous. By Arzela-Ascoli Theorem, there is a subsequence of $\{\mathrm{CE}^{\delta_n} g_n\}$, which converges uniformly on K_m. By passing to a subsequence, we may assume that $\mathrm{CE}^{\delta_n} g_n$ converges uniformly on each K_m. Let g_0 on D_0 be the limit function. Similarly, for any $z_1, \dots, z_l \in \mathbb{Z}^2$, there is a subsequence of $\{\mathrm{CE}^{\delta_n} \nabla_{z_1}^{\delta_n} \cdots \nabla_{z_l}^{\delta_n} g_n\}$ which converges uniformly on each K_m. By passing to a subsequence again, we may assume that for any $z_1,\dots,z_l\in\mathbb{Z}^2$, $\mathrm{CE}^{\delta_n}\nabla_{z_1}^{\delta_n}\cdots\nabla_{z_l}^{\delta_n}g_n$ converges to $g_0^{z_1,\dots,z_l}$ on D_0, uniformly on each K_m. It is easy to check that

$$g_0^{z_1,\dots,z_l}=(a_1\partial_x+b_1\partial_y)\cdots(a_l\partial_x+b_l\partial_y)g_0,$$

if $z_j=(a_j,b_j)$, $1\le j\le l$. Since g_n is δ_n-harmonic on K_m for n big enough, we have $(\nabla_1^{\delta_n}\nabla_{-1}^{\delta_n}+\nabla_i^{\delta_n}\nabla_{-i}^{\delta_n})g_n\equiv 0$ on K_m^n. Thus $(\partial_x^2+\partial_y^2)g_0=0$, which means that g_0 is harmonic.

Now suppose $x_n\in V(D^{\delta_n})\cap D\to I_2$ in the spherical metric. Since the spherical distance between K_1 and I_2 is positive, the probability that a random walk on D^{δ_n} started from x_n hits K_1 before I_2 tends to zero by Lemma 4.3.3. If n is big enough, K_1 is a subset of D_n and disconnects I_2 from I_1^n. We have proved that g_n is uniformly bounded on $\delta_n\mathbb{Z}^2\cap K_1$, if n is big enough. And by definition $g_n\equiv 1$ on $V(D^{\delta_n})\cap I_2$. By Markov property, we have $g_n(x_n)\to 1$. Since g_0 is the limit of $\mathrm{CE}^{\delta_n}g_n$, this implies that $g_0(z)\to 1$ as $z\in D_0$ and $z\to I_2$ in the spherical metric. Thus $g_0\circ J(z)\to 1$ as $z\in\mathbb{A}_{p_0}$ and $z\to\mathbf{C}_{p_0}$.

Now let us consider the behavior of u_n and u_0 near I_2. If $z\in D_n$ and $z\to I_2$ in the spherical metric, then $Q_n(z)\to\mathbf{C}_{p_n}$, and so $u_n(z)=\mathrm{Re}\,\mathbf{S}_{p_n}\circ Q_n(z)\to 1$. Using a plane Brownian motion instead of a random walk in the above argument, we conclude that $u_n(z)\to 1$ as $z\in D_n$ and $z\to I_2$ in the spherical metric, uniformly in n.

Suppose $\{v_n\}$, chosen at the beginning of this proof, has a subsequence that tends to I_2 in the spherical metric. By passing to a subsequence, we may assume that $v_n\to I_2$ in the spherical metric. From the result of the last two paragraphs, we see that $g_n(v_n)\to 1$ and $u_n(v_n)\to 1$. This contradicts the hypothesis that $|g_n(v_n)-u_n(v_n)|\ge\varepsilon_0$. Thus $\{v_n\}$ has a positive spherical distance from I_2. Since the domain bounded by α_1 and α_2 disconnects U_2 from I_1^0, and $\{v_n\}\subset U_2$, so $\{v_n\}$ has a positive spherical distance from I_1 too. Thus $\{v_n\}$ has a subsequence that converges to some $z_0\in D_0$. Again we may assume that $v_n\to z_0$. Then $u_0(z_0)=\lim u_n(v_n)$ and $g_0(z_0)=\lim g_n(v_n)$, and so $|u_0(z_0)-g_0(z_0)|\ge\varepsilon_0$. We will get a contradiction by proving that $g_0\equiv u_0$ in D_0.

Note that g_0 is non-negative, since each g_n is non-negative. We can find a Jordan curve β in D_0 which satisfies the following properties. It disconnects I_2 from I_1^0; it is the union of finite line segments which are parallel to either x or y axis; and it does not intersect $\cup_n \delta_n \mathbb{Z}^2$. From the fact that $\sum_{v\in V(D^{\delta_n})\cap I_2} \Delta_{D^{\delta_n}} g_n(v) = 0$, and the uniform convergence of $\nabla_1^{\delta_n} g_n$ to $\partial_x g_0$, and $\nabla_i^{\delta_n} g_n$ to $\partial_y g_0$ on some neighborhood of β, we have $\int_\beta \partial_{\mathbf{n}} g_0 ds = 0$, where $\mathbf{n}$ is the unit norm vector on β pointed towards I_1. Thus g_0 has a harmonic conjugate, and so does $g_0 \circ J$. We will prove $g_0 \circ J = \operatorname{Re} \mathbf{S}_{p_0}$, from which follows that $g_0 = u_0$. We have proved that $g_0 \circ J(z) \to 1$ as $\mathbb{A}_{p_0} \ni z \to \mathbf{C}_{p_0}$. It suffices to show that $g_0 \circ J(z) \to 0$ as $\mathbb{A}_{p_0} \setminus U \ni z \to \mathbf{C}_0$ for any neighborhood U of 1.

4.4.2 The existence of some sequences of crosscuts

For a doubly connected domain Ω and one of its boundary component X, we say that γ is a crosscut in Ω on X if γ is an open simple curve in D whose two ends approach two points (need not be distinct) of X in Euclidean distance. For such γ, $\Omega \setminus \gamma$ has two connected components, one is a simply connected domain, and the other is a doubly connected domain. Let $U(\gamma)$ denote the simply connected component of $D \setminus \gamma$. Then $\partial U(\gamma)$ is the union of γ and a subset of X.

Now Q_0 maps D_0 conformally onto $\mathbb{A}_{p_0}$, and $Q_0(I_1^0) = \mathbf{C}_0$. Similarly as Theorem 2.15 in [12], we can find a sequence of crosscuts $\{\gamma^k\}$ in D_0 on I_1^0 which satisfies
(i) for each k, $\overline{\gamma^{k+1}} \cap \overline{\gamma^k} = \emptyset$ and $U(\gamma^{k+1}) \subset U(\gamma^k)$;
(ii) $Q_0(\gamma^k)$, $k \in \mathbb{N}$, are mutually disjoint crosscuts in $\mathbb{A}_{p_0}$ on $\mathbf{C}_0$; and
(iii) $U(Q_0(\gamma^k))$, $k \in \mathbb{N}$, forms a neighborhood basis of 1 in $\mathbb{A}_{p_0}$.
Note that $U(Q_0(\gamma^k)) = Q_0(U(\gamma^k))$, so $U(Q_0(\gamma^{k+1})) \subset U(Q_0(\gamma^k))$, for all $k \in \mathbb{N}$. We will prove that there is some crosscut γ_n^k in each D_n on I_1^n such that γ_n^k and $Q_n(\gamma_n^k)$ converge to γ^k and $Q_0(\gamma^k)$, respectively, in the sense that we will specify.

Now fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Parameterize $\overline{\gamma^k}$ and $\overline{Q_0(\gamma^k)}$ as the images of the functions $a : [0,1] \to D \cup I_1^0$ and $b : [0,1] \to \mathbb{A}_{p_0} \cup \mathbf{C}_0$, respectively, so that $b(t) = Q_0(a(t))$, for $t \in (0,1)$. We may choose $s_1 \in (0, 1/2)$ such that the diameters of $a[0, s_1]$ and $a[1-s_1, 1]$ are both less than $\varepsilon/3$. There is $r_1 \in (0,\varepsilon) \cap (0, (1-e^{-p_0})/2)$ such that the curve $b[s_1, 1-s_1]$ and the balls $\overline{\mathbf{B}(b(0);r_1)}$ and $\overline{\mathbf{B}(b(1);r_1)}$ are mutually disjoint. Suppose γ^k is contained in $\mathbf{B}(0;M)$ for some $M>\varepsilon$. There is $C_M>0$ such that the spherical distance between any $z_1, z_2 \in \mathbf{B}(0;2M)$ is at least $C_M|z_1-z_2|$. So for every smooth curve γ in $\mathbf{B}(0;2M)$, we have $L^{\#}(\gamma) \ge C_M L(\gamma)$, where L and $L^{\#}$ denote the Euclidean length and spherical length, respectively. Let $r_2 = r_1 \exp(-72\pi^2/(C_M^2\varepsilon^2))$. Then we may choose $s_2 \in (0, s_1)$ such that $b[0,s_2] \subset \mathbf{B}(b(0);r_2)$ and $b[1-s_2,1] \subset \mathbf{B}(b(1);r_2)$.

For $j=0,1$, let Γ_j be the set of crosscuts γ in $\mathbb{A}_{p_0}$ on $\mathbf{C}_0$ such that

$$\mathbf{B}(b(j);r_2) \cap \mathbb{D} \subset U(\gamma) \subset \mathbf{B}(b(j);r_1).$$

Then the extremal length of Γ_j is less than

$$2\pi/(\ln r_1 - \ln r_2) = C_M^2\varepsilon^2/(36\pi).$$

If n is big enough, then $\mathbf{B}(b(j);r_1) \cap \mathbb{D} \subset \mathbb{A}_{p_n}$, so all $\gamma \in \Gamma_j$ are in $\mathbb{A}_{p_n}$. Then the extremal length of $Q_n^{-1}(\Gamma_j)$ is also less than $C_M^2\varepsilon^2/(36\pi)$. Since the spherical area of $Q_n^{-1}(\mathbb{A}_{p_n})$ is not bigger than that of $\mathbb{C}$, which is 4π, there is some $\beta_{n,j}$ in $Q_n^{-1}(\Gamma_j)$ of spherical length less than $C_M\varepsilon/3$. Since

$$J(b[s_2, 1-s_2]) = a[s_2, 1-s_2] \subset \gamma^k \subset \mathbf{B}(0;M),$$

and Q_n^{-1} converges to J uniformly on $b[s_2,1-s_2]$, so if n is big enough, then $Q_n^{-1}(b[s_2,1-s_2]) \subset \mathbf{B}(0;1.5M)$. Every curve in Γ_j intersects $b[s_2,1-s_2]$, so $\beta_{n,j} \in Q_n^{-1}(\Gamma_j)$ intersects $Q_n^{-1}(b[s_2,1-s_2]) \subset \mathbf{B}(0;1.5M)$. If $\beta_{n,j} \not\subset \mathbf{B}(0;2M)$, then there is a subarc γ of $\beta_{n,j}$ that is contained in $\mathbf{B}(0;2M)$ and connects $\partial\mathbf{B}(0;1.5M)$ with $\partial\mathbf{B}(0;2M)$. So $L^{\#}(\gamma) \ge C_M L(\gamma) \ge C_M M/2$. This is impossible since $L^{\#}(\gamma) \le L^{\#}(\beta_{n,j}) \le C_M\varepsilon/3 < C_M M/2$. Thus $\beta_{n,j} \subset \mathbf{B}(0;2M)$, and so $L(\beta_{n,j}) \le L^{\#}(\beta_{n,j})/C_M < \varepsilon/3$. Since $\beta_{n,j}$ has finite length, it is a crosscut in D_n on I_1^n. Let $s_{n,0}$ be the biggest s such that $Q_n^{-1}(b(s)) \in \beta_{n,0}$, and $s_{n,1}$ the biggest s such that $Q_n^{-1}(b(1-s)) \in \beta_{n,1}$. Then $s_{n,0}, s_{n,1} \in [s_2, s_1]$. Let $\beta'_{n,0}$ and $\beta'_{n,1}$ denote any one component of $\beta_{n,0} \setminus \{Q_n^{-1}(b(s_{n,0}))\}$

and $\beta_{n,1} \setminus \{Q_n^{-1}(b(1-s_{n,1}))\}$, respectively. Let

$$\gamma_n^k := Q_n^{-1}(b[s_{n,0}, 1-s_{n,1}]) \cup \beta'_{n,0} \cup \beta'_{n,1}.$$

Then γ_n^k is a crosscut in D_n on I_1^n. As $r_1 < \varepsilon$, the symmetric difference between $Q_n(\gamma_n^k)$ and $Q_0(\gamma^k)$ is contained in $\mathbf{B}(b(0);\varepsilon) \cup \mathbf{B}(b(1);\varepsilon)$. Since $b[s_{n,0}, 1-s_{n,1}]$ is contained in $b[s_2, 1-s_2]$, which is a compact subset of D_0, so if n is big enough, then the Hausdorff distance between $Q_n^{-1}(b[s_{n,0}, 1-s_{n,1}])$ and $a[s_{n,0}, 1-s_{n,1}]$ is less than $\varepsilon/3$. Now the Hausdorff distance between $Q_n^{-1}(b[s_{n,0}, 1-s_{n,1}])$ and γ_n^k is not bigger than the bigger diameter of $\beta'_{n,0}$ and $\beta'_{n,1}$, which is less than $\varepsilon/3$. And the Hausdorff distance between $a[s_{n,0}, 1-s_{n,1}]$ and γ^k is not bigger than the bigger diameter of $a[0, s_{n,0}]$ and $a[1-s_{n,1}, 1]$, which is also less than $\varepsilon/3$. So the Hausdorff distance between γ_n^k and γ^k is less than ε. Now we proved that we can choose crosscuts γ_n^k in D_n on I_1^n such that γ_n^k converges to γ^k, and the symmetric difference of $Q_n(\gamma_n^k)$ and $Q_0(\gamma^k)$ converges to the two end points of $Q_0(\gamma^k)$, respectively, both in the Hausdorff distance, as n tends to infinity.

4.4.3 Constructing hooks that hold the boundary

Now fix $k \geq 2$. We still parameterize $\overline{\gamma^k}$ and $\overline{Q_0(\gamma^k)}$ as the images of the functions $a : [0,1] \to D \cup I_1^0$ and $b : [0,1] \to \mathbb{A}_{p_0} \cup \mathbf{C}_0$, respectively, such that $b(t) = Q_0(a(t))$, for $t \in (0,1)$. Let Ω^k denote the domain bounded by $Q_0(\gamma^{k-1})$ and $Q_0(\gamma^{k+1})$ in $\mathbb{A}_{p_0}$. Then $\partial\Omega^k$ is composed of $Q_0(\gamma^{k-1})$, $Q_0(\gamma^{k+1})$, and two arcs on $\mathbf{C}_0$. Let ρ_0^k and ρ_1^k denote these two arcs such that $b(j) \in \rho_j^k$, $j = 0,1$. If n is big enough, from the convergence of $Q_n(\gamma_n^{k\pm1})$ to $Q_0(\gamma^{k\pm1})$, we have $\overline{Q_n(\gamma_n^{k-1})} \cap \overline{Q_n(\gamma_n^{k+1})} = \emptyset$, and $U(Q_n(\gamma_n^{k+1})) \subset U(Q_n(\gamma_n^{k-1}))$. Let Ω_n^k denote the domain bounded by $Q_n(\gamma_n^{k-1})$ and $Q_n(\gamma_n^{k+1})$ in $\mathbb{A}_{p_n}$. Then the boundary of Ω_n^k is composed of $Q_n(\gamma_n^{k-1})$, $Q_0(\gamma_n^{k+1})$, and two disjoint arcs on $\mathbf{C}_0$. If n is big enough, then each of these two arcs contains one of $b(0)$ and $b(1)$. Let $\rho_{n,0}^k$ and $\rho_{n,1}^k$ denote these two arcs so that $b(j) \in \rho_{n,j}^k$, $j = 0,1$. Now suppose $c : (-1,+1) \to \Omega^k$ is a crosscut in Ω^k with $c(\pm 1) \in Q_0(\gamma^{k\pm1})$. Then $c(-1,+1)$ divides Ω^k into two parts: Ω_0^k and Ω_1^k, so that $\rho_j^k \subset \partial\Omega_j^k$, $j = 0,1$. If n

is big enough, then $c(\pm 1) \in Q_n(\gamma_n^{k\pm 1})$, and $c(-1,+1) \subset \Omega_n^k$. Thus $c(-1,+1)$ also divides Ω_n^k into two parts: $\Omega_{n,0}^k$ and $\Omega_{n,1}^k$, so that $\rho_{n,j}^k \subset \partial\Omega_{n,j}^k$. Let λ_j ($\lambda_{n,j}$, resp.) be the extremal distance between $Q_0(\gamma^{k-1})$ ($Q_n(\gamma_n^{k-1})$, resp.) and $Q_0(\gamma^{k+1})$ ($Q_n(\gamma_n^{k+1})$, resp.) in Ω_j^k ($\Omega_{n,j}^k$, resp.), $j = 0, 1$. It is clear that $\lambda_{n,j} \to \lambda_j$ as $n \to \infty$, and $\lambda_j < \infty$. Thus $\{\lambda_{n,j}\}$ is bounded by some $E_k > 0$.

Since $\overline{\gamma^k} \cap \overline{\gamma^{k\pm 1}} = \emptyset$ and $\gamma_n^{k\pm 1}$ converges to $\gamma^{k\pm 1}$ in the Hausdorff distance, there is $d_k > 0$ such that the distance between γ^k and $\gamma_n^{k\pm 1}$ is greater than d_k, if n is big enough. For $x \in D_0$ and $r > 0$, let $\widetilde{\mathbf{B}}_0(x;r)$ and $\widetilde{\mathbf{B}}_n(x;r)$ denote the connected component of $\mathbf{B}(x;r) \cap D_0$ and $\mathbf{B}(x;r) \cap D_n$, respectively, that contains x. Since $D_n \to D_0$, it is easy to prove that $\widetilde{\mathbf{B}}_n(x;r) \to \widetilde{\mathbf{B}}_0(x;r)$. Let $e_k = d_k \exp(-2\pi E_k)$. Suppose $s_0 \in (0,1)$ is such that the diameter of $a(0, s_0)$ is less than e_k. By the construction of γ_n^k, we have $\Omega_n^k \to \Omega^k$, so $Q_n^{-1}(\Omega_n^k) \to Q_0^{-1}(\Omega^k)$. Now $a(s_0) \in \gamma^k \subset Q_0^{-1}(\Omega^k)$. Hence $a(s_0) \in Q_n^{-1}(\Omega_n^k)$ if n is big enough. Since the distance from $a(s_0)$ to $\gamma_n^{k\pm 1}$ is bigger than $d_k > e_k$, $\widetilde{\mathbf{B}}_n(a(s_0); e_k)$ is contained in $Q_n^{-1}(\Omega_n^k)$. We claim that $\widetilde{\mathbf{B}}_n(a(s_0); e_k) \subset Q_n^{-1}(\Omega_{n,0}^k)$, if n is big enough.

Since $a(0) \in \partial Q_0^{-1}(\Omega^k)$, $|a(0) - a(s_0)| < e_k$, and $Q_n^{-1}(\Omega_n^k) \to Q_0^{-1}(\Omega^k)$, so the distance from $a(s_0)$ to $\partial Q_n^{-1}(\Omega_n^k))$ is less than e_k, if n is big enough. Now choose $z_n \in \partial Q_n^{-1}(\Omega_n^k)$ that is the nearest to $a(s_0)$. Then the line segment $[a(s_0), z_n) \subset \widetilde{\mathbf{B}}_n(a(s_0); e_k)$. Hence $Q_n[a(s_0), z_n)$ is a simple curve in Ω_n^k such that $Q_n(z)$ tends to some $z_n' \in \partial\Omega_n^k$, as $z \in [a(s_0), z_n)$ and $z \to z_n$. Since $z_n \notin \gamma_n^{k\pm 1}$, $z_n' \notin Q_n(\gamma_n^{k\pm 1})$. Thus z_n' is on $\rho_{n,j}^k$ for some $j \in \{0, 1\}$. Since $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k)) \to Q_0(\widetilde{\mathbf{B}}_0(a(s_0); e_k)) \ni b(s_0)$, and $b(s_0) \in \Omega_{n,0}^k$, so if n is big enough, $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k))$ intersects $\Omega_{n,0}^k$. For such n, if $z_n' \in \rho_{n,1}^k$, then all curves in $Q_n^{-1}(\Omega_{n,0}^k)$ that go from γ_n^{k-1} to γ_n^{k-1} will pass $\widetilde{\mathbf{B}}_n(a(s_0); e_k)$. And so they all cross some annulus centered at $a(s_0)$ with inner radius e_k and outer radius greater than d_k. So the extremal distance between γ_n^{k-1} and γ_n^{k+1} in $Q_n^{-1}(\Omega_{n,j}^k)$ is greater than $(\ln d_k - \ln e_k)/(2\pi) = E_k$. However, by conformal invariance, this extremal distance is equal to $\lambda_{n,j}$, which is not bigger than E_k if n is big enough. Thus $z_n' \in \rho_{n,0}^k$ for n big enough. Similarly, $z_n' \in \rho_{n,0}^k$ and $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k)) \cap \overline{\Omega_{n,1}^k} \neq \emptyset$ can not happen at the same time when n is big enough. So if n is big enough, $Q_n(\widetilde{\mathbf{B}}_n(a(s_0); e_k))$ is contained in $\Omega_{n,0}^k$. Similarly, we let $s_1 \in (s_0, 1)$ be such that the diameter of $a(s_1,1)$ is less than e_k, then $Q_n(\widetilde{\mathbf{B}}_n(a(s_1);e_k)) \subset \Omega^k_{n,1}$, if n is big enough.

For $j=0,1$, $a(s_j)$ and $a(j)$ determine a square of side length $l_j = |a(j)-a(s_j)|$ with vertices $v_{0,j} := a(s_j)$, $v_{2,j}$, $v_{1,j}$, and $v_{3,j}$, in the clockwise order, so that $a(j)$ is on one middle line $[(v_{0,j}+v_{3,j})/2,(v_{1,j}+v_{2,j})/2]$. This square is contained in $\overline{\mathbf{B}(a(s_j);\sqrt{2}l_j)} \subset \mathbf{B}(a(s_j);0.8e_k)$, since $l_j < e_k/2$. And the union of line segments $[v_{0,j},v_{1,j}]$, $[v_{1,j},v_{2,j}]$ and $[v_{2,j},v_{3,j}]$ surrounds $\mathbf{B}(a(j);l_j/8)$.

For $j=0,1$, let N_j be the $l_j/20$-neighborhood of $[v_{0,j},v_{1,j}]\cup[v_{1,j},v_{2,j}]\cup[v_{2,j},v_{3,j}]$. Then $N_j \subset \mathbf{B}(a(s_j);e_k)$. Choose $q_j \in (0,l_j/30)$ such that $\overline{\mathbf{B}(a(s_j);q_j)} \subset Q_0^{-1}(\Omega^k)$. For $m=0,1,2,3$, let $W_{m,j} = \overline{\mathbf{B}(v_{m,j};q_j)}$. When n is big enough, $W_{0,j} \subset Q_n^{-1}(\Omega^k_n)$, and $\mathbf{B}(a(j);l_j/30)$ intersects $\partial Q_n^{-1}(\Omega^k_n)$. Suppose β_j is a curve in N_j which starts from $W_{0,j}$, and reaches $W_{1,j}$, $W_{2,j}$ and $W_{3,j}$ in the order. Then β_j disconnects a subset of $\partial Q_n^{-1}(\Omega^k_n)$ from ∞, if n is big enough. Since $Q_n^{-1}(\Omega^k_n)$ is a simply connected domain, β_j hits $\partial Q_n^{-1}(\Omega^k_n)$. Let β_j^n be the part of β_j before hitting $\partial Q_n^{-1}(\Omega^k_n)$. Then $\beta_j^n \subset \widetilde{\mathbf{B}}_n(a(s_j);e_k) \subset Q_n^{-1}(\Omega^k_{n,j})$, if n is big enough. So $Q_n(\beta_j^n)$ is a curve in $\Omega^k_{n,j}$ that tends to some point of $\partial\Omega^k_{n,j}$ at one end. This point is not on $Q_n(\gamma_n^{k\pm1})$, because the distance between γ^k and γ_n^{k+1} is greater than e_k. Hence $\overline{Q_n(\beta_j^n)}$ intersects $\rho^k_{n,j}$.

Suppose B is a closed ball in $Q_0^{-1}(\Omega^k)$. For $j=0,1$, let Π_j be a subdomain of $Q_0^{-1}(\Omega^k)$ that contains $B\cup W_{0,j}$ such that $\overline{\Pi_j}$ is a compact subset of $Q_0^{-1}(\Omega^k)$. Then Π_j is contained in $Q_n^{-1}(\Omega^k_n)$ for n big enough. For $x\in\delta_n\mathbb{Z}^2\cap B$, let $\mathcal{A}^x_{n,j}$ be the set of lattice paths of $\delta_n\mathbb{Z}^2$ that start from x, and hit $W_{0,j}$, $W_{1,j}$, $W_{2,j}$ and $W_{3,j}$ in the order before exiting $\Pi_j\cup N_j$. We may view $\beta\in\mathcal{A}^x_{n,j}$ as a continuous curve. Let β^{D_n} denote the part of $\beta\in\mathcal{A}^x_{n,j}$ before exiting $Q_n^{-1}(\Omega^k_n)$. Then β^{D_n} can be viewed as a lattice path on D^{δ_n}. We proved in the last paragraph that if n is big enough, $\overline{Q_n(\beta^{D_n})}$ intersects $\rho^k_{n,j}$, for any $\beta\in\mathcal{A}^x_{n,j}$, $x\in\delta_n\mathbb{Z}^2\cap B$, $j=0,1$. Thus for any $\beta_0\in\mathcal{A}^x_{n,0}$ and $\beta_1\in\mathcal{A}^x_{n,1}$, $\beta_0^{D_n}\cup\beta_1^{D_n}$ disconnects γ_n^{k-1} from γ_n^{k+1} in $Q_n^{-1}(\Omega^k_n)$.

4.4.4 The behaviors of $g_0\circ J$ outside any neighborhood of 1

Let $P^x_{n,j}$ be the probability that a random walk on $\delta_n\mathbb{Z}^2$ started from x belongs to $\mathcal{A}^x_{n,j}$. By Lemma 4.3.4, if n is big enough, then $P^x_{n,j}$ is greater than some $a_k>0$ for all $x \in \delta_n\mathbb{Z}^2 \cap B$, $j = 0, 1$. We may also choose n big enough such that $V(D^{\delta_n}) \cap B$ is non-empty, and $g_n(x)$ is less than some $b_k \in (0, \infty)$ for all $x \in \delta_n\mathbb{Z}^2 \cap B$. We claim that if n is big enough, then $g_n(x) \le \max\{b_k/a_k, 1\}$ for every $x \in \delta_n\mathbb{Z}^2 \cap (D_n \setminus U(\gamma_n^{k-1}))$. Suppose for infinitely many n, there are $x_n \in \delta_n\mathbb{Z}^2 \cap D_n \setminus U(\gamma_n^{k-1})$ such that $g_n(x_n) \ge M > \max\{b_k/a_k, 1\}$. Since g_n is discrete harmonic on $\delta_n\mathbb{Z}^2 \cap D_n$, and $g_n \le 1$ on the boundary vertices of D_n except at $P(w_n)$, the tip point of w_n, so there is a lattice path β_n in D_n that goes from x_n to $P(w_n)$ such that the value of g_n at each vertex of β_n is not less than M. By the construction of γ_n^{k+1}, if n is big enough, then $U(Q_n(\gamma_n^{k+1}))$ is some neighborhood of 1 in $\mathbb{A}_{p_n}$, and so $U(\gamma_n^{k+1})$ is some neighborhood of $P(w_n)$ in D_n. Thus β_n intersects both γ_n^{k-1} and γ_n^{k+1}. Choose $v_0 \in \delta_n\mathbb{Z}^2 \cap B$. For every $\rho_{n,0} \in \mathcal{A}_{n,0}^{v_0}$ and $\rho_{n,1} \in \mathcal{A}_{n,1}^{v_0}$, the path $\rho_{n,0}^{D_n} \cup \rho_{n,1}^{D_n}$ disconnects γ_n^{k-1} from γ_n^{k+1}. Therefore $\rho_{n,0}^{D_n} \cup \rho_{n,1}^{D_n}$ intersects β_n. This implies that for some $j_n \in \{0, 1\}$, for every $\rho \in \mathcal{A}_{n,j_n}^{v_0}$, we have ρ^{D_n} intersects β_n. Thus the probability that a random walk on $\delta_n\mathbb{Z}^2$ started from v_0 hits β_n before ∂D_n is greater than a_k. Let τ_n be the first time this random walk hits $\beta_n \cup \partial D_n$. Since g_n is non-negative, bounded, and discrete harmonic on $\delta_n\mathbb{Z}^2 \cap D_n$, so $g_n(v_0) = \mathbf{E}[g_n(\mathrm{RW}_{v_0}^x(\tau_n))] \ge a_k M > b_k$, which is a contradiction. So the claim is proved.

By passing to a subsequence depending on k, we can now assume the following. $U(\gamma_n^{k+1})$ is some neighborhood of $P(w_n)$ in D_n; the value of g_n on $\delta_n\mathbb{Z}^2 \cap D_n \setminus U(\gamma_n^{k+1})$ is bounded by some $M_k \ge 1$; $U(\gamma_n^{k+1}) \subset U(\gamma_n^k) \subset U(\gamma_n^{k-1})$; the spherical distance between γ_n^k and γ_n^{k-1} is greater than some $R_k > 0$; and the (Euclidean) distance between γ_n^k and γ_n^{k+1} is greater than δ_n. Since the end points of γ_n^k and γ_n^{k-1} are on I_1^n, the spherical diameter of I_1^n is at least R_k. Let R be the spherical distance between I_2 and α_2. Then the spherical distance between I_2 and I_1^n is at least R, as α_2 disconnects I_2 from I_1^n. Suppose $v \in V(D^{\delta_n}) \cap D_n \setminus U(\gamma_n^{k-1})$, and $dist^{\#}(v, I_1^n) = d < R/2$. Then $dist^{\#}(v, I_2) > R/2$. Let RW_v^n be a random walk on $\delta_n\mathbb{Z}^2$ started from v, and τ_n^k be the first time that RW_v^n leaves $D_n \setminus U(\gamma_n^k)$. Then $\mathrm{RW}_v^n(\tau_n^k)$ is either on I_2, or on I_1^n, or in $U(\gamma_n^k)$. In the first case, $g_n(\mathrm{RW}_v^n(\tau_n^k)) = 1$, and v should first exit $\mathbf{B}^{\#}(v, R/2)$ before hitting I_2. In the second and third cases, since $\mathrm{RW}_v^n(\tau_n^k - 1) \in D_n \setminus U(\gamma_n^k)$, and the Euclidean distance between γ_n^k and γ_n^{k+1} is greater than δ by construction, so

$[\mathrm{RW}_v^n(\tau_n^k-1),\mathrm{RW}_v^n(\tau_n^k)]$ does not intersect γ_n^{k+1}. Thus in the second case, $\mathrm{RW}_v^n(\tau_n^k)\ne P(w_n)$, and so $g_n(\mathrm{RW}_v^n(\tau_n^k))=0$. In the third case, $\mathrm{RW}_v^n(\tau_n^k)\in D_n\setminus U(\gamma_n^{k+1})$, so $g_n(\mathrm{RW}_v^n(\tau_n^k))\le M_k$; and RW_v^n first uses some edge that intersects γ_n^{k-1}, then uses some edge that intersects γ_n^k at time τ_n^k. So the spherical diameter of $\mathrm{RW}_v^n[0,\tau_n^k]$ is at least R_k. This implies that RW_v^n should exit $\mathbf{B}^\#(v;R_k/2)$ before hitting $U(\gamma_n^k)$. Let $R_k'=\min\{R/2,R_k/2\}$, then by Lemma 4.3.3,

$$\Pr[\mathrm{RW}_v^n(\tau_n^k)\notin I_1^n]\le C_0((\delta_n+d)/R_k')^{C_1},$$

for some absolute constants $C_0,C_1>0$. So we have $g_n(v)\le M_kC_0((\delta_n+d)/R_k')^{C_1}$.

Suppose $z\in D_0\setminus U(\gamma^{k-1})\setminus\gamma^{k-1}$, and $dist^\#(z,I_1^0)=d<R/4$. Choose $r\in(0,d/2)$ such that $\mathbf{B}^\#(z,r)$ is bounded and $\overline{\mathbf{B}^\#(z;r)}\subset D_0\setminus U(\gamma^{k-1})\setminus\gamma^{k-1}$. If n is big enough, then $\overline{\mathbf{B}^\#(z;r)}\subset D_n\setminus U(\gamma_n^{k-1})$, and the spherical distance from every $v\in\mathbf{B}^\#(z;r)$ to I_1^n is less than $2d<R/2$. Thus

$$g_n(v)\le M_kC_0((\delta_n+2d)/R_k')^{C_1},\ \ \forall v\in\delta_n\mathbb{Z}^2\cap\mathbf{B}^\#(z;r).$$

Since g_0 is the limit of g_n, $g_0(z)\le M_kC_0(2d/R)^{C_1}$. Thus for every $k\ge2$, $g_0(z)\to0$, as $z\in D_0\setminus U(\gamma^{k-1})\setminus\gamma^{k-1}$, and $z\to I_1$ in the spherical metric, and so $g_0\circ J(z)\to0$ as $z\in\mathbb{A}_{p_0}\setminus U(Q_0(\gamma^{k-1}))$, and $z\to\mathbf{C}_0$. Since $U(Q_0(\gamma^k))$, $k\in\mathbb{N}$, forms a neighborhood basis of 1 in $\mathbb{A}_{p_0}$, so for any $r>0$, $g_0\circ J(z)\to0$ if $z\in\mathbb{A}_{p_0}\setminus\mathbf{B}(1,r)$ and $z\to\mathbf{C}_0$. This is what we need at the end of 4.4.1. $\square$

4.5 Some discussions

Suppose an $\mathrm{HRLC}_2(D;w_0\to T)$ is proved to be the scaling limit of the corresponding LERW. Then we could obtain some geometric behaviors of the trace. First, the limit set of the trace in the conformal closure is a single point on the target set T. If T is an interior point or a prime end, then itself is the limit point. If T is a side arc then the distribution of the limit point is the hitting point of a plane Brownian motion (or Brownian excursion, resp.) in D started from w_0 conditioned to hit T if w_0 is an interior point (or a prime end, resp.). Here the Brownian excursion is the limit as $D \ni z \to w_0$ of a plane Brownian motion in D started from z conditioned to hit T. We will study more about Brownian excursion in the next chapter. Moreover, if T_1 is a subarc of T, then the trace conditioned to hit T_1 is a $\mathrm{HRLC}_2(D; w_0 \to T_1)$ trace. We may even have $\mathrm{HRLC}_2(D; w_0 \to T) = \int_T \mathrm{HRLC}_2(D; w_0 \to w) d\mu(w)$, where μ is the hitting distribution of an $\mathrm{HRLC}_2(D; w_0 \to T)$ trace at T.

From the reversibility of LERW in [3], we find that for any $p_1 \neq p_2 \in \widehat{D}$, the reversal of an $\mathrm{HRLC}_2(D; p_1 \to p_2)$ trace has the same lap as an $\mathrm{HRLC}_2(D; p_2 \to p_1)$ trace after a time-change. If the target is a side arc, the reversibility can only be stated for SLE_2. For a standard strip SLE_2 trace β, if we let $m + \pi i$ be the hitting point of β at $\pi i + \mathbb{R}$, then the reversal of $\beta - m$ has the same law as a strip $\mathrm{SLE}_2(\mathbb{S}_\pi; \pi i \to \mathbb{R})$ trace after a time-change. For a modulus p standard annulus SLE_2 trace β, let $\mathbf{x}_0$ and $\mathbf{x}_p$ be uniform random points on $\mathbf{C}_0$ and $\mathbf{C}_p$, respectively, then the reversal of an annulus $\mathrm{SLE}_2(\mathbb{A}_p; \mathbf{x}_0 \to \mathbf{C}_p)$ trace has the same law as an annulus $\mathrm{SLE}_2(\mathbb{A}_p; \mathbf{x}_p \to \mathbf{C}_0)$ trace after a time-change. Similarly, the reversal of a standard disc SLE_2 trace has the same law as a radial $\mathrm{SLE}_2(\mathbb{D}; \mathbf{x} \to 0)$ trace after a time-change.

We may also consider the reflection boundary condition. For example, suppose $0 \in \partial D$ and $(0, a] \subset D$ for some $a > 0$, $p_0 \in D \cap c\mathbb{Z}^2$, and I is a side of D that does not contain the prime end 0_+. Let D^{δ_n} be defined as usual for $\delta_n = c/n$. Let $\widetilde{\partial}_V D^{\delta_n} := \partial_V D^{\delta_n} \setminus I$, i.e., the set of boundary vertices not on I. Let X^{δ_n} be the LERW in $(D^{\delta_n}, \widetilde{\partial}_V D^{\delta_n} \cup \{p_0\})$ conditioned to hit p_0. As $n \to \infty$, the curve of X^{δ_n} should converge to a random continuous curve that can be described by an equation similar to that of $\mathrm{HRLC}_2(D; 0_+ \to p_0)$. To construct such a curve, we let $P^\xi_{t,W}$ in the definition of HRLC to be the modified Green function in $D \setminus W(K^\xi_t)$ with the pole at p_0 and with a reflection boundary I. This means that $P^\xi_{W,t}$ is a positive harmonic function in $D \setminus \{p_0\}$; it behaves like $-\ln|z - p_0|/(2\pi)$ for z near p_0; $P^\xi_{t,W}(z) \to 0$ as $z \to \hat{\partial} D \setminus I$; and $\partial_{\mathbf{n}} P^\xi_{t,W}(z) = 0$ for $z \in I$.

Chapter 5

Random Loewner chains and restriction measures

It is proved in [8] that the chordal and radial $\mathrm{SLE}_{8/3}$ satisfy the restriction property. For example, suppose β is a standard $\mathrm{SLE}_{8/3}$ trace, i.e., an $\mathrm{SLE}_{8/3}(\mathbb{H}; 0 \to \infty)$ trace. Let Ω be a simply connected subdomain of $\mathbb{H}$ that contains some neighborhoods of 0 and ∞ in $\mathbb{H}$. Then with a positive probability, β stays in Ω, and the law of β conditioned to stay in Ω is the same as an $\mathrm{SLE}_{8/3}(\Omega; 0 \to \infty)$ trace, after a time-change. If the trace is considered as a random set, then we don't need to worry about the time-change. So the law of that trace (set) is a conformally invariant restriction probability measure. The authors of [8] also studied other conformally invariant measures, which are not supported on simple paths, in simply connected domains. Since all simply connected domains are conformally equivalent to each other, so there is no much freedom to construct a restriction measure. The restriction measures other than chordal $\mathrm{SLE}_{8/3}$ trace can be constructed from some chordal SLE_κ, $0 < \kappa < 8/3$, by adding Brownian bubbles to the trace.

In this chapter we will imitate the work in [8], and study the restriction properties of HRLC_κ in a plane domain started from a prime end. Since a subdomain is in general not conformally equivalent to the whole domain, there could be a lot of different kinds of conformally equivalent restriction measure. We are only interested in those restriction measures that are most closed to HRLC.

5.1 Preparations

5.1.1 Brownian motion and Brownian excursion

We will use the notations and symbols in Section 3.1. It is known ([13]) that plane Brownian motions are conformally invariant up to a time-change. So a Brownian motion in a Riemann surface can be defined up to time-changes. For a finite Riemann surface D, we first consider a Brownian motion in its underlying surface R. The Brownian motion almost surely leaves D in a finite time, and hits a prime end of D. This Brownian motion stopped at the time it leaves D is called a Brownian motion in $(D, \hat{\partial} D)$. We will consider such a (stopped) Brownian motion as a random closed subset of D or $\widehat{D}$. So the its law is a measure on the space of closed subsets of D or $\widehat{D}$. We use $\mathrm{BM}(D; z \to \hat{\partial} D)$ to denote the law of a Brownian motion in $(D, \hat{\partial} D)$ started from $z \in D$. For any side arc I of D, the probability that the above Brownian motion hits I is the harmonic measure function of I in D valued at z, and so is positive. The law of a Brownian motion in $(D, \hat{\partial} D)$ conditioned to hit I is denoted by $\mathrm{BM}(D; z \to I)$.

Suppose w is a prime ends of D. We may find a decreasing sequence of side arcs I_n such that $\{w\} = \cap_n I_n$. Then the weak limit of $\mathrm{BM}(D; z \to I_n)$ as $n \to \infty$ exists and is independent of the choice of $\{I_n\}$. We use $\mathrm{BM}(D; z \to w)$ to denote the limit measure. A random set with this law is called a Brownian motion in $(D, \hat{\partial} D)$ started from z conditioned to hit w.

Suppose w is a prime end of D, I is a side arc of D, and $w \notin \overline{I}$. Then the weak limit of $\mathrm{BM}(D; z \to I)$ as $D \ni z \to w$ exists, and is denoted by $\mathrm{BE}(D; w \to I)$. A random set that has this law is called a Brownian excursion in D started from w conditioned to hit I.

Suppose $w_1 \neq w_2$ are two prime ends of D. Then the weak limit of $\mathrm{BM}(D; z \to w_2)$ as $D \ni z \to w_1$ exists, and is denoted by $\mathrm{BE}(D; w_1 \to w_2)$. It is also the weak limit of $\mathrm{BE}(D; w_1 \to I_n)$ as $n \to \infty$ for any decreasing sequence of side arcs I_n such that $\{w_2\} = \cap_n I_n$. A random set that has this law is called a Brownian excursion in D started from w_1 conditioned to hit w_2. In fact, we have $\mathrm{BE}(D; w_1 \to w_2) =$

$\mathrm{BE}(D; w_2 \to w_1)$.

Let $S(D)$ be the set of nonempty $F \in \mathrm{Cld}(D)$ which is a union of finitely many mutually disjoint closed Jordan discs. Since we could use sets in $S(D)$ to approximate any closed subset of D, so we have the following lemma.

Lemma 5.1.1 *Let μ_1 and μ_2 be two finite positive measures on $\mathcal{F}_D^{\mathcal{H}}$ with $|\mu_1| = |\mu_2|$. Suppose μ_1 and μ_2 agree on $\{\cap F = \emptyset\}$ for any $F \in S(D)$. Then $\mu_1 = \mu_2$.*

For $F \in S(D)$, $D \setminus F$ is still a finite Riemann surface. And all sides of D are sides of $D \setminus F$. Let $H(D, I; \cdot)$ denote the harmonic measure function of I in D. Then for any $F \in S(D)$,

$$\mathrm{BM}(D; z \to I)(\{\cap F = \emptyset\}) = \frac{H(D \setminus F, I; z)}{H(D, I; z)}.$$

Suppose w is a prime end of D and h maps a neighborhood U of w in $\widehat{D}$ conformally into $\overline{\mathbb{H}}$ such that $h(U \cap \hat{\partial} D) \subset \mathbb{R}$. Let $M(D, w, h; \cdot)$ denote the minimal function in D with the pole at w normalized by h. Then we have

$$\mathrm{BM}(D; z \to w)(\{\cap F = \emptyset\}) = \frac{M(D \setminus F, w, h; z)}{M(D, w, h; z)}.$$

If I is a side arc of D such that $w \notin \overline{I}$ then

$$\mathrm{BE}(D; w \to I)(\{\cap F = \emptyset\}) = \frac{\partial_y (H(D \setminus F, I; \cdot) \circ h^{-1})(h(w))}{\partial_y (H(D, I; \cdot) \circ h^{-1})(h(w))}.$$

Suppose $w_1 \neq w_2$ are two prime ends of D, and h_j maps a neighborhood U_j of w_j in $\widehat{D}$conformally into $\overline{\mathbb{H}}$ such that $h(U_j \cap \hat{\partial} D) \subset \mathbb{R}$. Then

$$\mathrm{BE}(D; w_1 \to w_2)(\{\cap F = \emptyset\}) = \frac{\partial_y (M(D \setminus F, I, h_2; \cdot) \circ h_1^{-1})(h_1(w_1))}{\partial_y (H(D, I, h_2; \cdot) \circ h_1^{-1})(h_1(w_1))}.$$

The Brownian motion and Brownian excursion have the restriction property. For example, suppose K has the law of $\mathrm{BM}(D; p \to I)$ and D' is a subdomain of D that contains p and a neighborhood of I. Then K conditioned to stay in D' has the law of $\mathrm{BM}(D'; p \to I)$. The case of Brownian excursion is similar.

5.1.2 Brownian bubble

Suppose w is a prime end of a finite Riemann surface D. As usual, let h map a neighborhood U of w conformally into $\overline{\mathbb{H}}$ so that $h(U \cap \hat{\partial} D) \subset \mathbb{R}$. Let $\xi = h(w)$. Choose a sequence $p_n \in U \cap D$ so that $p_n \to w$. Suppose $\mathbb{R} \ni r_m \searrow 0$ and $\{|z-\xi| < r_m\} \cap \overline{\mathbb{H}}$ is contained in the image of h. Let $U_m = h^{-1}(\{|z-\xi| < r_m\} \cap \mathbb{H})$. Let $\mu_{n,m}$ denote the measure $\mathrm{BM}(D; p_n \to w)/|h(p_n)-\xi|^2$ restricted to the subspace $\{\not\subset U_m\}$ of $\mathrm{Cld}(D)$. Let $z_n = h(p_n)$, $M_D(z) = M(D, w, h; h^{-1}(z))$ and $M_m(z) = M(U_m, w, h; h^{-1}(z))$. Then

$$|\mu_{n,m}| = \frac{M_D(z_n) - M_m(z_n)}{|z_n - \xi|^2 M_D(z_n)}.$$

As $z \to 0$ in $\mathbb{H}$, from the normalization of $M(D, w, h; \cdot)$, we have

$$M_D(z) = \frac{\operatorname{Im} z}{|z-\xi|^2}(1 + O(|z-\xi|^2)).$$

On the other hand, $M_D - M_m$ has no pole around ξ, so it is the real part of an analytic function in a neighborhood of ξ. So we have

$$M_D(z) - M_m(z) = \partial_y|_\xi (M_D - M_m) \operatorname{Im} z (1 + O(|z-\xi|),$$

as $z \to \xi$ in $\mathbb{H}$. Thus

$$\lim_{n\to\infty} |\mu_{n,m}| = \partial_y|_\xi (M_D - M_m).$$

So for any $m \in \mathbb{N}$, $\{\mu_{n,m} : n \in \mathbb{N}\}$ is uniformly bounded. So it has a subsequential weak limit $\mu_{0,m}$ with $|\mu_{0,m}| = \partial_y|_\xi (M_D - M_m)$.

Suppose $F \in \mathrm{S}(D)$. Then $\{\cap F = \emptyset\}$ is an open subset of $\mathrm{Cld}(D)$. And

$$\mu_{n,m}(\{\cap F = \emptyset\}) = \mathrm{BM}(D; p_n \to w)(\{\not\subset U_m, \cap F = \emptyset\})/|h(p_n) - \xi|^2$$

$$= (\mathrm{BM}(D; p_n \to w)(\{\not\subset U_m \setminus F\}) - \mathrm{BM}(D; p_n \to w)(\{\cap F \neq \emptyset\}))/|h(p_n) - \xi|^2.$$

So we have

$$\lim_{n\to\infty} \mu_{n,m}(\{\cap F = \emptyset\}) = \partial_y|_\xi(M_D - M_{m,F}) - \partial_y|_\xi(M_D - M_F),$$

where $M_{m,F}(z) = M(U_m \setminus F, w, h; h^{-1}(z))$ and $M_F(z) = M(D \setminus F, w, h; h^{-1}(z))$. Thus

$$\mu_{0,m}(\{\cap F = \emptyset\}) \leq \partial_y|_\xi(M_D - M_{m,F}) - \partial_y|_\xi(M_D - M_F).$$

We could find a decreasing sequence F_l in $\mathrm{S}(D)$ such that $F = \cap_l F_l$ and F is contained in the interior of each F_l. For each l, $\{\cap \mathrm{int} F_l = \emptyset\}$ is a closed subset of $\mathrm{Cld}(D)$, we have

$$\mu_{0,m}(\{\cap F = \emptyset\}) \geq \mu_{0,m}(\{\cap \mathrm{int} F_l = \emptyset\}) \geq \limsup_{n\to\infty} \mu_{n,m}(\{\cap \mathrm{int} F_l = \emptyset\})$$

$$\geq \partial_y|_\xi(M_D - M_{m,F_l}) - \partial_y|_\xi(M_D - M_{F_l}).$$

It can be proved that

$$\lim_{l\to\infty} \partial_y|_\xi(M_D - M_{m,F_l}) = \partial_y|_\xi(M_D - M_{m,F});$$

$$\lim_{l\to\infty} \partial_y|_\xi(M_D - M_{F_l}) = \partial_y|_\xi(M_D - M_F).$$

We conclude that

$$\mu_{0,m}(\{\cap F = \emptyset\}) = \partial_y|_\xi(M_D - M_{m,F}) - \partial_y|_\xi(M_D - M_F).$$

By Lemma 5.1.1, $\mu_{0,m}$ is uniquely determined, which means any subsequential weak limit of $\{\mu_{n,m} : n \in \mathbb{N}\}$ is the same $\mu_{0,m}$. So $\mu_{0,m}$ is the weak limit of $\mu_{n,m}$ as $n \to \infty$. If $m' < m$, then similarly as above, we have

$$\mu_{0,m}(\{\subset U_{m'}\}) = \partial_y|_\xi(M_D - M_m) - \partial_y|_\xi(M_D - M_{m'}) = |\mu_{0,m}| - |\mu_{0,m'}|.$$

Thus $\mu_{0,m}(\{\not\subset U_{m'}\} = |\mu_{0,m'}|$. Since $\mu_{n,m} \geq \mu_{n,m'}$ for all $n \in \mathbb{N}$, we have $\mu_{0,m} \geq \mu_{0,m'}$.

It follows that $\mu_{0,m}$ restricted to the set $\{\not\subset U_{m'}\}$ is no less than $\mu_{0,m'}$ restricted to $\{\not\subset U_{m'}\}$, which is $\mu_{0,m'}$ itself. However, from $\mu_{0,m}(\{\not\subset U_{m'}\} = |\mu_{0,m'}|$, we must have $\mu_{0,m}|_{\{\not\subset U_{m'}\}} = \mu_{0,m'}$. This is true for any $m' < m \in \mathbb{N}$.

Finally, note that $\{\{\not\subset U_m\}\}$ is an increasing subsequence of closed subsets of $\mathrm{Cld}(D)$, and the union of them is $\{\neq \emptyset\}$. So there is a unique measure $\mu_{0,0}$ on $\mathcal{F}_H^D$ so that $\mu_{0,0}(\{\emptyset\}) = 0$ and $\mu_{0,0}|_{\{\not\subset U_m\}} = \mu_{0,m}$ for any $m \in \mathbb{N}$. This $\mu_{0,0}$ is positive and σ-finite but not finite. It in fact does not depend on the choice of p_n or U_m. But it depends on the choice of h. We call $\mu_{0,0}$ the Brownian bubble measure in D hanging at w, normalized by h. Let it be denoted by $\mathrm{BB}(D, h; w)$. In fact, for any two normalizing function h_1 and h_2, we have

$$\mathrm{BB}(D, h_1; w) = |(h_2 \circ h_1^{-1})'(h_1(w))|^2 \mathrm{BB}(D, h_2; w).$$

If D' is a subdomain of D that contains a neighborhood of w in D, then $\mathrm{BB}(D, h; w)$ restricted in D' is equal to $\mathrm{BB}(D', h; w)$. And

$$\mathrm{BB}(D, h; w)(\{\not\subset D'\}) = \partial_y((M(D, w, h; \cdot) - M(D', w, h; \cdot)) \circ h^{-1})(h(w)).$$

The Brownian bubble defined here is similar to the Brownian bubble defined in [8] for the upper half plane. The main difference is that we don't fill in the holes of a Brownian bubble to make it a hull, so the information inside will not be lost.

5.2 A martingale for HRLC_2

Suppose L has the law of $\mathrm{HRLC}_\kappa(D; w \to I)$, where $\kappa > 0$ and D is a finite Riemann surface with a prime end w and a side arc I such that $w \not\in \overline{I}$. The case that the target is a prime end can be studied similarly. Let α be the side that contains w. Let W map a neighborhood Ω in $\mathbb{D} \setminus \{0\}$ of $\partial\mathbb{D}$ conformally onto a neighborhood Σ in D of α and satisfy $W(\partial\mathbb{D}) = \alpha$ and $W(1) = w$. Let T_Σ be the first t such that $L(t) \not\in \Sigma$ or $t = \Delta(L)$. Let L_Σ be that L restricted on $[0, T_\Sigma)$. Let $L_{\Sigma,W}$ be a time-change of L_Σ such that $(K_t := L_{\Sigma,W}(t))$ is a family of standard radial LE hulls. Let $[0, T_{\Sigma,W})$

be the range of $L_{\Sigma,W}$. Let ξ be the driving function with $\xi(0) = 0$ and φ_t be the corresponding LE maps. Then for some standard Brownian motion $B(t)$, we have

$$d\xi(t) = \sqrt{\kappa} dB(t) + (3 - \kappa/2) \frac{\partial_x \partial_y J_t(\xi(t))}{\partial_y J_t(\xi(t))} dt,$$

where $J_t := H_t \circ W \circ \varphi_t^{-1} \circ e^i$ and $H_t := H(D \setminus W(K_t), I; \cdot)$. Let $w(t)$ be the prime end determined by L_Σ at time t. Then $W \circ \varphi_t^{-1} \circ e^i$ maps a neighborhood of $\xi(t)$ conformally onto a neighborhood of $w(t)$ in $D \setminus W(K_t)$, and $w(t) = W \circ \varphi_t^{-1} \circ e^i(\xi(t))$. Let h_t be a local inverse of $W \circ \varphi_t^{-1} \circ e^i$ near $w(t)$. Then h_t is a normalizing function for minimal functions in $D \setminus W(K_t)$ with the pole at $w(t)$. We let M_t be the minimal function as above normalized by h_t. Suppose $F \in S(D)$. Let $T_{\Sigma,W,F}$ be the first t such that $W(K_t) \cap F \neq \emptyset$ or $t = T_{\Sigma,W}$. For $0 \leq t < T_{\Sigma,W,F}$, let $\widetilde{H}_t := H(D \setminus F \setminus W(K_t), I; \cdot)$. Let $\widetilde{J}_t := \widetilde{H}_t \circ W \circ \varphi_t^{-1} \circ e^i$ and $\widetilde{M}_t$ be the minimal function in $D \setminus F \setminus W(K_t)$ with the pole at $w(t)$, normalized by h_t. Then we have the following lemma.

Lemma 5.2.1

$$\partial_t H_t(z) = -2 \partial_y J_t(\xi(t)) \cdot M_t(z), \quad \textit{for} \ \ 0 \leq t < T_{\Sigma,W} \ \ \textit{and} \ \ z \in D;$$

$$\partial_t \widetilde{H}_t(z) = -2 \partial_y \widetilde{J}_t(\xi(t)) \cdot \widetilde{M}_t(z), \quad \textit{for} \ \ 0 \leq t < T_{\Sigma,W,F} \ \ \textit{and} \ \ z \in D \setminus F.$$

Proof. The proof is not immediate. But the picture is clear. The change of $H_t(z)$ comes from a disturbance at the prime end $w(t)$, which contributes the minimal function in $D \setminus W(K_t)$ with the pole at $w(t)$. So we omit the proof. $\square$

We may choose a family of conformal maps ψ_t such that $e^i \circ \psi_t = \varphi_t \circ e^i$, ψ_0 is an identity, and ψ_t is continuous in t. Then it satisfies the differential equation

$$\partial_t \psi_t(z) = \cot((\psi_t(z) - \xi(t))/2).$$

And $J_t \circ \psi_t = H_t \circ W \circ e^i$. Applying ∂_t on both sides of this equality, we have

$$\partial_t J_t \circ \psi_t(z) + \partial_x J_t(\psi_t(z))\text{Re }\cot(\frac{\psi_t(z) - \xi(t)}{2})+$$

$$+\partial_y J_t(\psi_t(z))\text{Im }\cot(\frac{\psi_t(z) - \xi(t)}{2}) = -2\partial_y J_t(\xi(t)) M_t \circ W \circ e^i(z).$$

Let $p = \psi_t(z)$ and $Q_t := M_t \circ W \circ e^i \circ \psi_t^{-1}$. Then

$$\partial_t J_t(p) + \partial_x J_t(p)\text{Re }\cot(\frac{p - \xi(t)}{2}) + \partial_y J_t(p)\text{Im }\cot(\frac{p - \xi(t)}{2}) = -2\partial_y J_t(\xi(t)) Q_t(p).$$

Suppose $\overline{J}_t$ and $\overline{Q}_t$ are some analytic functions in a neighborhood of $\xi(t)$ (or without the point $\xi(t)$) such that $J_t = \text{Im}\,\overline{J}_t$ and $Q_t = \text{Im}\,\overline{Q}_t$. Then $\xi(t)$ is a simple pole of $\overline{Q}_t$, and the principle part is $-1/(z - \xi(t))$ thanks to the normalization of M_t. The above displayed formula becomes

$$\partial_t \text{Im}\,\overline{J}_t(p) = -2\partial_y J_t(\xi(t))\text{Im}\,\overline{Q}_t(p) - \text{Im}\,(\partial_z \overline{J}_t(p)\cot(\frac{p - \xi(t)}{2})),$$

where ∂_z means the complex derivative. Applying ∂_z on both sides, we get

$$\partial_t \partial_z \overline{J}_t(p) = -2\partial_y J_t(\xi(t))\partial_z \overline{Q}_t(p) - \partial_z(\partial_z \overline{J}_t(p)\cot(\frac{p - \xi(t)}{2}))$$

$$= -2\partial_y J_t(\xi(t))\partial_z \overline{Q}_t(p) - \partial_z^2 \overline{J}_t(p)\cot(\frac{p - \xi(t)}{2}) + \frac{\partial_z \overline{J}_t(p)/2}{\sin^2((p - \xi(t))/2))}. \tag{5.2.1}$$

For $j = 1, 2, 3$, let $a_j(t) := \partial_z^j \overline{J}_t(\xi(t))$. So $a_j(t) = \partial_x^{j-1}\partial_y J_t(\xi(t))$. Suppose $b(t)$ is such that $\overline{Q}_t(p) = -1/(z - \xi(t)) + C + b(t)(p - \xi(t)) + O((p - \xi(t))^2)$. Then

$$\partial_z \overline{Q}_t(p) = \frac{1}{(p - \xi(t))^2} + b(t) + O(p - \xi(t));$$

$$\partial_z \overline{I}_t(p) = a_1(t) + a_2(t)(p - \xi(t)) + a_3(t)/2(p - \xi(t))^2 + O((p - \xi(t))^3);$$

$$\partial_z^2 \overline{J}_t(p) = a_2(t) + a_3(t)(p - \xi(t)) + O((p - \xi(t))^2).$$

Note that

$$\cot(\frac{p-\xi(t)}{2}) = \frac{2}{p-\xi(t)} - \frac{p-\xi(t)}{6} + O((p-\xi(t))^2);$$

$$\frac{1/2}{\sin^2((p-\xi(t))/2)} = \frac{2}{(p-\xi(t))^2} + \frac{1}{6} + O(p-\xi(t)).$$

Plugging these equalities into formula (5.2.1), we get

$$\partial_t\partial_y J_t(\xi(t)) = \partial_t\partial_z \overline{J}_t(\xi(t)) = (1/6 - 2b(t))a_1(t) - a_3(t).$$

From Ito's formula, we have

$$da_1(t) = d\partial_y J_t(\xi(t)) = \partial_t\partial_y J_t(\xi(t))dt + \kappa/2\partial_x^2\partial_y J_t(\xi(t))dt$$

$$+\partial_x\partial_y J_t(\xi(t))(\sqrt{\kappa}dB(t) + (3-\frac{\kappa}{2})\frac{\partial_x\partial_y J_t(\xi(t))}{\partial_y J_t(\xi(t))}dt)$$

$$= ((1/6 - 2b(t))a_1(t) - a_3(t))dt + \kappa/2a_3(t)dt$$

$$+(3-\frac{\kappa}{2})\frac{a_2(t)^2}{a_1(t)}dt + a_2(t)\sqrt{\kappa}dB(t).$$

Thus

$$d\ln a_1(t) = \frac{da_1(t)}{a_1(t)} - \kappa/2\frac{a_2(t)^2}{a_1(t)^2}dt = \frac{a_2(t)}{a_1(t)}\sqrt{\kappa}dB(t) +$$

$$+(\frac{1}{6} - 2b(t) - \frac{2-\kappa}{2}\frac{a_3(t)}{a_1(t)} + (3-\kappa)\frac{a_2(t)^2}{a_1(t)^2})dt.$$

Similarly, let $\widetilde{a}_j$ and $\widetilde{b}$ be defined for $\widetilde{J}_t$ and $\widetilde{M}_t$. Then we have

$$d\ln\widetilde{a}_1(t) = \frac{\widetilde{a}_2(t)}{\widetilde{a}_1(t)}\sqrt{\kappa}dB(t) + (\frac{1}{6} - 2\widetilde{b}(t) - \frac{2-\kappa}{2}\frac{\widetilde{a}_3(t)}{\widetilde{a}_1(t)}$$

$$+(3-\frac{\kappa}{2})\frac{a_2(t)}{a_1(t)}\cdot\frac{\widetilde{a}_2(t)}{\widetilde{a}_1(t)} - \frac{\kappa}{2}\frac{\widetilde{a}_2(t)^2}{\widetilde{a}_1(t)^2})dt.$$

Let

$$e(\kappa) = \frac{6-\kappa}{2\kappa},\ \ \alpha(\kappa) = \frac{(6-\kappa)(8-3\kappa)}{2\kappa},\text{ and }\ \beta(\kappa) = \frac{3(6-\kappa)(\kappa-2)}{2\kappa}. \tag{5.2.2}$$

Then we compute

$$d\left(\frac{\widetilde{a}_1(t)}{a_1(t)}\right)^{e(\kappa)} \Big/ \left(\frac{\widetilde{a}_1(t)}{a_1(t)}\right)^{e(\kappa)} = e(\kappa)(\frac{\widetilde{a}_2(t)}{\widetilde{a}_1(t)} - \frac{a_2(t)}{a_1(t)})\sqrt{\kappa}dB(t) - \alpha(\kappa)(\widetilde{b}(t) - b(t))dt$$

$$-\beta(\kappa)\left[(\widetilde{b}(t) - b(t)) - \frac{1}{6}(\frac{\widetilde{a}_3(t)}{\widetilde{a}_1(t)} - \frac{a_3(t)}{a_1(t)}) + \frac{1}{4}(\frac{\widetilde{a}_2(t)^2}{\widetilde{a}_1(t)^2} - \frac{a_2(t)^2}{a_1(t)^2})\right] dt. \tag{5.2.3}$$

If $\kappa = 2$, then $e(\kappa) = 1$, $\alpha(\kappa) = 2$, and $\beta(\kappa) = 0$. So we have

$$d\left(\frac{\widetilde{a}_1(t)}{a_1(t)}\right) \Big/ \left(\frac{\widetilde{a}_1(t)}{a_1(t)}\right) = -2(\widetilde{b}(t) - b(t))dt + (\frac{\widetilde{a}_2(t)}{\widetilde{a}_1(t)} - \frac{a_2(t)}{a_1(t)})\sqrt{\kappa}dB(t). \tag{5.2.4}$$

So we have the following Lemma

Lemma 5.2.2 *For $\kappa = 2$,*

$$\frac{\widetilde{a}_1(t)}{a_1(t)} \exp(-2\int_0^t (b(s) - \widetilde{b}(s))ds)$$

is a bounded continuous martingale for $0 \le t < T_{\Sigma,W,F}$.

Proof. From formula (5.2.4) we know this is a local martingale. Note that $\widetilde{a}_1(t)/a_1(t)$ $= \mathrm{BE}(D_t; w_t \to \alpha)(\{\cap F = \emptyset\} \le 1$; and $b(t) - \widetilde{b}(t) = \mathrm{BB}(D_t, h_t; w_t)(\{\cap F \ne \emptyset\}) \ge 0$. So the local martingale is bounded by 1. It must be a bounded continuous martingale. $\square$

Now we suppose $\kappa = 2$. So L has the law of $\mathrm{HRLC}_2(D; w \to I)$. From the above lemmas, we obtain the following proposition.

Proposition 5.2.1 *Let T_F be the first t such that $L(t) \cap F \ne \emptyset$ or $t = \Delta(L)$. For $0 \le t < T_F$, let*

$$p(t) = BE(D \setminus L(t); w_L(t) \to I)(\{\cap F = \emptyset\}),$$

$$q(t) = BB(D \setminus L(t), g_t; w_L(t))(\{\cap F \ne \emptyset\}),$$

where $w_L(t)$ is the prime end of $D \setminus L(t)$ determined by L at time t, and g_t maps a neighborhood of U_t of $w_L(t)$ in $D \setminus L(t)$ conformally into $\overline{\mathbb{H}}$ so that $g_t(U_t \cap \hat{\partial} D) \subset \mathbb{R}$), *and*

$$\partial_y(H(D \setminus L(t), \alpha; \cdot) \circ g_t^{-1})(g_t(w_{L_\Sigma}(t))) = 1.$$

Then $p(t)\exp(-2\int_0^t q(s)ds)$ *is a bounded continuous martingale for* $0 \le t < T_F$.

Proof. Note that L_Σ is a time-change of $L_{\Sigma,W}$, and the choice of g_t compensates the effect of the time-change. So from Lemma 5.2.2, $p(t)\exp(-2\int_0^t q(s)ds)$ is a bounded continuous martingale for $0 \le t < T_{\Sigma,F}$, where $T_{\Sigma,F}$ is the first t such that $L(t) \not\subset \Sigma$, or $L(t) \cap F \ne \emptyset$, or $t = \Delta(L)$. Since there is a sequence of Σ_n such that $T_F = \vee_n T_{\Sigma_n,F}$, so the proof is completed. $\square$

5.3 Adding Brownian bubbles to HRLC$_2$

We only consider HRLC$_2$ in a plane domain D. We may assume that D is a subdomain of $\mathbb{H}$ and $\mathbb{H} \setminus D$ is a compact subset of $\mathbb{H}$. Let I be as side arc of D. Suppose L has the law of HRLC$_2(D; 0 \to I)$. Let L_c on $[0, T_c)$ be a time change of L such that $(L_c(t))$ is a family of standard chordal LE hulls. Let ξ be the driving function, and φ_t be the corresponding LE maps. Then for some standard Brownian motion $B(t)$, we have

$$d\xi(t) = \sqrt{2}dB(t) + 2\frac{\partial_x\partial_y J_t(\xi(t))}{\partial_y J_t(\xi(t))}dt,$$

where $J_t := H_t \circ W \circ \varphi_t^{-1}$, $H_t := H(D_t, I; \cdot)$, and $D_t := D \setminus L_c(t)$.

Suppose $F \in S(D)$. Let $T_{c,F}$ be the first t such that $L_c(t) \cap F \ne \emptyset$ or $t = T_c$. For $0 \le t < T_{c,F}$, let

$$p(t) = \mathrm{BE}(D_t; w_{L_c}(t) \to I)(\{\cap F = \emptyset\}),$$

$$q(t) = \mathrm{BB}(D_t, \varphi_t; w_{L_c}(t))(\{\cap F \ne \emptyset\}),$$

where $w_{L_c}(t)$ is the prime end determined by L_c at time t. Similarly as Lemma 5.2.2, we have

Lemma 5.3.1 *For* $0 \le t < T_{c,F}$, $p(t)\exp(-2\int_0^t q(s)ds)$ *is a bounded martingale.*

Let β be the curve in D such that $L_c(t) = \beta(0,t]$. Then for $p \in D \setminus L_c(t)$, $p \to w_{L_c}(t)$ iff $p \to \beta(t)$. We may write $\beta(t)$ for $w_{L_c}(t)$. We have

Lemma 5.3.2 *If $\beta(t) \to z \in \partial F$ as $t \to T_{c,F}^-$, then $\lim_{t\to T_{c,F}^-} p(t) = 0$.*

Proof. This is similar to the proof of Lemma 6.3 in [8]. □

Lemma 5.3.3 *If $\beta(t) \to w_0 \in int\ I$ as $t \to T_{c,F}^-$, then $\lim_{t\to T_{c,F}^-} p(t) = 1$.*

Proof. Suppose γ is an open simple curve in D disconnecting w_0 from F in D and the two ends of γ converge to two different points of I. Let T_γ be the *last* t such that $\beta(t) \in \gamma$. Let U be a connected component of $D \setminus \gamma \setminus \beta(0, T_\gamma]$ that has a prime end w_0. Then U is simply connected, and the boundary of U in D is composed of a subarc of I that contains z_0, part of γ, and part of $\beta(0,T]$. Let $\widetilde{I} := I \cap \hat{\partial} U$. Suppose $t > T_\gamma$. By definition,

$$p(t) = \lim_{z\to\beta(t)} \mathrm{BM}(D_t; z \to I)(\{\cap F = \emptyset\}).$$

Let $U_t := U \setminus \beta(T_\gamma, t]$. Then we have

$$\mathrm{BM}(D_t; z \to I)(\{\cap F \neq \emptyset\}) = \frac{\mathrm{BM}(D_t; z \to \hat{\partial} D_t)(\{\cap F \neq \emptyset, \text{hits } I\})}{\mathrm{BM}(D_t; z \to \hat{\partial} D_t)(\{\text{hits } I\})}$$

$$\leq \frac{\mathrm{BM}(D_t; z \to \hat{\partial} D_t)(\{\not\subset U\})}{\mathrm{BM}(D_t; z \to \hat{\partial} D_t)(\{\text{hits } \widetilde{I}\})} \leq \frac{\mathrm{BM}(U_t; z \to \hat{\partial} U_t)(\{\text{hits } \gamma \cup \beta(0, T_\gamma]\})}{\mathrm{BM}(U_t; z \to \hat{\partial} U_t)(\{\text{hits } \widetilde{I}\})}.$$

The last inequality comes from the fact that $\mathrm{BM}(D_t; z \to \hat{\partial} D_t)$ stopped on leaving U_t has the same law as $\mathrm{BM}(U_t; z \to \hat{\partial} U_t)$. As $z \to \beta(t)$, the right-hand side of the above inequality tends to

$$\frac{1 - \mathrm{BE}(D_t; \beta(t) \to \partial U_t \setminus \beta[T_\gamma, t])(\{\text{hits } \widetilde{I}\})}{\mathrm{BE}(D_t; \beta(t) \to \partial U_t \setminus \beta[T_\gamma, t])(\{\text{hits } \widetilde{I}\})}.$$

To show that $\mathrm{BE}(D_t; \beta(t) \to I)(\{\cap F = \emptyset\})$ tends to 1 as $t \to T_c$, it suffices to prove that $\mathrm{BE}(D_t; \beta(t) \to \partial U_t \setminus \beta[T_\gamma, t])(\{\text{hits } \widetilde{I}\})$ tends to 1 as $t \to T_c$.

Note that U_t is simply connected, and $\partial U_t = \partial U \cup \beta(T_\gamma, t]$. The two sides of $\beta[T_\gamma, t]$ correspond two side arcs of $\widetilde{U_t}$ which have a common side point $\beta(t)$. Suppose Q_t maps U_t conformally onto $\mathbb{D}$ so that $Q_t(\beta(t)) = 1$ and the two sides of $\beta[T_\gamma, t]$ are mapped to the arc $< 1, +i >$ and $< 1, -i >$, respectively, where $< a, b >$ denotes the shorter arc of $\partial\mathbb{D}$ bounded by $a, b \in \partial\mathbb{D}$. Suppose $\widetilde{I}$ is mapped to $< e^{ir_+(t)}, e^{ir_-(t)} >$, where $\pi/2 < r_+(t) < r_-(t) < 3\pi/2$. By conformal invariance,

$$\mathrm{BE}(D_t; \beta(t) \to \partial U_t \setminus \beta[T_\gamma, t])(\{\text{hits } \widetilde{I}\})$$

$$= \mathrm{BE}(\mathbb{D}; 1 \to < +i, -i >)(\{\text{hits } < e^{ir_+(t)}, e^{ir_-(t)} >\}).$$

We now only need to show $e^{ir_\pm(t)} \to \pm i$ as $t \to T_c$. Write $\beta_t^\pm$ for $Q_t^{-1}(< 1, \pm i >)$ and $\alpha_t^\pm$ for $Q_t^{-1}(< \pm i, e^{ir_\pm(t)} >)$. As $t \to T_c$, every curve in U_t from $I_t^\pm$ to $\beta_t^\mp$ must cross an annulus centered at z_0 with modulus tends to ∞. So the extremal distance between $\alpha_t^\pm$ and $\beta_t^\mp$ in U_t tends to ∞. By conformal invariance, the extremal distance between $< 1, \pm i >$ and $< \mp i, e^{ir_\mp(t)} >$ in $\mathbb{D}$ tends to ∞ as $t \to T_c$. Thus the length of $< \mp i, e^{ir_\mp(t)} >$ tends to 0, i.e., $e^{ir_\pm(t)} \to \pm i$ as $t \to T_c$. $\square$

From the discussion of the last chapter, it is reasonable to assume that $\beta(t)$ converges to some interior point of I as $t \to \Delta(L)$. Combining the last three lemmas, we conclude

Proposition 5.3.1 $p(0) = E[1_{\{\beta(0,T_c) \cap F = \emptyset\}} \exp(-2\int_0^{T_c} q(s)ds)]$.

Consider a Poisson point process X on $\mathrm{Cld}(\mathbb{H}) \times [0, \infty)$ with mean $2\mathrm{BB}(\mathbb{H}, \mathrm{id}; 0) \times dt$, where dt is Lebesgue measure. Let

$$\widehat{X}_D := \{(\varphi_t^{-1}(K + \xi(t)), t) : (K, t) \in X, t \in [0, T_c), \varphi_t^{-1}(K + \xi(t)) \in D\}.$$

Roughly speaking, $\widehat{X}_D$ is a Poisson point process with mean $2\mathrm{BB}(D_t, \varphi_t; w_{L_c}(t)) \times dt$. Let Ξ be the union of $\beta(0, T_c)$ and all K such that $(K, t) \in \widehat{X}_D$ for some t.

Choose any $F \in \mathrm{S}(D)$. The probability that $\Xi \cap F = \emptyset$ is the probability that

$\varphi_t^{-1}(K+\xi(t)) \not\subset D$ or $\cap F=\emptyset$ for each $(K,t)\in X$ and $t\in[0,T_c)$, which is equal to

$$E[1_{\{\beta(0,T_c)\cap F=\emptyset\}}\exp(-2\int_0^{T_c}\mathrm{BB}(\mathbb{H}\setminus L_c(t),\varphi_t;w_{L_c}(t))(\{\subset D,\cap F\neq\emptyset\})dt)].$$

From the definition of Brownian bubbles, $\mathrm{BB}(\mathbb{H}\setminus L_c(t),\varphi_t;w_{L_c}(t))$ restricted to the event $\{\subset D\}$ is $\mathrm{BB}(D_t,\varphi_t;w_{L_c}(t))$. By Proposition 5.3.1, the above formula is equal to

$$E[1_{\{\beta(0,T_c)\cap F=\emptyset\}}\exp(-2\int_0^{T_c}\mathrm{BB}(D_t,\varphi_t;w_{L_c}(t))(\{\cap F\neq\emptyset\})dt)]$$
$$=\mathrm{BE}(D;0\to\alpha)(\{\cap F=\emptyset\}).$$

Theorem 5.3.1 *The law of* Ξ *is* $BE(D;0\to\alpha)$*.*

Proof. By Lemma 5.1.1, we only need to show that a.s. Ξ is a closed subset of D. For any $t\in[0,\infty)$, let Ξ^t denote the union of $\beta(0,\min\{T_c,t\}]$ and all K such that $(K,s)\in\widehat{X}_D$ for some $s\in[0,t]$, let Ξ_t be the union of $\beta(0,T_c]$ and all K such that $(K,s)\in\widehat{X}_D$ for some $s\in[t,T_c)$. A proof similar to that of Theorem 7.3 in [8] shows that Ξ^t is a.s. closed for any t. Since $\Xi=\Xi^t\cup\Xi_t$, so if Ξ is not a.s. closed, then there is some $F\in\mathrm{S}(D)$ such that with a positive probability $\beta(0,T_c)$ misses F but Ξ_t intersects F for all t. However, the probability that $\beta(0,T_c)$ misses F but Ξ_t intersects F is the value of

$$1-E[1_{\{\beta(0,T_c)\cap F=\emptyset\}}\exp(-2\int_t^{T_c}\mathrm{BB}(D_s,\varphi_s;w_s)(\{\cap F\neq\emptyset\})ds)]$$

$$=1-E[1_{\{\beta(0,T_c)\cap F=\emptyset\}}\mathrm{BE}(D_t;w_t\to\alpha)(\{\cap F=\emptyset\})]\to 0,\ \text{ as }\ t\to T_c^-.$$

This is because $p(t)\to 1$ on the event $\{\beta(0,T_c)\cap F=\emptyset\}$. So Ξ is closed. $\Box$

5.4 Other values of the parameter κ

In this section we only consider the HRLC_κ in a plane domain D from one prime end to another prime end, and the two prime ends are on the same side of D. We assume that D is a subdomain of $\mathbb{H}$ that contains some neighborhoods of 0 and ∞

in $\mathbb{H}$. Let $\mathcal{A}$ denote all domains of this kind. We study the $\mathrm{HRLC}_\kappa(D; 0 \to \infty)$ for $\kappa \in (0, 8/3]$. If L has the law of $\mathrm{HRLC}_\kappa(D; 0 \to \infty)$, then $\cup L(t)$ is a random simple curve in D started from 0. If D is a simply connected domain, then L is a chordal SLE, so the curve ends at ∞. Now we assume that L ends at ∞ for all $D \in \mathcal{A}$. With this assumption, HRLC_κ generates a measure on $\Gamma(D)$, the space of simple curves in D connecting 0 and ∞.

For any $D \in \mathcal{A}$, let M_D^0 be a minimal function in D with the pole at 0, normalized such that

$$M_D^0(z) + \operatorname{Im}\frac{1}{z} \to 0, \text{ as } z \to 0;$$

let M_D^∞ be a minimal function in D with the pole at ∞, normalized such that

$$M_D^\infty(z) - \operatorname{Im} z \to 0, \text{ as } z \to \infty.$$

Let

$$\mathrm{BB}(D) := -\partial_y(M_D^0(z) + \operatorname{Im}\frac{1}{z})|_{z=0};$$

$$\mathrm{Sh}(D) := \frac{\partial_x^2 \partial_y M_D^\infty(0)}{\partial_y M_D^\infty(0)} - \frac{3}{2}\left(\frac{\partial_x \partial_y M_D^\infty(0)}{\partial_y M_D^\infty(0)}\right)^2.$$

In fact, $\mathrm{BB}(D) = \mathrm{BB}(\mathbb{H}, \mathrm{id}; 0)(\{\not\subset D\}) > 0$. And if D is simply connected, we have $-\mathrm{BB}(D) = \mathrm{Sh}(D)/6$. Now let

$$J(D) := -\mathrm{BB}(D) - \mathrm{Sh}(D)/6.$$

One may check that $J(D)$ is conformally covariant in the sense that if $D, D' \in \mathcal{A}$, and f maps $(D; 0, \infty)$ conformally onto $(D'; 0, \infty)$, then $J(D) = f'(0)^2 J(D')$.

Suppose $\gamma \in \Gamma(D)$. We may parameterize γ such that $(\gamma(0, t], 0 \le t < \infty)$ is a family of standard chordal LE hulls. Let ξ^γ and φ_t^γ be the driving function and LE maps. Let $D_t^\gamma := D \setminus \gamma(0, t]$ and $\Omega_t^\gamma := \varphi_t^\gamma(D_t^\gamma) - \xi^\gamma(t)$. Then $\Omega_t^\gamma \in \mathcal{A}$. Now we assume that for any $\gamma \in \Gamma(D)$, $\int_0^\infty J(\Omega_t^\gamma) dt$ exists and is finite. Let

$$\triangle_D(\gamma) := \exp(\int_0^\infty J(\Omega_t^\gamma) dt), \text{ and } \square_D(\gamma) := \exp(\int_0^\infty -\mathrm{BB}(\Omega_t^\gamma) dt).$$

Then $\triangle_D$ is conformally invariant in the sense that if $D, D' \in \mathbb{A}$, and f maps $(D; 0, \infty)$ conformally onto $(D'; 0, \infty)$, we have $\triangle_D(\gamma) = \triangle_{D'}(f(\gamma))$ for any $\gamma \in \Gamma(D)$. For $c > 0$, let X_C be a Poisson point process on $\mathrm{Cld}(\mathbb{H}) \times [0, \infty)$ with mean $c\mathrm{BB}(\mathbb{H}, \mathrm{id}; 0) \times dt$. Let

$$\Xi_\gamma^c := \cup\{(\varphi_t^\gamma)^{-1}(K + \xi^\gamma(t)), t) : (K, t) \in X_c, t \in [0, \infty)\} \cup \gamma.$$

Then Ξ_γ^c is called the union of γ with a Poisson cloud of Brownian bubbles in $\mathbb{H}$ with density c. And the probability that Ξ_γ^c stays in D is equal to $\square_D(\gamma)^c$. We now use the symbols $e(\kappa)$, $\alpha(\kappa)$ and $\beta(\kappa)$ in (5.2.2). Suppose $D' \subset D$ are two domains in $\mathcal{A}$. Using an argument similar to that of the last section, we get

$$\left(\frac{\mathrm{BE}(D')}{\mathrm{BE}(D)}\right)^{e(\kappa)} = \int_{\Gamma(D')} \left(\frac{\square_{D'}}{\square_D}\right)^{\alpha(\kappa)} \left(\frac{\triangle_{D'}}{\triangle_D}\right)^{\beta(\kappa)} d\mu_\kappa^D, \tag{5.4.1}$$

where μ_κ^D is the measure on $\Gamma(D)$ given by $\mathrm{HRLC}_\kappa(D; 0 \to \infty)$, and $\mathrm{BE}(D) := \mathrm{BE}(\mathbb{H}; 0 \to \infty)(\{\subset D\})$. If $\kappa = 8/3$, then $e(\kappa) = 5/8$, $\alpha(\kappa) = 0$ and $\beta(\kappa) = 5/4$. Let $D_1 \subset D_2$ be two domains of $\mathcal{A}$ and $D' \subset D_1$ is a simply connected domain in $\mathcal{A}$. Then $\triangle_{D'} \equiv 1$. From (5.4.1), we have

$$\left(\frac{\mathrm{BE}(D')}{\mathrm{BE}(D_j)}\right)^{5/8} = \int_{\Gamma(D')} \frac{d\mu_{8/3}^{D_j}}{\triangle_{D_j}^{5/4}}, \quad \text{for} \quad j = 1, 2.$$

Let $\widetilde{\nu}_{8/3}^D$ be defined by $d\widetilde{\nu}_{8/3}^D = \triangle_D^{-5/4} d\mu_{8/3}^D$. Since $\mu_{8/3}^D$ and $\triangle_D$ are conformally invariant, so is $\widetilde{\nu}_{8/3}^D$. Assume it is a bounded measure. Then we have

$$\widetilde{\nu}_{8/3}^{D_1}(D')/\widetilde{\nu}_{8/3}^{D_2}(D') = (\mathrm{BE}(D_1)/\mathrm{BE}(D_2))^{-5/8}.$$

Note that two bounded measure on $\Gamma(D)$ are equal iff they agree on the set $\{\subset D'\}$ for all simply connected $D' \in \mathcal{A}$. So we find that $\widetilde{\nu}_{8/3}^{D_2}$ restricted to $\Gamma(D_1)$ is equal to $(\mathrm{BE}(D_1)/\mathrm{BE}(D_2))^{5/8}\widetilde{\nu}_{8/3}^{D_1}$. If we let $\nu_{8/3}^D$ be equal to $\widetilde{\nu}_{8/3}^D/|\widetilde{\nu}_{8/3}^D|$. Then $\nu_{8/3}^D$ is a conformally invariant restriction probability measure.

For $\kappa \in (0, 8/3)$, we have $\alpha(\kappa) > 0$. From (5.4.1), we may guess that

$$\mathrm{BE}(D_1)^{e(\kappa)}(\Box_{D_1})^{-\alpha(\kappa)}(\triangle_{D_1})^{-\beta(\kappa)}d\mu_\kappa^{D_1} = \mathrm{BE}(D_2)^{e(\kappa)}(\Box_{D_2})^{-\alpha(\kappa)}(\triangle_{D_2})^{-\beta(\kappa)}d\mu_\kappa^{D_2} \tag{5.4.2}$$

on $\Gamma(D_1)$. Now let ν_κ^D be a probability measure so that $d\nu_\kappa^D = C(\triangle_{D_1})^{-\beta(\kappa)}d\mu_\kappa^D$ for some $C > 0$. Then ν_κ^D is conformally invariant. And for some $C > 0$, we have

$$(\Box_{D_1})^{-\alpha(\kappa)}d\nu_\kappa^{D_1} = C(\Box_{D_2})^{-\alpha(\kappa)}d\nu_\kappa^{D_2}.$$

Let γ_j be a random curve with the law of $\nu_\kappa^{D_j}$. Consider a Poisson point process X on $\mathrm{Cld}(\mathbb{H}) \times [0, \infty)$ with mean $\alpha(\kappa)\mathrm{BB}(\mathbb{H}, \mathrm{id}; 0) \times dt$. Let

$$\widehat{X}_j^\gamma := \{((\varphi_t^{\gamma_j})^{-1}(K + \xi(t)), t) : (K, t) \in X, t \in [0, \infty), (\varphi_t^\gamma)^{-1}(K + \xi^{\gamma_j}(t)) \in D_j\}.$$

Let Ξ_j be the union of γ_j and all K such that $(K, t) \in \widehat{X}_j^\gamma$ for some t. Then for any $D' \in \mathbb{A}$ and $D' \subset D_1$, the probability that Ξ_j stays in D' is equal to

$$\int_{\Gamma(D')} \left(\frac{\Box_{D'}}{\Box_{D_j}}\right)^{b(\kappa)} d\nu_\kappa^{D_j}.$$

So there is a $C = C(D_1, D_2) > 0$ such that the probability that Ξ_1 stays in D' is equal to C times the probability that Ξ_2 stays in D', for any $D' \in \mathbb{A}$ and $D' \subset D_1$. So Ξ_2 conditioned to stay in Ξ_1 has the same law as Ξ_1. Thus by adding a Poisson cloud of Brownian bubbles with density $\alpha(\kappa)$ to a random curve with the law of ν_κ^D, we obtain a conformally invariant restriction Probability measure. If $D = \mathbb{H}$, then after filling all holes, this measure agrees with the restriction measure constructed in [8] from a chordal SLE_κ.

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