Time-reversal of Multiple-force-point SLE\(_{\kappa}(\rho)\) with All Force Points Lying on the Same Side

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Abstract

We define intermediate SLE\(_{\kappa}(\rho)\) and reversed intermediate SLE\(_{\kappa}(\rho)\) processes using Appell-Lauricella multiple hypergeometric functions, and use them to describe the time-reversal of multiple-force-point chordal SLE\(_{\kappa}(\rho)\) curves in the case that all force points are on the boundary and lie on the same side of the initial point, and \(\kappa\) and \(\rho = (\rho_1, \ldots, \rho_m)\) satisfy that either \(\kappa \in (0, 4]\) and \(\sum_{j=1}^{k} \rho_j > -2\) for all \(1 \leq k \leq m\), or \(\kappa \in (4, 8)\) and \(\sum_{j=1}^{k} \rho_j \geq \frac{\kappa}{2} - 2\) for all \(1 \leq k \leq m\).

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1 Introduction

The Schramm–Loewner evolution (SLE), first introduced by Oded Schramm in 1999 ([15]), is a one-parameter ($\kappa \in (0, \infty)$) family of measures on non-self-crossing curves, which has received a lot of attention over the past two decades. It has been shown that, modulo time parametrization, a number of discrete random paths on grids have SLE with different parameters as their scaling limits. We refer the reader to Lawler’s textbook [3] for basic properties of SLE.

Based on the convergence of various discrete lattice models to SLE, Rohde and Schramm conjectured (cf. [13]) that, for $\kappa \in (0, 8)$, chordal SLE$_\kappa$ (growing in a simply connected domain from one boundary point to another) satisfies reversibility, i.e., the time reversal of a chordal SLE$_\kappa$ curve is also a chordal SLE$_\kappa$ curve, modulo a time reparametrization.

The conjecture was first proved for $\kappa \in (0, 4]$ in [23], which constructed a commutation coupling of two chordal SLE$_\kappa$ curves growing towards each other, and used the coupling to show that the two curves are time-reversal of each other. The conjecture for $\kappa \in (4, 8)$ was proved by [7] using the celebrated imaginary geometry theory.

People have also worked on the reversibility property of other types of SLE. Radial SLE grows in a simply connected domain from a boundary point to an interior point, and obviously does not satisfy reversibility. However, one may consider its close relative: whole-plane SLE, which grows in the Riemann sphere from one (interior) point to another point. Conditionally on an initial segment of a whole-plane SLE$_\kappa$ curve, the rest of the curve is a radial SLE$_\kappa$ curve in the remaining domain. The reversibility of whole-plane SLE$_\kappa$ was first proved for $\kappa \in (0, 4]$ in [20], and later for $\kappa \in (4, 8)$ in [6]. The work [20] also describes the time-reversal of radial SLE$_\kappa$ for $\kappa \in (0, 4]$ although the reversibility does not hold.

SLE$_\kappa(\rho)$ is another important type of SLE, whose growth is affected by some additional marked points, called force points, besides the target. They were introduced in [5] for the construction of restriction measures, and were later used in the series [9, 8, 7, 6] as building blocks of the imaginary geometry.

The paper [21] proves the reversibility of a single-force-point chordal SLE$_\kappa(\rho)$ in the case that $\kappa \in (0, 4]$, $\rho \geq \kappa/2 - 2$, and that the only force point lies on the boundary, and is degenerate, i.e., lies immediately next to the initial point. A new process called intermediate SLE$_\kappa(\rho)$ was introduced there to describe the time-reversal of chordal SLE$_\kappa(\rho)$ in the case that the boundary force point is not degenerate. An intermediate SLE$_\kappa(\rho)$ is a two-force-point process defined using a hypergeometric function, and is different from the SLE$_\kappa(\rho)$ in [5]. It is also proved in [21] that intermediate SLE$_\kappa(\rho)$ satisfies reversibility for $\kappa \in (0, 4]$ and $\rho \geq \kappa/2 - 2$. 


The intermediate SLE was later called hypergeometric SLE or hSLE in [11] and [17]. The latter paper [17] extends the reversibility of intermediate SLE\(\kappa_\rho\) to \(\kappa \in (0, 8)\) and \(\rho \geq \kappa / 2 - 2\) in the case that both force points are not degenerate, and proved that intermediate SLE\(\kappa(2)\) is the marginal law of a single curve in a multiple 2-SLE\(\kappa\) configuration.

The papers [8, 7] established the reversibility of chordal SLE\(\kappa_\rho\) in the case that \(\kappa \in (0, 4)\) and \(\rho_1, \rho_2 > -2\), or \(\kappa \in (4, 8)\) and \(\rho_1, \rho_2 \geq \frac{\kappa}{2} - 4\), and that the two boundary force points are both degenerate, one on each side. But those papers did not provide description of the time-reversal of chordal SLE\(\kappa_\rho\) in the case that any force point is not degenerate.

The current paper studies the time-reversal of multiple-force-point chordal SLE\(\kappa_\rho\) in the case that all force points are boundary points and lie on the same side of the initial point. The first result of this form was obtained in [22] for \(\kappa = 4\), where it was shown that if the force points \(v = (v_1, \ldots, v_m)\) are ordered such that \(v_j\) is closer to the initial point than \(v_k\) when \(j < k\), and if the corresponding force values \(\rho = (\rho_1, \ldots, \rho_m)\) satisfy that \(\sum_{j=1}^k \rho_j \geq 0\) for all \(1 \leq k \leq m\), then the time-reversal of a chordal SLE\(4(\rho)\) curve is a chordal SLE\(4(-\rho, -\rho_\infty)\) curve, where \(\rho_\infty = -\sum_{j=1}^m \rho_j\). The value \(-\rho_j\) force point for the time-reversal is still \(v_j\), \(1 \leq j \leq m\), and the value \(-\rho_\infty\) force point for the time-reversal lies immediately next to the initial point of the reversal curve, i.e., the terminal point of the original curve.

Below are the main theorems of the paper, which extend the results of [21].

**Theorem 1.1.** Let \(v_1 > \cdots > v_m \in (-\infty, 0) \cup \{0^\pm\}\) or \(v_1 < \cdots < v_m \in (0, +\infty) \cup \{0^\pm\}\). Let \(\sigma \in \{+, -\}\) be the sign of \(v_j\)’s. Suppose \(\kappa\) and \(\rho_1, \ldots, \rho_m\) satisfy either

(I) \(\kappa \in (0, 4]\) and for any \(1 \leq k \leq m\), \(\sum_{j=1}^k \rho_j > -2\); or

(II) \(\kappa \in (4, 8]\) and for any \(1 \leq k \leq m\), \(\sum_{j=1}^k \rho_j \geq \frac{\kappa}{2} - 2\).

Let \(\eta\) be a chordal SLE\(\kappa_\rho\) curve in \(\mathbb{H}\) from 0 to \(\infty\) with force points \(v_1, \ldots, v_m\). Let \(J(z) = -1/z\). Let \(\rho_\infty = -\sum_{j=1}^m \rho_j\). Then the time-reversal of \(J \circ \eta\) may be reparametrized by half-plane capacity and become a chordal Loewner curve, whose law is absolutely continuous w.r.t. that of a chordal SLE\(\kappa(\rho, \rho_\infty)\) curve in \(\mathbb{H}\) from 0 to \(\infty\) with force points \(J(v_1), \ldots, J(v_m), 0^{-\sigma}\). Here we use the convention that \(J(0^\pm) = \mp \infty\).

**Theorem 1.2.** Let \(\kappa, \rho_1, \ldots, \rho_m, \rho_\infty, v_1, \ldots, v_m, \sigma, \text{ and } J\) be as in Theorem 1.1. Let \(v_0 \in (v_m, +\infty) \cup \{+\infty\}\) if \(\sigma = +\); and \(\in (-\infty, v_m)\) if \(\sigma = -\). Let \(v_j^r = J(v_j)\) and \(\rho_j^r = -\rho_j\), \(j \in \{1, \ldots, m, \infty\}\). Here we use the convention that \(J(0^\pm) = \mp \infty\) and \(J(\pm \infty) = 0^\mp\). Let \(\eta^r\) be an iSLE\(\kappa_\rho\) curve (Definition 3.3) in \(\mathbb{H}\) from 0 to \(\infty\) with force points \(v_1, \ldots, v_m, v_\infty\). Let \(\eta^r\) be an iSLE\(\kappa_\rho\) curve (Definition 3.3) in \(\mathbb{H}\) from 0 to \(\infty\) with force points \(v_1^r, \ldots, v_m^r, v_\infty^r\). Then up to a time-change, the law of the time-reversal of \(J(\eta)\) agrees with the law of \(\eta^r\), which is absolutely continuous w.r.t. that of a chordal SLE\(\kappa_\rho\) curve in \(\mathbb{H}\) from 0 to \(\infty\) with force points \(v_1^r, \ldots, v_m^r, v_\infty^r\).

The iSLE\(\kappa_\rho\) and iSLE\(\kappa_\rho^r\) (shorthands for intermediate SLE\(\kappa_\rho\) and reversed intermediate SLE\(\kappa_\rho\), respectively) processes will be defined using Appell-Lauricella multiple hypergeometric functions. When \(v_\infty = \sigma \cdot \infty\), the \(\eta\) in Theorem 1.2 agrees with the \(\eta\) in Theorem 1.1. So
Theorem 1.1 is a special case of Theorem 1.2 and we have a description of the law of $\eta'$ in Theorem 1.1. Unless $\kappa = 4$, an $\text{iSLE}_\kappa'(\rho)$ curve is not a chordal $\text{SLE}_\kappa'(\rho')$. So the time-reversal of a chordal $\text{SLE}_\kappa(\rho)$ may not be a chordal $\text{SLE}_\kappa'(\rho')$ curve.

The proof of Theorem 1.2 in the case that $\kappa \in (0, 4]$ uses the stochastic coupling technique introduced in [23, 22]. We will construct a commutation coupling of an $\text{iSLE}_\kappa(\rho)$ curve with an $\text{iSLE}_\kappa'(\rho')$ curve, and use the commutation relation to prove that the two curves are time-reversal of each other. The proof in the case that $\kappa \in (4, 8)$ uses the reversibility of chordal SLE$_\kappa$ established in [7]. We will show that when none of the force points is degenerate, the laws of both $\eta$ and $\eta'$ are absolutely continuous w.r.t. that of a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to $\infty$, and the Radon-Nikodym derivatives are related by the map $J$. We will then extend the result to the case that some force points are degenerate using commutation couplings.

The definitions of $\text{iSLE}_\kappa(\rho)$ and $\text{iSLE}_\kappa'(\rho)$ are valid for all $\kappa \in (0, 8)$ and $\rho = (\rho_1, \ldots, \rho_m)$ satisfying that $\sum_{j=1}^{k} \rho_j > \max\{-2, \frac{k}{2} - 4\}$ for $1 \leq k \leq m$. We believe that both theorems should hold if Condition (II) is weakened to $\kappa \in (4, 8)$ and $\sum_{j=1}^{k} \rho_j > \frac{k}{2} - 4$ for $1 \leq k \leq m$. Actually, we believe that the theorems should also hold if there are force points on both sides. The expected extension of Theorem 1.1 is the following conjecture.

**Conjecture 1.3.** Suppose $v_1^- > \cdots > v_{m_\cdot}^- \in (-\infty, 0) \cup \{0^-\}$ and $v_1^+ < \cdots < v_{m_\cdot}^+ \in (0, +\infty) \cup \{0^+\}$. Let $\kappa$ and $\rho_j^\pm$, $1 \leq j \leq m_\cdot$, $\sigma \in \{+, -\}$, satisfy that $\kappa \in (0, 8)$ and $\sum_{j=1}^{k} \rho_j^\sigma > \max\{-2, \frac{k}{2} - 4\}$ for all $1 \leq k \leq m_\cdot$ and $\sigma \in \{+, -\}$. Let $\rho^\sigma = (\rho_1^\sigma, \ldots, \rho_{m_\cdot}^\sigma)$ and $v^\sigma = (v_1^\sigma, \ldots, v_{m_\cdot}^\sigma)$, $\sigma \in \{+, -\}$. Let $\eta$ be a chordal SLE$_\kappa(\rho^+, \rho^-)$ curve in $\mathbb{H}$ from 0 to $\infty$ with force points $(\underline{v}^+, \underline{v}^-)$. Let $\rho^\infty_\cdot = -\sum_{j=1}^{m_\cdot} \rho_j^\sigma$, $\sigma \in \{+, -\}$. Then the time-reversal of $J(\eta)$ may be reparametrized by half-plane capacity and become a chordal Loewner curve, whose law is absolutely continuous w.r.t. that of a chordal SLE$_\kappa(-\rho^+, -\rho^+, -\rho^-)$ curve with force points $J(v_1^-), \ldots, J(v_{m_\cdot}^-), 0^-, J(v_1^-), \ldots, J(v_{m_\cdot}^-), 0^+$.

The conjecture is known to be true (cf. [22, Theorem 5.5]) in the case that $\kappa = 4$ and $\rho_j^\pm$ satisfy that $\sum_{j=1}^{k} \rho_j^\pm \geq 0$ for all $1 \leq k \leq m_\pm$. In that case, the time-reversal is exactly a chordal SLE$_4(-\rho^+, -\rho^+, -\rho^-, -\rho^-)$ curve. For other $\kappa$, we have not found the correct definitions of two-sided $\text{iSLE}_\kappa(\rho^+, \rho^-)$ and $\text{iSLE}_\kappa'(\rho^+, \rho^-)$ curves to make the extension of Theorem 1.2 holds, even in the simplest case that $m_+ = m_- = 1$.

Below is the outline of the rest of the paper. In the next section, we recall $\mathbb{H}$-hulls, chordal Loewner equation, chordal $\text{SLE}_\kappa(\rho)$, and multiple hypergeometric functions. In Section 3 we define $\text{iSLE}_\kappa(\rho)$ and $\text{iSLE}_\kappa'(\rho)$ curves, and study some basic properties. In Section 4 we construct a commutation coupling of an $\text{iSLE}_\kappa(\rho)$ curve with an $\text{iSLE}_\kappa'(\rho)$ curve. We prove the main theorems in the last section.

## 2 Preliminary

We first fix some notation. We write $x \lor y = \max\{x, y\}$ and $x \land y = \min\{x, y\}$ for $x, y \in \mathbb{R}$. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $z_0 \in \mathbb{C}$ and $S \subset \mathbb{C}$, let
rad\(_{z_0}(S) = \sup\{|z - z_0| : z \in S \cup \{z_0\}\} \). Let \(D \subseteq \mathbb{C}\) be a simply connected domain. The conformal radius of \(D\) w.r.t. any \(z_0 \in D\) is defined by \(\text{crad}_{z_0}(D) = 1/|g'_{z_0}(0)|\) if \(g_{z_0}\) maps \(D\) conformally onto \(\mathbb{D}\) such that \(g_{z_0}(z_0) = 0\). We write \(\text{crad}_{z_0}^{(4)}(D)\) for \(\text{crad}_{z_0}(D)/4\). Then for \(x > y \in \mathbb{R}\), \(\text{crad}_{x}^{(4)}(\mathbb{C} \setminus (-\infty, y]) = |x - y|\). By Koebel’s 1/4 theorem, we have \(\text{dist}(z_0, D^c)/4 \leq \text{crad}_{z_0}^{(4)}(D) \leq \text{dist}(z_0, D^c)\). The boundary Poisson kernel w.r.t. \(z \not\in \partial D\) at which \(\partial D\) is smooth is defined by \(H_D(z, w) = \frac{|g'(z)||g'(w)|}{(g(z) - g(w))^2}\), where \(g\) maps \(D\) conformally onto \(\mathbb{H}\) such that \(g(z), g(w) \neq \infty\). The value does not depend on the choice of \(g\).

2.1 \(\mathbb{H}\)-hulls

A relatively closed subset \(K\) of \(\mathbb{H}\) is called an \(\mathbb{H}\)-hull if \(K\) is bounded and \(\mathbb{H} \setminus K\) is a simply connected domain. If \(S\) is a bounded subset of \(\mathbb{H}\) such that \(\overline{S} \cup \mathbb{R}\) is connected, then the unbounded connected component of \(\mathbb{H} \setminus S\) is a simply connected domain, whose complement in \(\mathbb{H}\) is an \(\mathbb{H}\)-hull, which is called the \(\mathbb{H}\)-hull generated by \(S\), and denoted by \(\text{Hull}(S)\). For an \(\mathbb{H}\)-hull \(K\), there is a unique conformal map \(g_K\) from \(\mathbb{H} \setminus K\) onto \(\mathbb{H}\) such that \(g_K(z) = z + \frac{c}{\overline{z}} + O(\frac{1}{|z|^2})\) as \(z \to \infty\) for some \(c \geq 0\). The constant \(c\), denoted by \(h\text{cap}(K)\), is called the \(\mathbb{H}\)-capacity of \(K\), which is zero iff \(K = \emptyset\). We write \(h\text{cap}_2(K)\) for \(h\text{cap}(K)/2\).

If \(K_1 \subset K_2\) are two \(\mathbb{H}\)-hulls, then the quotient hull \(K_2/K_1\) is defined as \(g_{K_1}(K_2 \setminus K_1)\), which is also an \(\mathbb{H}\)-hull, and we have \(g_{K_2} = g_{K_2/K_1} \circ g_{K_1}\) and \(h\text{cap}(K_2) = h\text{cap}(K_2/K_1) + h\text{cap}(K_1)\). From \(h\text{cap} \geq 0\) we see that \(h\text{cap}(K_1)\), \(h\text{cap}(K_2/K_1) \leq h\text{cap}(K_2)\). If \(K_1 \subset K_2 \subset K_3\) are \(\mathbb{H}\)-hulls, then \(K_2/K_1 \subset K_3/K_1\) and \((K_3/K_1)/(K_2/K_1) = K_3/K_2\).

Let \(K\) be a non-empty \(\mathbb{H}\)-hull. Let \(K^{\text{doub}} = \overline{K} \cup \{\overline{z} : z \in K\}\), where \(\overline{K}\) is the closure of \(K\), and \(\overline{z}\) is the complex conjugate of \(z\). By Schwarz reflection principle, there is a compact set \(S_K \subset \mathbb{R}\) such that \(g_K\) extends to a conformal map from \(\mathbb{C} \setminus K^{\text{doub}}\) onto \(\mathbb{C} \setminus S_K\). Let \(\overline{\text{min}(K \cap \mathbb{R})}\), \(b_K = \text{max}(\overline{\text{K} \cap \mathbb{R}}\), \(c_K = \text{min}S_K\), \(d_K = \text{max}S_K\). Then \(g_K\) maps \(\mathbb{C} \setminus (K^{\text{doub}} \cup [a_K, b_K])\) conformally onto \(\mathbb{C} \setminus [c_K, d_K]\). Below is an important example.

Example 2.1. For \(x_0 \in \mathbb{R}\) and \(r > 0\), \(H := \{z \in \mathbb{H} : |z - x_0| \leq r\}\) is an \(\mathbb{H}\)-hull with \(g_H(z) = z + \frac{r^2}{z-x_0}\), \(h\text{cap}(H) = r^2\), \(a_H = x_0 - r\), \(b_H = x_0 + r\), \(H^{\text{doub}} = \{z \in \mathbb{C} : |z - x_0| \leq r\}\), \(c_H = x_0 - 2r\), \(d_H = x_0 + 2r\).

The next proposition combines Lemmas 5.2 and 5.3 of [24].

Proposition 2.2. If \(L \subset K\) are two non-empty \(\mathbb{H}\)-hulls, then \([a_L, b_L] \subset [a_K, b_K] \subset [c_K, d_K]\), \([c_L, d_L] \subset [c_K, d_K]\), and \([c_{K/L}, d_{K/L}] \subset [c_K, d_K]\).

Proposition 2.3. Let \(x_0 \in \mathbb{R}\) and \(r > 0\). If \(K\) is an \(\mathbb{H}\)-hull with \(\text{rad}_{x_0}(K) \leq r\), then \(h\text{cap}(K) \leq r^2\), \(\text{rad}_{x_0}(S_K) \leq 2r\), and \(|g_K(z) - z| \leq 3r\) for any \(z \in \mathbb{C} \setminus K^{\text{doub}}\).

Proof. We have \(K \subset H := \{z \in \mathbb{H} : |z - x_0| \leq r\}\). So \(h\text{cap}(K) \leq h\text{cap}(H) = r^2\). From Proposition 2.2 \(S_K \subset [c_K, d_K] \subset [c_H, d_H] = [x_0 - 2r, x_0 + 2r]\). So \(\text{rad}_{x_0}(S_K) \leq 2r\). Since \(g_K(z) = z\) is analytic on \(\mathbb{C} \setminus K^{\text{doub}}\) and tends to 0 as \(z \to \infty\), by the maximum modulus
principle,
\[
\sup_{z \in \mathbb{C} \setminus K^{doub}} |g_K(z) - z| \leq \limsup_{z \to K^{doub}} |(g_K(z) - x_0) - (z - x_0)| \leq 2r + r = 3r,
\]
where the second inequality follows from the facts that \(z \to K^{doub}\) implies that \(g_K(z) \to S_K\), \(\text{rad}_{x_0}(S_K) \leq 2r\), and \(\text{rad}_{x_0}(K^{doub}) \leq r\).
\[Q.E.D.\]

Let \(f_K = g_K^{-1}\). By [14] Lemma C.1, there is a measure \(\mu_K\) supported on \(S_K\) with \(|\mu_K| = \text{hc}(K)\) such that for any \(w \in \mathbb{C} \setminus S_K\),
\[
f_K(w) - w = \int_{S_K} \frac{-1}{w - y} d\mu_K(y).
\]
Differentiating the equality about \(w\), we get
\[
f_K'(w) - 1 = \int_{S_K} \frac{1}{(w - y)^2} d\mu_K(y).
\]
From this formula we see that \(f_K'(w) \geq 1\) on \(\mathbb{R} \setminus S_K\), and is decreasing on \((d_K, \infty)\) and increasing on \((-\infty, c_K)\). So \(f_K'(w) \in (0, 1)\) on \(\mathbb{R} \setminus K\), is increasing on \((b_K, \infty)\) and decreasing on \((-\infty, a_K)\). Moreover, we have the following proposition.

**Proposition 2.4.** Let \(K\) be an \(\mathbb{H}\)-hull contained in \(\{|z| \leq R\}\). If \(|z| \geq 7R\), then \(|\log \frac{|g_K(z)|}{|z|}| \leq 1.5 \frac{R^2}{|z|^2}\), \(|\log |g_K'(z)|| \leq 2.25 \frac{R^2}{|z|^2}\), and \(|Sg_K(z)| \leq 35 \frac{R^2}{|z|^2}\), where \(Sg_K\) is the Schwarzian derivative of \(g_K\), i.e., \(Sg_K = \frac{g''_K}{g_K} - \frac{3}{2} \frac{g'_K}{g_K}^2\).

**Proof.** Suppose \(|w| \geq 6R\). Since \(\mu_K\) is supported by \(S_K \subset [c_K, d_K] \subset [-2R, 2R]\), and \(|\mu_K| = \text{hc}(K) \leq R^2\), by (2.1),
\[
|f_K(w) - w| = \left| \int_{-2R}^{2R} \frac{-1}{w - y} d\mu_K(y) \right| \leq \frac{R^2}{|w| - 2R} \leq \frac{R^2}{6R - 2R} = \frac{R}{4},
\]
Thus, \(f_K(\{|w| = 6R\})\) is a Jordan curve contained in \(\{|w| < 6.25R\}\). Since \(f_K\) maps \(\{|w| > 6R\}\) onto the exterior of \(f_K(\{|w| = 6R\})\), which contains \(\{|w| > 6.25R\}\), we see that \(g_K = f_K^{-1}\) maps \(\{|z| < 2R\}\) into \(\{|z| > 6.25R\}\).

Suppose now \(|z| \geq 7R\). Then \(|g_K(z)| \geq 6R\), and by (2.3), \(|g_K(z) - z| \leq \frac{R}{4}\). So \(|g_K(z) - 2R \geq |z| - \frac{R}{4} \geq \frac{19}{25} |z|\). By (2.3) again, \(|g_K(z) - z| \leq \frac{R^2}{|g_K(z) - 2R|} \leq \frac{28R^2}{19|z|^2}\), which implies that \(|\frac{|g_K(z)|}{|z|} - 1| \leq \frac{28R^2}{19|z|^2} \leq \frac{1}{33}\). Since \(|\log x| \leq 1.016 |x - 1|\) if \(|x - 1| < \frac{1}{33}\), we get \(|\log \frac{|g_K(z)|}{|z|} | \leq 1.016 \frac{28R^2}{19|z|^2}\). From (2.2) we get \(|g_K'(z) - 1| \leq \frac{R}{|g_K(z) - 2R|} \leq \frac{28R^2}{19|z|^2}\) \((\frac{R}{19|z|^2})^2 \leq (\frac{4}{19})^2\). Since \(f_K'(g_K(z)) = 1/g_K'(z)\), and \(|\log x| \leq 1.03 |x - 1|\) if \(|x - 1| < \frac{4}{19}\), we get \(|\log g_K'(z)| \leq 1.03(\frac{28R^2}{19|z|^2})^2 \leq 2.25 \frac{R^2}{|z|^2}\).
Differentiating \((2.2)\) further w.r.t. \(z\) twice and then replacing \(w\) by \(g_K(z)\) and using \(|g_K(z)| - 2R \geq \frac{19}{28}|z|\) and \(|z| \geq 7R\), we get

\[
|f''_K(g_K(z))| \leq \frac{2 \cdot 28^3 R^2}{19^3|z|^3} \leq \frac{8 \cdot 28^2 R}{19^3|z|^2}, \quad |f'''_K(g_K(z))| \leq \frac{6 \cdot 28^4 R^2}{19^4|z|^4}.
\]

Using that \(|1/f'_K(g_K(z))| \leq |g'_K(z)| \leq e^{2.25 R^2/|z|^2} \leq e^{2.25} \leq 1.05\) and the chain rule for Schwarzian derivative, we get

\[
|Sg_K(z)| = |Sf_K(g_K(z))| : |g'_K(z)|^2 \leq 1.5 \cdot 1.05^4 |f''_K(g_K(z))|^2 + 1.05^3 |f'''_K(g_K(z))|.
\]

Combining the above two displayed formulas, we get \(|Sg_K(z)| \leq 35 R^2/|z|^4\).

The following proposition is essentially Lemma 2.8 in [4].

**Proposition 2.5.** Let \(\phi\) be a conformal map, which maps a real open interval containing \(x_0\) into \(\mathbb{R}\), and satisfies \(\phi'(x_0) > 0\). Then

\[
\lim_{H \to x_0} \frac{\text{hcap}(\phi(H))}{\text{hcap}(H)} = |\phi'(x_0)|^2,
\]

where \(H \to x_0\) means that \(\text{rad}_{x_0}(H) \to 0\) with \(H\) being a nonempty \(\mathbb{H}\)-hull.

**Definition 2.6.** For \(w \in \mathbb{R}\), let \(\mathbb{R}_w = (\mathbb{R} \setminus \{w\}) \cup \{w^+, w^-\}\). Let \(K\) be an \(\mathbb{H}\)-hull. Let the interval \([a^K_w, b^K_w]\) be the convex hull generated by \(w\) and \(K \cap \mathbb{R}\). Then \(g_K\) maps \(C \setminus (K\text{doub} \cup [a^K_w, b^K_w])\) conformally onto \(C \setminus [c^K_w, d^K_w]\) for some interval \([c^K_w, d^K_w]\). We define \(g^K_w\) from \(\mathbb{R}_w \cup \{+\infty, -\infty\}\) onto \([-\infty, c^K_w) \cup (d^K_w, +\infty]\) such that \(g^K_w(\pm \infty) = \pm \infty\); if \(x \in \mathbb{R} \setminus [a^K_w, b^K_w]\), \(g^K_w(x) = g_K(x)\); if \(x \in [a^K_w, w) \cup \{w^+\}\), \(g^K_w(x) = c^K_w\); and if \(x \in (w, b^K_w) \cup \{w^+\}\), \(g^K_w(x) = d^K_w\).

**Remark 2.7.** The maps \(g^K_w\) will be useful in describing force point processes for SLE\(_{\kappa}(\rho)\). Note that if \(K = 0\), \(a^K_w = b^K_w = c^K_w = d^K_w = w\); if \(w \in [a^K_w, b^K_w]\), then \(a^K_w = a^K_w\), \(b^K_w = b^K_w\), \(c^K_w = c^K_w\), and \(d^K_w = d^K_w\). It is clear that \(g^K_w\) is increasing. Since \(g^K_w = g_K\) and \(g^K_w' \in (0, 1]\) on \((-\infty, a^K_w) \cup (b^K_w, \infty)\), and \(g^K_w\) maps \([a^K_w, w) \cup \{w^+\}\) and \((w, b^K_w) \cup \{w^+\}\) respectively to \(c^K_w\) and \(d^K_w\), we see that \(g_K\) is a contraction on \((-\infty, w) \cup \{w^+\}\) and \((w, \infty) \cup \{w^+\}\).

### 2.2 Chordal Loewner equation

Let \(\hat{w} \in C([0, T], \mathbb{R})\) for some \(T \in (0, \infty]\). The chordal Loewner equation driven by \(\hat{w}\) is

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - \hat{w}(t)}, \quad 0 \leq t < T; \quad g_0(z) = z.
\]

For every \(z \in \mathbb{C}\), let \(\tau_z\) be the first time that the solution \(t \mapsto g_t(z)\) blows up; if such time does not exist, then set \(\tau_z = \infty\). For \(t \in [0, T)\), let \(K_t = \{z \in \mathbb{H} : \tau_z \leq t\}\). It turns out that for each \(t \geq 0\), \(K_t\) is an \(\mathbb{H}\)-hull with \(\text{hcap}(K_t) = 2t\), \(K_t\text{doub} = \{z \in \mathbb{C} : \tau_z \leq t\}\), and \(g_t\) agrees with
$g_{K_t}$. We call $g_t$ and $K_t$ the chordal Loewner maps and hulls, respectively, driven by $\hat{w}$. Since we write $\operatorname{hcp}_2(K)$ for $\operatorname{hcp}(K)/2$, $\operatorname{hcp}_2(K_t) = t$ for all $t$.

If for every $t \in [0,T)$, $f_{K_t} = g_{K_t}^{-1}$ extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$, and $\eta(t) := f_{K_t}(\hat{w}(t))$, $0 \leq t < T$, is continuous in $t$, then we say that $\eta$ is the chordal Loewner curve driven by $\hat{w}$. Such $\eta$ may not exist in general. When it exists, we have $\eta(0) = \hat{w}(0) \in \mathbb{R}$, and $K_t = \operatorname{Hull}(\eta([0,t]))$ for all $t$, and we say that $K_t$, $0 \leq t < T$, are generated by $\eta$.

Let $u$ be a continuous and strictly increasing function on $[0,T)$ such that $u(0) = 0$. Suppose that $g_t$ and $K_t$, $0 \leq t < T$, satisfy that $g_{u^{-1}(t)}$ and $K_{u^{-1}(t)}$, $0 \leq t < u(T)$, are chordal Loewner maps and hulls, respectively, driven by $\hat{w} \circ u^{-1}$. Then we say that $g_t$ and $K_t$, $0 \leq t < T$, are chordal Loewner maps and hulls, respectively, driven by $\hat{w}$ with speed $du$, and call $(K_{u^{-1}(t)})$ the normalization of $(K_t)$. If $(K_t)$ are generated by a curve $\eta$, i.e., $K_t = \operatorname{Hull}(\eta([0,t]))$ for all $t$, then $\eta$ is called a chordal Loewner curve driven by $\hat{w}$ with speed $du$, and $\eta \circ u^{-1}$ is called the normalization of $\eta$. If $u$ is absolutely continuous, we also say that the speed is $u'$. In this case, the $g_t$ satisfy the differential equation $\partial_t g_t(z) = \frac{2u'(t)}{g_t(z) - \hat{w}(t)}$. The original Loewner maps and hulls then have speed 1.

The following proposition is a slight variation of Theorem 2.6 of [3].

**Proposition 2.8.** The $\mathbb{H}$-hulls $K_t$, $0 \leq t < T$, are chordal Loewner hulls with some speed if and only if for any fixed $a \in [0,T)$, $\lim_{\delta \to 0} \sup_{0 \leq t \leq a} \operatorname{diam}(K_{t+\delta}/K_t) = 0$. Moreover, the driving function $\hat{w}$ satisfies that $\{\hat{w}(t)\} = \bigcap_{\delta > 0} K_{t+\delta}/K_t$, $0 \leq t < T$; and the speed is $du$, where $u(t) = \operatorname{hcp}_2(K_t)$, $0 \leq t < T$.

**Proposition 2.9.** Suppose $K_t$, $0 \leq t < T$, are chordal Loewner hulls driven by $\hat{w}$ with some speed. Then for any $t_0 \in (0,T)$ and $t \in [0,t_0]$, $c_{K_{t_0}} \leq \hat{w}(t) \leq d_{K_{t_0}}$.

**Proof.** Let $t_0 \in (0,T)$. If $0 \leq t < t_0$, by Propositions 2.8 and 2.8, $\hat{w}(t) \in [a_{K_{t_0}}/K_t,b_{K_{t_0}}/K_t] \subset [c_{K_{t_0}}/K_t,d_{K_{t_0}}/K_t] \subset [c_{K_{t_0}},d_{K_{t_0}}]$. By the continuity of $\hat{w}$, we also have $\hat{w}(t_0) \in [c_{K_{t_0}},d_{K_{t_0}}]$. \hfill $\Box$

We now cite [13, Proposition 2.13] below, which is a corollary of [9, Lemma 2.5] and [8, Lemma 3.3].

**Proposition 2.10.** Let $K_t$ and $\eta(t)$, $0 \leq t < T$, be chordal Loewner hulls and curve driven by $\hat{w}$ with speed $q$. Suppose the Lebesgue measure of $\eta([0,T)) \cap \mathbb{R}$ is 0. Let $w = \hat{w}(0)$, and $x \in \mathbb{R}_w$. Define $X(t) = g_{K_t}^w(x)$, $0 \leq t < T$. Then the set of $t$ such that $X(t) \neq \hat{w}(t)$ is zero, and $X$ is absolutely continuous with $X'(t) = 1_{\{X(t) \neq \hat{w}(t)\}} \frac{2q(t)}{X(t) - \hat{w}(t)}$ almost everywhere on $[0,T)$.

### 2.3 Chordal SLE$_\kappa$ and SLE$_\kappa(\rho)$ processes

If $\hat{w}(t) = \sqrt{\kappa}B(t)$, $0 \leq t < \infty$, where $\kappa > 0$ and $B(t)$ is a standard Brownian motion, then the chordal Loewner curve $\eta$ driven by $\hat{w}$ is known to exist (cf. [13]), and is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from $0$ to $\infty$. It satisfies $\eta(0) = 0$ and $\lim_{t \to \infty} \eta(t) = \infty$. The behavior of $\eta$ depends on $\kappa$: if $\kappa \in (0,4]$, $\eta$ is simple and intersects $\mathbb{R}$ only at 0; if $\kappa \geq 8$, $\eta$ is space-filling, i.e., $\mathbb{H} = \eta(\mathbb{R}_+)$; if $\kappa \in (4,8)$, $\eta$ is neither simple nor space-filling. If $D$ is a simply connected domain
with two distinct marked boundary points (or more precisely, prime ends (cf. \([1]\))) \(a\) and \(b\), the chordal SLE\(_\kappa\) curve in \(D\) from \(a\) to \(b\) is defined to be the conformal image of a chordal SLE\(_\kappa\) curve in \(\mathbb{H}\) from 0 to \(\infty\) under a conformal map from \(\mathbb{H}; (0, \infty)\) onto \((D; a, b)\).

For any \(\kappa > 0\), chordal SLE\(_\kappa\) satisfies Domain Markov Property (DMP): if \(\eta\) is a chordal SLE\(_\kappa\) curve in \(D\) from \(a\) to \(b\), and \(\tau\) is a stopping time for \(\eta\), then conditionally on the part of \(\eta\) before \(\tau\) and the event that \(\eta\) does not end at the time \(\tau\), the part of \(\eta\) after \(\tau\) is a chordal SLE\(_\kappa\) curve from \(\eta(\tau)\) to \(b\) in the connected component of \(D \setminus \eta([0, \tau])\) whose boundary contains \(b\).

The SLE\(_\kappa(p)\) processes, first appeared in \([5]\), are natural variants of SLE\(_\kappa\), where one keeps track of additional marked points, often called force points, which may lie on the boundary or interior. For the generality needed here, all force points will lie on the boundary. We now review the definition and properties of SLE\(_\kappa(p)\) developed in \([9]\).

Let \(\kappa > 0\), \(n \in \mathbb{N}\), \(\rho_1, \ldots, \rho_n \in \mathbb{R}\), \(w \in \mathbb{R}\), \(v_1, \ldots, v_n \in \mathbb{R}_w \cup \{+\infty, -\infty\}\). Recall that \(\mathbb{R}_w = (\mathbb{R} \setminus \{w\}) \cup \{w^+, w^-\}\). We require that for \(\sigma \in \{+, -\}\), \(\sum\{j: v_j = w^\sigma\} \rho_j > -2\). The chordal SLE\(_\kappa(p, \ldots, p)\) process in \(\mathbb{H}\) started from \(w\) with force points \(v_1, \ldots, v_n\) is the chordal Loewner process driven by the function \(\hat{w}(t), 0 \leq t < T\), which drives chordal Loewner maps \(g_t\) and hulls \(K_t\), and satisfies the following system of SDE:

\[
d\hat{w}(t) = \sqrt{\kappa} dB(t) + \sum_{j=1}^n 1_{\{\hat{w}(t) \neq \hat{\nu}_j(t)\}} \frac{\rho_j}{\hat{w}(t) - \hat{\nu}_j(t)} dt, \quad \hat{w}(0) = w, \tag{2.4}
\]

where \(B\) is a standard Brownian motion, and \(\hat{\nu}_j(t) = g^w_{K_t}(v_j), 1 \leq j \leq n\). Here we used Definition 2.6. The SDE should be understood as an integral equation, i.e., \(\hat{w}(t) - w - \sqrt{\kappa} B(t)\) equals the Lebesgue integral of the summation from 0 to \(t\). The solution exists uniquely up to the first time \(T\) (called a continuation threshold) that \(\sum\{j: \hat{\nu}_j(t) = c_{K_t}\} \rho_j \leq -2\) or \(\sum\{j: \hat{\nu}_j(t) = d_{K_t}\} \rho_j \leq -2\), whichever comes first. If there does not exist a continuation threshold, then the lifetime is \(\infty\).

The \(\hat{\nu}_j\) is called the force point function started from \(v_j\). If \(v_j = +\infty\) or \(-\infty\), then \(\hat{\nu}_j\) is constant \(+\infty\) or \(-\infty\), and the term \(\frac{\rho_j}{\hat{w}(t) - \hat{\nu}_j(t)}\) is constant 0, which means that the force point \(+\infty\) or \(-\infty\) does not play a role. If \(v_j \notin \{+\infty, -\infty\}\), then \(\hat{\nu}_j\) satisfies the ODE:

\[
d\hat{\nu}_j(t) = 1_{\{\hat{w}(t) \neq \hat{\nu}_j(t)\}} \frac{2}{\hat{\nu}_j(t) - \hat{w}(t)}, \quad \hat{\nu}_j(0) = v_j, \quad 1 \leq j \leq n. \tag{2.5}
\]

This equation should also be understood as an integral equation, which means that \(\hat{\nu}_j\) is absolutely continuous. If \(v_j > w\), then \(\hat{\nu}_j \geq \hat{w}\), and \(\hat{\nu}_j\) is increasing; if \(v_j < w\), then \(\hat{\nu}_j \leq \hat{w}\), and \(\hat{\nu}_j\) is decreasing. Here the sets \(\{\hat{\nu}_j \neq \hat{w}\}\) have Lebesgue measure zero. So we may omit the factors \(1_{\{\hat{w}(t) \neq \hat{\nu}_j(t)\}}\) in (2.4) and (2.5).

A chordal SLE\(_\kappa(p)\) process generates a chordal Loewner curve \(\eta\) in \(\mathbb{H}\) started from \(w\) up to the continuation threshold. If no force point is swallowed by the process at any time, this fact follows from the existence of chordal SLE\(_\kappa\) curve and Girsanov Theorem. The existence of the curve in the general case was proved in \([9]\). The chordal SLE\(_\kappa(p)\) curve \(\eta\) satisfies the following DMP. If \(\tau\) is a stopping time for \(\eta\), then conditionally on the process before \(\tau\) and the event that \(\tau\) is less than the lifetime of \(\eta\), \(\hat{w}(\tau + \cdot)\) and \(\hat{\nu}_j(\tau + \cdot), 1 \leq j \leq n\) are the driving function.
and force point functions for a chordal SLE$_\kappa(\rho)$ curve $\eta^\tau$ started from $\hat{w}(\tau)$ with force points at $\hat{v}_1(\tau), \ldots, \hat{v}_n(\tau)$. Moreover, $\eta(\tau + \cdot) = f_{K(\tau)}(\eta^\tau)$, where $K(\tau) := \text{Hull}(\eta([0, \tau]))$. Here if for some $j$, $\hat{v}_j(\tau) = \hat{w}(\tau)$, then $\hat{v}_j(\tau)$ as a force point for $\eta^\tau$ is treated as $\hat{w}(\tau)^+$ if $v_j \geq w^+$, or $\hat{w}(\tau)^-$ if $v_j \leq w^-$. If two force points $v_j$ and $v_k$ are equal, we may treat them as a single force point with force value $\rho_j + \rho_k$. By merging the force points and removing $+\infty$ and $-\infty$, we may assume that the force points $v_1, \ldots, v_n$ are mutually distinct finite numbers. We now relabel them by $v_j^{(\sigma)}$, $1 \leq j \leq n_\sigma$, $\sigma \in \{+, -\}$, such that $v_1^{(-)} < \cdots < v_1^{(1)} \leq w^- < w^+ \leq v_1^{(+)} < \cdots < v_1^{(+)}$, where $n_-$ or $n_+$ could be 0. Then $\hat{v}_1^{(1)} \leq \cdots \leq \hat{v}_1^{(-)} \leq \hat{w} \leq \hat{v}_1^{(+)} \leq \cdots \leq \hat{v}_1^{(+)}$ throughout the life period. Let $\rho_j^{(\pm)}$, $1 \leq j \leq n_\pm$, denote the corresponding force values. If for any $\sigma \in \{-, +\}$ and $1 \leq k \leq n_\sigma$, $\sum_{j=1}^k \rho_j^{(\sigma)} > -2$, then the process will never reach a continuation threshold, and so its lifetime is $\infty$, in which case $\lim_{t \to \infty} \eta(t) = \infty$. For $\sigma \in \{-, +\}$ and $k \in \{1, \ldots, n_\sigma\}$, if $\sum_{j=1}^k \rho_j^{(\sigma)} > \frac{\kappa}{2} - 2$, then a.s. $\eta$ stays at a positive distance from $v_k^{(\sigma)}$; if $\sum_{j=1}^k \rho_j^{(\sigma)} \geq \frac{\kappa}{2} - 2$, then a.s. $\eta$ does not hit the open interval between $v_k^{(\sigma)}$ and $v_{k+1}^{(\sigma)}$ (with $v_{n_\sigma+1}^{(\sigma)}$ understood as $\sigma \cdot \infty$). If for some $t_0$, $\hat{w}(t_0) = v_k^{(\sigma)}(t_0)$, then $\hat{v}_j^{(\sigma)}(t) = v_k^{(\sigma)}(t)$ for all $1 \leq j \leq k$ and $t \geq t_0$ in the life period, which means that the force point processes $\hat{v}_j^{(\sigma)}$ for $1 \leq j \leq k$ merge after $t_0$.

The following proposition will be needed. Recall the one quarter conformal radius $\text{crad}_z^{(4)}(D)$ and boundary Poisson kernel $H_D(z, w)$ defined in Section 2.

**Proposition 2.11.** Let $\kappa \in (0, 8)$ and $\rho_1, \ldots, \rho_m, \rho_{m+1} \in \mathbb{R}$ satisfy $\sum_{j=1}^{m+1} \rho_j = 0$, and for any $1 \leq k \leq m$, $\sum_{j=1}^k \rho_j \geq \frac{\kappa}{2} - 2$. Suppose $w > v_1 > \cdots > v_m > v_{m+1} \in \mathbb{R}$ or $w < v_1 < \cdots < v_m < v_{m+1} \in \mathbb{R}$. We also write $\rho_\infty$ for $\rho_{m+1}$, and $v_\infty$ for $v_{m+1}$. Let $\mathbb{P}_0$ denote the law of the chordal SLE$_\kappa$ curve in $\mathbb{H}$ from $w$ to $\infty$. Let $\mathbb{P}_1$ be the law of a chordal SLE$_\kappa(\rho_1, \ldots, \rho_m, \rho_{m+1})$ curve in $\mathbb{H}$ started from $w$ with force points $v_1, \ldots, v_m, v_{m+1}$. Then

$$\frac{d\mathbb{P}_1}{d\mathbb{P}_0} = \frac{E_0}{Z} \prod_{j=1}^m \left( \text{crad}_{v_\infty}^{(4)}(\Omega_j(\infty)) \cdot \frac{\rho_j^{(\kappa-4)}}{4\kappa} H_D(\tau_j, v_\infty) \cdot \frac{\rho_j^{(\kappa+\kappa-4)}}{4\kappa} \right) \prod_{1 \leq j < k \leq m} H_D(v_j, v_k) \cdot \frac{\rho_j \rho_k}{4\kappa},$$

(2.6)

where $E_0$ is the event that $\mathbb{H} \setminus \eta$ contains a connected component, denoted by $D_\infty$, which contains a neighborhood of the line segment $[v_\infty, v_1]$ in $\mathbb{H}$; $\Omega_j(\infty)$ is the connected component of $\mathbb{C} \setminus ([v_j, w] \cup \eta \cup \{z \in \mathbb{C} : \Im z \in \eta\})$ which contains $v_\infty$, $1 \leq j \leq m$; and $Z > 0$ is given by

$$Z := \prod_{j=1}^m \left( \frac{w - v_j}{w - v_\infty} \right) \prod_{j=1}^m |v_j - v_\infty| \cdot \frac{\rho_j \rho_\infty}{4\kappa} \prod_{1 \leq j < k \leq m} |v_j - v_k| \cdot \frac{\rho_j \rho_k}{4\kappa}.$$

**Proof.** By symmetry, we may assume that $w > v_1 > \cdots > v_m > v_{m+1}$. By [15], before any force point is separated by $\eta$ from $\infty$, the law $\mathbb{P}_1$ can be obtained by tilting the law $\mathbb{P}_0$ by the
local martingale defined by
\[
N(t) = \frac{1}{Z} \prod_{j=1}^{m+1} g_t(v_j) - \frac{\rho_j}{4\pi} \prod_{j=1}^{m+1} \left| \hat{w}(t) - \hat{v}_j(t) \right|^{\frac{\rho_j}{2\pi}} \prod_{1 \leq j < k \leq m+1} \left| \tilde{v}_j(t) - \tilde{v}_k(t) \right|^{\frac{\rho_j \rho_k}{4\pi}}, \quad (2.7)
\]
where \( g_t \) are the chordal Loewner maps, \( \hat{w} \) is the driving function, \( \hat{v}_j(t) = g_t(v_j) \), and the constant \( Z > 0 \) is such that \( N(0) = 1 \). More specifically, this means that, if \( \tau \) is a stopping time such that \( \tau \leq \min\{T_j : 1 \leq j \leq m+1\} = T_1 \), where \( T_j \) is the first time that \( v_j \) is swallowed by the process, and \( N(t) \), \( 0 \leq t < \tau \), is bounded by a uniform constant, then
\[
P_1 = N(\tau) \cdot P_0 \quad \text{on} \quad F_\tau. \quad (2.8)
\]
Here if \( \tau = T_1 \), then \( N(\tau) \) is understood as \( N(T_1) := \lim_{t \uparrow T_1} N(t) \), which \( P_0 \)-a.s. converges. This can be also checked directly using Girsanov Theorem and Itô’s formula, (12). For every \( n \in \mathbb{N} \), let \( \tau_n = \inf(T_1 \cup \{ t \geq 0 : N(t) \geq n \}) \). Then (2.8) holds for each \( \tau_n \). Since \( F_{T_1} \cap \{ T_1 = \tau_n \} \subset F_{\tau_n} \), we then get \( P_1 = N(T_1) \cdot P_0 \) on \( F_{T_1} \cap \{ T_1 = \tau_n \} \). Since this holds for any \( n \in \mathbb{N} \), we get \( P_1 = N(T_1) \cdot P_0 \) on \( F_{T_1} \cap E_B \), where \( E_B := \bigcup_{n=1}^{\infty} \{ T_1 = \tau_n \} = \{ \sup_{0 \leq t < T_1} N(t) < \infty \} \), i.e., the event that \( N \) is bounded on \( [0, T_1) \). Since \( P_0 \)-a.s. \( \lim_{T_1} N(t) \) converges, we have \( P_0[E_B] = 1 \).

For \( 0 \leq t < T_1 \) and \( 1 \leq j \leq m \), let \( D_t = \mathbb{H} / \text{Hull}(\eta([0, t])) \), and let \( \Omega_j(t) \) denote the union of \( D_t \), its reflection about \( \mathbb{R} \), and the interval \((-\infty, v_j)\). Since \( g_t \) maps \( D_t \) conformally onto \( \mathbb{H} \), we have \( H_{D_t}(v_j, v_k) = g_t(v_j)g_t(v_k)/|\tilde{v}_j(t) - \tilde{v}_k(t)|^2, 1 \leq j < k \leq m + 1 \). Since \( g_t \) maps \( \Omega_j(t) \) conformally onto \( \mathbb{C} \setminus \{ \tilde{v}_j(t), \infty \} \), we get \( \text{crad}_{\infty}^{(4)}(\Omega_j(t)) = |\tilde{v}_j(t) - \tilde{v}_j(\infty)|/g_t(v_j) \).

Let \( \hat{x}_j = \hat{w} - \hat{v}_j \), and \( R_j = \frac{\rho_j}{2\pi} \). Since \( \rho_\infty = -\sum_{j=1}^{m} \rho_j \), \( N(t) \) equals
\[
\frac{1}{Z} \prod_{j=1}^{m} \left( R_j(t) - \frac{\rho_j}{2\pi} \text{crad}_{\infty}^{(4)}(\Omega_j(t)) - \frac{\rho_j}{2\pi} H_{D_t}(v_j, v_\infty) - \frac{\rho_j^{(\kappa-4)}}{2\pi} H_{D_t}(v_j, v_\infty) - \frac{\rho_j^{(\kappa-4)}}{2\pi} H_{D_t}(v_j, v_\infty) \right) \prod_{1 \leq j < k \leq m} H_{D_t}(v_j, v_k) \cdot \frac{\rho_j \rho_k}{4\pi}.
\]
Suppose \( E_0 \) occurs. Then \( T_1 = \cdots = T_m = T_\infty \). Let \( t \uparrow T_1 = T_\infty \). Since \( D_t \rightarrow D_\infty \) and \( \Omega_j(t) \rightarrow \Omega_j(\infty) \) in the Carathéodory topology, we have \( \text{crad}_{\infty}^{(4)}(\Omega_j(t)) \rightarrow \text{crad}_{\infty}^{(4)}(\Omega_j(\infty)) \), \( 1 \leq j \leq m \), and \( H_{D_t}(v_j, v_k) \rightarrow H_{D_\infty}(v_j, v_k), 1 \leq j < k \leq m + 1 \).

For \( 0 \leq t < T_\infty \), let \( P_t \) denote the set of prime ends of \( D_t \), which lie on either \([w, \infty)\) or the right side of \( \eta([0, t]) \), i.e., it is the image of \( [\hat{w}(t), \infty) \) under \( g_t^{-1} \). Suppose \( \kappa \in (0, 4] \). Then \( T_\infty = \infty \). Let \( L > |v_\infty - w| \), and \( \xi_t \) be the connected component of \( \{ |z - w| = L \} \cap D_t \) whose closure contains \( w - L \). Since \( \eta(t) \rightarrow \infty \) as \( t \rightarrow \infty \), there is \( N > 0 \) such that \( |\eta(t) - w| > L \) for \( t \geq N \). For those \( t \), since \( \xi_t \) separates \([v_\infty, v_j)\) from \( P_t \) in \( D_t \), the extremal distance (cf. 1) between \([v_\infty, v_j)\) and \( P_t \) in \( D_t \) is by comparison principle at least \( \log(L/|v_\infty - w|)/\pi \). Thus, the extremal distance between \([v_\infty, v_j)\) and \( P_t \) in \( D_t \) tends to \( \infty \) as \( t \uparrow T_\infty \). Suppose \( \kappa \in (4, \infty) \). Then \( T_\infty < \infty \) and \( \eta(T_\infty) \notin \langle 0, \infty \rangle \). Let \( \varepsilon \in (0, |\eta(T_\infty) - v_\infty|) \), and \( \xi_t \) be the connected component of \( \{ |z - \eta(T_\infty)| = \varepsilon \} \cap D_t \) whose closure contains \( \eta(T_\infty) + \varepsilon \). Then there is \( \delta > 0 \) such that \( |\eta(t) - \eta(T_\infty)| < \varepsilon \) for \( t \in [T_\infty - \delta, T_\infty) \). For those \( t \), since \( \xi_t \) separates \([v_\infty, v_j)\) from \( P_t \) in \( D_t \), the extremal distance between \([v_\infty, v_j)\) and \( P_t \) in \( D_t \) is at least \( \log(|\eta(T_\infty) - v_\infty|/\varepsilon)/\pi \). So we again get that the extremal distance between \([v_\infty, v_j)\) and \( P_t \) in \( D_t \) tends to \( \infty \) as \( t \uparrow T_\infty \).
Since $g_t$ maps $D_t$ conformally onto $\mathbb{H}$, and takes $P_t$ and $[v_\infty, v_j]$ to $[\hat{w}(t), \infty)$ and $[\hat{v}_\infty(t), \hat{v}_j(t)]$, respectively, by conformal invariance, we get $R_j(t) = \hat{w}(t) - \hat{v}_j(t) / \hat{w}(t) - v_\infty(t) \to 1$ as $t \uparrow T_\infty$.

On the event $E_0$, since $T_1 = T_\infty$, $N(T_1) = \lim_{t \to T_\infty} N(t)$ equals the RHS of (2.6). This implies that $E_0 \subset E_B$. By the assumptions on $\kappa$ and $\rho_j$’s, we know that $\mathbb{P}_1$ is supported by $E_0$. Since $\mathbb{P}_1 = N(T_1) \cdot \mathbb{P}_0$ on $\mathcal{F}_{T_1} \cap E_B$, and $\mathcal{F}_{T_1}$ agrees with $\mathcal{F}_{T_\infty}$ on the event $E_0$, we see that $d(\mathbb{P}_1|\mathcal{F}_{T_\infty})/d(\mathbb{P}_0|\mathcal{F}_{T_\infty})$ is given by the RHS of (2.6). If $\kappa \in (0, 4]$, then $\mathbb{P}_0$-a.s. $T_\infty = \infty$, and so (2.6) holds. If $\kappa \in (4, 8)$, then $\mathbb{P}_0$-a.s. $T_\infty < \infty$. If $\eta$ follows the law $\mathbb{P}_0$, then by the DMP for chordal SLE$_\kappa$, conditionally on $\mathcal{F}_{T_\infty}$, the part of $\eta$ after $T_\infty$ is a chordal SLE$_\kappa$ from $\eta(T_\infty)$ to $\infty$ in $H_{T_\infty} := \mathbb{H} \setminus \text{Hull}(\eta([0, T_\infty]))$. If $\eta$ follows the law $\mathbb{P}_1$, then by the DMP for chordal SLE$_\kappa(\rho)$ and the fact that $\sum_{j=1}^m \rho_j + \rho_\infty = 0$, conditionally on $\mathcal{F}_{T_\infty}$, the part of $\eta$ after $T_\infty$ is also a chordal SLE$_\kappa$ from $\eta(T_\infty)$ to $\infty$ in $H_{T_\infty}$. So the absolute continuity between $\mathbb{P}_1$ and $\mathbb{P}_0$ on $\mathcal{F}_{T_\infty}$ extends to $\mathcal{F}_\infty$ with the same Radon-Nikodym derivative, and we again have (2.6). \qed

**Remark 2.12.** We may express $d\mathbb{P}_1/d\mathbb{P}_0$ in the above theorem in terms of a conformal map from $D_\infty$ onto $\mathbb{H}$. Suppose $w > v_1 > \cdots > v_m > v_{m+1}$. Suppose $\partial D_\infty \cap \mathbb{R} = [x_L, x_R]$. Let $g_s$ be a conformal map from $D_\infty$ onto $\mathbb{H}$ such that $g_s(x_L) = \infty$. By Schwarz reflection principle, $g_s$ extends to a conformal map defined on the union of $D_\infty$, its reflection about $\mathbb{R}$, and $(x_L, x_R)$. Then $H_{D_\infty}(v_j, v_k) = g_s'(v_j)g_s'(v_k)/g_s(v_j) - g_s(v_k)|^2$, $1 \leq j < k \leq m + 1$. Since $g_s$ maps $\Omega_j(\infty)$ conformally onto $\mathbb{C} \setminus [g_s(v_j), \infty)$, we get $\text{crad}_{v_j}(\Omega_j(\infty)) = |g_s(v_j) - g_s(v_{m+1})/g_s'(v_{m+1})|$, $1 \leq j \leq m$. Combining these formulas with the equality $\sum_{j=1}^m \rho_j = 0$, we get

$$
d\mathbb{P}_1/d\mathbb{P}_0 = \prod_{j=1}^{m+1} \frac{g_s'(v_j)}{|w - v_j|^2} \prod_{1 \leq j < k \leq m+1} \left( \frac{|g_s(v_j) - g_s(v_k)|}{|v_j - v_k|} \right)^{\rho_j \rho_k / 2\kappa}. \tag{2.9}
$$

**2.4 Multiple hypergeometric functions**

Let $\alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathbb{R}$, and $\gamma \notin \{0, -1, -2, \ldots\}$. We use the Pochhammer symbol $(\alpha)_n$ to denote the rising factorial, i.e., $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1)$ for $n \geq 1$. Note that $(1)_n = n!$. Write $\underline{\beta} = (\beta_1, \ldots, \beta_m)$. Let $F(\alpha, \beta_1, \ldots, \beta_m; x)$, $x = (x_1, \ldots, x_m)$, be the (first) Appell-Lauricella multiple hypergeometric function defined by (cf. [10])

$$
F(\alpha, \underline{\beta}; x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{(\alpha)_{n_1+\cdots+n_m} (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}}{(\gamma)_{n_1+\cdots+n_m} (1)_{n_1} \cdots (1)_{n_m}} x_1^{n_1} \cdots x_m^{n_m}. \tag{2.10}
$$

Using Stirling’s formula, one easily see that the series itself as well as the series of the partial derivatives to any order converge absolutely and uniformly on $[-r, r]^m$ for any $r \in (0, 1)$. Thus, $F$ is $C^\infty$ on $(-1, 1)^m$, and one may differentiate the series term by term. For $1 \leq j \leq m$, let $\underline{\epsilon}_j$ denote the vector in $\mathbb{R}^m$, whose $j$-th component is 1, and other components are 0. Straightforward calculation shows that for any $1 \leq j \leq m$,

$$
\partial_{x_j} F(\alpha, \underline{\beta}; x) = \frac{\alpha \beta_j}{\gamma} F(\alpha + 1, \underline{\beta} + \underline{\epsilon}_j, \gamma + 1; x). \tag{2.11}
$$
Since we may change the order of summation, we find that for any \( \Lambda \subset \{1, \ldots, m\} \),
\[
F(\alpha, \beta, \gamma; \mathbf{x}) = \sum_{n \in \mathbb{N}^m} \frac{(\alpha)_{|n|}}{\prod_{j \in \Lambda}(\beta_j)_{n_j}} \prod_{j \in \Lambda} x_j^{n_j} F(\alpha + |n|, \beta, \gamma + |n|; \mathbf{x}|_{\Lambda^c}),
\]
(2.12)
where \( \mathbb{N} := \mathbb{N} \cup \{0\} \), \( |n| = \sum_{j \in \Lambda} n_j \) for \( n \in \mathbb{N}^m \), and \( \Lambda^c := \{1, \ldots, m\} \setminus \Lambda \). The equality holds even in the case \( \Lambda = \emptyset \) or \( \Lambda = \{1, \ldots, m\} \). In the former case, there is only one term in the summation, and the equality is trivial; in the latter case, the \( F \)-functions on the RHS of (2.12) are understood as constant 1, and the equality reduces to the definition (2.10).

Let \( F = F(\alpha, \beta_1, \ldots, \beta_m, \gamma; \cdot) \). By (2.10), we have
\[
F(0, x_2, \ldots, x_m) = F(\alpha, \beta_2, \ldots, \beta_m, \gamma; x_2, \ldots, x_m);
\]
(2.13)
\[
F(x_1, \ldots, x_{m-2}, x_{m-1}, x_{m-1}) = F(\alpha, \beta_1, \ldots, \beta_{m-2}, \beta_{m-1} + \beta_m, \gamma; x_1, \ldots, x_{m-1}).
\]
(2.14)
If \( \gamma > \alpha + \beta_m \), by Stirling’s formula, the series (2.10) converges uniformly on \([0, r]^{m-1} \times [0, 1]\) for any \( r \in (0, 1) \). Thus, \( F \) extends continuously from \([0, 1]^m\) to \([0, 1]^{m-1} \times [0, 1]\), and by and Gauss’s Theorem,
\[
F(x_1, \ldots, x_{m-2}, x_{m-1}) = \sum_{n \in \mathbb{N}^{m-1}} \frac{(\alpha)_{|n|}}{\prod_{j=1}^{m-1}(\beta_j)_{n_j}} \prod_{j=1}^{m-1} x_j^{n_j} F(\alpha + |n|, \beta, \gamma + |n|, 1)
\]
\[
= \sum_{n \in \mathbb{N}^{m-1}} \frac{(\alpha)_{|n|}}{\prod_{j=1}^{m-1}(\beta_j)_{n_j}} \Gamma(\gamma + |n|) \Gamma(\gamma - \alpha - \beta_m) \Gamma(\gamma - \alpha)
\]
\[
\times \frac{\Gamma(\gamma - \alpha - \beta_m)}{\Gamma(\gamma - \beta_m)} F(\alpha, \beta_1, \ldots, \beta_{m-1}, \gamma - \beta_m; x_1, \ldots, x_{m-1}).
\]
(2.15)

We are going to derive some PDEs for the multiple hypergeometric functions. Some of them can be found in the literature. But for completeness, we will provide detailed proofs. For \( \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( n = (n_1, \ldots, n_m) \in \mathbb{N}^m \), we define \( |n| = n_1 + \cdots + n_m \) and \( x^n = x_1^{n_1} \cdots x_m^{n_m} \). We may express \( F(\mathbf{x}) \) as \( \sum_{n \in \mathbb{N}^m} A_n x^n \), where \( A_n = \frac{(\alpha)_{|n|}}{\prod_{j=1}^{m}(\beta_j)_{n_j}} \). Then
\[
A_{n''} = \frac{(\alpha + |n|)(\beta_j + n_j)}{(\gamma + |n|)(1 + n_j)} A_n.
\]
(2.16)
Let \( \partial_j \) denote the partial differential operator \( x_j \partial_{x_j} \). Then
\[
\partial_j F(\mathbf{x}) = \sum_{n \in \mathbb{N}^m} n_j A_n x^n.
\]
(2.17)
So
\[
(\alpha + \theta_1 + \cdots + \theta_m)(\beta_j + \theta_j)F(\mathbf{x}) = \sum_{n \in \mathbb{N}^m} (\alpha + |n|)(\beta_j + n_j) A_n x^n;
\]

13
\[ \theta_j(\gamma - 1 + \theta_1 + \cdots + \theta_m)F(x) = \sum_{n \in \mathbb{N}^m} n_j(\gamma - 1 + |n|)A_n x^n \]
\[ = \sum_{n \in \mathbb{N}^m : n_j \geq 1} n_j(\gamma - 1 + |n|)A_n x^n = x_j \sum_{n \in \mathbb{N}^m} (n_j + 1)(\gamma + |n|)A_{n+e_j} x^m. \]

By (2.16), \( F \) satisfies the PDE (cf. [10, Formula (56) of Chapter 9]) \( \mathcal{L}_j F = 0, 1 \leq j \leq m, \) where
\[ \mathcal{L}_j := -\alpha + \theta_1 + \cdots + \theta_m)(\beta_j + \theta_j) + \frac{1}{x_j} \theta_j(\gamma - 1 + \theta_1 + \cdots + \theta_m) \]
\[ = \sum_{k=1}^m x_k(1 - x_j) \partial_{x_j} \partial_{x_k} + [\gamma - (\alpha + 1)x_j] \partial_{x_j} - \beta_j \sum_{k=1}^m x_k \partial_{x_k} - \alpha \beta_j. \]

From (2.17) we also know that for \( 1 \leq j \neq k \leq m, \)
\[ \partial_{x_k}(\beta_j + \theta_j)F(x) = \sum_{n \in \mathbb{N}^m : n_k \geq 1} n_k(\beta_j + n_j)A_n x^{n-k} = \sum_{n \in \mathbb{N}^m} (1 + n_k)(\beta_j + n_j)A_{n+e_k} x^n. \]

From (2.16) we know that \( (1 + n_k)(\beta_j + n_j)A_{n+e_k} = (1 + n_j)(\beta_k + n_k)A_{n+e_j}. \) So \( F \) satisfies the PDE \( \mathcal{L}_{j,k} F = 0 \) for \( 1 \leq j \neq k \leq m, \) where
\[ \mathcal{L}_{j,k} := \partial_{x_k}(\beta_j + \theta_j) - \partial_{x_j}(\beta_k + \theta_k) = (x_j - x_k) \partial_{x_j} \partial_{x_k} + \beta_j \partial_{x_k} - \beta_k \partial_{x_j}. \]

If \( j = k, \) the equality \( \mathcal{L}_{j,k} F = 0 \) trivially holds. Now we let
\[ \mathcal{L} = \sum_{j=1}^m \frac{1 - x_j}{x_j} \mathcal{L}_j + \sum_{j=1}^m \sum_{k=1}^m \frac{1}{x_j} \mathcal{L}_{j,k} \]
\[ = \sum_{j=1}^m \sum_{k=1}^m (1 - x_j)(1 - x_k) \partial_{x_j} \partial_{x_k} - \sum_{j=1}^m \alpha \beta_j \left( \frac{1}{x_j} - 1 \right) \]
\[ + \sum_{j=1}^m (1 - x_j) \left[ \gamma - \sum_{k=1}^m \beta_k - (\alpha + 1) + \sum_{k=1}^m \beta_k \left( \frac{1}{x_k} - 1 \right) \right] \partial_{x_j}, \quad (2.18) \]
and
\[ \mathcal{L}^r = \sum_{j=1}^m x_j(1 - x_j) \mathcal{L}_j + \sum_{j=1}^m \sum_{k=1}^m x_j x_k^2 \mathcal{L}_{j,k} \]
\[ = \sum_{j=1}^m \sum_{k=1}^m x_j x_k (1 - x_j)(1 - x_k) \partial_{x_j} \partial_{x_k} - \sum_{j=1}^m \alpha \beta_j x_j (1 - x_j) \]
\[ + \sum_{j=1}^m x_j (1 - x_j) \left[ \gamma - \sum_{k=1}^m \beta_k - (\alpha + 1)x_j + \sum_{k=1}^m \beta_k (1 - x_k) \right] \partial_{x_j}, \quad (2.19) \]
In the above equalities, we used 
\[ \sum_{j=1}^{m} \sum_{k=1}^{m} (x_j - x_k) \partial_{x_j} \partial_{x_k} = \sum_{j=1}^{m} \sum_{k=1}^{m} (x_j - x_k) x_j x_k (1 - x_j - x_k) \partial_{x_j} \partial_{x_k} = 0. \]

Then we have 
\[ \mathcal{L} F = 0 \quad \text{on } (0,1)^m; \quad \mathcal{L}' F = 0 \quad \text{on } (-1,1)^m. \quad (2.20) \]

We now study the positiveness and continuation of the multiple hypergeometric function. We make some assumptions on the parameters.

**Definition 2.13.** For \( m \in \mathbb{N} \), we say that \( \alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathbb{R} \) satisfy the parameter assumption if \( \gamma > 0 \lor \alpha \land \gamma > (0 \lor \alpha) + \sum_{j=k}^{m} \beta_j \) for any \( 1 \leq k \leq m \).

From now on, we fix \( \alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathbb{R} \), and let \( F = F(\alpha, \beta_1, \ldots, \beta_m, \gamma; \cdot) \). Let \( \Delta_m \) denote the set \( \{ \mathbf{x} \in \mathbb{R}^m : 0 \leq x_1 \leq \ldots \leq x_m < 1 \} \).

**Lemma 2.14.** If \( \alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathbb{R} \) satisfy the parameter assumption, \( F \) is positive on \( \Delta_m \).

**Proof.** We prove by induction on \( m \), and use the idea in the proof of Lemma 3.1 of [17]. First, consider the case \( m = 1 \). In this case, the multiple hypergeometric function reduces to a single-variable hypergeometric function \( {}_2F_1(\alpha, \beta_1, \gamma; x_1) \), and \( \Delta_1 = [0,1) \). Since \( \gamma > \alpha + \beta_1 \), by Gauss’s Theorem (cf. [10, Formula (20) of Section 1.2]), \( F \) extends continuously to \([0,1] \) with \( F(1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta_1)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta_1)} \). Since \( \gamma, \gamma - \alpha, \gamma - \beta_1, \gamma - \alpha - \beta_1 > 0 \), we get \( F(1) > 0 \). We also note that \( F(0) = 1 > 0 \).

If \( \alpha \land \beta_1 \geq 0 \), then we have \( F \geq 1 \) on \([0,1) \) since every term in (2.10) is nonnegative for \( x_1 \in [0,1) \), and the first term \( (n_1 = 0) \) is 1. Now suppose \( \alpha \land \beta_1 < 0 \). Let \( n \) be the smallest integer such that \( n + (\alpha \land \beta_1) \geq 0 \). We write \( F_j = F(\alpha + j, \beta_1 + j, \gamma + j; \cdot) \). By (2.11),
\[ F_j' = \frac{(\alpha + j)(\beta_1 + j)}{\gamma + j} F_{j+1}, \quad j \geq 0. \quad (2.21) \]

We claim that for any \( 0 \leq j \leq n - 1 \), \( \alpha + j, \beta_1 + j, \gamma + j \) satisfy the parameter assumption. In fact, since \( \alpha, \beta_1, \gamma \) satisfy the parameter assumption, the only way that \( \alpha + j, \beta_1 + j, \gamma + j \) could fail to satisfy the parameter assumption is that \( (\gamma + j) \leq (\alpha + j) + (\beta_1 + j) \), which implies that \( \gamma \leq (\alpha + \beta_1) + j \). Since \( j < n \), by the definition of \( n \), we have \( j < -(\alpha \land \beta_1) \). So we get \( \gamma < (\alpha + \beta_1) - (\alpha \land \beta_1) = \alpha \lor \beta_1 \), which contradicts that \( \gamma > \alpha \lor \beta_1 \). By this claim and the statement in the last paragraph, for each \( 0 \leq j \leq n - 1 \), \( F_j \) extends continuously to \([0,1] \), and is positive at 0 and 1. Since \( \alpha + n, \beta_1 + n \geq 0 \) and \( \gamma + n > 0 \), by (2.10) \( F_n > 0 \) on \([0,1] \). By (2.21), \( F_{n-1} \) is monotone on \([0,1] \). Since \( F_{n-1}(1), F_{n-1}(0) > 0 \), in either case \( F_{n-1} > 0 \) on \([0,1] \). Then we may use the same argument to show that \( F_{n-2} > 0 \) on \([0,1] \) (if \( n \geq 2 \)). Iterating the argument, we get \( F = F_0 > 0 \) on \([0,1] \).

Suppose \( m \geq 2 \) and the lemma holds for \( m - 1 \). First assume that \( \alpha \land \beta_1 \geq 0 \). By (2.12),
\[ F(\mathbf{x}) = \sum_{n_1=0}^{\infty} \frac{(\alpha)_{n_1} (\beta_1)_{n_1} x_1^{n_1} F(\alpha + n_1, \beta_1, \ldots, \beta_m, \gamma + n_1; x_2, \ldots, x_m)}. \quad (2.22) \]
Since for any $n_1 \geq 0$, $\alpha + n_1, \beta_2, \ldots, \beta_m, \gamma + n_1$ satisfy the parameter assumption, by induction hypothesis, $F(\alpha + n_1, \beta_2, \ldots, \beta_m, \gamma + n_1; \cdot)$ is positive on $\Delta_{m-1}$. Since $\alpha, \beta_1 \geq 0$ and $\gamma > 0$, every term in the series of (2.22) is nonnegative, and the first term $(n_1 = 0)$ is positive on $\Delta_m$. Thus, $F$ is positive on $\Delta_m$. Now we assume that $\alpha \land \beta_1 < 0$. Let $n$ be the first integer such that $n + (\alpha \land \beta_1) \geq 0$. For each $j$, let $F_j = F(\alpha + j, \beta_1 + j, \beta_2, \ldots, \beta_m, \gamma + j; \cdot)$. By (2.11),

$$\partial_{x_j} F_j = \frac{(\alpha + j)(\beta_1 + j)}{\gamma + j} F_{j+1}, \quad j \geq 0. \quad (2.23)$$

By (2.12) we get

$$F_n(x) = \sum_{n_1=0}^{\infty} \frac{(\alpha + n_1)(\beta_1 + n_1)\beta_2 \ldots \beta_m}{(\gamma + n_1)(1)} x_1^{n_1} F(\alpha + n_1, \beta_2, \ldots, \beta_m, \gamma + n_1; x_2, \ldots, x_m). \quad (2.24)$$

Since $\alpha + n + n_1, \beta_2, \ldots, \beta_m, \gamma + n + n_1$ satisfy the parameter assumption, by induction hypotheses, $F(\alpha + n + n_1, \beta_2, \ldots, \beta_m, \gamma + n + n_1; \cdot)$ is positive on $\Delta_{m-1}$. Since $\alpha + n, \beta_1 + n \geq 0$ and $\gamma + n > 0$, every term in the series of (2.24) is nonnegative, and the first term is positive on $\Delta_m$. Thus, $F_n > 0$ on $\Delta_m$. From (2.23) we then know that $F_{n-1}$ is monotone in $x_1$ on $\Delta_m$. Now for every fixed $(x_2, \ldots, x_m) \in \Delta_{m-1}$, the $x_1$ such that $(x_1, x_2, \ldots, x_m) \in \Delta_m$ is $[0, x_2]$. By (2.13),(2.14),

$$F_j(0, x_2, \ldots, x_m) = F(\alpha + j, \beta_1 + j, \beta_2, \beta_3, \ldots, \beta_m, \gamma + j; x_2, \ldots, x_m).$$

Since $\alpha, \beta_1, \beta_2, \ldots, \beta_m, \gamma$ satisfy the parameter assumption, so do $\alpha + j, \beta_1 + j, \beta_2, \beta_3, \ldots, \beta_m, \gamma + j$. Thus by induction hypothesis, $F_j(0, x_2, \ldots, x_m) > 0$ for $(x_2, \ldots, x_m) \in \Delta_{m-1}$. We claim that for any $0 \leq j \leq n - 1$, $\alpha + j, \beta_1 + j, \beta_2, \beta_3, \ldots, \beta_m, \gamma + j$ satisfy the parameter assumption. This holds because the only way that they could fail to satisfy the parameter assumption is $\gamma + j < (\alpha + j) + (j + \sum_{s=1}^{m} \beta_s)$, i.e., $\gamma \leq (\gamma + \sum_{s=1}^{m} \beta_s)$. Since $j < n$, by the definition of $n$, $j < -(\alpha + \beta_1)$. So $\gamma < \alpha + \sum_{s=1}^{m} \beta_s - \alpha + \beta_1 = \alpha \lor \beta_1 + \sum_{s=1}^{m} \beta_s$, which contradicts that $\gamma > (\sum_{s=1}^{m} \beta_s) \lor (\alpha + \sum_{s=2}^{m} \beta_s)$. So the claim is proved, which implies by induction hypothesis that $F_j(x_2, x_2, \ldots, x_m) > 0$ for $(x_2, \ldots, x_m) \in \Delta_{m-1}$ and $0 \leq j \leq n - 1$. Since $F_{n-1}$ is monotone in $x_1$ on $\Delta_m$ and is positive when $x_1 \in [0, x_2]$, we see that $F_{n-1} > 0$ on $\Delta_m$. Applying the same argument to $F_{n-1}$, we get $F_{n-2} > 0$ on $\Delta_m$. Iterating, we get $F = F_0 > 0$ on $\Delta_m$.

**Lemma 2.15.** If $0, \beta_1, \ldots, \beta_m, \gamma$ satisfy the parameter assumption, and $\gamma > \alpha$, then (i) $F$ is positive on $\Delta_m$; and (ii) $F$ is monotone in $x_j$ on $\Delta_m$ for every $1 \leq j \leq m$.

**Proof.** (i) If $\alpha \leq 0$, then $\alpha, \beta_1, \ldots, \beta_m, \gamma$ satisfy the parameter assumption, so by Lemma 2.14, $F > 0$ on $\Delta_m$. Now suppose $\alpha > 0$. Let $\Lambda$ denote the set of all $j$ such that $\beta_j > 0$. Order the elements in $\{1, \ldots, m\} \setminus \Lambda$ by $t_1 < \cdots < t_k$, where $k = m - |\Lambda|$. Since $\gamma > \alpha \lor 0$, and $\beta_{t_1}, \ldots, \beta_{t_k} \leq 0$, we see that for any $n \geq 0$, $\alpha + n, \beta_{t_1}, \ldots, \beta_{t_k}, \gamma + n; \cdot$ satisfy the parameter assumption. So by Lemma 2.14, $F(\alpha + n, \beta_{t_1}, \ldots, \beta_{t_k}, \gamma + n; \cdot) > 0$ on $\Delta_k$. Since $\gamma, \beta_j, j \in \Lambda$, are positive, for any $n \in \mathbb{N}^{\Lambda}$, $\prod_{j \in \Lambda} (\alpha + \beta_{t_j} + n_j; \cdot) > 0$. By (2.12), $F > 0$ on $\Delta_m$.

(ii) By (2.11), $\partial_{x_j} F = \frac{\alpha \beta_j}{\gamma + \beta_j} F_1$, where $F_1 := F(\alpha + 1, \beta + \epsilon_j, \gamma + 1; \cdot)$. Note that $\alpha + 1, \beta + \epsilon_j, \gamma + 1$ satisfy the assumption of the lemma. By (i) $F_1 > 0$ on $\Delta_m$. So the conclusion holds. \qed
Theorem 2.16. If \( \alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathbb{R} \) satisfy the parameter assumption, then \( F \) extends to a positive continuous function on \( \overline{\Delta_m} = \{(x_1, \ldots, x_m) : 0 \leq x_1 \leq \cdots < x_m \leq 1\} \).

Proof. We prove the theorem by induction. If \( m = 1 \), then \( \overline{\Delta_1} = [0, 1] \). The statement holds by Lemma 2.14 and Gauss’s Theorem. Note that \( F(1) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha - \beta_1)} > 0 \). Now suppose \( m \geq 2 \), and the statement holds for \( m - 1 \). Let \( S_L = \{x \in \Delta_m : x_m = x_{m-1}\} \) and \( S_U = \Delta_{m-1} \times \{1\} \).

We define \( P_L \) and \( P_U \) by

\[
P_L(x_1, \ldots, x_{m-1}, x_m) = (x_1, \ldots, x_{m-1}, x_{m-1}), \quad P_U(x_1, \ldots, x_{m-1}, x_m) = (x_1, \ldots, x_{m-1}, 1).
\]

Then for each \( x \in \overline{\Delta_m}, P_L(x) \in \overline{S_L}, P_U(x) \in \overline{S_U}, \) and \( x \in [P_L(x), P_U(x)]. \)

By (2.14) and induction hypothesis, \( F|_{S_{m-1}} \) extends to a positive continuous function on \( \overline{S_L} \), and we denote it by \( F_L \). Since \( \gamma > \alpha + \beta_m \), by (2.15) and the induction hypothesis, \( F|_{S_{m-1}} \) extends to a positive continuous function \( F_U \) on \( \overline{S_U} \).

Suppose \( (x^n) \) is a sequence in \( \Delta_m \) with \( x^n \rightarrow x^0 \in \overline{\Delta_m} \). To prove the existence of the continuation of \( F \) on \( \overline{\Delta_m} \), we need to show that \( (F(x^n)) \) converges. If \( x_{m-1}^0 < 1 \), then this is true since \( F \) extends continuously to \( [0, 1]^{m-2} \times [0, 1] \supset \Delta_m \cup S_T \ni x^0 \). Now suppose \( x_{m-1}^0 = 1 \). Then \( P_U(x^n), P_L(x^n) \rightarrow x^0 \). So we have \( F(P_U(x^n)) = F(U(P_U(x^n)) \rightarrow F_U(x^0) \) and \( F(P_L(x^n)) = F_L(P_L(x^n)) \rightarrow F_L(x^0) \). By Lemma 2.15 (ii), \( F \) is monotone in \( x_m \) on \( \Delta_m \). So \( F(x^n) \) lies between \( F(P_U(x^n)) \) and \( F(P_L(x^n)) \).

For the existence of the limit, it remains to show that \( F_U = F_L \) on \( \overline{S_U} \cap \overline{S_L} = \overline{\Delta_{m-2}} \times \{1\} \times \{1\} \), which follows easily from (2.14, 2.15).

Finally, since \( F_U \) and \( F_L \) are respectively positive on \( \overline{S_U} \) and \( \overline{S_L} \), and \( F \) is monotone in \( x_m \), we conclude that \( F \) is positive on \( \overline{\Delta_m} \).

\[ \square \]

3 Intermediate SLE\( _\kappa(\rho) \) Processes

3.1 Forward curves

Fix \( \kappa \in (0, 8) \). Let \( m \in \mathbb{N}, N_m = \{n \in \mathbb{N} : 1 \leq n \leq m\} \), and \( N_m^\infty = N_m \cup \{\infty\} \). Let \( \rho_j \in \mathbb{R}, j \in N_m^\infty \), satisfy that \( \sum_{j \in N_m^\infty} \rho_j = 0 \), and for \( k \in N_m, \sum_{j=1}^k \rho_j > \max\{-2, \frac{\kappa}{2} - 4\} \). Let \( w \in \mathbb{R} \).

Let \( v_1, \ldots, v_m, v_\infty \in \mathbb{R}_w \cup \{+\infty, -\infty\} \) be such that either \( w^- \geq v_1 \geq \cdots \geq v_m \geq v_\infty \) or \( w^+ \leq v_1 \leq \cdots \leq v_m \leq v_\infty \). Let \( (\rho_1, \ldots, \rho_m) \) and \( \underline{v} = (v_1, \ldots, v_m, v_\infty) \). We are going to define an intermediate SLE\( _\kappa(\rho) \) curve in \( \mathbb{H} \) from \( w \) to \( \infty \) with force points \( \underline{v} \). By symmetry, we only need to deal with the case that \( w^- \geq v_1 \geq \cdots \geq v_m \geq v_\infty \).

Let \( \eta \) be a chordal SLE\( _\kappa(\rho, \rho_\infty) \) curve in \( \mathbb{H} \) started from \( w \in \mathbb{R} \) with force points \( \underline{v} \). Since \( \sum_{j=1}^k \rho_j > -2 \) for \( 1 \leq k \leq m \), and \( \sum_{j \in N_m^\infty} \rho_j = 0 \), there is no continuation threshold for \( \eta \), so the lifetime of \( \eta \) is \( \infty \), and \( \eta(t) \rightarrow \infty \) as \( t \rightarrow \infty \). Since \( \sum_{j \in N_m^\infty} \rho_j = 0 > \frac{\kappa}{2} - 4 \) and \( \sum_{j=1}^k \rho_j > \frac{\kappa}{2} - 4 \) for \( 1 \leq k \leq m \), \( \eta \) a.s. does not visit any of \( v_j, j \in N_m^\infty \), which is different from \( w^- \) or \( -\infty \). Let \( (K_t) \) be the chordal Loewner hulls generated by \( \eta \), let \( \tilde{w} \) be the chordal Loewner driving function for \( \eta \), and let \( \tilde{v}_j \) be the force point function started from \( v_j, j \in N_m^\infty \). Then \( \tilde{v}_j(t) = g^w_{K_t}(v_j) \) (Definition 2.6), and \( \tilde{w} \geq \tilde{v}_1 \geq \cdots \tilde{v}_m \geq \tilde{v}_\infty \). Moreover, for some standard
Brownian motion $B$, $\tilde{w}$ and $\tilde{v}_j$ satisfy the SDE:

$$d\tilde{w}(t) = \sqrt{\kappa} dB(t) + \sum_{k=1}^{m} \left( \frac{\rho_k}{\tilde{w}(t) - \tilde{v}_k(t)} - \frac{\rho_k}{\tilde{w}(t) - \tilde{v}_\infty(t)} \right) dt, \quad \tilde{w}(0) = w; \quad (3.1)$$

$$d\tilde{v}_j(t) = \frac{2}{\tilde{v}_j(t) - \tilde{w}(t)} dt, \quad \tilde{v}_j(0) = v_j, \quad j \in \mathbb{N}_m. \quad (3.2)$$

For $j \in \mathbb{N}_m$, let $\tilde{x}_j = \tilde{w}_j - \tilde{v}_j$. Then $0 \leq \tilde{x}_1 \leq \cdots \leq \tilde{x}_m \leq \tilde{x}_\infty$. If $v_j = -\infty$, then $\tilde{x}_j \equiv +\infty$; otherwise $\tilde{x}_j$ is finite and satisfies the SDE

$$d\tilde{x}_j(t) = \sqrt{\kappa} dB(t) + \sum_{k=1}^{m} \left( \frac{\rho_k}{\tilde{x}_k(t) - \tilde{v}_\infty(t)} \right) dt + \frac{2}{\tilde{x}_j(t)} dt. \quad (3.3)$$

For $j \in \mathbb{N}_m$, let $T_j$ denote the first time that $\tilde{x}_j = 0$. Then $T_1 \leq \cdots \leq T_m \leq T_\infty$. In the case that $v_j = w^-$, we have $T_j = 0$. Define continuous processes $I_j$, $j \in \mathbb{N}_m$, on $[0, \infty]$ by

$$I_j(t) = \exp \left( \int_0^t 1_{\{\tilde{x}_j \neq 0\}}(s) \left( \frac{2}{\tilde{x}_\infty(s)^2} - \frac{2}{\tilde{x}_j(s)\tilde{x}_\infty(s)} \right) ds \right). \quad (3.4)$$

Note that the set of $t$ such that any $\tilde{x}_j(t)$ equals 0 has Lebesgue measure zero. Since $0 \leq \tilde{x}_j \leq \tilde{x}_\infty$, $I_j$ is nonnegative and decreasing. Since $\tilde{x}_j = \tilde{x}_\infty$ after $T_\infty$, $I_j$ is constant on $[T_\infty, \infty]$. If $v_j = v_\infty$, then $\tilde{x}_j \equiv \tilde{x}_\infty$, and so $I_j \equiv 1$. If $v_\infty = -\infty$, then $I_j \equiv 1$ for all $j$. Now suppose $v_j \neq v_\infty$ and $v_\infty \neq -\infty$. For $0 \leq t < T_\infty$, we define $\Omega_j(t)$ to be the union of $\mathbb{H} \setminus K_t$, $(-\infty, v_j)$, and the reflection of $\mathbb{H} \setminus K_t$ about $\mathbb{R}$. Then $g_t$ maps $\Omega_j(t)$ conformally onto $\mathbb{C} \setminus \{\tilde{v}_j(t), \infty\}$, and takes $v_\infty \in \Omega_j(t)$ to $\tilde{v}_\infty(t)$. By chordal Loewner equation and (3.2),

$$\frac{dg_t^{(v_j)}(v_j)}{g_t^{(v_j)}(v_j)} = -\frac{2}{\tilde{x}_j(t)^2} dt, \quad j \in \mathbb{N}_m; \quad \frac{d|\tilde{v}_j(t) - \tilde{v}_\infty(t)|}{|\tilde{v}_j(t) - \tilde{v}_\infty(t)|} = -\frac{2}{\tilde{x}_j(t)\tilde{x}_\infty(t)} dt, \quad j \in \mathbb{N}_m. \quad (3.5)$$

So we get

$$I_j(t) = \frac{|\tilde{v}_j(t) - \tilde{v}_\infty(t)|}{g_t^{(v_\infty)}(v_j - v_\infty)} = \frac{\text{crad}_t^{(4)}(\Omega_j(t))}{\text{crad}_0^{(4)}(\Omega_j(0))}, \quad 0 \leq t < T_\infty. \quad (3.6)$$

Since $\text{dist}(v_\infty, \eta) > 0$, $\mathbb{H} \setminus \eta$ contains a connected component, denoted by $D_\infty$, whose boundary contains $v_\infty$. For $1 \leq j \leq m$, let $\Omega_j(\infty)$ denote the union of $D_\infty$, the reflection of $D_\infty$ about $\mathbb{R}$, and the real interval $(\eta(T_\infty), v_j)$ if $T_\infty < \infty$ or $(-\infty, v_j)$ if $T_\infty = \infty$. Then $\Omega_j(\infty)$ is a simply connected domain containing $v_\infty$. As $t \uparrow T_\infty$, $\Omega_j(t) \to \Omega_j(\infty)$ in the Carathéodory topology. So $\text{crad}_t^{(4)}(\Omega_j(t)) \to \text{crad}_\infty^{(4)}(\Omega_j(\infty)) \in (0, \infty)$, which implies that $I_j(T_\infty) = \lim_{t \to T_\infty} I_j(t)$ is a finite positive number.

For $j \in \mathbb{N}_m$, define $R_j = \tilde{x}_j/\tilde{x}_\infty$ on $[0, T_\infty)$, and $R_j \equiv 1$ on $[T_\infty, \infty]$. Here if $v_\infty = -\infty$, then $R_j$ is understood as constant 0 if $v_j \neq -\infty$, and constant 1 if $v_j = -\infty$. Then $0 \leq R_1 \leq
\[ \cdots \leq R_m \leq 1. \] If \( v_\infty = w^- \), then \( T_\infty = 0 \), and all \( R_j \equiv 1 \). Suppose now \( v_\infty \notin \{ w^-, -\infty \} \). Then \( T_\infty > 0 \), and each \( R_j \) satisfies the following SDE up to \( T_\infty \)
\[ dR_j = \frac{1 - R_j}{x_\infty} \sqrt{\kappa} dB + \frac{1 - R_j}{x_\infty^2} \left[ \frac{2}{R_j} + 2 - \kappa + \sum_{k=1}^m \rho_k \left( \frac{1}{R_k} - 1 \right) \right] dt. \tag{3.7} \]

Since \( \rho_1 + \cdots + \rho_m + \rho_\infty = 0 > \frac{5}{2} - 4 \), either \( \kappa \in (0, 4), T_\infty = \infty \), and \( \lim_{t \to T_\infty} \eta(t) = \infty \); or \( \kappa \in (4, 8), T_\infty < \infty \), and \( \eta(T_\infty) \in (-\infty, v_\infty) \). Using the same extremal distance argument as in the proof of Proposition 2.11 except with \([v_\infty, v_j \wedge \min(\eta([0, t]) \cap \mathbb{R})]\) in place of \([v_\infty, v_j]\), we get \( R_j(t) \to 1 \) as \( t \uparrow T_\infty \). Thus, \( R_j \) is continuous on \([0, \infty) \). Also note that in any case, (3.7) holds throughout \([0, \infty) \) because of the factor \( 1 - R_j \) on its RHS.

Define parameters
\[ \alpha = 1 - \frac{4}{\kappa}, \quad \beta_j = \frac{2\rho_j}{\kappa}, \quad j \in \mathbb{N}_m, \quad \gamma = \frac{4}{\kappa} + \sum_{k=1}^\infty \beta_k, \tag{3.8} \]
and \( F = F(\alpha, \beta_1, \ldots, \beta_k, \gamma; \cdot) \). Let \( \underline{R}(t) = (R_1(t), \ldots, R_m(t)) \in \underline{\Sigma}_m \), and
\[ M(t) = \frac{F(\underline{R}(t))}{F(\underline{R}(0))} \prod_{j=1}^m I_j(t)^{\alpha \beta_j}, \quad t \in [0, \infty]. \tag{3.9} \]

**Lemma 3.1.** \( M \) is a uniformly integrable positive continuous martingale.

**Proof.** It is easy to see that \( \alpha, \beta_1, \ldots, \beta_k, \gamma \) satisfy the parameter assumption in Definition 2.13. By Theorem 2.16, \( F \) extends to a positive continuous function on \( \Sigma_m \). Since \( R \) is continuous and takes values in \( \Sigma_m \), \( F(R) \) is positive and continuous. We also know that \( I_j, 1 \leq j \leq m \), are positive and continuous. So \( M \) is positive and continuous.

Now we prove the martingale property. If all \( v_j \)'s are equal to \( v_\infty \), then all \( R_j \)'s and \( I_j \)'s are constant 1, and so is \( M \). If \( v_\infty = -\infty \), then all \( I_j \)'s are constant 1, and \( R \) is constant, and so \( M \) is again constant 1. Now we suppose that not all \( v_j \), \( 1 \leq j \leq m \), are equal to \( v_\infty \), and \( v_\infty \neq -\infty \). Let \( m' \) be the biggest \( j \leq m \) such that \( v_j \neq v_\infty \). Then \( v_j \neq v_\infty \), \( 1 \leq j \leq m' \), and \( v_j = v_\infty \), \( m' + 1 \leq j \leq m \). So \( \eta \) is a chordal \( \text{SLE}_\kappa(\rho_1, \ldots, \rho_{m'}, \rho'_\infty) \) curve in \( \mathbb{H} \) started from \( w \) with force points \( v_1, \ldots, v_{m'}, v_\infty \), where \( \rho'_\infty = \rho_\infty + \sum_{j=m'+1}^m \rho_j = -\sum_{j=1}^{m'} \rho_j \). We have \( I_j = R_j = 1 \) for \( m' + 1 \leq j \leq m \). Let \( \tilde{\gamma} = \frac{4}{\kappa} + \sum_{j=1}^{m'} \beta_j \) and \( \tilde{F} = F(\alpha, \beta_1, \ldots, \beta_{m'}, \tilde{\gamma}; \cdot) \). Then \( \tilde{R} = (R_1, \ldots, R_{m'}) \). Then \( M(t) = \frac{\tilde{F}(\tilde{R}(t))}{\tilde{F}(\tilde{R}(0))} \prod_{j=1}^{m'} I_j(t)^{\alpha \beta_j}, \) which is the \( M \) defined for the chordal \( \text{SLE}_\kappa(\rho_1, \ldots, \rho_{m'}, \rho'_\infty) \) curve. So by replacing \( m \) by \( m' \) we may assume below that \( v_j \neq v_\infty \) for all \( 1 \leq j \leq m \).

Since \( \underline{R}(t) \in [0, 1]^m \) for \( t < T_\infty \), from (2.18, 2.20, 3.4, 3.7) and Itô’s formula, we see that \( M(t) \) is a local martingale up to \( T_\infty \). Here we used the fact that the set of \( t \) such that any \( R_j(t) \) equals 0 has Lebesgue measure zero. Since \( M \) is constant on \([T_\infty, \infty) \), it is a local martingale throughout \([0, \infty) \). To show that \( M \) is uniformly integrable, it suffices to show that \( \sup_{0 \leq t < T_\infty} M(t) \) is integrable. By Theorem 2.16, \( |\log(F(\underline{R}(t)))| \) is bounded by a constant
 depending only on $\kappa$ and $\rho_j$’s. So we only have to control the size of $\prod_{j=1}^m I_j^{-\frac{\alpha \rho_j}{\kappa}}$. From (3.6) and (3.4), we easily get

$$I_k \geq I_j \geq \frac{|v_k - v_\infty|}{|v_j - v_\infty|} I_k, \quad \text{on } [0, T_\infty), \quad 1 \leq j \leq k \leq m. \quad (3.10)$$

Let $\rho_\Sigma = \sum_{j=1}^m \rho_j$. By (3.10), it now suffices to show that $\sup_{0 \leq t < T_\infty} I_m(t)^{\frac{\alpha \rho_\Sigma}{\kappa}}$ is integrable. Since $I_m$ is decreasing, if $\alpha \rho_\Sigma \geq 0$, then $\sup_{0 \leq t < T_\infty} I_m(t)^{\frac{\alpha \rho_\Sigma}{\kappa}}$ is bounded by 1, and so is integrable. Now we assume that $\alpha \rho_\Sigma < 0$. Then $I_m(t)^{\frac{\alpha \rho_\Sigma}{\kappa}}$ is increasing.

Let $\tau_n = T_\infty \wedge \inf \{ t \in [0, T_\infty) : M(t) \geq n \}, n \in \mathbb{N}$. Then $(\tau_n)$ is an increasing sequence of stopping times tending to $T_\infty$, and for each $n$, $M(t \wedge \tau_n)$ is a bounded martingale. By Optional Stopping Theorem, $\mathbb{E}[M(\tau_n)] = M(0) = 1$. By Theorem 2.16 and (3.10), $M(\tau_n) \propto I_m(\tau_n)^{\frac{\alpha \rho_\Sigma}{\kappa}}$, with the implicit constants depending only on $\kappa, \rho_1, \ldots, \rho_m, v_1, \ldots, v_m, v_\infty$. Thus, $\mathbb{E}[I_m(\tau_n)^{\frac{\alpha \rho_\Sigma}{\kappa}}]$ is bounded by a constant. Since $I_m(t)^{\frac{\alpha \rho_\Sigma}{\kappa}}$ is increasing, and $\tau_n \uparrow T_\infty$, by monotone convergence theorem, $\mathbb{E}[\sup_{0 \leq t < T_\infty} I_m(t)^{\frac{\alpha \rho_\Sigma}{\kappa}}] < \infty$. So the proof is done.

By this lemma, we know that $\mathbb{E}[M(T_\infty)] = M(0) = 1$. So we may define another probability measure by weighting the law of $\eta$ by $M(T_\infty)$.

**Definition 3.2.** A (forward) intermediate SLE$_\kappa(\rho)$ (iSLE$_\kappa(\rho)$ for short) curve in $\mathbb{H}$ from $w$ to $\infty$ with force points $v$ is a random curve $\eta$, whose law is absolutely continuous w.r.t. that of a chordal SLE$_\kappa(\rho, \rho_\infty)$ curve in $\mathbb{H}$ from $w$ to $\infty$ with force points $v$, and the Radon-Nikodym derivative is $M(T_\infty)$. We extend the definition to general simply connected domains via conformal maps.

We now describe some properties of the iSLE$_\kappa(\rho)$ curve. Because of the absolute continuity, it satisfies every almost sure property of the chordal SLE$_\kappa(\rho, \rho_\infty)$ curve. For example, it a.s. ends at its target, and does not visit any of its force points not immediately next to any of its endpoints. If $\kappa \leq 4$, the curve is simple, does not visit the boundary arc between its two endpoints which does not contain any force point, and does not visit the boundary arc between its target point and its last force point which does not contain its initial point. In the case that the domain is $\mathbb{H}$, and the force points are on the left of the initial point $w$, these two boundary arcs that will not be visited are $\langle w, +\infty \rangle$ and $\langle -\infty, v_\infty \rangle$.

There are some degenerate cases. If all $v_j$’s are equal to $v_\infty$, then since $\sum_{j=1}^m \rho_j + \rho_\infty = 0$, and $M$ is constant 1, the iSLE$_\kappa(\rho)$ curve is just a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from $w$ to $\infty$. If $v_\infty = -\infty$, then $M$ is again constant 1, and the iSLE$_\kappa(\rho)$ curve is a chordal SLE$_\kappa(\rho)$ curve with force points $v_1, \ldots, v_m$. So Theorem 1.1 is a special case of Theorem 1.2.

Now we assume that not all $v_j$’s are equal to $v_\infty$, and $v_\infty \neq -\infty$. By merging force points as we did in the proof of Theorem 3.1, we may assume that $v_j \neq v_\infty$ for all $j \in \mathbb{N}_m$. We now derive a formula of $M(T_\infty)$ in terms of conformal radius. Since $\Omega_j(t) \rightarrow \Omega_j(\infty)$ in the Carathéodory topology, by (3.6) we have $I_j(t) \rightarrow crad_{v_\infty}^{(4)}(\Omega_j(\infty))/crad_{v_j}^{(4)}(\Omega_j(0))$ as $t \uparrow T_\infty$. 20
Recall that $R_j(t) \to 1$ as $t \uparrow T_\infty$. Let $1 = (1, \ldots, 1) \in \mathbb{R}^m$. Then we get
\[
M(T_\infty) = \frac{F(1)}{F(R(0))} \prod_{j \in \mathbb{N}_m} \frac{\text{crad}_{\infty}^{(4)}(\Omega_j(\infty))}{\text{crad}_{\infty}^{(4)}(\Omega_j(0))} \frac{(r_j^{(k-4)})}{2k}. \tag{3.11}
\]

### 3.2 Reversed curves

Let $\kappa, \rho_1, \ldots, \rho_m, \rho_\infty, \rho$ be as in Section 3.1. Let $w^r \in \mathbb{R}$ and $v_1^r, \ldots, v_m^r, v_\infty^r \in \mathbb{R}_+ \cup \{+\infty, -\infty\}$ be such that either $(w^r)^+ \leq v_\infty^r \leq v_m^r \leq \cdots \leq v_1^r$ or $(w^r)^- \geq v_\infty^r \geq v_m^r \geq \cdots \geq v_1^r$. Let $\normalsize r^r = (v_1^r, \ldots, v_m^r, v_\infty^r)$. We will define a reversed intermediate SLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from $w^r$ to $\infty$ with force points $\normalsize r^r$. By symmetry, we only need to deal with the case that $(w^r)^+ \leq v_\infty^r < v_m^r \leq \cdots \leq v_1^r$.

Let $\rho_j^r = -\rho_j$, $j \in \mathbb{N}_m$, and $\rho_j^r = (\rho_j^r, \ldots, \rho_m^r)$. Let $\eta^r$ be a chordal SLE$_\kappa(\rho^r, \rho_\infty^r)$ curve in $\mathbb{H}$ started from $w^r$ with force points $\normalsize r^r$. By the assumptions on $\rho_j$’s, we have $\rho_\infty^r = \sum_{j=1}^m \rho_j^r > -2$ and for any $k \in \mathbb{N}_m$, $\rho_\infty^r + \sum_{j=k}^m \rho_j^r = \sum_{j=k}^{m-1} \rho_j > -2$. So there is no continuation threshold for $\eta^r$. Thus, the lifetime of $\eta^r$ is $\infty$, and $\eta^r(t) \to \infty$ as $t \to \infty$. Similarly, we have $\rho_\infty^r > \frac{k}{2} - 4$ and for any $k \in \mathbb{N}_m$, $\rho_\infty^r + \sum_{j=k}^m \rho_j^r > \frac{k}{2} - 4$. So $\eta^r$ a.s. does not visit any of its force point other than $(w^r)^+$ or $+\infty$.

Let $(K_t^r)$ be the chordal Loewner hulls generated by $\eta^r$, let $\hat{w}^r$ be the driving function, and let $\hat{\nu}_j^r$ be the force point function started from $v_j^r$, $j \in \mathbb{N}^\infty_m$. Then $\hat{\nu}_j^r(t) = g_{K_t^r}(v_j^r)$, $\hat{w}^r \leq \hat{v}_j^r \leq \hat{v}_m^r \leq \cdots \leq \hat{v}_1^r$, and for some standard Brownian motion $B^r$, $\hat{w}^r$ and $\hat{\nu}_j^r$, $j \in \mathbb{N}^\infty_m$, satisfy the SDE:
\[
d\hat{w}^r(t) = \sqrt{\kappa}dB^r(t) + \sum_{k \in \mathbb{N}_m} \left( \frac{\rho_k^r}{\hat{w}^r(t) - \hat{v}_k^r(t)} - \frac{\rho_k^r}{\hat{w}^r(t) - \hat{v}_\infty^r(t)} \right) dt, \quad \hat{w}^r(0) = w^r; \tag{3.12}
\]
\[
d\hat{\nu}_j^r(t) = \frac{2}{\hat{v}_j^r(t) - \hat{w}^r(t)} dt, \quad \hat{\nu}_j^r(0) = v_j^r, \quad j \in \mathbb{N}^\infty_m. \tag{3.13}
\]

For $j \in \mathbb{N}^\infty_m$, let $\hat{x}^r_j = \hat{w}^r_j - \hat{\nu}_j^r$. Then $0 \geq \hat{x}^r_\infty \geq \hat{x}^r_m \geq \cdots \geq \hat{x}^r_1$, and each finite function $\hat{x}^r_j$ ($v_j^r \neq +\infty$) satisfies the SDE
\[
d\hat{x}^r_j(t) = \sqrt{\kappa}dB^r(t) + \sum_{k \in \mathbb{N}_m} \left( \frac{\rho_k^r}{\hat{x}^r_k(t)} - \frac{\rho_k^r}{\hat{x}^r_\infty(t)} \right) dt + \frac{2}{\hat{x}^r_j(t)} dt. \tag{3.14}
\]

For $j \in \mathbb{N}^\infty_m$, let $T^r_j$ denote the first time that $\hat{x}^r_j = 0$. Then $T^r_\infty \leq T^r_m \leq \cdots \leq T^r_1$. Now Equation (3.5) holds here with additional superscripts “$r$”.

Define $I^r_j$, $j \in \mathbb{N}_m$, on $[0, \infty]$ by
\[
I^r_j(t) = \exp \left( \int_0^t 1_{\{\hat{x}^r_j \neq 0\}}(s) \left( \frac{2}{\hat{x}^r_j(s)} - \frac{2}{\hat{x}^r_j(s)\hat{v}_j^r(s)} \right) ds \right). \tag{3.15}
\]

Since $\hat{x}^r_j \geq \hat{x}^r_\infty \geq 0$, $I^r_j$ is continuous and decreasing. Since $\hat{x}^r_j = \hat{x}^r_\infty$ on $[T^r_j, \infty)$, $I^r_j$ takes constant value on $[T^r_j, \infty)$. If $v_j^r = v_\infty^r$, then $\hat{\nu}_j^r \equiv \hat{v}_j^r$, and so $I^r_j \equiv 1$. If $v_j^r = +\infty$, we also
get $I_j^r \equiv 1$. Let $\Lambda$ denote the set of $j \in \mathbb{N}_m$ such that $+\infty > v_j^r > v_\infty^r$. By chordal Loewner equation and (3.13), we find that, for $j \in \Lambda$, up to $T_j^r$,

$$I_j^r(t) = \frac{|\tilde{v}_j^r(t) - \tilde{v}_\infty^r(t)|}{(g_{K_f})'(v_j^r)|v_j^r - v_\infty^r|} \geq \frac{\text{crad}_{v_j^r}^r(\Omega^r(t))}{\text{crad}_{v_j^r}^r(\Omega^r(0))} \geq \frac{|v_j^r - v_\infty^r| \land \text{dist}(v_j^r, \eta^r([0,t]))}{4|v_j^r - v_\infty^r|}, \quad (3.16)$$

where $\Omega^r(t)$ is the union of $\mathbb{H} \setminus K_f^r$, the interval $(v_\infty^r \lor \max(K_f^r \cap \mathbb{R}), \infty)$, and the reflection of $\mathbb{H} \setminus K_f^r$ about $\mathbb{R}$. The second “=” follows from the fact that $g_{K_f}$ maps $\Omega^r(t)$ conformally onto $\mathbb{C} \setminus (-\infty, \tilde{v}_\infty^r(t))$, and takes $v_j^r$ to $\tilde{v}_j^r(t)$; and the “$\geq$” follows from Koebe’s $1/4$ theorem. Since $I_j^r$ stays constant on $[T_j^r, \infty)$, and $\eta^r$ does not get closer to $v_j^r$ after $T_j^r$, we get

$$1 = I_j^r(0) \geq I_j^r(t) \geq (1 \land \text{dist}(v_j^r, \eta^r([0,t]))/|v_j^r - v_\infty^r|)/4, \quad t \in [0, \infty), \quad j \in \Lambda. \quad (3.17)$$

Since $I_j^r \equiv 1$ for $j \in \mathbb{N}_m \setminus \Lambda$, and $\text{dist}(v_j^r, \eta^r) > 0$, we get $I_j^r > 0$ on $[0, \infty]$ for all $j \in \mathbb{N}_m$.

For $1 \leq j < m$, define $R_j^r$ on $[0, \infty]$ such that $R_j^r = \tilde{x}_j^r/\tilde{x}_\infty^r$ on $[0, T_j^r)$, and $R_j^r \equiv 1$ on $[T_j^r, \infty)$. Here if $v_j^r = v_\infty^r = +\infty$, then $R_j^r$ is understood as constant $1$; and if $v_j^r = +\infty > v_\infty^r$, then $R_j^r$ is understood as constant $0$. Then $0 \leq R_j^r \leq \cdots \leq R_m^r \leq 1$. Let $R^r = (R_1^r, \ldots, R_m^r) \in \mathbb{S}_m$. If $v_j^r \neq +\infty$, $R_j^r$ satisfies the following SDE up to $T_j^r$:

$$dR_j^r = \frac{R_j^r(1 - R_j^r)}{\tilde{x}_\infty^r} \sqrt{\kappa} dB^r + \frac{R_j^r(1 - R_j^r)}{(\tilde{x}_\infty^r)^2} \left[ 2 + (2 - \kappa)R_j^r + \sum_{k=1}^m \rho_k(1 - R_k^r) \right] dt. \quad (3.18)$$

The same extremal distance argument as before shows that in the case that for $j \in \Lambda$, as $t \uparrow T_j^r$, $R_j^r \to 1$. Thus, for all $j \in \mathbb{N}_m$, $R_j^r$ is continuous on $[0, \infty)$. Also note that in any case (3.18) holds throughout $[0, \infty)$ because of the factor of $R_j^r(1 - R_j^r)$ on the RHS.

Let $F$ be the multiple hypergeometric function as in the last subsection. Define the $M^r$ on $[0, \infty]$ by

$$M^r(t) = \frac{F(R^r(t))}{F(R^r(0))} \prod_{j \in \mathbb{N}_m} I_j^r(t)^{\rho_j} = \frac{F(R^r(t))}{F(R^r(0))} \prod_{j \in \Lambda} I_j^r(t)^{\rho_j(\kappa - 4)/(2\kappa)}. \quad (3.19)$$

**Lemma 3.3.** $M^r$ is a positive continuous local martingale.

**Proof.** The continuity and positiveness of $M$ follows from the continuity and positiveness of $F(R^r)$ and $I_j^r$, $1 \leq j \leq m$. Here we use the continuity of $R^r$ and the continuity and positiveness of $F$ on $\mathbb{S}_m$. Now we check the local martingale property of $M^r$.

If $v_j^r = v_\infty^r$, then $R_j^r$ is constant $1$. If $v_j^r = +\infty > v_\infty^r$, then $R_j^r$ is constant $0$. Thus, if $\Lambda = \emptyset$, then $F(R^r)$ is constant, and so is $M^r$. Suppose now $\Lambda = \{j \in \mathbb{N}_m : m_1 \leq j \leq m_2\} \neq \emptyset$, where $m_1 \leq m_2 \in \mathbb{N}_m$. By (3.19) and (3.15), $F(0, \ldots, 0, x_{m_1}, \ldots, x_{m_2}, 1, \ldots, 1)$ equals a constant times $\tilde{F}(x_{m_1}, \ldots, x_{m_2})$, where $F$ is the multiple hypergeometric function $F(\alpha, \beta_{m_1}, \ldots, \gamma) - \sum_{k=m_k+1}^{m_2} \beta_k$. So $M^r(t) = \tilde{F}(F(R^r(0)) \prod_{j \in \Lambda} I_j^r(t)^{\rho_j}) \prod_{j \in \Lambda} I_j^r(t)^{\rho_j(\kappa - 4)/(2\kappa)}$.

For $j \in \Lambda$, we have $R_j^r(t) < 1$ before $T_{m_2}^r$. By (3.19), (3.15), (3.18) and Itô’s formula, we find that $M^r$ is a local martingale up to $T_{m_2}^r$. Conditionally on $w^r(t)$, $t \leq T_{m_2}^r$, the process
\( \bar{w} := \bar{w}(T^r_{m_2} + \cdot) \) is the driving function of a chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve in \( \mathbb{H} \) started from \( \bar{w}^r(T^r_{m_2}) \) with force points \( \bar{v}^r_j(T^r_{m_2}) \) and force point processes \( \tilde{v}^r_j := \tilde{v}^r_j(T^r_{m_2} + \cdot), j \in \mathbb{N}_m. \) If we define \( \tilde{M}^r \) for this process, then from what we have proved, \( \tilde{M}^r \) is a local martingale up to the first time that any force point \( \tilde{v}^r_j(T^r_{m_2}) \), which lies strictly between \( \bar{w}^r(T^r_{m_2}) \) and \( +\infty \), is separated from \( \infty \). Moreover, \( \tilde{M}^r = M^r(T^r_{m_2} + \cdot)/M^r(T^r_{m_2}). \) So \( M^r \) is a local martingale at least up to \( T^r_{m_2-1}. \) Repeating this argument, we conclude that \( M^r \) is a local martingale up to \( T^r_1. \) Since every \( R^r_j \) is constant 1 on \( [T^r_1, \infty] \), and every \( I^r_j \)'s takes (random) constant value on \( [T^r_1, \infty], \) so does \( M^r. \) Thus, \( M^r \) is a local martingale throughout \([0, \infty]. \) \( \square \)

**Definition 3.4.** A reversed intermediate SLE\(_\kappa(\rho) \) (iSLE\(_\kappa(\rho) \) for short) curve in \( \mathbb{H} \) from \( w^r \) to \( \infty \) with force points \( v^r \) is a random curve, whose law is obtained by locally weighting the law of a chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve in \( \mathbb{H} \) started from \( w^r \) with force points \( v^r \) by the positive continuous local martingale \( M^r \) (which is then a supermartingale) as in Lemma \[\text{A.1}\]. We extend the definition to general simply connected domains via conformal maps.

**Remark 3.5.** By Lemma \[\text{A.1}\] the law of the iSLE\(_\kappa(\rho) \) curve is absolutely continuous w.r.t. the chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve if and only if \( M^r \) is uniformly integrable w.r.t. the latter law, and then the Radon-Nikodym derivative is \( M^r(\infty). \) We will see that this holds if \( \kappa \) and \( \rho_1, \ldots, \rho_m \) satisfies Condition (I) or (II) in Theorem \[\text{I.1}\].

We now describe some properties of the iSLE\(_\kappa(\rho) \) curve. Because of the local absolute continuity, it satisfies every local almost sure property of the chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve. For example, before the end of its lifetime, it a.s. does not visit any of its force points not immediately next to its initial point. If \( \kappa \leq 4 \), then the curve is simple. If, in addition, its law is (globally) absolutely continuous w.r.t. that of the chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve, then it a.s. do not accumulate at any of its force points not immediately next to any of its endpoints. The following lemma provides us the converse statement.

**Lemma 3.6.** Let \( \mathbb{P}_r \) denote the law of an iSLE\(_\kappa(\rho) \) curve in \( \mathbb{H} \) from \( w^r \) to \( \infty \) with force points \( v^r. \) Let \( \mathbb{P}_c \) denote the law of a chordal SLE\(_\kappa(\bar{\rho}^r, \rho^r_\infty) \) curve in \( \mathbb{H} \) started from \( w^r \) with force points \( v^r. \) Let \( \mathcal{F}^r \) be the filtration. Let \( S = \{v_j : j \in \Lambda\}. \) Then we have the following.

(i) Let \( \tau \) be an \( \mathcal{F}^r \)-stopping time such that \( \text{dist}(\eta([0, \tau)), S) \) is bounded from below by a positive constant, then \( M^r(\cdot \land \tau) \) is uniformly bounded. If \( \mathbb{P}_c \) is supported by the space of curves whose lifetimes are strictly greater than \( \tau, \) then so is \( \mathbb{P}_r. \)

(ii) \( \mathbb{P}_r \) restricted to the event \( \{\text{dist}(\eta, S) > 0\} \) is absolutely continuous w.r.t. \( \mathbb{P}_c. \)

(iii) If \( \mathbb{P}_r \) is supported by \( \{\text{dist}(\eta, S) > 0\}, \) then \( \mathbb{P}_r \ll \mathbb{P}_c. \)

(iv) \( \mathbb{P}_r \) is supported by the set of curves that have zero spherical distance from \( S \cup \{\infty\}. \)

Here we use the convention that if \( S = \emptyset, \) then \( \text{dist}(\eta, \emptyset) = \infty. \)
Proof. (i) By (3.19)[3.17] and the fact that $F$ is continuous and positive on the compact set $\Delta_m$, $M'(\cdot \wedge \tau)$ is uniformly bounded. If $\mathbb{P}_c[T_\Sigma > \tau] = 1$, then by Lemma A.1 (iii), $\mathbb{P}_r[T_\Sigma > \tau] = \mathbb{E}_c[1_{\tau < \infty}]M'(\tau) = \mathbb{E}_c[M'(\tau)] = M'(0) = 1$.

(ii) For each $n \in \mathbb{N}$, let $\tau_n$ be the first $t$ such that $\text{dist}(\eta([0, t]), S) \leq 1/n$, which satisfies the assumption in (i). By (i) and Lemma A.1 (iv), $\mathbb{P}_c$ restricted to $\mathcal{F}_{\tau_n}^r$ is absolutely continuous w.r.t. $\mathbb{P}_r$ restricted to $\mathcal{F}_{\tau_n}^r$. Since $\mathcal{F}_{\tau_n}^r$ agrees with $\mathcal{F}_\infty^r$ on the event $\{\tau_n = \infty\}$, the restriction of $\mathbb{P}_r$ to $\{\tau_n = \infty\}$ is absolutely continuous w.r.t. $\mathbb{P}_c$. Since $\{\text{dist}(\eta, S) > 0\} = \bigcup_{n \in \mathbb{N}}\{\tau_n = \infty\}$, we get (ii). Finally, (iii) and (iv) follow immediately from (ii) and the fact that $\mathbb{P}_c$ is supported by the curves that end at $\infty$.

Remark 3.7. In the case that $m = 1$, the iSLE$_\kappa(\rho)$ and iSLE$_\kappa^\epsilon(\rho)$ curves both agree with the intermediate SLE$_\kappa(\rho)$ curve defined in [21]. So Theorems 1.1 and 1.2 extend the reversibility results there. For $m \geq 2$, an iSLE$_\kappa^\epsilon(\rho)$ curve is in general different from an iSLE$_\kappa(\rho)$ curve.

3.3 Driving functions

Define $G_j$, $1 \leq j \leq m$, on $\Delta_m$ by

$$G_j(x) = 1_{\{x_j \neq 1\}}x_j : \frac{\partial x_j F(x)}{F(x)}.$$  \hspace{1cm} (3.20)

We know that $\partial x_j F$ is well defined on $(-1, 1)^m$. Since by (2.15), $F(x_1, \ldots, x_m, 1, \ldots, 1)$ equals some constant times $F(\alpha_1, \beta_1, \ldots, \beta_m, \gamma - \sum_{k=m+1}^{m} \beta_k; x_1, \ldots, x_m)$, $\partial x_j F$ is also well defined on $\Delta_m \cap \{x \in \mathbb{R}^m : x_j < 1\}$. Since $F$ is positive on $\Delta_m$, $G_j$ is well defined on $\Delta_m$. By Girsanov Theorem we see that the driving function $\hat{w}$ for the iSLE$_\kappa(\rho)$ curve in $\mathbb{H}$ from $w$ to $\infty$ with force points $v_1, \ldots, v_m, v_\infty$, which generates chordal Loewner hulls ($K_t$), satisfies the SDE

$$d\hat{w}(t) = \sqrt{\kappa}dB(t) + \sum_{j=1}^{m} \left( \frac{1}{\hat{w}(t) - \hat{v}_j(t)} - \frac{1}{\hat{w}(t) - \hat{v}_\infty(t)} \right) \rho_j + \kappa G_j(R(t)) dt,$$  \hspace{1cm} (3.21)

where $B$ is a standard Brownian motion; $\hat{v}_j(t) = g_{R_t}^\omega(v_j)$, $j \in \mathbb{N}^\infty$; $R = (R_1, \ldots, R_m)$; and $R_j(t) = \frac{\hat{w}(t) - \hat{v}_j(t)}{\hat{w}(t) - \hat{v}_\infty(t)}$ before the first time that $\hat{w}(t) = \hat{v}_\infty(t)$, and equals 1 after that time.

Similarly, the driving function $\hat{w}^\epsilon$ for an iSLE$_\kappa^\epsilon(\rho)$ curve in $\mathbb{H}$ from $w^\epsilon$ to $\infty$ with force points $v_1^\epsilon, \ldots, v_m^\epsilon, v_\infty^\epsilon$, which generates chordal Loewner hulls ($K_t^\epsilon$), satisfies the SDE

$$d\hat{w}^\epsilon(t) = \sqrt{\kappa}dB^\epsilon(t) - \sum_{j=1}^{m} \left( \frac{1}{\hat{w}^\epsilon(t) - \hat{v}_j^\epsilon(t)} - \frac{1}{\hat{w}^\epsilon(t) - \hat{v}_\infty^\epsilon(t)} \right) \rho_j + \kappa G_j(R^\epsilon(t)) dt,$$  \hspace{1cm} (3.22)

where $B^\epsilon$ is a standard Brownian motion; $\hat{v}_j^\epsilon(t) = g_{R^\epsilon_t}^{\omega^\epsilon}(v_j^\epsilon)$, $j \in \mathbb{N}^\infty$; $R^\epsilon = (R_1^\epsilon, \ldots, R_m^\epsilon)$; and $R_j^\epsilon(t) = \frac{\hat{w}^\epsilon(t) - \hat{v}_j^\epsilon(t)}{\hat{w}^\epsilon(t) - \hat{v}_\infty^\epsilon(t)}$ before the first time that $\hat{w}^\epsilon(t) = \hat{v}_\infty^\epsilon(t)$, and equals 1 after that time.
From the SDEs for driving functions, we see that both iSLE_κ(ρ) and iSLE'_κ(ρ) processes satisfy DMP. We now provide a proof for the DMP of iSLE_κ(ρ). Suppose by symmetry that \( \tilde{v}_j \leq \tilde{w} \) for all \( j \). We claim that, for any \( j \in \mathbb{N}_m^\infty \) and \( \tau, t \geq 0 \),

\[
\tilde{v}_j(\tau + t) = \begin{cases} 
g_{\hat{\kappa}_{\tau+t}/\kappa}(\tilde{v}_j(\tau)), & \text{if } \tilde{v}_j(\tau) < \tilde{w}(\tau); 
g_{\hat{\kappa}_{\tau+t}/\kappa}(\tilde{w}(\tau)^-), & \text{if } \tilde{v}_j(\tau) = \tilde{w}(\tau). \end{cases}
\] (3.23)

If \( \tau \cdot t = 0 \), the statement is trivial. Suppose now \( \tau, t > 0 \). Let \( \tilde{v}_j(\tau + t) \) denote the RHS of (3.23). Since \( \tilde{w}(\tau) \in \hat{K}_{\tau+t}/K_{\tau} \) and \( \tilde{w}(\tau) \geq c_{\kappa} \), we see that \( g_{\hat{\kappa}_{\tau+t}/\kappa} \) maps \( C \backslash ((K_{\tau+t}/K_{\tau})_{\text{doub}} \cup [\tilde{v}_j(\tau), \infty)) \) conformally onto \( \mathbb{C} \backslash [\tilde{v}_j(\tau + t), \infty) \). By the definition of \( \hat{\tau}_j \), \( g_{\hat{\kappa}_{\tau+t}} \) and \( g_{\hat{\kappa}_j} \) maps \( C \backslash (K_{\tau+t}^\text{doub} \cup [\tilde{v}_j, \infty)) \) conformally onto \( \mathbb{C} \backslash [\tilde{v}_j(\tau + t), \infty) \) and \( \mathbb{C} \backslash ((K_{\tau+t}/K_{\tau})_{\text{doub}} \cup [\tilde{v}_j(\tau), \infty)) \), respectively. Thus, \( g_{\hat{\kappa}_{\tau+t}/\kappa} \) maps \( \mathbb{C} \backslash ((K_{\tau+t}/K_{\tau})_{\text{doub}} \cup [\tilde{v}_j(\tau), \infty)) \) conformally onto \( \mathbb{C} \backslash [\tilde{v}_j(\tau + t), \infty) \). So we have \( \tilde{v}_j(\tau + t) = \tilde{v}_j(\tau + t) \), as desired.

Suppose now \( \tau \) is an \( F \)-stopping time. On the event \( \tau < \infty \), define \( B^\tau(t) = B(\tau + t) - B(\tau) \), \( \hat{\omega}^\tau = \hat{w}(\tau) - \cdot \), and \( \hat{v}_j^\tau = \hat{v}_j(\tau + \cdot) \), \( j \in \mathbb{N}_m^\infty \). Then \( B^\tau \) is a Brownian motion conditionally on \( F_\tau \) and the event \( \{ \tau < \infty \} \); and \( \hat{w}^\tau, \hat{v}_j^\tau, j \in \mathbb{N}_m^\infty \), and \( B^\tau \) solve (3.21). The chordal Loewner hulls generated by \( \hat{w}^\tau \) are \( K_\tau^\tau := K_{\tau+t}/K_{\tau}, t \geq 0 \). By (3.23), \( \hat{v}_j^\tau(t) = g_{\hat{\kappa}_{\tau}}((\hat{v}_j^\tau(0)) \), where \( \hat{v}_j(0) = \hat{v}_j(\tau) \) is understood as \( \hat{w}(\tau)^- \) if \( \hat{v}_j(\tau) = \hat{w}(\tau) \). Thus, \( \hat{w}^\tau \) generates a chordal Loewner curve \( \eta^\tau \), whose law conditional on \( F_\tau \) is an iSLE_κ(ρ) curve in \( \mathbb{H} \) from \( \hat{w}(\tau) \) to \( \infty \) with force points \( \hat{v}_j(\tau) \), \( j \in \mathbb{N}_m^\infty \), where if any \( \hat{v}_j(\tau) \) equals to \( \hat{w}(\tau) \), then as a force point it is treated as \( \hat{w}(\tau)^- \). Since \( g_{\hat{\kappa}_{\tau}} \) maps \( \mathbb{H} \) conformally onto \( \mathbb{H} \backslash K_{\tau} \), and maps \( \hat{w}(\tau) \) to \( \eta(\tau) \), and \( \eta^\tau(t) \) to \( \eta(\tau + \cdot) \), the conditional law of \( \eta(\tau + \cdot) \) given \( F_\tau \) and the event \( \{ \tau < \infty \} \) is an iSLE_κ(ρ) curve in \( \mathbb{H} \backslash K_{\tau} \) from \( \eta(\tau) \) to \( \infty \) with force points \( \{ \hat{v}_j \} \cup \eta([0, \tau]) \cap \mathbb{R} \), \( j \in \mathbb{N}_m^\infty \). A similar statement with max in place of min holds if \( \hat{v}_j \geq \hat{w} \) for all \( j \).

At the end of this subsection, we describe the driving function for a forward or reversed intermediate \( \text{SLE}_\kappa(\rho) \) curves in \( \mathbb{H} \) when the target is not \( \infty \). Let \( \kappa, \rho_j \) and \( \rho_\tau^j \), \( j \in \mathbb{N}_m^\infty \), \( \rho \) and \( \rho^\tau \) be as above. Let \( w_- < w_+ \in \mathbb{R} \). Let \( v_0 \leq v_1 \leq \cdots \leq v_1 \in \{ w_-, w_+ \} \cup (w_-, w_+) \), and \( \tilde{v} = (v_1, \ldots, v_m, v_\infty) \). Let \( \eta \) be an \( \text{SLE}_\kappa(\rho) \) curve in \( \mathbb{H} \) from \( w_+ \) to \( w_- \) with force points \( \tilde{v} \). Then the part of \( \eta \) up to the first time that it separates \( w_- \) from \( \infty \) is a chordal Loewner curve with some speed. After normalization, we make this part of \( \eta \) a chordal Loewner curve (with speed 1), and call it an \( \text{SLE}_\kappa(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_+ \) to \( w_- \) with force points \( \tilde{v} \). We similarly define an \( \text{SLE}_\kappa^r(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_- \) to \( w_+ \) with force points \( \tilde{v} \). Following the argument in [16] we obtain the proposition below. We leave the proof to the interested reader.

**Proposition 3.8.** (i) The driving process \( \tilde{\hat{w}}_+ \) of an \( \text{SLE}_\kappa(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_+ \) to \( w_- \) with force points \( \tilde{v} \), which generates chordal Loewner hulls \( (K_+)(t) \), satisfies the SDE

\[
d\tilde{\hat{w}}_+ = \sqrt{\kappa}dB_+ + \kappa - 6 \tilde{w}_+ - \tilde{w}_- dt + \sum_{j=1}^m \left( \frac{1}{\tilde{w}_+ - \tilde{v}_j} - \frac{1}{\tilde{w}_+ - \tilde{v}_\infty} \right) |\rho_j + \kappa G_j(R^\tau)| dt, \] (3.24)
where \( B_+ \) is a standard Brownian motion; \( \hat{w}_+(t) = g_{K_+(t)}^{w_+}(w_-) \) and \( \hat{v}_j^+(t) = g_{K_+(t)}^{w_+}(v_j) \), \( j \in \mathbb{N}_m \); \( R^+ = (R_1^+, \ldots, R_m^+) \), and \( R_j^+ = \frac{\hat{w}_+ - \hat{v}_j^+}{\hat{w}_+ - \hat{v}_j^+} \) before the first time that the denominator vanishes, and equals 1 after that time.

(ii) The driving process \( \hat{w}_- \) of an \( \text{iSLE}_\kappa^+(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_- \) to \( w_+ \) with force points \( v \), which generates chordal Loewner hulls \( (K_-(t)) \), satisfies the SDE

\[
d\hat{w}_- = \sqrt{\kappa} dB - \frac{\kappa - 6}{\hat{w}_- - \hat{w}_+} dt - \sum_{j=1}^{m} \left( \frac{1}{\hat{w}_- - \hat{v}_j^+} - \frac{1}{\hat{w}_- - \hat{v}_j^+} \right) \left( \rho_j + \kappa G_j(R^-) \right) dt, \tag{3.25}
\]

where \( B_- \) is a standard Brownian motion; \( \hat{w}_-^+(t) = g_{K_-(t)}^{w_-}(w_+) \) and \( \hat{v}_j^-(t) = g_{K_-(t)}^{w_-}(v_j) \), \( j \in \mathbb{N}_m \); \( R^- = (R_1^-, \ldots, R_m^-) \), and \( R_j^- = \frac{\hat{w}_- - \hat{v}_j^-}{\hat{w}_- - \hat{v}_j^-} \) before the first time that the denominator vanishes, and equals 1 after that time.

### 4 Commutation Coupling

We are going to construct a commutation coupling of an \( \text{iSLE}_\kappa(\rho) \) curve with an \( \text{iSLE}_\kappa^+(\rho) \) curve in the sense of [2]. More specifically, we will prove the following theorem.

**Theorem 4.1.** Let \( \kappa \in (0,8) \). Let \( \rho_1, \ldots, \rho_m, \rho_\infty \in \mathbb{R} \) satisfies that \( \sum_{j=1}^{k} \rho_j > (-2) \lor (\frac{\kappa}{2} - 4) \) for any \( k \in \mathbb{N}_m \), and \( \sum_{j=1}^{m} \rho_j = 0 \). Let \( w_+ > w_- \in \mathbb{R} \). Let \( v_1 > \cdots > v_m > v_\infty \in (w_-, w_+ \cup \{w_-, w_+\}) \). Then there is a pair of random curves \( \eta_+(t_+) \), \( 0 \leq t_+ < T_+ \), and \( \eta_-(t-) \), \( 0 \leq t_- < T_- \), defined on the same probability space such that \( \eta_+ \) is an \( \text{iSLE}_\kappa(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_+ \) to \( w_- \) with force points \( v \), \( \eta_- \) is an \( \text{iSLE}_\kappa^+(\rho) \) curve in \( \mathbb{H} \) under chordal coordinate from \( w_- \) to \( w_+ \) with force points \( v \), and they commute with each other in the following sense. Let \( F_\pm \) and \( K_\pm(\cdot) \) be the filtration and chordal Loewner hulls, respectively, generated by \( \eta_\pm \).

(i) If \( \tau_- \) is an \( F^- \)-stopping time, then conditionally on \( F^-_{\tau_-} \) and the event that \( \tau_- < T_- \), up to a time-change, the law of \( \eta_+ \) up to the time that it hits \( \eta_-(0, \tau_-) \) is that of an \( \text{iSLE}_\kappa(\rho) \) curve in \( \mathbb{H} \setminus K_-(\tau_-) \) from \( w_+ \) to \( \eta_-(\tau_-) \) with force points \( v_j \lor \max(\eta_-(0, \tau_-) \cap \mathbb{R}) \), \( j \in \mathbb{N}_m^\infty \), up to the time that it hits \( \eta_-(0, \tau_-) \) or separates \( \eta_-(0, \tau_-) \) from \( \infty \).

(ii) If \( \tau_+ \) is an \( F^+ \)-stopping time, then conditionally on \( F^+_{\tau_+} \) and the event that \( \tau_+ < T_+ \), up to a time-change, the law of \( \eta_- \) up to the time that it hits \( \eta_+(0, \tau_+) \) is that of an \( \text{iSLE}_\kappa^+(\rho) \) curve in \( \mathbb{H} \setminus K_+(\tau_+) \) from \( w_- \) to \( \eta_+(\tau_+) \) with force points \( v_j \land \min(\eta_+(0, \tau_+) \cap \mathbb{R}) \), \( j \in \mathbb{N}_m^\infty \), up to the time that it hits \( \eta_+(0, \tau_+) \) or separates \( \eta_+(0, \tau_+) \) from \( \infty \).

Let \( \mathbb{P}_+ \) denote the marginal law of \( \eta_+ \) in the theorem. We call the joint law of \( \eta_+ \) and \( \eta_- \) a (global) commutation coupling of \( \mathbb{P}_+ \) and \( \mathbb{P}_- \). We now introduce local commutation couplings.
For $\sigma \in \{+, -, \}$, let $\Xi_\sigma$ denote the space of crosscuts $\xi$ in $\mathbb{H}$, which have positive distance from $\{w_+, w_-\}$, and separate $w_\sigma$ from both $\infty$ and $w_{-\sigma}$. So for each $\xi \in \Xi_\sigma$, Hull($\xi$) contains a neighborhood of $w_\sigma$ in $\mathbb{H}$, and does not have $w_{-\sigma}$ on its closure. Let $\Xi$ be the set of $(\xi_+, \xi_-) \in \Xi_+ \times \Xi_-$ such that Hull($\xi_+$) $\cap$ Hull($\xi_-$) $= \emptyset$. For each $\sigma \in \{+, -, \}$ and $\xi \in \Xi_\sigma$, let $\tau_\xi^\sigma$ denote the first $t$ such that $\eta_\sigma(t) \in \overline{\xi}$. If such $t$ does not exist, then we set $\tau_\xi^\sigma = T_\sigma$.

For $(\xi_+, \xi_-) \in \Xi$, a coupling of a curve $\eta_+$ with law $\mathbb{P}_+$ and a curve $\eta_-$ with law $\mathbb{P}_-$ is called a locally commutation coupling within $(\xi_+, \xi_-)$ if Theorem 4.1(i) holds up to $\tau_\xi^+$ with the additional assumption that $\tau_+ \leq \tau_\xi^+$, and Theorem 4.1(ii) holds up to $\tau_\xi^-$ with the additional assumption that $\tau_- \leq \tau_\xi^-$. This section is devoted to the proof of this theorem. The construction of the coupling follows the procedure in [23, 22]. We first study how two deterministic/random chordal Loewner curves interact with each other. Then use that to construct local commutation couplings, and finally extend the local couplings to a global commutation coupling.

4.1 Deterministic ensemble

Let $w_+, w_-$ and $v_j$, $j \in \mathbb{N}_0^\infty$, be as in Theorem 4.1. Let $\eta_+(t)$, $0 \leq t < T_+$, and $\eta_-(t)$, $0 \leq t < T_-$, be two chordal Loewner curves in $\mathbb{H}$ with $\eta_+(0) = w_+$, which respectively generate chordal Loewner hulls $(K_+(t))$ and $(K_-(t))$. Suppose further that for $\sigma \in \{+, -, \}$, $\eta_\sigma$ does not visit $\{v_j : j \in \mathbb{N}_0^\infty \} \setminus \{w_-, w_+\}$, $w_{-\sigma} \notin K_\sigma(t)$ for $0 \leq t < T_\sigma$, and that the Lebesgue measure of $\eta_\sigma \cap [\mathbb{R}]$ is zero. We remark here that these properties are almost surely satisfied if $\eta_\sigma$ follows the law $\mathbb{P}_\sigma$. Let $\hat{\eta}_+$ and $\hat{\eta}_-$ be their driving functions. Then $\hat{\eta}_\pm(0) = w_\pm$, and we have chordal Loewner equations:

$$\partial_t g_{K_\pm}(t)(z) = \frac{2}{g_{K_\pm}(t)(z) - \hat{\eta}_\pm(t)}, \quad g_{K_\pm}(0)(z) = z. \tag{4.1}$$

Let

$$\mathcal{D} = \{(t_+, t_-) \in [0, T_+) \times [0, T_-) : \overline{K_+(t_+)} \cap \overline{K_-(t_-)} = \emptyset\}. \tag{4.2}$$

For $\sigma \in \{+, -, \}$, let $T^\sigma_\sigma : [0, T_{-\sigma}) \to [0, T_\sigma)$ be such that $T^\sigma_\sigma(t_{-\sigma})$ is the supremum of $t_\sigma$ such that $(t_+, t_-) \in \mathcal{D}$. For a function $X$ defined on $\mathcal{D}$ and $s \in [0, T_{-\sigma})$ (resp. $[0, T_\sigma)$), we use $X|_{s}^+$ (resp. $X|_{s}^-$) to denote the function obtained from $X$ by restricting the first (resp. second) variable to be $s$. For example, $X|_{-\sigma}^-$ is the function $t \mapsto X(t, s)$ with definition domain $[0, T^\sigma_{-\sigma}(s))$. We also view $X|_{0}^\sigma$ as functions defined on $\mathcal{D}$. For example, $X|_{0}^\sigma(t_+, t_-) = X(0, t_-)$.

For each $(t_+, t_-) \in \mathcal{D}$, we define $K(t_+, t_-) = K_+(t_+) \cup K_-(t_-)$, which is an $\mathbb{H}$-hull, and

$$K_{\sigma, t_{-\sigma}}(t_\sigma) = K(t_+, t_-) / K_{-\sigma}(t_{-\sigma}) = g_{K_{-\sigma}(t_{-\sigma})}(K_{\sigma}(t_\sigma)), \quad \sigma \in \{+, -, \}. \tag{4.3}$$

Then we have

$$g_{K_{+, t_-}(t_+)} \circ g_{K_-(t_-)} = g_{K_+(t_+), t_-} = g_{K_{-, t_+}(t_-)} \circ g_{K_+(t_+)}. \tag{4.3}$$

Let $\eta_{\sigma, t_{-\sigma}}(t_\sigma) = g_{K_{-\sigma}(t_{-\sigma})}(\eta_\sigma(t_\sigma))$. Then $(K_{\sigma, t_{-\sigma}}(t_\sigma))$ are the chordal Loewner hulls generated by $\eta_{\sigma, t_{-\sigma}}$, $\sigma \in \{+, -, \}$. 

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Fix \( \sigma \neq \nu \in \{+, -\} \) and \( t_\nu \in [0, T_\nu). \) Let \( m_{\sigma,t_\nu}(t_\sigma) = \text{hcap}_2(K_{\sigma,t_\nu}(t_\sigma)). \) Since \( g_{K_{\nu}(t_\nu)} \) maps \( \mathbb{H} \setminus K_{\nu}(t_\nu) \) conformally onto \( \mathbb{H} \), by Proposition 2.8 \( \eta_{\sigma,t_\nu}(t_\sigma), 0 \leq t_\sigma < T_\sigma^D(t_\nu), \) is a chordal Loewner curve with speed \( d \) function with speed \( z \), we get

\[
K_{\sigma,t_\nu}(t'_\sigma) = g_{K_{\sigma,t_\nu}(t_\sigma)}(K_{\sigma,t_\nu}(t'_\sigma)) = g_{K_{\sigma,t_\nu}(t_\sigma)}(K_{\sigma}(t'_\sigma) \setminus K_{\sigma}(t_\sigma)) = g_{K_{\nu,t_\nu}(t_\sigma)}(K_{\nu}(t'_\sigma) \setminus K_{\nu}(t_\sigma)).
\]

By Proposition 2.8, \( \nu, t_\sigma > t_\nu \), we get \( \nu, t_\sigma > t_\nu \), we get \( \nu, t_\nu \) and \( \hat{w}_\sigma(t_\nu) \)

Thus, the chordal Loewner driving function with speed \( d m_{\sigma,t_\nu} \) for \( \eta_{\sigma,t_\nu} \) is

\[
W_{\sigma}(t_+, t_-) := g_{K_{\nu,t_\nu}(t_\nu)}(\hat{w}_\sigma(t_\nu)), \quad 0 \leq t_\sigma < T_\sigma^D(t_\nu).
\]

Note that \( W_{\sigma}|_{t_\nu} = \hat{w}_\sigma. \) Since \( \text{hcap}_2(K_{\nu}(t'_\nu)/K_{\nu}(t_\nu)) = t'_\nu - t_\nu \), and \( \text{hcap}_2(K_{\sigma,t_\nu}(t'_\sigma)/K_{\sigma,t_\nu}(t_\sigma)) = m_{\sigma,t_\nu}(t'_\sigma) - m_{\sigma,t_\nu}(t_\sigma) \), we use Proposition 2.5 to conclude that \( m_{\sigma,t_\nu} \) has a right-hand derivative at \( t_\sigma \), which is equal to \( g'_{K_{\nu,t_\nu}}(\hat{w}_\sigma(t_\sigma))^2. \) Since \( g_{K_{\nu,t_\sigma}} \) and \( \hat{w}_\sigma \) are continuous in \( t_\sigma \), the right-hand derivatives are actually two-sided derivatives. We now define

\[
A_{\sigma,n}(t_+, t_-) = g^{(n)}_{K_{\nu,t_\nu}}(\hat{w}_\sigma(t_\nu)), \quad n = 1, 2, 3; \quad A_{\sigma,S} = A_{\sigma,3}/A_{\sigma,1} = 3/2 \big( A_{\sigma,2}/A_{\sigma,1} \big)^2.
\]

where the superscript \( (n) \) stands for \( n \)-th derivative. So \( A_{\sigma,S}(t_+, t_-) \) is the Schwarzian derivative of \( g_{K_{\nu,t_\sigma}} \) at \( \hat{w}_\sigma(t_\nu) \).

Then we get

\[
\partial_{t_\sigma} g_{K_{\sigma,t_\nu}(t_\sigma)}(z) = \frac{2A_{\sigma,1}(t_+, t_-)^2}{g_{K_{\sigma,t_\nu}(t_\sigma)}(z) - W_{\sigma}(t_+, t_-)^2}; \quad \text{(4.6)}
\]

Let \( X_{\sigma,\nu} = W_{\sigma} - W_{\nu} \) and \( X_{\sigma,\nu}^A = A_{\sigma,1}/X_{\sigma,\nu}. \) Setting \( z = \hat{w}_\nu(t_\nu) \) in (4.6), and using (4.4), we get

\[
\partial_{t_\nu} W_{\nu} = -2A_{\sigma,1}^2/X_{\sigma,\nu} = -2A_{\sigma,1}X_{\sigma,\nu}^A.
\]

Differentiating (4.6) w.r.t. \( z \), we get

\[
\partial_{t_\nu} \partial_{t_\nu} \log(g'_{K_{\sigma,t_\nu}(t_\sigma)}(z)) = \frac{\partial_{t_\nu} g'_{K_{\sigma,t_\nu}(t_\sigma)}(z)}{g'_{K_{\sigma,t_\nu}(t_\sigma)}(z)} = -\frac{2A_{\sigma,1}(t_+, t_-)^2}{(g_{K_{\sigma,t_\nu}(t_\sigma)}(z) - W_{\sigma}(t_+, t_-))^2}.
\]

Setting \( z = \hat{w}_\nu(t_\nu) \) in (4.8), we get by (4.4, 4.5)

\[
\partial_{t_\nu} A_{\nu,1}/A_{\nu,1} = -2(X_{\sigma,\nu}^A)^2.
\]

Differentiating (4.8) further w.r.t. \( z \) twice and setting \( z = \hat{w}_\nu(t_\nu) \), we get by (4.4, 4.5)

\[
\partial_{t_\nu} A_{\nu,S} = -12(X_{\sigma,\nu}^A)^2(X_{\sigma,\nu}^A)^2.
\]

Define \( I_{S} \) on \( D \) by

\[
I_{S}(t_+, t_-) = \exp \left( -12 \int_0^{t_+} \int_0^{t_-} X_{\sigma,\nu}^A(s_+, s_-) X_{\sigma,\nu}^A(s_+, s_-) ds_- ds_+ \right).
\]

(4.11)
By (4.10) and that $A_{\sigma,S}|_{v_0} = 1$ we get
\[ \partial_{t_{\sigma}} I_S \bigg| I_S = A_{\sigma,S} \] (4.12)
Differentiating (4.3) w.r.t. $t_{\sigma}$, using (4.1,4.6,4.4,4.5) and setting $\zeta = g_{K_{\sigma}}(z)$, we get
\[ \partial_{t_{\sigma}} g_{K_{\sigma},t_{\sigma}}(t_{\sigma}) = \frac{2g'_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{\zeta - w_{\sigma}(t_{\sigma})} - \frac{2g'_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{g_{K_{\sigma},t_{\sigma}}(t_{\sigma})} \left( \zeta - g_{K_{\sigma},t_{\sigma}}(t_{\sigma}) (w_{\sigma}(t_{\sigma})) \right)^2. \] (4.13)
Differentiating the above formula w.r.t. $\zeta$, we get
\[ \partial_{t_{\sigma}} g'_{K_{\sigma},t_{\sigma}}(t_{\sigma}) = \frac{2g''_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{\zeta} - \frac{2g''_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{(\zeta - w_{\sigma}(t_{\sigma}))^2} + \frac{2g'_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{g_{K_{\sigma},t_{\sigma}}(t_{\sigma})} \left( \zeta - g_{K_{\sigma},t_{\sigma}}(t_{\sigma}) (w_{\sigma}(t_{\sigma})) \right)^2 \] (4.14)
Sending $\zeta \to \bar{w}_{\sigma}(t_{\sigma})$ in (4.13) and (4.14) respectively, we get
\[ \partial_{t_{\sigma}} g_{K_{\sigma},t_{\sigma}}(t_{\sigma})(\zeta = \bar{w}_{\sigma}(t_{\sigma})) = -3A_{\sigma,2}(t_+, t_-); \] (4.15)
\[ \frac{\partial_{t_{\sigma}} g'_{K_{\sigma},t_{\sigma}}(t_{\sigma})}{g_{K_{\sigma},t_{\sigma}}(t_{\sigma})} \bigg|_{\zeta = \bar{w}_{\sigma}(t_{\sigma})} = \frac{1}{2} \left( A_{\sigma,2}(t_+, t_-) \right)^2 - \frac{4}{3} A_{\sigma,1}(t_+, t_-). \] (4.16)
Recall the $g_{\nu}^{w}$ in Definition 2.6. For $j \in \mathbb{N}^\infty$ and $\sigma \in \{+, -\}$, we call $\bar{v}_{\sigma}(t_{\sigma}) := g_{K_{\sigma}}(v_j) = 0 \leq t_{\sigma} < T_{\sigma}$, the force point process started from $v_j$ driven by $\eta_{\sigma}$. We are going to define the force point process started from $v_j$ jointly driven by $\eta_+$ and $\eta_-$, which is a function $V_j$ defined on $D$. We need the following proposition.

**Proposition 4.2.** For any $(t_+, t_-) \in D$ and $v \in (w_-, w_+) \cup \{w_+, w_\pm\}$,
\[ g_{K_{\sigma}^{-}, t_+}(0) \circ g_{K_{\sigma}^{-}, t_-}(0) = g_{K_{\sigma}^{+}, t_+}(0) \circ g_{K_{\sigma}^{+}, t_-}(0). \] (4.17)

**Proof.** Suppose $v \in (w_-, w_+) \cup \{w_+, w_\pm\}$. Since $K_{\sigma}(t_+) \cap K_{\sigma}(t_-) = \emptyset$, there are three cases. Case 1. $v \notin K_{\sigma}(t_+) \cup K_{\sigma}(t_-)$. In this case, $g_{K_{\sigma}^{-}, t_+}(v) = g_{K_{\sigma}^{-}, t_-}(v)$, which is not contained in the closure of $g_{K_{\sigma}^{-}, t_+}(K_{\sigma}(t_+)) = K_{\sigma}(t_+)$, and so $g_{K_{\sigma}^{-}, t_+}(v) \circ g_{K_{\sigma}^{-}, t_-}(v) = g_{K_{\sigma}^{+}, t_+}(v) \circ g_{K_{\sigma}^{+}, t_-}(v)$. Symmetrically, the RHS equals $g_{K_{\sigma}^{-}, t_+}(v) \circ g_{K_{\sigma}^{-}, t_-}(v)$, which is not contained in the closure of $g_{K_{\sigma}^{-}, t_+}(K_{\sigma}(t_+)) = K_{\sigma}(t_+)$. Thus,
\[ g_{K_{\sigma}^{-}, t_+}(v) \circ g_{K_{\sigma}^{-}, t_-}(v) = g_{K_{\sigma}^{+}, t_+}(v) \circ g_{K_{\sigma}^{+}, t_-}(v) = \lim_{x \neq g_{K_{\sigma}^{+}, t_+}(t_+)} g_{K_{\sigma}^{+}, t_+}(v) \circ g_{K_{\sigma}^{+}, t_-}(v). \]
On the other hand, since $v \notin K_{\sigma}(t_-)$, $g_{K_{\sigma}^{-}, t_-}(v) = g_{K_{\sigma}^{-}, t_-}(v)$, which is contained in the closure of $g_{K_{\sigma}^{-}, t_-}(K_{\sigma}(t_-)) = K_{\sigma}(t_-)$, and is less than $g_{K_{\sigma}^{-}, t_-}(w_+) = \eta_{t_+}(-0)$. Thus,
\[ g_{K_{\sigma}^{-}, t_+}(v) \circ g_{K_{\sigma}^{-}, t_-}(v) = g_{K_{\sigma}^{+}, t_+}(v) \circ g_{K_{\sigma}^{+}, t_-}(v). \]

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where we used $g_{K_{τ}(t_+)}(a_{K_{τ}(t_+)}) = a_{K_{τ}(t_+)}$. Combining the above two displayed formulas with (4.3), we get (4.17) in Case 2. The last case, i.e., $v ∈ K_2(t_2)$, is symmetric.

Because of the proposition, we define $g_{K_{τ}(t_+)}(w_+, w_-)\bigcap D = \bigcap_{\nu,j} \{ \nu_j \}$, and (4.18, 4.19), for any $\eta \in D$, $\eta$ is absolutely continuous with $\nu_j$. If $\nu_j = \nu_j$: if $\nu_j$ is absolutely continuous with $\nu_j$, $\nu_j$: if $\nu_j$ is absolutely continuous with $\nu_j$, $\nu_j$: if $\nu_j$ is absolutely continuous with $\nu_j$, $\nu_j$: if $\nu_j$ is absolutely continuous with $\nu_j$. By Proposition 2.10 and (4.18, 4.19), for any $\nu_j \in [0, T_\nu)$, $V_j(t_\nu)$ is absolutely continuous with $\nu_j$. By (4.7) and (4.20), we get

\begin{align*}
\partial_{\nu_j} Y_{\nu_j} &= -2A_{\nu_j}^2/X_{\nu_j}^2, \quad \partial_{\nu_j} X_{\nu_j} = -2X_{\nu_j}^2/\nu_j, \quad \text{a.e.} \quad (4.21)
\end{align*}

Define $Y_{\nu_j}$ on $D$ by

\begin{align*}
Y_{\nu_j}(t_+, t_-) &= \begin{cases} X_{\nu_j}(t_+, t_-)/X_{\nu_j}(t_\nu), & \text{if } \nu_j(t_\nu) \neq \nu_j(t_\nu); \\
g_{K_{\nu_\nu}(t_\nu)}(\nu_j(t_\nu)) = A_{\nu_\nu}(t_+, t_-), & \text{if } \nu_j(t_\nu) = \nu_j(t_\nu). 
\end{cases} 
\end{align*}

Recall that $\nu_j(t_\nu) = W_j(t_\nu) \notin K_{\nu_\nu}(t_\nu)$, and (4.22, 4.19), $Y_{\nu_j}$ is well defined, continuous, and positive on $D$. By (4.9, 4.21),

\begin{align*}
\partial_{\nu_j} Y_{\nu_j}/Y_{\nu_j} &= -2X_{\nu_j}^2/X_{\nu_j}^2, \quad \text{a.e.} 
\end{align*}

We then define $E_{\nu_j}$ on $D$ by $E_{\nu_j} = \frac{Y_{\nu_j}}{Y_{\nu_j}(t_\nu)}$. Let $X_{\nu_j}^4 = X_{\nu_j}^4/0$ and $X_{\nu_j}^4 = X_{\nu_j}^4/0$. By (4.23),

\begin{align*}
\partial_{\nu_j} E_{\nu_j}/E_{\nu_j} &= -2X_{\nu_j}^4/X_{\nu_j}^4 + 2X_{\nu_j}^4/X_{\nu_j}^4, \quad \text{a.e.} \quad (4.24)
\end{align*}

If $\{v_j, v_k\} \notin K_{\nu}(t_\nu)$, then $\{\nu_j(t_\nu), \nu_k(t_\nu)\} \notin K_{\nu_\nu}(t_\nu)$, and we define $Y_{\nu_\nu}$ on $D$ by

\begin{align*}
Y_{\nu_\nu}(t_+, t_-) &= \begin{cases} X_{\nu_\nu}(t_+, t_-)/X_{\nu_\nu}(t_\nu), & \text{if } \nu_j(t_\nu) \neq \nu_k(t_\nu); \\
g_{K_{\nu_\nu}(t_\nu)}(\nu_\nu(t_\nu)), & \text{if } \nu_j(t_\nu) = \nu_k(t_\nu). 
\end{cases} 
\end{align*}

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By (4.19), $Y_{j,k}^{\sigma}$ is well defined, continuous, and positive on the set of $(t_+, t_-) \in \mathcal{D}$ such that $\{v_j, v_k\} \not\subset \mathcal{K}_\nu(t_\nu)$. By (4.21)

$$\partial_t Y_{j,k}^{\nu}/Y_{j,k}^{\nu} = -2X_{\sigma,j}^{A;X_{\sigma,k}^{A}} \ a.e.; \quad \partial_t Y_{j,k}^{\sigma}/Y_{j,k}^{\sigma} = -2X_{\sigma,j}^{A;X_{\sigma,k}^{A}} + 2X_{\sigma,j}^{A;X_{\sigma,k}^{A}} \ a.e. \quad (4.26)$$

We then define $E_{j,k}$ on $\mathcal{D}$ by

$$E_{j,k}(t_+, t_-) = \begin{cases} Y_{j,k}^+(t_+, t_-)/Y_{j,k}^+(0, t_-), & \text{if } \{v_j, v_k\} \not\subset \mathcal{K}_+(t_+) \ ; \\ Y_{j,k}^-(t_+, t_-)/Y_{j,k}^-(t_+, 0), & \text{if } \{v_j, v_k\} \not\subset \mathcal{K}_+(t_+). \end{cases} \quad (4.27)$$

The $E_{j,k}$ is well defined because if $\{v_j, v_k\} \not\subset \mathcal{K}_+(t_+)$ and $\{v_j, v_k\} \not\subset \mathcal{K}_-(t_-)$ both hold, then if $v_j \neq v_k$, both lines of the RHS of (4.27) equal $X_{\sigma,j}^{(A;X_{\sigma,k}^{A})}$, and if $v_j = v_k$, both lines equal $g_k^{(A;X_{\sigma,j}^{A})}$. Moreover, $E_{j,k}$ is positive and continuous on $\mathcal{D}$ because both $Y_{j,k}^+$ and $Y_{j,k}^-$ are positive and continuous on their respective domains. By (4.26), we have

$$\partial_t E_{j,k}/E_{j,k} = -2X_{\sigma,j}^{A;X_{\sigma,k}^{A}} + 2X_{\sigma,j}^{A;X_{\sigma,k}^{A}} \ a.e. \quad (4.28)$$

**Proposition 4.3.** Let $(\xi_+, \xi_-) \in \Xi$. There is $C \in (1, \infty)$ depending only on $\xi_+, \xi_-$ such that the restrictions of $A_{\sigma,1}$, $X_{\sigma,\tau}$, $I_{\sigma}$, $E_{\sigma,j}$, and $E_{j,k}$, $\sigma \in \{+,-\}$, $j, k \in \mathbb{N}^\infty$, to $[0, \tau_+^\xi) \times [0, \tau_-^\xi)$, are all bounded from above by $C$ and from below by $1/C$.

**Proof.** Throughout the proof, a constant is a number depending only on $\xi_+, \xi_-$. By symmetry, assume that $\sigma = +$. Fix $(t_+, t_-) \in [0, \tau_+^\xi) \times [0, \tau_-^\xi)$. Let $x_\pm$ be the endpoint of $\xi_\pm$ that lies on $(w_-, w_+)$, and $x_0 = (x_+ + x_-)/2$. Since $K(t_+, t_-) \subset \text{Hull}(\xi_+ \cup \xi_-)$, we have $1 \geq g^0_k(t_+, t_-)(x_0) > 0$, which implies that $|\log g^0_k(t_+, t_-)(x_0)|$ is bounded by a constant. By (4.3) and that $K(t_+, 0) = K_+(t_+)$, $|\log g^0_k(t_+, t_-)(g^0_k(t_+, t_+)(x_0))|$ is bounded by a constant. Since $g^0_k(t_+, t_-) \in (0, 1]$, and is increasing on $[g^0_k(t_+, t_-), \infty)$, we see that $|\log g^0_k(t_+, t_-)|$ is bounded by a constant on $[g^0_k(t_+, t_+)(x_0), \infty) = I_+$. By Proposition 2.9, $\tilde{w}_+(t_+) \in I_+$. Since $A_{\sigma,1}(t_+, t_-) = g^0_k(t_+, t_-)(\tilde{w}_+(t_+))$, we see that $|\log A_{\sigma,1}(t_+, t_-)|$ is bounded by a constant.

The quantity $X_{\sigma,\tau}(t_+, t_-) = W_+(t_+, t_-) - W_-(t_+, t_-)$ is bounded from above by $d_K(t_+, t_-)$, which is further bounded by the constant $d_{\text{Hull}(\xi_+ \cup \xi_-)} - c_{\text{Hull}(\xi_+ \cup \xi_-)}$ by Proposition 2.2. For the lower bound, pick any $x_1 < x_2 \in (x_-, x_+)$. Then $X_{\sigma,\tau}(t_+, t_-) \geq g_k(t_+, t_-)(x_2) - g_k(t_+, t_-)(x_1)$, which is further bounded from below by the positive constant $g_{\text{Hull}(\xi_+ \cup \xi_-)}(x_2) - g_{\text{Hull}(\xi_+ \cup \xi_-)}(x_1)$ due to the fact that $d_{\text{Hull}(\xi_+ \cup \xi_-)}/d_{\text{Hull}(\xi_+ \cup \xi_-)}([x_1, x_2]) \in (0, 1]$. From what we have proved, $|X_{\sigma,\tau}^A| = |A_{\sigma,1}/X_{\sigma,\tau}^A|$ and $|X_{\sigma,\tau}^A| = |A_{\sigma,1}/X_{\sigma,\tau}^A|$ are uniformly bounded by a constant on $[0, t_+] \times [0, t_-]$. We also know that $t_\pm$ is bounded by the constant $\text{Hcap}_2(\text{Hull}(\xi_\pm))$. By (4.11), we see that $|\log I_{\sigma}(t_+, t_-)|$ is bounded by a constant.

For $E_{\sigma,j}$, consider two cases. Case 1. $v_j \geq x_0$. We have seen that $|\log g^0_k(t_+, t_-)|$ is bounded by a constant on $I_+ = [g_k(t_+, t_-)(x_0), \infty)$, and $\tilde{w}_+(t_+) \in I_+$. Since $\tilde{w}_+(t_+) = g^0_k(t_+, t_+)(v_j) \in I_+$, by (4.22), $|\log Y_{\sigma,j}(t_+, t_-)|$ is bounded by a constant.
Case 2. \(v_j \leq x_0\). Let \(y_+ = (x_0 + x_+) / 2\). Then

\[
\widehat{v}_j^+(t_+) = g_{K_+(t_+)}(v_j) \leq g_{K_+(t_+)}(x_0) < g_{K_+(t_+)}(y_+) \leq c_{K_+(t_+)} \leq \widehat{w}_+(t_+),
\]

which implies by (4.13, 4.4, 4.19) that \(V_j(t_+, t_-) \leq g_{K_+(t_+)}(x_0) < g_{K_+(t_+)}(y_+) \leq W_+(t_+, t_-)\). Since \(g'_{K_+(t_+)}(x_0) > 0\), \(g_{K_+(t_+)}(y_+)-g_{K_+(t_+)}(x_0)\) is bounded from below by the positive constant \(g_{\text{Hall}}(\xi_+ \cup \xi_-)(y_+)-g_{\text{Hall}}(\xi_+ \cup \xi_-)(x_0)\). So \(X_{+, j}(t_+ + t_-) - V_j(t_+, t_-)\) is bounded from below by a constant. On the other hand, by Proposition 2.2 \(X_{+, j}(t_+ + t_-)\) is bounded from above by the constant \(d_{\text{Hall}}(\xi_+ \cup \xi_-)-c_{\text{Hall}}(\xi_+ \cup \xi_-)\). These properties are also satisfied by \(X_{+, j}(t_+, 0)\). By (4.22), \(|\log Y_{+, j}(t_+, t_-)|\) is bounded by a constant.

Thus, in both cases, \(|\log Y_{+, j}(t_+, t_-)|\) is bounded by a constant. This property is also satisfied by \(|\log Y_{+, j}(0, t_-)|\). Since \(E_{+, j}(t_+, t_-) = \frac{Y_{+, j}(t_+, t_-)}{e^{Y_{+, j}(0, t_-)}}, \) \(|\log E_{+, j}(t_+, t_-)|\) is also bounded by a constant.

For \(E_{+, j, k}\), by symmetry and relabeling, we may assume that \(v_j \geq v_k \vee x_0\). Let \(y_+ = (x_0 + x_+) / 2\). Consider two cases. Case 1. \(v_k \geq y_+\). Using the proof of Case 1 for \(E_{+, j}\) with \(y_+\) in place of \(x_0\), we see that \(|\log g'_{K_-(t_-)}(y_+)\) is bounded by a constant on \([g_{K_+(t_+)}(y_+), \infty)\), which contains both \(\widehat{v}_j^+(t_+))\) and \(\widehat{v}_k^+(t_+))\). So by (4.25), \(|\log Y_{+, j, k}(t_+, t_-)|\) is bounded by a constant. Case 2. \(v_k \leq y_-\). Using the proof of Case 2 for \(E_{+, j}\) with \(x_0\) and \(y_-\) in place of \(y_+\) and \(x_0\), respectively, we see that \(g_{\text{Hall}}(\xi_+ \cup \xi_-)(x_0) - g_{\text{Hall}}(\xi_+ \cup \xi_-)(y_-) \leq X_{+, j}(t_+ + t_-) \leq d_{\text{Hall}}(\xi_+ \cup \xi_-)-c_{\text{Hall}}(\xi_+ \cup \xi_-)\), which implies that \(|\log Y_{+, j, k}(t_+, t_-)|\) is again bounded by a constant. Thus, \(|\log E_{+, j, k}(t_+, t_-)|\) is also bounded by a constant by (4.27).

For \(j \in \mathbb{N}_m\), define \(R_j\) on \(D\) by \(R_j = \frac{X_{+, j}}{X_{+, \infty}} \cdot \frac{X_{+, \infty}}{X_{+, -j}}\) if \(X_{+, \infty}X_{-, j} \neq 0\); and \(R_j = 1\) if \(X_{+, \infty}X_{-, j} = 0\). Let \(\mathcal{R} = (R_1, \ldots, R_m)\). It is clear that \(0 \leq R_1 \leq \cdots \leq R_m \leq 1\). So \(\mathcal{R}\) takes values in \(\Delta_m\).

**Lemma 4.4.** Every \(R_j\) is continuous on \(D\), and so is \(\mathcal{R}\).

**Proof.** Fix \(j \in \mathbb{N}_m\). Let \(T_{\infty}^+\) be the first time that \(\eta_+\) reaches \((-\infty, v_{\infty})\). We understand \(T_{\infty}^+\) as \(T_+^+\) if such time does not exist, and at \(0\) if \(v_{\infty} = w_+\). Similarly, let \(T_{\infty}^-\) be the first time that \(\eta_+\) reaches \((v_{\infty}, \infty)\). Let \(D_j = D \cap \{0, T_{\infty}^+ \times [0, T_j^-]\}\). Then \(X_{+, \infty}X_{-, j} \neq 0\) on \(D_j\), and so \(R_j\) is continuous on \(D_j\). If \(T_+^+ < T_+\), then \(\widehat{v}_+^+(T_+^+) = \widehat{v}_+^+(T_+^-)\), which implies that \(\widehat{v}_j^+ = \widehat{v}_j^-\) on \([T_+^+, T_+^-]\). By (4.19), we see that, on \(D \cap \{t_+ + t_- : t_+ \geq T_{\infty}^+\}, V_j \equiv V_{\infty},\) which implies that \(R_j \equiv 1\). Similarly, we have \(R_j \equiv 1\) on \(D \cap \{t_+ + t_- : t_- \geq T_{\infty}^-\}\).

If \(T_+^+\) or \(T_+^-\) equals \(0\), then \(R_j\) is constant \(1\), and its continuity is trivial. Suppose \(T_{\infty}^+\) and \(T_{\infty}^-\) are both positive. Then \((T_{\infty}^+, T_{\infty}^-) \notin D\) because \(K_+(T_{\infty}^+)\) and \(K_-(T_{\infty}^-)\) both contain \([v_{\infty}, v_j]\). It suffices to show that (i) if \(0 < T_+^+ < T_+\), then as \(t_+ \uparrow T_+^+\), \(R_j(t_+, t_-) \to 1\) uniformly in \(t_- \in [0, T_{\infty}^+(T_+^+)]\); and (ii) if \(0 < T_-^- < T_-\), then as \(t_- \uparrow T_-^-\), \(R_j(t_+, t_-) \to 1\) uniformly in \(t_+ \in [0, T_{\infty}^-(T_-^-)]\). They follow from an extremal distance argument shown below.

(i) Since \(\eta_+\) does not visit \(v_{\infty}\), \(x_0 := \eta_+(T_{\infty}^+) \in (w_-, v_{\infty})\). Let \(0 < \delta < |x_0 - v_{\infty}|\). Suppose \(t_+ < T_{\infty}^+\) is such that \(\text{diam}(\eta_+(t_+, T_{\infty}^+)) < \delta\). Then for \(t_- \in [0, T_{\infty}^+(T_+^+)]\), any curve in
\[ \mathbb{H} \setminus K(t_+, t_-) \] connecting the line segment \([v_\infty, v_j \wedge \min(\eta_+(\{0, t_+\}) \cap \mathbb{R})]\), denoted by \(I\), and the union of the right side of \(\eta_+(\{0, t_+\})\), \([w_+, \infty]\), \((-\infty, w_-]\), and the left side of \(\eta_-(\{0, t_-\})\), denoted by \(U\), must cross the semi-annulus \(\{z \in \mathbb{H} : \delta < |z - x_0| < |v_\infty - x_0|\}\). By comparison principle of extremal length, the extremal distance between \(I\) and \(U\) in \(\mathbb{H} \setminus K(t_+, t_-)\) is at least \(\log(|v_\infty - x_0|/\delta)\). Since \(g_K(t_+, t_-)\) maps \(\mathbb{H} \setminus K(t_+, t_-)\) conformally onto \(\mathbb{H}\), and maps \(I\) and \(U\) respectively to \([V_\infty(t_+, t_-), V_f(t_+, t_-)]\) and \((-\infty, W_-(t_+, t_-)) \cup [W_+(t_+, t_-), \infty)\), the extremal distance between the latter two sets in \(\mathbb{H}\), which can be expressed as a function \(f\) of \(R_j(t_+, t_-)\), is at least \(\log(|v_\infty - x_0|/\delta)\). Since the function \(f\) is bounded on \((0, 1 - \varepsilon)\) for any \(\varepsilon > 0\), we then finish the proof of (i). The proof of (ii) is similar. \(\Box\)

### 4.2 Stochastic ensemble

We adopt the assumption and notation in the previous subsection. Let \(\kappa > 0\) and \(\rho_j, j \in \mathbb{N}_m^\infty\), be as in Theorem 4.1. Let \(\tilde{\rho}_j = \rho_j/\kappa, j \in \mathbb{N}_m^\infty\).

Suppose \(\tilde{w}_+(t_+)\) and \(\tilde{w}_-(t_-)\) are independent semimartingales with quadratic variation being \(\langle \tilde{w}_\pm \rangle_t = \kappa t, 0 \leq t < T_\ast\). Let \(F^\pm\) be the filtration generated by \(\tilde{w}_\pm\). Fix \(\sigma \neq \nu \in \{+,-\}\) and two \(F^\nu\)-stopping times \(\tau_\nu\) and \(\tau_\nu'\) with \(\tau_\nu < \tau_\nu' < T_\nu\). Since \(F^+\) and \(F^-\) are independent, \(\tilde{w}_\sigma(t_\sigma)\) is also an \((F^\sigma_t \times F^\nu_{t_\nu})_{t_\sigma \geq 0}\)-semimartingale. We will repeatedly apply Itô’s formula in this subsection, where the underlying filtration is always \((F^\sigma_t \times F^\nu_{t_\nu})_{t_\sigma \geq 0}\), the time parameter \(t_\nu\) is fixed to be \(\tau_\nu\), and the time parameter \(t_\sigma\) runs from 0 to \(T_\sigma^D(\tau_\nu)\). By (4.4) and (4.15), \(W_\sigma\) satisfies the SDE

\[
\partial_\sigma W_\sigma = A_{\sigma, 1} \partial \tilde{w}_\sigma + \left(\frac{\kappa}{2} - 3\right) A_{\sigma, 2} \partial t_\sigma. \tag{4.29}
\]

By (4.5), we get

\[
\frac{\partial A_{\sigma, 1}}{A_{\sigma, 1}} = \frac{A_{\sigma, 2}}{A_{\sigma, 1}} \frac{\partial \tilde{w}_\sigma}{A_{\sigma, 1}} + \frac{1}{2} \left(\frac{A_{\sigma, 2}}{A_{\sigma, 1}}\right)^2 \frac{\partial t_\sigma}{A_{\sigma, 1}} + \left(\frac{\kappa}{2} - \frac{4}{3}\right) \frac{A_{\sigma, 3}}{A_{\sigma, 1}}.
\]

Let \(A_{\sigma, 2} = A_{\sigma, 2}/A_{\sigma, 1}\) and

\[
b = \frac{6 - \kappa}{2\kappa}, \quad c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}.
\]

The previous formula implies that

\[
\partial_\sigma A_{\sigma, 1}^b/A_{\sigma, 1} = b A_{\sigma, 1}^b \partial \tilde{w}_\sigma + (c/6) \cdot A_{\sigma, 1} \partial t_\sigma. \tag{4.30}
\]

From (4.7), we get

\[
\partial_\sigma X_{\sigma, \nu}/X_{\sigma, \nu} = X_{\sigma, \nu}^A \partial \tilde{w}_\sigma - \kappa b X_{\sigma, \nu} A_{\sigma}^{21} \partial t_\sigma + 2(X_{\sigma, \nu}^A)^2 \partial t_\sigma; \tag{4.31}
\]

\[
\partial_\sigma X_{\sigma, j}/X_{\sigma, j} = X_{\sigma, j}^A \partial \tilde{w}_\sigma - \kappa b X_{\sigma, j} A_{\sigma}^{21} \partial t_\sigma + 2(X_{\sigma, j}^A)^2 \partial t_\sigma. \tag{4.32}
\]

Here (4.31) holds throughout, and (4.32) holds up to the time that \(X_{\sigma, j} = 0\).
Defining positive continuous functions $E_+\ E_-, E_{+,\cdot}$ on $D$ by $E_\sigma = \frac{A_{\sigma,1}}{A_{\sigma,10}}$, $\sigma \in \{+, -, \}$, and $E_{+,-(t_+,t_-)} = \frac{X_{+,-(t_+,t_-)}X_{+,-(0,0)}}{X_{+,-(t_+,t_-)}X_{+,-(0,0)}}$, By (4.30,4.31),
\[
\partial_\sigma E_\sigma^{b}\big/ E_\sigma^{b} = -2b((X_{\sigma',\nu}^{A})^2 - (X_{\sigma',\nu}^{A,0})^2) \partial_\sigma; \\
\partial_\sigma E_\sigma^{b}\big/ E_\sigma^{b} = bA_{\sigma}^{2,1}\partial_\sigma + (c/6) \cdot A_{\sigma,\sigma,\partial t_\sigma}. \quad (4.33)
\]
\[
\partial_\sigma E_{+,\cdot}^{b}\big/ E_{+,\cdot}^{b} = -2b(X_{\sigma',\nu}^{A} - X_{\sigma',\nu}^{A,0}) \partial_\sigma + 2c \cdot 2b X_{\sigma',\nu}^{A} A_{\sigma}^{2,1} \partial_\sigma \\
+ 2b((X_{\sigma',\nu}^{A})^2 - (X_{\sigma',\nu}^{A,0})^2) \partial_\sigma - 4c \cdot 2b X_{\sigma',\nu}^{A,0}(X_{\sigma',\nu}^{A} - X_{\sigma',\nu}^{A,0}) \partial_\sigma. \quad (4.34)
\]
Let $E_{S,b} = I_{S}^{-1/2} E_{+,\cdot}^{b} E_{+,\cdot}^{b}$. Combining (4.33,4.34) with (4.12), we get
\[
\partial_\sigma E_{S,b}/E_{S,b} = (bA_{\sigma}^{2,1} - 2b(X_{\sigma',\nu}^{A} - X_{\sigma',\nu}^{A,0})) \partial_\sigma + 2c b X_{\sigma',\nu}^{A,0} \partial_\sigma. \quad (4.35)
\]
Recall the $R_j$, $E_{+,j}$, and $E_{j,b}$ defined before. Since $R_j = (\frac{A_{\sigma,1}}{X_{\sigma,\infty}} \cdot \frac{X_{\sigma,0}}{X_{\sigma,j}})^{\gamma - 1}$, by (4.21,4.32), $R_j$ satisfies the following SDE up to the time that it equals 1:
\[
\partial_\sigma R_j/R_j = \sigma(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) \partial_\sigma + \sigma(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A})(-\kappa b A_{\sigma}^{2,1} \partial_\sigma + 2X_{\sigma,j}^{A} \partial_\sigma) \\
+ \sigma(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A})((2 - \kappa/2) X_{\sigma,j}^{A}) + (2 - \kappa/2) X_{\sigma,\infty}^{A}) \partial_\sigma. \quad (4.36)
\]
Since $E_{\sigma,j} = \frac{Y_{\sigma,j}}{Y_{\sigma,j}^{\infty}}$ and $Y_{\sigma,j} = \frac{X_{\sigma,j}^{A}}{X_{\sigma,j}^{A,0}}$ (the generic case), by (4.24,4.32), for $j \in N_m$,
\[
\partial_\sigma (E_{\sigma,j}/E_{\sigma,j}^{\infty}) = (X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) \partial_\sigma - \kappa b X_{\sigma,j}^{A} A_{\sigma}^{2,1} \partial_\sigma + 2(X_{\sigma,j}^{A} X_{\sigma,j}^{A} - X_{\sigma,j}^{A,0} X_{\sigma,j}^{A,0}) \partial_\sigma \\
+ (2(X_{\sigma,j}^{A})^2 - 2(X_{\sigma,j}^{A,0})^2) \partial_\sigma - \kappa X_{\sigma,j}^{A,0}(X_{\sigma,j}^{A} - X_{\sigma,j}^{A,0}) \partial_\sigma. \quad (4.37)
\]
Recall that $\tilde{\rho}_j = \rho_j/\kappa$. Define
\[
E_{j,\infty,\tilde{\rho}} = \prod_{s \in \{+, -\}} \left( \frac{E_{\sigma,j}}{E_{\sigma,\infty}} \right)^{\tilde{\rho}_j} = \left( \frac{E_{\sigma,j}}{E_{\sigma,\infty}} \right)^{\sigma \tilde{\rho}_j} \cdot 1 \leq j \leq m.
\]
By (4.37),
\[
\partial_\sigma E_{j,\infty,\tilde{\rho}} = \tilde{\rho}_j([X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}] - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A}) \partial_\sigma - \sigma \tilde{\rho}_j \kappa b A_{\sigma}^{2,1}(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) \partial_\sigma \\
+ 2\sigma \tilde{\rho}_j[X_{\sigma,j}^{A} (X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) - X_{\sigma,j}^{A,0} (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0})] \partial_\sigma \\
+ 2\sigma \tilde{\rho}_j[((X_{\sigma,j}^{A})^2 - (X_{\sigma,j}^{A,0})^2) - (X_{\sigma,j}^{A,0})^2 \partial_\sigma \\
- \sigma \tilde{\rho}_j[X_{\sigma,j}^{A} (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) + X_{\sigma,j}^{A,0} (X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A})] \partial_\sigma \\
- \kappa \sigma \tilde{\rho}_j(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A})(X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) - (X_{\sigma,j}^{A,0})^2 \partial_\sigma \\
- \kappa \sigma \tilde{\rho}_j((X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A})((X_{\sigma,j}^{A} - X_{\sigma,\infty}^{A}) - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) \partial_\sigma
\]

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\[+(\kappa/2)\sigma\tilde{p}_j(\sigma\tilde{p}_j - 1) [(X_{\sigma,j}^A - X_{\sigma,\infty}^A) - (X_{\sigma,j}^{A:0} - X_{\sigma,\infty}^{A:0})]^2 \partial t_\sigma. \quad (4.38)\]

Let \(\alpha, \beta, \gamma\) be defined by (3.8). Then they satisfy the parameter assumption. Let \(F\) be the hypergeometric function \(F(\alpha, \beta_1, \ldots, \beta_m, \gamma; \cdot)\). By Theorem 2.16, it extends to a positive continuous function on \(\sum_m\). So \(F(R)\) is a positive continuous function defined on \(D\). Since \(F\) is smooth in \((-1,1)^m\), \(F(R)\) is a local martingale up to the first time that \(R\) exits \((-1,1)^m\). Recalling the \(G_j\) defined by (3.20). Combining (2.18, 2.19, 2.20) with (4.36), we see that \(F(R)\) satisfies the following SDE up to the first time that \(R\) exits \((-1,1)^m\):

\[
\frac{\partial \sigma F(R)}{F(R)} = \sigma \sum_j (X_{\sigma,j}^A - X_{\sigma,\infty}^A) G_j(R) \partial \tilde{w}_\sigma - \sigma \sum_j \kappa b A^{2:1}_\sigma (X_{\sigma,j}^A - X_{\sigma,\infty}^A) G_j(R) \partial t_\sigma \]

\[-\sigma \sum_j (X_{\sigma,j}^A - X_{\sigma,\infty}^A) \left(-2\kappa b X_{\sigma,\nu}^A + \sigma \sum_k \rho_k (X_{\sigma,k}^A - X_{\sigma,\infty}^A) \right) G_j(R) \partial t_\sigma \]

\[-\sigma \sum_j \frac{\beta_j (\kappa - 4)}{2\kappa} (X_{\sigma,j}^A - X_{\sigma,\infty}^A) [2X_{\sigma,\nu}^A + \sigma (X_{\sigma,j}^A - X_{\sigma,\infty}^A) - (X_{\sigma,j}^A + X_{\sigma,\infty}^A)] \partial t_\sigma. \quad (4.39)\]

We now argue that \(F(R)\) satisfies (4.39) throughout \([0, T_\sigma^D(\tau_\nu)]\). We need to deal with the case that some \(R_j\) equals 1. In fact, if on some interval there is \(m' < m\) such that \(R_j\) equals 1 for \(m' + 1 \leq j \leq m\) and \(R_j < 1\) for \(1 \leq j \leq m'\), then by (2.15), \(F(R)\) equals some constant times \(\tilde{F}(R)\), where \(\tilde{R} = (R_1, \ldots, R_{m'})\) and \(\tilde{F}\) is the hypergeometric function \(F(\alpha, \beta_1, \ldots, \beta_{m'}, \gamma - \sum_{j=m'+1}^m \beta_m; \cdot)\). So on that interval \(\frac{\partial \sigma F(R)}{F(R)} = \frac{\partial \sigma F(\tilde{R})}{F(\tilde{R})}\), and we may get (4.39) by applying (2.18, 2.19, 2.20) to \(\tilde{F}\) and using the facts that for \(j \leq m'\), \(R_j\) satisfies (4.36), and the terms on the RHS of (4.39) for \(j > m'\) vanish because \(G_j(R)\) and \(X_{\sigma,j}^A - X_{\sigma,\infty}^A\) both vanish.

Define another positive continuous function \(F_R\) on \(D\) by \(F_R(t_+, t_-) = \frac{F(R(t_+, t_-)) F(R(0,0))}{F(R(t_+, 0)) F(R(0,t_-))}\). By (4.39), \(F_R\) satisfies the following SDEs:

\[
\frac{\partial \sigma F_R}{F_R} = \sigma \sum_j \left[ (X_{\sigma,j}^A - X_{\sigma,\infty}^A) G_j(R) - (X_{\sigma,j}^{A:0} - X_{\sigma,\infty}^{A:0}) G_j(R) \right] \partial \tilde{w}_\sigma \]

\[-\sigma \sum_j \kappa b (A^{2:1}_\sigma - 2(X_{\sigma,\nu}^A - X_{\sigma,\nu}^{A:0})) (X_{\sigma,j}^A - X_{\sigma,\infty}^A) G_j(R) \partial t_\sigma \]

\[-\sigma \sum_j (X_{\sigma,j}^A - X_{\sigma,\infty}^A) \left( \sum_k \rho_k (X_{\sigma,k}^A - X_{\sigma,\infty}^A) \right) G_j(R) \partial t_\sigma \]

\[+\sigma \sum_j (X_{\sigma,j}^{A:0} - X_{\sigma,\infty}^{A:0}) \left( \sum_k \rho_k (X_{\sigma,k}^{A:0} - X_{\sigma,\infty}^{A:0}) \right) G_j(R) \partial t_\sigma \]

\[-\sigma \sum_j \frac{\beta_j (\kappa - 4)}{2\kappa} (X_{\sigma,j}^A - X_{\sigma,\infty}^A) [2X_{\sigma,\nu}^A + \sigma (X_{\sigma,j}^A - X_{\sigma,\infty}^A) - (X_{\sigma,j}^A + X_{\sigma,\infty}^A)] \partial t_\sigma. \]
\[ + \sigma \sum_j \rho_j \frac{(\kappa - 4)}{2\kappa} (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) (X_{\sigma,j}^{A,0} + \sigma (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) - (X_{\sigma,j}^{A,0} + X_{\sigma,\infty}^{A,0})) \partial t_{\sigma} \]

\[ - \kappa \sum_{j,k} (X_{\sigma,k}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R_0^{\nu}) \left( (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R) - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R_0^{\nu}) \right) \partial t_{\sigma}. \]  

(4.40)

Here the terms in the last line of (4.40) come from the quadratic covariation, and other terms on the RHS of (4.40) come from the difference between the RHS of (4.39) and the function obtained by replacing the \( \tau_\nu \) by 0 in the RHS of (4.39). Note that \( A_{\sigma}^{2,1} |_0 = 0 \).

Define \( M \) on \( D \) by

\[ M = F_R E_{S,b} \prod_{j \in \mathbb{N}_m} [E_{j,\infty,\tilde{\rho}} \times (E_{j,\infty,\infty}/E_{j,j,\infty}^{2} \rho_j^{(4-\kappa)}) \prod_{j,k \in \mathbb{N}_m} E_{j,k}^{\rho_j \rho_k}]. \]  

(4.41)

**Lemma 4.5.** (i) The function \( M \) is positive and continuous on \( D \), and takes value 1 on \( \{0, T_+\} \times \{0\} \) and \( \{0\} \times [0, T_-] \). (ii) For any \( (\xi_+, \xi_-) \in \Xi \), \( |\log M| \) on \( [0, \tau_{\xi_+}^+] \times [0, \tau_{\xi_-}] \) is uniformly bounded by a constant depending only on \( \xi_+, \xi_- \).

**Proof.** (i) This holds because every factor on the RHS of (4.41) is positive and continuous on \( D \). (ii) This follows from Proposition 4.3 and the fact that \( F \) is continuous and positive on the compact set \( \overline{\Sigma_m} \). \( \square \)

We may now calculate that \( M \) satisfies the following SDE:

\[ \frac{\partial_{\sigma} M}{M} = \left( \sigma \sum_j (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R) - \sigma \sum_j (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R_0^{\nu}) \right) \]

\[ + b A_\sigma^{2,1} - 2 b (X_{\sigma,j}^{A,0} - X_{\sigma,\nu}^{A,0}) + \sigma b \sum_j \tilde{\rho}_j [(X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0})] \times \]

\[ \times \left( \partial_{\sigma} \tilde{w}_\sigma + 2 \kappa b X_{\sigma,\nu}^{A,0} \partial t_{\sigma} - \sigma \sum_k (X_{\sigma,k}^{A,0} - X_{\sigma,\infty}^{A,0}) [\tilde{\rho}_k + \kappa G_j(R_0^{\nu})] \partial t_{\sigma} \right). \]  

(4.42)

The computation is tedious but straightforward. First, we note that the coefficients of \( \partial_{\sigma} \tilde{w}_\sigma \) in the SDEs (4.35, 4.38, 4.40) sum up to the coefficients of \( \partial_{\sigma} \tilde{w}_\sigma \) in (4.42). Since \( \partial(\tilde{w}_\sigma) = \kappa \partial t_\sigma \), the SDEs contribute the following covariance terms:

\[ \left( \sigma \kappa b (A_\sigma^{2,1} - 2 (X_{\sigma,j}^{A,0} - X_{\sigma,\nu}^{A,0})) + \sum_k \tilde{\rho}_k [(X_{\sigma,k}^{A,0} - X_{\sigma,\nu}^{A,0}) - (X_{\sigma,k}^{A,0} - X_{\sigma,\infty}^{A,0})] \times \right. \]

\[ \left. \times \sum_j [(X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R) - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) G_j(R_0^{\nu})] \partial t_{\sigma} \right] \]

\[ + b (A_\sigma^{2,1} - 2 (X_{\sigma,j}^{A,0} - X_{\sigma,\nu}^{A,0})) \sum_j \tilde{\rho}_j [(X_{\sigma,j}^{A,0} - X_{\sigma,\nu}^{A,0}) - (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0})] \partial t_{\sigma}. \]  

(4.43)
By (4.28) and the fact that \( \sum_{j \in \mathbb{N}_m^c} \rho_j = 0 \), we have

\[
\frac{\partial \prod_j (E_{j,j} E_{\infty,\infty} / E_{j,\infty}^2)^{\frac{\rho_j(4-k)}{2\kappa}}} {\prod_j E_{j,j} E_{\infty,\infty} / E_{j,\infty}^2} = -\sum_j \rho_j (4 - \kappa) \left( (X_{\sigma,j} - X_{\sigma,\infty})^2 - (X_{\sigma,j}^0 - X_{\sigma,\infty}^0)^2 \right) \partial \sigma; \quad (4.44)
\]

\[
\frac{\partial \prod_{j,k \in \mathbb{N}_m^c} E_{j,k}^{\rho_{j,k}}} {\prod_{j,k \in \mathbb{N}_m^c} E_{j,k}^{\rho_{j,k}}} = -\frac{1}{2\kappa} \left( \sum_j \rho_j (X_{\sigma,j}^A - X_{\sigma,\infty}^A) \right)^2 \partial \sigma + \frac{1}{2\kappa} \left( \sum_j \rho_j (X_{\sigma,j}^A - X_{\sigma,\infty}^A) \right)^2 \partial \sigma. \quad (4.45)
\]

It remains to show that the sum of the coefficients of \( \partial \sigma \) in (4.35, 4.38, 4.40, 4.43, 4.44, 4.45) is equal to the sum of the coefficients of \( \partial \sigma \) in (4.42). For that purpose, the interested reader may first compare all terms containing the factor \( G_{j,i}(\mathbb{R}) \) or \( G_{j,i}(\mathbb{R}_{\infty}^c) \), and then all remaining terms containing the factor \( A_{2}^{\infty,1}, X_{\sigma,\nu}, \) or \( X_{\sigma,\nu}^A \), and finally all other terms.

### 4.3 Construction of the couplings

Suppose \( \eta_+ \) follows the law \( \mathbb{P}_+ \), \( \eta_- \) follows the law \( \mathbb{P}_- \), and \( \eta_+ \) and \( \eta_- \) are independent. Then they almost surely satisfy the assumptions in the previous subsections, and we then adopt the notation there.

By Proposition 3.8 \( \hat{w}_+ \) and \( \hat{w}_- \) satisfy (3.24) and (3.25) for a pair of independent Brownian motions \( B_+ \) and \( B_- \). Let \( \sigma \neq \nu \in \{+, -\} \). We may rewrite the SDEs as:

\[
d\hat{w}_\sigma = \sqrt{\kappa} dB_\sigma - 2\kappa b X_{\sigma,\nu}^{A,0} dt_{\sigma} + \sigma \sum_j (X_{\sigma,j}^{A,0} - X_{\sigma,\infty}^{A,0}) [\rho_j + \kappa G_{j,i}(\mathbb{R}_{\infty}^c)] dt_{\sigma}. \quad (4.46)
\]

Combining (4.42) and (4.46), we obtain the following lemma.

**Lemma 4.6.** Let \( \sigma \neq \nu \in \{+, -\} \). Then for any \( \mathcal{F}' \)-stopping times \( \tau_\nu \) and \( \tau'_\nu \) with \( \tau_\nu \leq \tau'_\nu \) and \( \tau_\nu < T_\nu \), \( M|_{\tau_\nu}' \) is an \( (\mathcal{F}_{\tau_\nu}^r \vee \mathcal{F}_{\tau_\nu}^l)' \)-local martingale up to \( T_\sigma^D(\tau_\nu) \).

Let \( \mathbb{P}_i \) denote the law of \( (\eta_+, \eta_-) \). Since \( \eta_+ \) and \( \eta_- \) are independent, \( \mathbb{P}_i = \mathbb{P}_+ \times \mathbb{P}_- \) is the independent coupling of \( \mathbb{P}_+ \) and \( \mathbb{P}_- \). Fix \( \xi = (\xi_+, \xi_-) \in \Xi \). Combining Lemmas 4.6 and 4.5 we find that for any \( \sigma \neq \nu \in \{+, -\} \), \( t_\sigma \mapsto M(t_\sigma \wedge \tau_{\xi_+}^+, t_\sigma \wedge \tau_{\xi_-}^-) \) is a bounded \( (\mathcal{F}_{t_\sigma}^r \vee \mathcal{F}_{t_\sigma}^l)' \)-martingale. Since \( M(t_\sigma \wedge \tau_{\xi_+}^+, t_\sigma \wedge \tau_{\xi_-}^-) \) as \( t_\sigma \to \infty \), by dominated convergence theorem, we get \( M(t_\sigma \wedge \tau_{\xi_+}^+, t_\sigma \wedge \tau_{\xi_-}^-) = \mathbb{E}[M(\tau_{\xi_+}^+, \tau_{\xi_-}^-)] \mathcal{F}_{t_\sigma}^r \vee \mathcal{F}_{t_\sigma}^l \) for \( t_\sigma \geq 0 \). This means that \( (t_\sigma \wedge \tau_{\xi_+}^+, t_\sigma \wedge \tau_{\xi_-}^-) \) is an \( (\mathcal{F}_{t_\sigma}^r \vee \mathcal{F}_{t_\sigma}^l)' \)-martingale closed by \( M(\tau_{\xi_+}^+, \tau_{\xi_-}^-) \). In particular, we have \( \mathbb{E}[M(\tau_{\xi_+}^+, \tau_{\xi_-}^-)] = M(0, 0) = 1 \). So we may define another probability measure \( \mathbb{P}_k^c \) by \( d\mathbb{P}_k^c = M(\tau_{\xi_+}^+, \tau_{\xi_-}^-) d\mathbb{P}_i \).

Suppose now \( (\eta_+, \eta_-) \) follows the law \( \mathbb{P}_k^c \) instead of \( \mathbb{P}_i \). We now describe the properties of \( (\eta_+, \eta_-) \). By the martingale property of \( M \), we have \( \mathbb{E}[M(\tau_{\xi_+}^+, \tau_{\xi_-}^-)] = M(0, 0) = 1 \), which implies that \( \mathbb{P}_k^c \) is also a coupling of \( \mathbb{P}_+ \) and \( \mathbb{P}_- \).
Fix $\sigma \neq \nu \in \{+,-\}$. Let $\tau_\nu$ be an $F_\nu^\nu$-stopping time with $\tau_\nu \leq \tau_\nu^\nu$. By the martingale property of $M$, we see that for any $t_\sigma \geq 0$,

$$d(P_{(\xi_+^\nu, \xi_-^\nu)}^\nu|F_{t_\sigma \wedge \tau_\nu^\nu} \cap F_{\tau_\nu^\nu}) = M|\tau_\nu^\nu(t_\sigma \wedge \tau_\nu^\nu) \eta^\nu_{\tau_\nu^\nu} \wedge \nu^\nu_{\tau_\nu^\nu} \vee F_{\tau_\nu^\nu}.$$  

By (4.42, 4.46) and Girsanov Theorem, there is an $(F_{t_\sigma \wedge \tau_\nu^\nu} \cap F_{\tau_\nu^\nu})_{t_\sigma \geq 0}$-Brownian motion $\tilde{B}_{t_\sigma}^\nu$ under $P_{(\xi_+^\nu, \xi_-^\nu)}^\nu$ such that $\tilde{w}_\sigma$ satisfies the following SDE up to $\tau_\nu^\nu$:

$$d\tilde{w}_\sigma = \sqrt{k}d\tilde{B}_{t_\sigma}^\nu + \left(\nu \cdot 2\kappa b X_{\sigma, \nu}^A|_{\tau_\nu^\nu} + \kappa \sum_j \left(X_{\sigma, j}^A|_{\tau_\nu^\nu} - X_{\sigma, \infty}^A|_{\tau_\nu^\nu}\right)|p_j + \kappa G_j(R_{\nu}^\nu)\right)dt_\sigma.$$

By (4.29), $W_\sigma$ satisfies the following SDE up to $\tau_\nu^\nu$ (with the variable $t_\nu$ fixed being $\tau_\nu$):

$$dW_\sigma = A_{\sigma, 1}^{2, 1}|_{\tau_\nu^\nu} + \left(\frac{(\kappa - 6)}{W_\nu - W_\nu} - A_{\sigma, 1}^{2, 1}\right)|p_j + \kappa G_j(R_{\nu}^\nu)dt_\sigma.$$

Note that $\eta_\nu, \tau_\nu$ and $(K_{\sigma, \tau_\nu}(\cdot))$ are chordal Loewner curve and chordal Loewner hulls, respectively, driven by $W_\nu|_{\tau_\nu^\nu}$ with speed $A_{\sigma, 1}^{2, 1}|_{\tau_\nu^\nu}$, the Brownian motion $\tilde{B}_{t_\sigma}^\nu$ is independent of $F_{\tau_\nu^\nu}$, and the processes $W_\nu|_{\tau_\nu^\nu}$ and $V_j|_{\tau_\nu^\nu}$ are force point processes for this family of Loewner hulls started from $\tilde{w}_\nu(\tau_\nu)$ and $\hat{v}_j(\tau_\nu)$. By Proposition 3.8 we see that, conditionally on $F_{\tau_\nu^\nu}$, after a reparametrization by half-plane capacity, the law of the part of $\eta_{\sigma, \tau_\nu}$ up to the time that it hits $g_{K_{\nu_1}(\tau_\nu)}(\nu_1)$ agrees with that of an iSLE (if $\sigma = +$) or iSLE (if $\sigma = -$) curve in $\mathbb{H}$ under chordal coordinate from $g_{K_{\nu_1}(\tau_\nu)}(w_\sigma)$ to $\tilde{w}_\nu(\tau_\nu)$ with force points $\hat{v}_j(\tau_\nu)$, $j \in \mathbb{N}_\infty$, up to the same hitting time. Applying the conformal map $g_{K_{\nu_1}(\tau_\nu)}^{-1}$, we then conclude that $\eta_\nu$ satisfies Theorem 4.1 (i) (if $\sigma = +$) or Theorem 4.1 (ii) (if $\sigma = -$) up to $\tau_\nu^\nu$ with the additional assumption that $\tau_\nu \leq \tau_\nu^\nu$. So $P_{(\xi_+^\nu, \xi_-^\nu)}^\nu$ is a local commutation coupling of $P_+$ and $P_-$ within $\xi$.

**Lemma 4.7.** Let $\eta_+$ and $\eta_-$ be two random Loewner curves started from $w_+$ and $w_-$, respectively. Let $(\xi_+, \xi_-) \in \Xi$. Let $\sigma \neq \nu \in \{+,-\}$. Suppose that the law of $\eta_\nu$ restricted to $F_{\tau_\nu^\nu}^\nu$ agrees with $P_{(\xi_+, \xi_-)}^\nu$, and conditionally on $F_{\tau_\nu^\nu}^\nu$, $\eta_\sigma$ satisfies Theorem 4.1 (i) up to $\tau_\nu^\nu$ if $\sigma = +$, or Theorem 4.1 (ii) up to $\tau_\nu^\nu$ if $\sigma = -$. Then the law of $\eta_\nu$ restricted to $F_{\tau_\nu^\nu}^\nu$ agrees with $P_\sigma$.

**Proof.** We know the law of $\eta_\nu|_{[0, \tau_\nu^\nu]}$ and the conditional law of $\eta_\sigma|_{[0, \tau_\nu^\nu]}$ given $\eta_\nu|_{[0, \tau_\nu^\nu]}$, which together determine the joint law of $\eta_\sigma|_{[0, \tau_\nu^\nu]}$ and $\eta_\nu|_{[0, \tau_\nu^\nu]}$. So the joint law of $\eta_+$ and $\eta_-$ restricted to $F_{\tau_\nu^\nu}^+ \vee F_{\tau_\nu^\nu}^-$ is also determined, which has to agree with the local commutation coupling $P_{(\xi_+, \xi_-)}^\nu$. So the law of $\eta_\sigma$ restricted to $F_{\tau_\nu^\nu}^\nu$ agrees with $P_\sigma$. \(\square\)

We now use the local commutation couplings to construct a global commutation coupling, and finish the proof of Theorem 4.1. First, we observe that, for any $(\xi_+, \xi_-) \in \Xi$, if any coupling $P$ of $P_+$ and $P_-$ agrees with $P_{(\xi_+, \xi_-)}^\nu$ on $F_{\tau_\nu^\nu}^+ \vee F_{\tau_\nu^\nu}^-$, then $P$ is also a local commutation coupling of $P_+$ and $P_-$ within $(\xi_+, \xi_-)$.  

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Let $ξ^k = (ξ^k_+, ξ^k_-) ∈ Ξ$, $1 ≤ k ≤ n$. By [23] Theorem 6.1, there is a bounded positive continuous $(F^+_i, F^-_j)_{(t_i,t_j) ∈ R^2}$ martingale $M^n(t_+, t_-)$, $t_+, t_- ≥ 0$, such that $M^n(t_0) = M^n(0, t) = 1$ for any $t ≥ 0$, and for any $1 ≤ k ≤ n$, $M^n$ agrees with $M$ on $[0, τ^+_k] × [0, τ^-_k]$. Moreover, $M^n$ takes random constant value, denoted by $M^n(∞)$; if $t_+ ≥ τ^+_k$ and $t_- ≥ τ^-_k$, for all $1 ≤ k ≤ n$. So we have $E^x[M^n(∞)] = M^n(0, 0) = 1$, and may define another probability measure $P^n$ by $dP^n = M^n(∞) dP^n$. By the martingale property of $M^n$, $P^n$ is also a coupling of $P_+$ and $P_-$, and for any $1 ≤ k ≤ n$,

$$d(P^n | F^+_i, F^-_j, \tau^+_k, \tau^-_k) / d(P^n | F^+, F^-, \tau^+_k, \tau^-_k) = M^n(\tau^+_k, \tau^-_k) = M(\tau^+_k, \tau^-_k).$$

Thus, $P^n$ agrees with $P^n_c$ on $F^+_k, \tau^+_k, F^-_k, \tau^-_k$ for $1 ≤ k ≤ n$, which implies that $P^n$ is a local commutation coupling of $P_+$ and $P_-$ within $ξ^k$ for any $1 ≤ k ≤ n$.

We may pick a countable subset $Ξ^*$ of $Ξ$ such that for every $(ξ^+, ξ^-) ∈ Ξ$, there is $(ξ^*_+, ξ^*_-) ∈ Ξ^*$ such that for $σ ∈ \{+, -, 0\}$, Hull$(ξ_σ) ⊂$ Hull$(ξ^*_σ)$, which then implies that $τ^*_σ ≤ τ^κ_σ$. Enumerate $Ξ^*$ by $ξ^k : k ∈ N$. By the previous paragraph, for each $n ∈ N$, there is a coupling $P^n$ of $P_+$ and $P_-$, which is a local commutation coupling of $P_+$ and $P_-$ within $ξ^k$ for any $1 ≤ k ≤ n$. We let $P^∞$ be a subsequential weak limit of the sequence $P^n$ in some suitable topology. Then $P^∞$ is still a coupling of $P_+$ and $P_-$, and for every $k ∈ N$, it is a local commutation coupling of $P_+$ and $P_-$ within $ξ^k$. Finally, if $(ξ^+, ξ^-)$ follows the law $P^∞$, then Theorem 4.1 (i) and (ii) both hold.

This is because for $σ ≠ ν ∈ \{+, -, 0\}$ and $τ^∗ < T_ν$, $T_ν^D(τ^∗) = sup{τ^∗_σ : (ξ^+, ξ^-) ∈ Ξ^*, τ^∗ < τ^κ_σ}$.  

5 Proofs of the Main Theorems

In this section, we will prove Theorem 1.2 which contains Theorem 1.1 as a special case. We work on the cases $κ ∈ (0, 4]$ and $κ ∈ (4, 8)$ separately. Let $N_m = \{1, \ldots, m\}$, $N^∞_m = N_m ∪ \{∞\}$, $ρ = (ρ_1, \ldots, ρ_m)$, $ρ^*_j = -ρ_j$, $j ∈ N^∞_m$, $ρ^∗_j = (ρ^*_1, \ldots, ρ^*_m)$, $ν = (v_1, \ldots, v_m, v_∞)$, $v^*_j = J(v_j)$, $j ∈ N^∞_m$, and $v^∗_j = (v^*_1, \ldots, v^*_m, v^*_∞)$. By symmetry, we assume that $σ = -$.

We only need to show that the time-reversal of $J(η)$ has the same law as $η^T$ after a time-change because the absolute continuity statement then follows from Lemma 3.6 (iii) and the fact that $η$ a.s. does not visit any force point other than $0^±$ and $±∞$.

5.1 The simple curve case

Proof of Theorem 1.2 in the case $κ ≤ 4$. Since $κ ≤ 4$ and $σ = -$, $η$ a.s. does not intersect $(0, ∞)$. Let $f(z) = 1/(1 - z)$. Let $u_j = f(v_j)$, $j ∈ N^∞_m$, and $u = (u_1, \ldots, u_m, u_∞)$. We use the convention that $f(0^-) = 1^−$ and $f(−∞) = 0^+$. Then $u_1 > \cdots > u_m > u_∞ ∈ (0, 1) ∪ \{0^+, 1^-\}$, and $f(η)$ does not intersect $−∞, 0$. So $f(η)$ does not separate from $∞$ before it ends. Thus, we may reparametrize the complete $f(η)$ by half-plane capacity to get an isLE$_κ(ρ)$ curve in $H$ under chordal coordinate from $f(0) = 1$ to $f(∞) = 0$ with force points $u$. 

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Similarly, \( \eta^* \) a.s. does not intersect \((-\infty, 0)\). Let \( f^*(z) = f \circ J(z) = z/(z + 1) \). Then \( u_j = f^*(v_j) \), \( j \in \mathbb{N}_m^\infty \), and \( f^*(\eta^*) \) does not intersect \((1, \infty)\). So \( f^*(\eta^*) \) does not separate 1 from \( \infty \) before it ends. Thus, we may reparametrize the complete \( f^*(\eta^*) \) by half-plane capacity to get an iSLE\(_\kappa^\infty(\rho)\) curve in \( \mathbb{H} \) under chordal coordinate from \( f^*(0) = 0 \) to \( f^*(\infty) = 1 \) with force points \( u_j \). Thus, \( f(\eta) \) and \( f^*(\eta^*) \) have the same laws as the \( \eta_+ \) and \( \eta_- \) in Theorem 4.1 respectively, with \( u \) in place of \( \eta^* \). It now suffices to show that the \( \eta_+ \) and \( \eta_- \) in Theorem 4.1 are time-reversal of each other in the case \( \kappa \leq 4 \).

Suppose \( \tau_- < T_- \) is an \( F^-\)-stopping time. Then given \( F_{\tau_-}^- \), up to a time-change, the part of \( \eta_+ \) up to \( T^+_{\tau_-}(\tau_-) \), which is the first time that it intersects \( \overline{K_-(\tau_-)} \) or separates \( K_-(\tau_-) \) from \( \infty \), is an iSLE\(_\kappa(\rho)\) curve in \( \mathbb{H} \setminus \overline{K_-(\tau_-)} \) from \( w_+ \) to \( \eta_-(\tau_-) \) with force points \( v_j \vee \max\{K_-(\tau_-) \cap \mathbb{R}\} \), \( j \in \mathbb{N}_m^\infty \), also up to \( T^+_{\tau_-}(\tau_-) \). Since \( \eta_+ \) a.s. ends at \( w_- \), if the iSLE\(_\kappa(\rho)\) curve \( \eta_+|_{[0,T^+_{\tau_-}(\tau_-)]} \) in \( \mathbb{H} \setminus K_-(\tau_-) \) does not land at its target \( \eta_-(\tau_-) \), then \( \eta_+(T^+_{\tau_-}(\tau_-)) \) belongs to one of the following boundary arcs of \( \mathbb{H} \setminus K_-(\tau_-) \): (i) \( P_R \), the part of \( \partial K_-(\tau_-) \) on the right of \( \eta_-(\tau_-) \), (ii) \( P_L \), the part of \( \partial K_-(\tau_-) \) on the left of \( \eta_-(\tau_-) \), and (iii) \( P_\mathbb{R} \), the real interval \(( -\infty, \min\{K_-(\tau_-) \cap \mathbb{R}\}] \). The \( g_{K_-(\tau_-)} \)-image of \( \eta_+|_{[0,T^+_{\tau_-}(\tau_-)]} \) is an iSLE\(_\kappa(\rho)\) curve in \( \mathbb{H} \) from \( W_+(0,\tau_-) \) to \( W_-(0,\tau_-) \) with force points \( V_j(0,\tau_-) \), \( j \in \mathbb{N}_m^\infty \), up to some time. Since \( W_- \leq V_\infty \leq \cdots \leq V_1 \leq W_+ \), this curve a.s. does not visit the intervals \( (W_-(0,\tau_-),V_\infty(0,\tau_-)] \) and \(( -\infty, W_-(0,\tau_-)] \), which are respectively the \( g_{K_-(\tau_-)} \)-images of \( P_R \) and \( P_L \cup P_\mathbb{R} \). So \( \eta_+ \) a.s. does not visit \( P_R \cup P_L \cup P_\mathbb{R} \) at \( T^+_{\tau_-}(\tau_-) \), which implies that \( \eta_+ \) a.s. visits \( \eta_-(\tau_-) \) at \( T^+_{\tau_-}(\tau_-) \).

Consider countably many \( F^-\)-stopping times: \( q \wedge \tau^+_{\xi^-} \), where \( q \in \mathbb{Q}_+ \) and \( \xi^- \in \Xi^* \), which is the projection of \( \Xi^* \) to \( \Xi_- \). Then a.s. \( \eta_+ \) visits \( \eta_-(q \wedge \tau^+_{\xi^-}) \) for every \( q \in \mathbb{Q}_+ \) and \( \xi^- \in \Xi^*_- \). By the denseness of \( \mathbb{Q}_+ \) in \( \mathbb{R}_+ \) and the continuity of \( \eta_+ \) and \( \eta_- \), we know that a.s. \( \eta_-([0,\tau^+_{\xi^-}]) \subset \eta_+([0,T_+]) \) for every \( \xi^- \in \Xi^*_- \), which further implies that a.s. \( \eta_-([0,T_-]) \subset \eta_+([0,T_+]) \). Since \( \eta_+ \) a.s. does not visit \( (w_+,\infty) \), \( \eta_- \) does not either. So \( \eta_- \) is a time-change of a complete iSLE\(_\kappa(\rho)\) curve. Since \( \eta_+ \) a.s. does not visit any of its force points other than \( w^+_\pm \) or \( w^-_\pm \), \( \eta_- \) has the same property. By Lemma 3.6 (iii), the law of \( \eta_- \) is absolutely continuous w.r.t. that of a chordal iSLE\(_\kappa(\rho^*,\rho^I_\kappa)\) curve in \( \mathbb{H} \) under chordal coordinate from \( w_- \) to \( w_+ \) with force points \( v \). Thus, \( \eta_- \) a.s. ends at \( w_+ \), and we get \( \eta_-([0,T_-]) = \eta_+([0,T_+]) \). From this we then conclude that \( \eta_- \) is a time-reversal of \( \eta_+ \).

\[ \square \]

5.2 The non-simple curve case

The argument in the previous subsection does not work for \( \kappa \in (4,8) \) because for a commuting pair of nonsimple curves, if we condition on a part of one curve, the first point that the second curve will hit the given part of the first curve may not be the tip point.

**Proof of Theorem 1.2 in the case that \( \kappa \in (4,8) \).** We now have \( \sum_{j=1}^k \rho_j \geq \frac{\kappa}{2} - 2 \) for any \( 1 \leq k \leq m \). Let \( P_2 \) and \( P_2^* \) denote the laws of \( \eta \) and \( \eta^* \), respectively, in the theorem. Let \( R \) denote the space of chordal Loewner curves \( \gamma \) from 0 to \( \infty \), such that the time-reversal of \( J(\gamma) \) could be parametrized to be a chordal Loewner curve, which will be denoted by \( J_*(\gamma) \). We also use
$J_\ast$ to denote the pushforward map induced by $J_*$. Our goal is to show that $\mathbb{P}_2$ is supported by $\mathcal{R}$, and $J_\ast(\mathbb{P}_2) = \mathbb{P}_2^\ast$.

We first consider the case that all force points take values in $(-\infty,0)$, i.e., there are no degenerate force points. Let $\mathbb{P}_0$ denote the law of a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to $\infty$. By reversibility of chordal SLE$_\kappa$ for $\kappa \in (4,8)$ (cf. [12]), $\mathbb{P}_0$ is supported by $\mathcal{R}$, and $J_\ast(\mathbb{P}_0) = \mathbb{P}_0$. We will use an idea in [12], which is to show that the both $\mathbb{P}_2$ and $\mathbb{P}_2^\ast$ are absolutely continuous w.r.t. $\mathbb{P}_0$, and the Radon-Nikodym derivatives are related by the map $J_\ast$.

Let $\mathbb{P}_1$ denote the law of the chordal SLE$_{\kappa}(\rho, \rho_\infty)$ curve in $\mathbb{H}$ from 0 to $\infty$ with force points $\gamma$. By Proposition 2.11, $\mathbb{P}_1 \ll \mathbb{P}_0$, and $d\mathbb{P}_1/d\mathbb{P}_0$ is given by (2.6). By the definition of iSLE$_{\kappa}(\rho)$ curve, $\mathbb{P}_2 \ll \mathbb{P}_1$, and $d\mathbb{P}_2/d\mathbb{P}_1 = M(T_\infty)$, which is given by (3.11). Thus, $\mathbb{P}_2 \ll \mathbb{P}_0$, and so $\mathbb{P}_2$ is supported by $\mathcal{R}$. Let $E_0$ be the set of $\gamma \in \mathcal{R}$ such that $\gamma \cap [v_\infty, v_1] = \emptyset$. For $\gamma \in E_0$, let $D_\infty(\gamma)$ be the connected component of $\mathbb{H} \setminus \gamma$, whose boundary contains $[v_\infty, v_1]$. Let $\tilde{\rho}_j = \rho_j/\kappa$, $R_j(0) = v_j/v_\infty$, $1 \leq j \leq m$, and $R(0) = (R_1(0), \ldots, R_m(0))$. Combining (2.6) with (3.11), we get

$$
\frac{d\mathbb{P}_2}{d\mathbb{P}_0} = \frac{1}{F(R(0))} \prod_{j=1}^{m} R_j(0)^{-\frac{\rho_j}{\kappa}} \left( \frac{H_{D_\infty}(v_j, v_\infty)}{|v_j - v_\infty|^2} \right)^{-\frac{\rho_j(\mu_\kappa + \kappa - 4)}{4\kappa}} \prod_{1 \leq j < k \leq m} \left( \frac{H_{D_\infty}(v_j, v_k)}{|v_j - v_k|^2} \right)^{-\frac{\rho_j \rho_k}{4\kappa}}. \tag{5.1}
$$

Let $\mathbb{P}_1^\ast$ denote the law of the chordal SLE$_{\kappa}(\rho^\ast, \rho_\infty)$ curve in $\mathbb{H}$ from 0 to $\infty$ with force points $\gamma^\ast$. Since $\rho_\infty = \sum_{j=1}^{m} \rho_j \geq \frac{\kappa}{2} - 2$, and for any $2 \leq k \leq m$, $\rho^\ast_{k} + \sum_{j=k}^{m} \rho^\ast_{j} = \sum_{j=1}^{k-1} \rho_j \geq \frac{\kappa}{2} - 2$, by Proposition 2.11, $\mathbb{P}_1^\ast \ll \mathbb{P}_0$. Let $E_0^\ast$ be the set of $\gamma \in \mathcal{R}$, which do not intersect $[v_\infty^\ast, v_1^\ast]$. For $\gamma \in E_0^\ast$, let $D_\infty^\ast(\gamma)$ be the connected component of $\mathbb{H} \setminus \gamma$, whose boundary contains $[v_\infty^\ast, v_1^\ast]$. Let $g_\ast$ be a conformal map from $D_\infty^\ast$ onto $\mathbb{H}$ such that max$(\partial D_\infty^\ast \cap \mathbb{R})$ is mapped to $\infty$. By (2.9),

$$
\frac{d\mathbb{P}_1^\ast}{d\mathbb{P}_0} = \frac{1}{F_\ast} \prod_{j \in \mathbb{N}_\infty} \left( \frac{g_\ast(v_j^\ast) - g_\ast(v_k^\ast)}{|v_j^\ast - v_k^\ast|} \right)^{\frac{\rho^\ast_j}{4\kappa}} \prod_{j < k \in \mathbb{N}_\infty} \left( \frac{g_\ast(v_j^\ast) - g_\ast(v_k^\ast)}{|v_j^\ast - v_k^\ast|} \right)^{\frac{\rho^\ast_j \rho^\ast_k}{4\kappa}}, \tag{5.2}
$$

We now express $d\mathbb{P}_2^\ast/d\mathbb{P}_0$ in terms of boundary Poisson kernel and conformal radius, but in a way different from (2.6). First, we have $H_{D_\infty^\ast}(v_j^\ast, v_k^\ast) = \frac{g_\ast(v_j^\ast)}{|g_\ast(v_j^\ast) - g_\ast(v_k^\ast)|^2}$. Just as $\gamma_0^\ast$ is supported by the union of $\partial D_\infty^\ast \cap \mathbb{R}$, the boundary intersects $[v_\infty^\ast, v_1^\ast]$. So we have $\text{crad}^{(4)}(\Omega_\infty^\ast) = \frac{\rho^\ast_j}{4\kappa} - \frac{\rho^\ast_j}{4\kappa} + 2 \frac{\rho^\ast_j}{4\kappa}$. By (5.2) and that $\sum_{j \in \mathbb{N}_\infty} \rho^\ast_j = 0$,

$$
\frac{d\mathbb{P}_2^\ast}{d\mathbb{P}_0} = \frac{1}{\mathcal{Z}^\ast} \prod_{j=1}^{m} \left( \text{crad}^{(4)}(\Omega_\infty^\ast) - \frac{\rho^\ast_j}{4\kappa} H_{D_\infty^\ast}(v_j^\ast, v_\infty^\ast) \right) \prod_{1 \leq j < k \leq m} H_{D_\infty^\ast}(v_j^\ast, v_k^\ast)^{-\frac{\rho^\ast_j \rho^\ast_k}{4\kappa}},
$$

where $\mathcal{Z}^\ast > 0$ is a constant given by

$$
\mathcal{Z}^\ast := \prod_{j=1}^{m} \frac{v_j^\ast}{v_\infty^\ast} \prod_{j=1}^{m} \frac{v_j^\ast - v_\infty^\ast}{|v_j^\ast - v_\infty^\ast|} \prod_{1 \leq j < k \leq m} \frac{|v_j^\ast - v_k^\ast|}{|v_j^\ast - v_k^\ast|^2}.
$$
Recall the definition of $M^r$ in (3.19) and the formula for $I_j^r$ in (3.16). On the event $E_0^r$, for $j \in \mathbb{N}_m$, as $t \uparrow T_\infty = T_j$, $R_j^r(t) \to 1$ and $\text{crad}_{v_j^r}^{(4)}(\Omega_j^r) \to \text{crad}_{v_j^r}^{(4)}(\Omega_\infty^r)$ because $\Omega_j^r$ tends to $\Omega_T = \Omega_\infty^r$ in the Carathéodory topology. Since $\rho_j = -\rho_j^r$, we get

$$M^r(\infty) = M^r(T_j^r) = \frac{F(1)}{F(R_j^r(0))} \prod_{j=1}^m \left( \frac{\text{crad}_{v_j^r}^{(4)}(\Omega_\infty^r)}{|v_j^r - v_k^r|} - \frac{\rho_j^{r}(\infty)}{2\pi}, \right),$$
onumber

on $E_0^r$. Since $R_j^r(0) = v_{\infty}^r/v_j^r$, we get

$$\frac{dP_j^r}{dP_0}M^r(\infty) = \frac{1}{E_j^r}F(\frac{1}{F(R_j^r(0))}) \prod_{j=1}^m R_j^r(0) \frac{H_{D^{\infty}}(v_j^r, v_{\infty}^r)}{|v_j^r - v_\infty^r|^{-2}} \prod_{1 \leq j < k \leq m} \left( \frac{H_{D^{\infty}}(v_j^r, v_k^r)}{|v_j^r - v_k^r|^{-2}} - \frac{\rho_j^{r}(\infty)}{4\pi} \right).$$

(5.3)

We compare (5.1) with (5.3). Note that $E_0^r = J_\infty(E_0)$, $J(D^{\infty}(\gamma)) = D^{\infty}(J_\infty(\gamma))$ for $\gamma \in E_0$, $R^r(0) = R(0)$ and $\rho_j^r = -\rho_j$. By conformal covariance of boundary Poisson kernel, we get

$$H_{D^{\infty}}(J_\infty(\gamma))(v_j^r, v_k^r)/|v_j^r - v_k^r|^{-2} = H_{D^{\infty}}(v_j, v_k)/|v_j - v_k|^{-2}, \quad 1 \leq j < k \leq m + 1.$$ 

So we get $(dP_j^r/dP_0) \cdot M^r(\infty) = (dP_2/dP_0) \circ J_\infty^{-1}$. Since $J_\infty(P_0) = P_0$ and $P_2$ is a probability measure, we get $E_0[M^r(\infty)] = E_0[(dP_j^r/dP_0) \cdot M^r(\infty)] = E_0[dP_2/dP_0] = 1$. Since $M^r$ is a positive supermartingale w.r.t. $P_1$, and $M^r(0) = 1$, we then conclude that $M^r$ is a uniformly integrable martingale w.r.t. $P_1$. By Definition 3.4 and Lemma A.1, we have $P_2 \ll P_1$, and $dP_2/dP_1 = M^r(\infty)$. Thus, $P_2 \ll P_0$, and $dP_2/dP_0 = (dP_1^r/dP_0)^* \cdot M^r(\infty) = (dP_2/dP_0) \circ J_\infty^{-1}$. Since $J_\infty(P_0) = P_0$, we then conclude that $J_\infty(P_2) = P_2$. This finishes the proof of the case that none of the $v_j$'s takes values $0^\pm$ or $\pm \infty$.

Now suppose $v_1 = 0^-$ and all other $v_j$'s including $v_\infty$ lie in $(-\infty, 0)$. Let $0 < l < \min\{|v_j| : j \in \mathbb{N}_m, j > 1\}$. Let $\tau_1$ be the first time that $\eta$ reaches $\{z = l\}$. By DMP of iSLK($\rho$) curve, conditionally on $F_{\tau_1}$, $\eta(\tau_1 + \cdot)$ is an iSLK($\rho$) curve in $\mathbb{H} \setminus K_{\tau_1}$ from $\eta(\tau_1)$ to $\infty$ with force points $0^-, v_2, \ldots, v_m, v_\infty$. Note that $\eta$ does not visit $(-\infty, 0]$ during the time interval $(0, \tau_1]$. So $\eta([0, \tau_1])$ does not separate any of $v_2, \ldots, v_m, v_\infty$, from $\infty$. Thus, none of the force points for the iSLK($\rho$) curve in $\mathbb{H} \setminus K_{\tau_1}$ from $\eta(\tau_1)$ to $\infty$ is degenerate. By the reversibility result we have derived, the conditional law given $F_{\tau_1}$ of the time-reversal of $\eta$ up to the time of hitting $\eta(\tau_1)$ is that of an iSLK($\rho$) curve in $\mathbb{H} \setminus K_{\tau_1}$ from $\eta(\tau_1)$ to $\infty$ with force points $0^-, v_2, \ldots, v_m, v_\infty$. In particular, this implies that a.s. the time-reversal of $J(\eta)$ up to hitting the circle $\{z = 1/l\}$ can be parametrized to be a chordal Loewner curve. By letting $l \downarrow 0$, we see that the time-reversal of the complete $J(\eta)$ a.s. can be parametrized to be a chordal Loewner curve. So $\eta$ a.s. belongs to $\mathcal{R}$, and we may define $J_\infty(\eta)$. Given $F_{\tau_1}$, the part of $J_\infty(\eta)$ up to the time that it reaches $J(\eta(\tau_1))$ is then an iSLK($\rho$) curve in $J(\mathbb{H} \setminus K_{\tau_1})$ from 0 to $J(\eta(\tau_1))$ with force points $+\infty, v_2, \ldots, v_m, v_\infty$.

Fix $n \in \mathbb{N}$. Let $f_n(z) = 1/(1 - nz)$ be a M"obius automorphism of $\mathbb{H}$, which maps 0 and $\infty$ to 1 and 0, respectively. Let $u_j = f_n(v_j), j \in \mathbb{N}_m$, and $y = (u_1, \ldots, u_m, u_\infty)$. Then $u_j$'s lie on $(0, 1)$ except that $u_1 = 1^-$. We may reparametrize the part of $f_n(\eta)$ up to the time that it hits

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its target 0 or separates 0 from ∞ by half-plane capacity, and get a chordal Loewner curve η+, which is an iSLEκ(ρ) curve in H under chordal coordinate from 1 to 0 with force points u. Let f_κ^*(u) = f_κ ∘ J(z) = z/(z + n). We may reparametrize the part of f_κ^*(J_κ(η)) up to the time that it hits its target 1 or separates 1 from ∞, and get a chordal Loewner curve η−.

Let ξ^+_κ = f_κ((z ∈ H : |z| = l) ∈ Ξ+. By the relation between η and J_κ(η) derived earlier, we know that, for any ξ− ∈ Ξ− such that (ξ^+_κ, ξ−) ∈ Ξ+, given the part of η+ up to τ^+_κ, up to a time-change, the part of η− up to τ^−_κ is an iSLEκ(ρ) curve in H \ Hull(η+, ([0, τ^+_κ])) from 0 to η+(τ^+_κ) with force points 1−, u_2, ..., u_m, u_∞, also up to τ^−_κ. By Lemma 4.7, the part of η− up to τ^−_κ is an iSLEκ(ρ) curve in H from 0 to 1 with force points u_2, up to τ^−_κ. Since this holds for any ξ− ∈ Ξ− such that (ξ^+_κ, ξ−) ∈ Ξ, we then conclude that the part of η− up to hitting τ^−_κ is an iSLEκ(ρ) curve in H under chordal coordinate from 0 to 1 with force points u up to the same hitting time. By letting l ↓ 0 and using the definition of η−, we know that the part of f_κ^*(J_κ(η)) up to the time that it hits 1 or separates 1 from ∞ is an iSLEκ(ρ) curve in H from 0 to 1 with force points u up to the same time. Since f_κ^* maps −n, ∞, and v_j^− to ∞, 1, and u_j, respectively, the part of J_κ(η) up to the first time that it separates −n from ∞ is an iSLEκ(ρ) curve in H from 0 to ∞ with force points v^− up to the same time. Letting n → ∞, we conclude that the whole J_κ(η) is an iSLEκ(ρ) curve in H from 0 to ∞ with force points v^−. So J_κ(P_2) = P_2.

Finally, we consider the case v_∞ = −∞. It suffices to show that the law of J_κ(η^r) is P_2 in the case that v^∞ is degenerate, i.e., 0+. We have proved that this is true if v^∞ is not degenerate. Let l ∈ (0, v^∞) and let τ_l^r be the first time that η^r hits {z = l}. By Lemma 3.6 (iv), τ_l^r is strictly less than the lifetime of η^r. By DMP of iSLEκ(ρ), conditionally on F_lτ^r, η^r(τ_l^r + ·) is an iSLEκ(ρ) curve in H \ K_r(τ_l^r) from η^r(τ_l^r) to ∞ with force points v_l^−, ..., v_m^−, 0+. Since ρ^r ≥ n/2 − 2, none of the force points for the iSLEκ(ρ) curve in H \ K_r(τ_l^r) is degenerate.

By the reversibility result we have derived, the conditional law given F_lτ^r of the time-reversal of η^r up to the time of hitting η^r(τ_l^r) is that of an iSLEκ(ρ) curve in H \ K_r(τ_l^r) from ∞ to η^r(τ_l^r) with force points v_l^−, ..., v_m^−, 0+. In particular, this implies that a.s. η^r ∈ R, and we may define J_κ(η^r). Given F_lτ^r, the part of J_κ(η^r) up to the time that it reaches J(η^r(τ_l^r)) is then an iSLEκ(ρ) curve in J(H \ K_r(τ_l^r)) from 0 to J(η^r(τ_l^r)) with force points v_1, ..., v_m, −∞.

Fix n ∈ N. Let f_n^*(z) = n/z + 2, which maps 0 and ∞ to 0 and 1, respectively. Let u_j = f_n^*(v_j^−), j ∈ N^m, and u = (u_1, ..., u_m, u_∞). Then u_j’s lie on (0, 1) except that u_∞ = 0+. We may reparametrize the part of f_n^*(η^r) up to the time that it hits its target 1 or separates 1 from ∞ by half-plane capacity, and get a chordal Loewner curve η−, which is an iSLEκ(ρ) curve in H from 0 to 1 with force points u. Let f_n(z) = f_n^* ∘ J(z) = n/z. We may reparametrize the part of f_n(J_κ(η^r)) up to the time that it hits its target 0 or separates 0 from ∞ by half-plane capacity, and get a chordal Loewner curve η−.

Let ξ^l = f_n((z ∈ H : |z| = l) ∈ Ξ−. By the relation between η^r and J_κ(η^r) derived earlier, we know that, for any ξ− ∈ Ξ− such that (ξ^l, ξ−) ∈ Ξ, given the part of η− up to τ^−_l, up to a
time-change, the part of \( \eta_+ \) up to \( \tau^+_{\xi^+} \) is an iSLE\(_\infty(\rho)\) curve in \( \mathbb{H} \setminus \text{Hull}(\eta_-(\{0, \tau^\infty_{\xi^+}\})) \) from 1 to \( \eta_-(\tau^\infty_{\xi^+}) \) with force points \( u \). By Lemma 4.7, the part of \( \eta_+ \) up to \( \tau^+_{\xi^+} \) is an iSLE\(_\infty(\rho)\) curve in \( \mathbb{H} \) from 1 to 0 with force points \( u \) up to the same hitting time. Since this holds for any \( \xi^+ \in \Xi_+ \) such that \( (\xi^+, \xi^+) \in \Xi \), we then conclude that the part of \( \eta_+ \) up to hitting \( \xi^+ \) is an iSLE\(_\infty(\rho)\) curve in \( \mathbb{H} \) from 1 to 0 with force points \( u \) up to the same hitting time. Thus, \( (3.22) \). Note that \( v \) does not change after that time. Then the driving function \( \tilde{w}^r \) for \( \eta^r \) satisfies the SDE

\[
d\tilde{w}^r = \sqrt{\kappa}dB^r + \frac{\rho_1}{\tilde{w}^r - v^r_{\infty}} dt - \left( \frac{1}{\tilde{w}^r - v^r_{\infty}} - \frac{1}{\tilde{w}^r_{\infty} - \tilde{v}^r_{\infty}} \right) [\rho_2 + \kappa G_\tau(R^r)] dt, \tag{5.4}
\]

where \( B^r \) is a standard Brownian motion, \( G_\tau(x) = xF_\tau^\prime(x)/F_\tau(x) \), and \( F_\tau \) is the (single-variable) hypergeometric function \( F(1 - \frac{4}{\kappa}, \frac{2\rho_2}{\kappa}, \frac{2\rho_1 + 2\rho_2 + 4}{\kappa}; \cdot) \).

**Proof.** We apply Theorem 1.2 to the case that \( m = 2, v_1 = 0^\sigma, v_2 = v, v_{\infty} = \sigma_{\infty} \), and derive a statement about the law of \( \eta^r \). Then the driving function \( \tilde{w}^r \) of \( \eta^r \) solves the SDE (3.22). Note that \( v^r_{\infty} = J(v^r) = -\sigma_{\infty} \), and so \( R^r_1 \equiv 0 \). The \( R^r_2 \) in Theorem 1.2 agrees with the \( R^r \) here. Also note that for the function \( F \) in Theorem 1.2 by (2.13) we have \( F(0, \cdot) = F_\tau \). Thus, \( G_1(R^r_1, R^r_2) \equiv 0 \) and \( G_2(R^r_1, R^r_2) \equiv G_\tau(R^r \). Then (3.22) reduces to (5.4).

**Remark 5.2.** The \( \eta^r \) in the corollary is an SLE\(_\infty(\rho)\) process with two force points, and may be defined using a single-variable hypergeometric function. But it is different from the intermediate SLE\(_\infty(\rho)\) process in [21], which is also defined using a single-variable hypergeometric function.

## Appendices

### A  Laws of Stochastic Processes with Random Lifetime

This appendix can be viewed as a supplement of [19], Section 2, and we use the setup there as follows. Let \( S \) be a Polish space, and \( \Sigma = \bigcup_{T \in (0, \infty]} C([0, T), S) \). For each \( f \in \Sigma \), let \( T_\Sigma(f) \)
be such that \([0, T_\Sigma(f))\) is the domain of \(f\). Let \(\Sigma_t = \{f \in \Sigma : T_\Sigma(f) > t\}, 0 \leq t < \infty\), and \(\Sigma_\infty = \bigcap_{0 \leq t < \infty} \Sigma_t = C([0, \infty), S)\). For \(0 \leq t < \infty\), let \(\mathcal{F}_t\) be the \(\sigma\)-algebra generated by the family \(\{f \in \Sigma_s : f(s) \in U\}\) over all \(s \in [0, t]\) and \(U \in \mathcal{B}(S)\), and let \(\mathcal{F}_\infty = \vee_{0 \leq t < \infty} \mathcal{F}_t\). A probability measure on \((\Sigma, \mathcal{F}_\infty)\) is viewed as the law of a continuous \(S\)-valued stochastic process with random lifetime. For two probability measures \(\mu\) and \(\nu\) on \(\Sigma\), we say that \(\nu\) is locally absolutely continuous w.r.t. \(\mu\), and write \(\nu \ll \mu\), if for every \(t \geq 0\), \(\nu|_{\mathcal{F}_t \cap \Sigma_t} \ll \mu|_{\mathcal{F}_t \cap \Sigma_t}\), which means that for any \(A \in \mathcal{F}_t\) with \(A \subseteq \Sigma_t\), \(\mu(A) = 0\) implies that \(\nu(A) = 0\). Let \(M_t\) be the Radon-Nikodym derivative of \(\nu|_{\mathcal{F}_t \cap \Sigma_t}\) against \(\mu|_{\mathcal{F}_t \cap \Sigma_t}\). We call \((M_t)\) the density process. It is clear that \(\nu \ll \mu\) implies that \(\nu < \mu\).

Now suppose that \(\mu\) and \(\nu\) are probability measures on \(\Sigma\), \(\mu\) is supported by \(\Sigma_\infty\), and \(\nu \ll \mu\) with \((M_t)\) being the density process. Then each \(M_t\) is \(\mu\)-integrable, and for any \(t_2 > t_1 \geq 0\) and \(A \in \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}\),

\[
\int_A M_{t_2} d\mu = \int_{A \cap \Sigma_{t_2}} M_{t_2} d\mu = \nu(A \cap \Sigma_{t_2}) \leq \nu(A \cap \Sigma_{t_1}) = \int_{A \cap \Sigma_{t_1}} M_{t_1} d\mu = \int_A M_{t_1} d\mu.
\]

So \((M_t)\) is a nonnegative supermartingale w.r.t. \(\mu\). The following lemma provides us the existence of \(\nu\) given the measure \(\mu\) and the supermartingale \(M\).

**Lemma A.1.** Let \(\mu\) be a probability measure supported by \(\Sigma_\infty\). Let \((M_t)_{0 \leq t < \infty}\) be a nonnegative right-continuous \((\mathcal{F}_t)\)-supermartingale w.r.t. \(\mu\) such that \(\int M_0 d\mu = 1\). Then there exists a unique probability measure \(\nu\) on \(\Sigma\) such that \(\nu \ll \mu\), and \(M\) is the density process. Moreover, we have the following.

(i) The \(\nu\) is supported by \(\Sigma_\infty\) if and only if \(M\) is a martingale w.r.t. \(\mu\).

(ii) For any \((\mathcal{F}_t)\)-stopping time \(\tau\), \(\nu \ll \mu\) on \(\mathcal{F}_\tau \cap \{T_\Sigma > \tau\}\), and \(M_\tau\) is the Radon-Nikodym derivative.

(iii) For any \((\mathcal{F}_t)\)-stopping time \(\tau\), \(\nu \ll \mu\) on \(\mathcal{F}_\tau\) if and only if \(M(t \wedge \tau), t \geq 0\), is a uniformly integrable martingale (w.r.t. \(\mu\)); and then \(d\nu/\nu|_{\mathcal{F}_\tau} = M_\tau\). Here on the event \(\tau = \infty\), \(M_\tau\) is understood as \(M_\infty := \lim_{t \to \infty} M_{t \wedge \tau}\), which \(\mu\)-a.s. converges. In particular, \(\nu \ll \mu\) if and only if \(M\) is a uniformly integrable martingale; and then \(d\nu/\mu = M_\infty\).

We say that the measure \(\nu\) is constructed by locally weighting the measure \(\mu\) by \(M\).

**Proof.** Add an extra element, denoted by \(*\), to \(S\), and write \(S_*\) for the union \(S \cup \{\ast\}\). For \(\Lambda \subseteq [0, \infty]\), an element \(f \in S^\Lambda_*\) is called ordered if there do not exist \(\lambda_1 < \lambda_2 \in \Lambda\) such that \(f(\lambda_1) = \ast\) and \(f(\lambda_2) \in S\). For any finite set \(\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\} \subseteq [0, \infty)\), we define a measure \(\nu_\Lambda\) on \(S^\Lambda_*\) by the following. For \(0 \leq k \leq n\), let \(\pi_{\Lambda,k} : \Sigma_\infty \to S^\Lambda_*\) be defined by \(\pi_{\Lambda,k}(f) = (f(t_0), f(t_1), \ldots, f(t_k), \ast, \ldots, \ast)\). For any measurable subset \(A\) of \(S^\Lambda_*\), define

\[
\nu_\Lambda(A) = \sum_{k=0}^{n-1} \int_{\pi_{\Lambda,k}^{-1}(A)} (M_{t_k} - M_{t_{k+1}}) d\mu + \int_{\pi_{\Lambda,n}^{-1}(A)} M_{t_n} d\mu.
\]
Since $M$ is a nonnegative supermartingale and $\int M_t \, d\mu = 1$, $\nu_\Lambda$ is a probability measure. Since $\pi_{\Lambda,n}$ is the projection $\pi_\Lambda$ from $\Sigma_\infty$ onto $S^\Lambda$, and $\pi_{\Lambda,k}^{-1}(S^\Lambda) = \emptyset$ for $k < n$, we have $\nu_\Lambda|_{S^\Lambda} \ll \pi_\Lambda(\mu)$, and $M_{t_n}$ is the Radon-Nikodym derivative. We also see that $\nu_\Lambda$ is supported by the set of ordered elements of $S^\Lambda$ since every $\pi_{\Lambda,k}$ takes values in ordered elements.

We now check that $\{\nu_\Lambda : 0 \in \Lambda \subset [0, \infty), |\Lambda| < \infty\}$ is a consistent family. Let $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$. Suppose $\Lambda' = \Lambda \cup \{s\} \subset [0, \infty)$ and $s \notin \Lambda$. Let $A \subset S^\Lambda$ be measurable, and $A' = A \times S^\{s\} \subset S^{\Lambda'}$. We need to show that $\nu_{\Lambda'}(A') = \nu_\Lambda(A)$. First, suppose $s > t_n$. Then for each $0 \leq k \leq n$, $\pi_{\Lambda',k}(A') = \pi_{\Lambda,k}(A)$, and $\pi_{\Lambda',n+1}(A') = \pi_{\Lambda,n}(A)$. So

$$\nu_{\Lambda'}(A') = \sum_{k=0}^{n-1} \int_{\pi_{\Lambda,k}(A')} (M_{t_k} - M_{t_{k+1}}) \, d\mu + \int_{\pi_{\Lambda,k}(A')} M_s \, d\mu.$$

Next, suppose $t_{k_0 - 1} < s < t_{k_0}$ for some $1 \leq k_0 \leq n$. Then for any $k < k_0$, $\pi_{\Lambda',k}(A') = \pi_{\Lambda,k}(A)$, and for any $k \geq k_0$, $\pi_{\Lambda',k}(A') = \pi_{\Lambda,k-1}(A)$. Let $t'_k = t_k$ for $0 \leq k < k_0$, $t'_{k_0} = s$, and $t'_{k} = t_{k-1}$ for $k_0 < k \leq n + 1$. Then $\Lambda' = \{0 = t'_0 < t'_1 < \cdots < t'_{n+1}\}$. So

$$\nu_{\Lambda'}(A') = \sum_{k=0}^{n} \int_{\pi_{\Lambda,k}(A')} (M_{t_k} - M_{t_{k+1}}) \, d\mu + \int_{\pi_{\Lambda,k}(A')} M_s \, d\mu.$$

By Kolmogorov extension theorem, there is an $S^\Lambda$-valued process $(Z_t)_{0 \leq t < \infty}$ (defined on some probability space) such that for any finite set $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\} \subset [0, \infty)$, the joint distribution of $(Z_{t_0}, Z_{t_1}, \ldots, Z_{t_n})$ is $\nu_\Lambda$. We now restrict our attention to $(Z_p)_{p \in \mathbb{Q}^+}$. By the properties of $\nu_\Lambda$ we know that for any $p_1 < p_2 \in \mathbb{Q}$, if $Z_{p_1} = *$ then a.s. $Z_{p_2} = *$. Thus, by excluding an event with probability zero, we may assume that $(Z_p)_{p \in \mathbb{Q}^+}$ takes values in ordered elements. Let $T_\Sigma = \inf\{p \in \mathbb{Q}_+ : Z_p = \ast\}$. Then $T_\Sigma$ is a random number such that $Z_t \in S$ for $t \in [0, T_\Sigma) \cap \mathbb{Q}_+$ and $Z_t = *$ for $t \in (T_\Sigma, \infty) \cap \mathbb{Q}_+$.

Suppose $(Y_t)_{t \geq 0}$ is a continuous process with law $\mu$. Let $t_0 \in \mathbb{Q}_+$. By the property of $\nu_\Lambda$, the law of $(Z_p)_{p \in [0,t_0]\cap \mathbb{Q}_+}$ restricted to the event that $Z_{t_0} \in S$, is absolutely continuous w.r.t. that of $(Y_p)_{p \in [0,t_0]\cap \mathbb{Q}_+}$, and the Radon-Nikodym derivative is $M_{t_0}$. Since $Y$ is continuous on $[0, \infty)$, this implies that on the event that $Z_{t_0} \in S$, a.s. $(Z_p)_{p \in [0,t_0]\cap \mathbb{Q}_+}$ extends to a (random) continuous function $Z'(t_0)$ on $[0, t_0]$. By excluding an event with probability zero, we may assume that this is always true for every $t_0 \in \mathbb{Q}_+$. We may define a continuous function $Z'$ on $[0, T_\Sigma)$ such that
for any $p \in \mathbb{Q}_+$, on the event $\{T_{\Sigma} > p\}$, which is contained in $\{Z_p \in S\}$, we define $Z' = Z(p)$ on $[0, p]$. There is no contradiction in the definition because whenever $p_1 < p_2 \in \mathbb{Q}_+$, on the event $\{T_{\Sigma} > p_2\}$, which is contained in $\{T_{\Sigma} > p_1\}$, we have $Z(p_1) = Z(p_2)|_{[0,p_1]}$. Then $Z'$ is a continuous stochastic process with a random lifetime $T_{\Sigma}$.

Let $\nu$ be the law of $Z'$. We claim that $\nu$ is the measure that we need. Fix $t_\ast \geq 0$. We need to show that $\nu(A) = \int_A M_{t_\ast} \, d\mu$ for any $A \in \mathcal{F}_{t_\ast} \cap \Sigma_{t_\ast}$. For every finite set $\Lambda \subset [0, \infty)$, let $\pi_\Lambda$ denote the natural projection from $S^{[0,\infty]}_\ast$ onto $S^\Lambda_\ast$. We naturally embed $\Sigma$ into $S^\Lambda_\ast$ by understanding the value of $f(t)$ for $t \geq T_{\Sigma}(f)$ as $\ast$. So $\pi_\Lambda$ is also a mapping from $\Sigma$ into $S^\Lambda_\ast$. First, assume that there is $\Lambda \subset \mathbb{Q} \cap [0, t_\ast]$ with $0 \in \Lambda$ and $|\Lambda| < \infty$ such that $A = \pi^{-1}_\Lambda(A_\Lambda) \cap \Sigma_{t_\ast}$, for some $A_\Lambda \in \mathcal{B}(S^\Lambda)$. Let $(p_m)_{m \in \mathbb{N}}$ be a sequence in $\mathbb{Q} \cap (t_\ast, \infty)$ such that $p_m \downarrow t_\ast$. By the definition of $T_{\Sigma}$, we see that $Z' \in \Sigma_{t_\ast}$, i.e., $T_{\Sigma} > t_\ast$, if and only if there is some $m$ such that $Z_{p_m} \in S$. Also note that $Z'(t) = Z(t)$ for $t \in \Lambda$ because $\Lambda \subset \mathbb{Q}$. Thus,

$$\nu(A) = \mathbb{P}[Z' \in A] = \mathbb{P}[\bigcup_{m=1}^{\infty} (\pi_\Lambda(Z) \in A_\Lambda, Z(p_m) \in S)] \leq \lim_{m \to \infty} \mathbb{P}[\pi_{\Lambda \cup \{p_m\}}(Z) \in A_\Lambda \times S]$$

For each $m \in \mathbb{N}$, since the law of $\pi_{\Lambda \cup \{p_m\}}(Z)$ is $\nu_{\Lambda \cup \{p_m\}}$, whose restriction to $S^{\Lambda \cup \{p_m\}}$ is absolutely continuous w.r.t. $\pi_{\Lambda \cup \{p_m\}}(\mu)$ with Radon-Nikodym derivative $M_{p_m}$, we get

$$\mathbb{P}[\pi_{\Lambda \cup \{p_m\}}(Z) \in A_\Lambda \times S] = \nu_{\Lambda \cup \{p_m\}}(A_\Lambda \times S) = \int_{\pi^{-1}_{\Lambda \cup \{p_m\}}(A_\Lambda \times S)} M_{p_m} \, d\mu = \int_{\pi^{-1}_\Lambda(A_\Lambda)} M_{p_m} \, d\mu.$$

Here the last equality follows from that $\mu$ is supported by $\Sigma_{t_\ast}$. Since $p_m \downarrow t_\ast$, by right-continuity of $M$ and Fatou’s lemma,

$$\int_{\pi^{-1}_\Lambda(A_\Lambda)} M_{t_\ast} \, d\mu \leq \liminf_{m \to \infty} \int_{\pi^{-1}_\Lambda(A_\Lambda)} M_{p_m} \, d\mu.$$

On the other hand, since $M$ is an $(\mathcal{F}_t)$-supermartingale w.r.t. $\mu$, for all $m \in \mathbb{N}$,

$$\int_{\pi^{-1}_\Lambda(A_\Lambda)} M_{p_m} \, d\mu \leq \int_{\pi^{-1}_\Lambda(A_\Lambda)} M_{t_\ast} \, d\mu.$$

Combining the last four displayed formulas and the fact that $\mu$ is supported by $\Sigma_{t_\ast} \subset \Sigma_{t_\ast}$, we get $\nu(A) = \int_A M_{t_\ast} \, d\mu$. This holds for any $A$ in the $\pi$-family

$$\{\pi^{-1}_\Lambda(A_\Lambda) \cap \Sigma_{t_\ast} : A_\Lambda \in \mathcal{B}(S^\Lambda), \Lambda \subset [0, t_\ast] \cap \mathbb{Q}, |\Lambda| < \infty\},$$

which generates the $\sigma$-algebra $\mathcal{F}_{t_\ast} \cap \Sigma_{t_\ast}$ in $\Sigma_{t_\ast}$ thanks to the continuity of $f \in \Sigma_{t_\ast}$ on $[0, t_\ast]$. By Dynkin’s $\pi - \lambda$ theorem, we get $\nu(A) = \int_A M_{t_\ast} \, d\mu$ for any $A \in \mathcal{F}_{t_\ast} \cap \Sigma_{t_\ast}$. The uniqueness of such $\nu$ also follows from Dynkin’s $\pi - \lambda$ theorem.

(i) If $\nu$ is supported by $\Sigma_{t_\ast}$, then for any $t_2 \geq t_1 \geq 0$ and $A \in \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$, we get $\nu(A) = \nu(A \cap \Sigma_{t_2}) = \int_{A \cap \Sigma_{t_2}} M_{t_j} \, d\mu = \int_A M_{t_j} \, d\mu$, $j = 1, 2$. So $\mathbb{E}_\mu[M_{t_2} | \mathcal{F}_{t_1}] = M_{t_1}$, i.e., $M$ is a martingale.
On the other hand, if $M$ is a martingale, then for any $t \geq 0$, $\nu(\Sigma_t) = \int M_t d\mu = \int M_0 d\mu = 1$. So $\nu(\Sigma_\infty) = \lim_{t \to \infty} \nu(\Sigma_t) = 1$, i.e., $\nu$ is supported by $\Sigma_\infty$.

(ii) Let $\tau$ be an $(\mathcal{F}_t)$-stopping time. Since $M$ is right-continuous and adapted, it is progressive. So $M_\tau$ on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable. First assume that $\tau$ takes values in $\mathbb{Q}_+$. Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$. Then for any $t \in \mathbb{Q}_+$, $A \cap \{\tau = t\} \in \mathcal{F}_t \cap \{T_\Sigma > t\}$. So we have

$$\nu(A) = \sum_{t \in \mathbb{Q}_+} \nu(A \cap \{\tau = t\}) = \sum_{t \in \mathbb{Q}_+} \int_{A \cap \{\tau = t\}} M_t d\mu = \sum_{t \in \mathbb{Q}_+} \int_{A \cap \{\tau = t\}} M_\tau d\mu = \int_A M_\tau d\mu.$$

Next, we do not assume that $\tau$ takes values in $\mathbb{Q}_+$, but assume that there is a deterministic number $N \in \mathbb{N}$ such that $\tau < N$. For each $n \in \mathbb{N}$, define $\tau_n$ such that if $\tau \in [\frac{k-1}{2^n}, \frac{k}{2^n})$ for some $k \in \mathbb{N}$, then $\tau_n = \frac{k}{2^n}$. Then each $\tau_n$ is a bounded stopping time taking values in $\mathbb{Q}_+$, and $\tau_n \downarrow \tau$.

Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$, and $A_n = A \cap \{T_\Sigma > \tau_n\}$. Then $A = \bigcup_n A_n$, and $A_n \in \mathcal{F}_{\tau_n} \cap \{T_\Sigma > \tau_n\}$ for each $n$. So we have $\nu(A) = \lim_{n \to \infty} \nu(A_n) = \lim_{n \to \infty} \int_{A_n} M_{\tau_n} d\mu = \lim_{n \to \infty} \int_A M_{\tau_n} d\mu$, where the last "=" follows from that $\mu$ is supported by $\Sigma_\infty$. By right-continuity of $M$ and Fatou’s lemma, $\lim_{n \to \infty} \int_A M_{\tau_n} d\mu \geq \int_A M_\tau d\mu$. Applying Optional Stopping Theorem to the right-continuous supermartingale $M$ and the bounded stopping times $\tau \leq \tau_n$, we get $\int_A M_{\tau_n} d\mu \leq \int_A M_\tau d\mu$ for each $n$. So $\lim_{n \to \infty} \int_A M_{\tau_n} d\mu = \int_A M_\tau d\mu$, and we then get $\nu(A) = \int_A M_\tau d\mu$. Finally, we do not assume that $\tau$ is uniformly bounded. Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$. Then for any $N \in \mathbb{N}$, $\tau \wedge N$ is a uniformly bounded stopping time, and $A \cap \{\tau \leq N\} \in \mathcal{F}_{\tau \wedge N} \cap \{T_\Sigma > \tau \wedge N\}$. So $\nu(A \cap \{\tau \leq N\}) = \int_{A \cap \{\tau \leq N\}} M_{\tau \wedge N} d\mu = \int_{A \cap \{\tau \leq N\}} M_\tau d\mu$. By monotone convergence theorem, we get $\nu(A) = \lim_{N \to \infty} \nu(A \cap \{\tau \leq N\}) = \int_A M_\tau d\mu$, as desired.

(iii) First, suppose $\nu \ll \mu$ on $\mathcal{F}_\tau$ with $\zeta = d(\nu|\mathcal{F}_\tau)/d(\mu|\mathcal{F}_\tau)$. Then for any $t \geq 0$, $\nu \ll \mu$ on $\mathcal{F}_{\tau \wedge t}$ with $d(\nu|\mathcal{F}_{\tau \wedge t})/d(\mu|\mathcal{F}_{\tau \wedge t}) = \mathbb{E}_{\mu}[\zeta|\mathcal{F}_{\tau \wedge t}]$. By (ii), $d(\nu|\mathcal{F}_{\tau \wedge t})/d(\mu|\mathcal{F}_{\tau \wedge t}) = M(\tau \wedge t)$ on $\Sigma_{\tau \wedge t}$. Since $\mu$ is supported by $\Sigma_\infty \subset \Sigma_{\tau \wedge t}$, we get $M_{\tau \wedge t} = \mathbb{E}_{\mu}[\zeta|\mathcal{F}_{\tau \wedge t}]$ for all $t \geq 0$. Thus, $M_{\tau \wedge t}, t \geq 0$, is a uniformly integrable martingale.

Next, suppose that $M_{\tau \wedge t}$, $t \geq 0$, is a uniformly integrable martingale. Then $\lim_{t \to \infty} M_{\tau \wedge t}$ converges a.s. to $M_\tau$, and for any $t \geq 0$, $\mathbb{E}_\mu[M_{\tau \wedge t}] = M_{\tau \wedge t}$. Define a measure $\nu_\tau$ on $(\Sigma, \mathcal{F}_\tau)$ by $d\nu_\tau = M(\tau) d\mu$. Since $\mathbb{E}_\mu[M_0] = \mathbb{E}_\mu[0] = 1$, $\nu_\tau$ is a probability measure. For any $t \geq 0$, $\nu_\tau \ll \mu$ on $\mathcal{F}_{\tau \wedge t}$, and $d(\nu_\tau|\mathcal{F}_{\tau \wedge t})/d(\mu|\mathcal{F}_{\tau \wedge t}) = \mathbb{E}_\mu[M_{\tau \wedge t}|\mathcal{F}_{\tau \wedge t}] = M_{\tau \wedge t}$. On the other hand, by (ii) $d(\nu|\mathcal{F}_{\tau \wedge t})/d(\mu|\mathcal{F}_{\tau \wedge t}) = M_{\tau \wedge t}$ on $\Sigma_{\tau \wedge t}$. Since $\nu_\tau$ and $\nu$ are both probability measures, they must agree on $\mathcal{F}_{\tau \wedge t}$. Since $\bigcup_{t \geq 0} \mathcal{F}_{\tau \wedge t}$ is a $\pi$-family, by Dynkin’s $\pi - \lambda$ theorem, $\nu_\tau$ and $\nu$ agree on $\bigcup_{t \geq 0} \mathcal{F}_{\tau \wedge t} = \mathcal{F}_\tau$. By the definition of $\nu_\tau$, we get that $d(\nu|\mathcal{F}_\tau)d(\mu|\mathcal{F}_\tau) = M_\tau$.

The last statement of (iii) follows from the above equivalence by choosing $\tau = \infty$. \qed

References


[8] Jason Miller and Scott Sheffield. Imaginary geometry II: reversibility of SLE$_\kappa(\rho_1; \rho_2)$ for $\kappa \in (0,4)$. *Ann. Probab.*, 44(3):1647-722, 2016.


