Research Statement

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The Schramm-Loewner evolution (SLE), first introduced by Oded Schramm ([12]), is a one-parameter ($\kappa \in (0,\infty)$) family of random non-self-crossing curves, which has received a lot of attention over the last decade. The definition uses Loewner's differential equation with the driving function being the rescaled Brownian motion. It was shown that SLE describes the limits of a number of models from statistical physics so answering the question of conformal invariance for them.

My research focuses on studying the geometric properties of SLE curves, which will help people get better understanding of those lattice models that converge to SLE. My works can be roughly divided into the following categories.

1 SLE in Multiply Connected Domains

SLE were originally defined only in simply connected domains. In my Ph.D Thesis [14], I extended the definition of SLE to multiply connected domains. The idea is to first define SLE in each simply connected chart of the multiply connected domain using the known types of SLE (growing in the simply connected subdomain) so that the SLE in different charts are consistent, and then glue these SLE together to get a global SLE. In [15], I studied SLE in doubly connected domains called annulus SLE, which can be defined using annulus Loewner equations. I proved that annulus SLE_6 satisfies the locality property as regular SLE_6, and annulus SLE_2 is the scaling limit of loop-erased random walk in doubly connected domains. In [16], I studied some properties of annulus SLE, especially how the curve interacts with the boundary of the domain. Since there is no marked point in my definition, an annulus SLE curve in a doubly connected domain starts from a boundary point, and ends at a random point in the other boundary component. In [17], I followed the approach in [4] and proved that the loop-erased random walk in any finitely connected domain started from a boundary point converges to the SLE_2 in the domain as defined in [14]. In the subsequent paper [18], I proved the convergence of the loop-erased random walk in a finitely connected domain started from an interior point.

2 Reversibility and Duality

There were two major conjectures about SLE: reversibility and duality. The reversibility conjecture says that, for $\kappa \in (0,8]$, a chordal SLE_κ curve (which runs from one boundary point to another boundary point in a simply connected domain) that goes from a to b is the reversal of a chordal SLE_κ curve that goes from b to a in the same domain. The duality conjectures says that, for $\kappa \in (4,\infty)$, the outer boundary of an SLE_κ curve looks similar to an $\mathrm{SLE}_{16/\kappa}$ curve.

My work [19] solved the reversibility conjecture for $\kappa \in (0,4]$. The main idea is that, instead of studying the reversal of SLE directly, I constructed a coupling of two chordal SLE_κ curves: one from a to b, the other from b to a, such that these two SLE curves commute with each other, which means that if we condition on the past of one curve at a stopping time, the other curve becomes an SLE_κ curve in the remaining domain aiming at the tip of the first curve. Then it is easy to see that the two SLE curves in the commutation coupling overlap with each other, and the reversibility conjecture is proved. A stochastic coupling technique was introduced in [19] to construct the coupling of two SLE_κ curves.

Shortly after, the coupling technique was used in my papers [20] and [21] to prove the duality conjecture. The way that I proved the duality conjecture is to use the stochastic coupling technique to construct a commutation coupling of an SLE_κ curve ($\kappa > 4$) with an $\mathrm{SLE}_{16/\kappa}$ curve, such that in the coupling, the $\mathrm{SLE}_{16/\kappa}$ curve is the outer boundary of the SLE_κ curve.

Later in [22] I used the coupling technique to prove the reversibility of $\mathrm{SLE}(\kappa;\rho)$ curves, which are natural variants of regular SLE. In [23] I used the coupling technique to couple a planar Brownian motion with an SLE_2 curve, and showed that SLE_2 can be obtained by erasing loops on a planar Brownian motion.

After that, I worked on the reversibility of whole-plane SLE, which describes an interior-to-interior curve in the Riemann sphere $\widehat{\mathbb{C}}$. In my work [25], I proved that, for $\kappa \in (0,4]$, a whole-plane SLE_κ curve satisfies reversibility. The paper combines the above coupling technique and the annulus Loewner equation introduced earlier in [15], and also uses some new idea. The main idea is to couple two whole-plane SLE_κ curves growing towards each other so that they overlap. The annulus Loewner equation comes into play because after both curves grow for a while, the remaining domain becomes doubly connected. The driving function for the curve growing in the remaining domain using the annulus Loewner equation can not be a regular Brownian motion because we have an extra marked point that the curve has to end at. We need to find a suitable force function to control the influence of the target point on the driving function. The force function must satisfy certain PDE in order for the two opposite curves to commute with each other. The PDE was solved using a Feynman-Kac formula.

Later, I collaborated with Steffen Rohde to study backward SLE ([27]). A backward SLE is defined using the backward Leowner equation, which differs from the (forward) Loewner equation by a minus sign. A backward SLE process does not naturally generate a curve as a forward SLE does. For $\kappa \in (0,4]$, a backward chordal SLE_κ generates a welding of $\mathbb{R}$, which is a random automorphism ϕ of $\mathbb{R}$ that satisfies $\phi^{-1} = \phi$, $\phi(0) = 0$, and $\phi(\pm\infty) = \mp\infty$. The main result in [27] is that the backward chordal SLE_κ welding satisfies the following symmetry: Let ϕ be a backward SLE_κ welding and $h(x) = -1/x$. Then $h \circ \phi \circ h$ has the same law as ϕ. This work illustrates another important application of the stochastic coupling technique, which is used to construct a coupling of two backward SLE_κ processes such that the weldings $\phi^{(1)}$ and $\phi^{(2)}$ generated by the two processes satisfy that $\phi^{(1)} = h \circ \phi^{(2)} \circ h$.

Since the backward SLE welding is a quite different object from the forward SLE curve, we derived in [27] some fundamental results in Complex Analysis to study the backward SLE process and its welding. Those results are used in constructing the coupling of backward SLE processes, and are also interesting on their own. Furthermore, from the computation used in that paper, we made an interesting observation that a backward SLE_κ process may be viewed as a forward SLE with parameter $-\kappa$. This observation extends the range of the parameter κ for SLE from $(0,\infty)$ to $(-\infty,0)\cup(0,\infty)$.

3 Tips and Decomposition of SLE

The symmetry property of backward SLE welding was later found useful in studying forward SLE curves in my subsequent work [26], which combines the symmetry of backward SLE, the conformal removability of SLE_κ curves for $\kappa\in(0,4)$ ([3, 11]), and the forward/backward SLE symmetry. In that paper, I proved that, for $\kappa\in(0,4)$, a whole-plane $\mathrm{SLE}(\kappa;\kappa+2)$ curve (a natural variant of regular whole-plane SLE) stopped at a deterministic (capacity) time satisfies reversibility. This result has a different flavor from other reversibility results of SLE because it is concerned with a stopped SLE curve instead of a complete SLE curve. An immediate application of this reversibility result is that, for $\kappa\in(0,4)$, the tip of a chordal or radial SLE_κ curve at a deterministic capacity time looks similar to the initial part of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve, which agrees in law with the final part of a whole-plane $\mathrm{SLE}_\kappa(\kappa+2)$ curve by the reversibility of whole-plane $\mathrm{SLE}_\kappa(\rho)$ process ([10]). From the SLE coordinate changes ([13]), we see that, for $\kappa<4$, a chordal SLE_κ curve stopped at a deterministic capacity time looks like a chordal $\mathrm{SLE}_\kappa(-8)$ curve with the force point being the tip of the former curve.

In my later paper [29], I extended the above result from $\kappa\in(0,4)$ to all $\kappa>0$. That paper does not use symmetry of backward SLE or conformal removability of SLE, but uses only technique from stochastic processes. One main result of the paper states that, the following two methods generate the same measure on the space of curve-point pairs: (i) first sample a point z in a subdomain U of the half plane according to certain density function with close-form formula, and then sample a chordal $\mathrm{SLE}_\kappa(-8)$ curve that ends at z, and finally extend the curve by a chordal SLE_κ curve from z to ∞ in the remaining domain; (ii) first same a chordal SLE_κ curve γ from 0 to ∞, and then sample a point on γ according to the capacity time that γ spends in U. One immediate corollary is that we get the expectation of the capacity time that a chordal SLE_κ curve spends in a given subset U of the half plane, which is equal to the integral of the above density function on U.

Another major result of [29] involves two-sided radial SLE, Green's function and natural parametrization for SLE. Let $\kappa\in(0,8)$. A two-sided radial SLE_κ curve in the half plane from 0 to ∞ that passes through a given point z_0 has two arms: the first arm is a chordal $\mathrm{SLE}_\kappa(\kappa-8)$ curve that grows from 0 to z_0, and the second arm is a regular chordal SLE_κ curve from z_0 to ∞ in the remaining domain given the first arm. The Green's function for a chordal SLE_κ curve γ is $G(z)=\lim_{r\downarrow 0} r^{d-2}\mathbb{P}[\mathrm{dist}(z,\gamma)<r]$, where $d=1+\frac{\kappa}{8}$ is the Hausdorff dimension of SLE_κ ([1]). The natural parametrization for SLE_κ was introduced in [6, 7] as candidates of the limit

of discrete lattice paths with there natural length, and was proved later in [5] to agree with the d-dimensional Minkowski content of the SLE curve. It is stated in [29] that the following two methods generate the same measure on the space of curve-point pairs: (i) first sample a point z in a subdomain U of the half plane according to the Green's function for SLE_κ, and then sample a two-sided radial SLE_κ curve γ from 0 to ∞ that passes through z; (ii) first same a chordal SLE_κ curve γ, and then sample a point on $\gamma \cap U$ according to the d-dimensional Minkowski measure on γ restricted to U. Thus, if one samples a point on a chordal SLE_κ curve according to its d-dimensional Minkowski content measure, then he sees a two-sided radial SLE_κ near that point. The result extends Field's recent work [2] from $\kappa \in (0,4]$ to $\kappa \in (0,8)$.

4 Multi-point Green's Functions for SLE

In a series of papers [28, 31], I collaborated with my postdoc Mohammad A. Rezaei to study the multi-point Green's functions. Let $\kappa \in (0,8)$, γ be a chordal SLE_κ curve in a domain D, and $z_1, \dots, z_n$ be distinct points in D. The Green's function for γ valued at $(z_1, \dots, z_n)$ is defined to be

$$G(z_1, \dots, z_n) = \lim_{r_1, \dots, r_n \downarrow 0} \prod_{k=1}^{n} r_k^{d-2} \mathbb{P}[\text{dist}(z_k, \gamma) < r_k, 1 \le k \le n], \tag{1}$$

provided that the limit exists. The limit was previously only known to converge for $n = 1, 2$.

In [28], we obtained an upper bound of $\mathbb{P}[\text{dist}(z_k, \gamma) < r_k, 1 \le k \le n]$ in terms of a function that can be easily handled. Using that upper bound, we concluded that, for any bounded subdomain U of the half plane, the d-dimensional Minkowski content of a chordal SLE_κ curve γ restricted in U has finite moments of any order.

In [31], we proved that the upper bound in [28] for $\mathbb{P}[\text{dist}(z_k, \gamma) < r_k, 1 \le k \le n]$ is also its lower bound, up to a multiplicative constant depending only on n. In addition, we proved that the n-point Green's function exists for any $n \in \mathbb{N}$. We followed the approach of [8], where the existence of 2-point Green's function is proved. It takes some significant work to extend their result to all $n \ge 3$ because the argument in [8] uses the close-form formula of 1-point Green's function, while there is no close-form formulas for n-point Green's function when $n \ge 2$. The argument in [31] relies on a series of careful estimates on the SLE curve, which may be useful in the future. At the same time, we derived the convergence rate of the rescaled probability to the n-point Green's function, which was previously known only for $n = 1$. The upper bound and lower bound for $\mathbb{P}[\text{dist}(z_k, \gamma) < r_k, 1 \le k \le n]$ then give us the up-to-constant sharp bound of the multi-point Green's function.

5 Arm Exponents for SLE

I had a joint paper with Hao Wu on arm exponents for SLE, which would give the alternating half-plane arm exponents for the critical lattice models, which converge to SLE. Let γ be a chordal SLE_κ curve in the upper half plane growing from 0. Let D be a small semi-disc centered at 1 with radius $\varepsilon > 0$, and L be the half-infinite interval $(-\infty, -1]$. For $\kappa > 4$, we considered the event that γ visits D and L alternatively for n times, and estimated the probability of that event as $\varepsilon \to 0$. In the paper, we derived exponents α_n^+ and proved that the above probability is comparable to $\varepsilon^{\alpha_n^+}$ when ε is small.

The same setting does not work for $\kappa \le 4$ because in that case the SLE_κ curve γ does not intersect L. To solve this issue, we used a half-infinite strip $(-\infty, -1] \times (0, h]$ in place of the half-infinite interval $(-\infty, -1]$, where $h > 0$ is small but does not go to 0 as $\varepsilon \to 0$. In this case, we also derived exponents α_n^+ and proved that the probability that γ visits D and L back and forth for n times is comparable to $\varepsilon^{\alpha_n^+}$ when ε is small.

6 Future Goals

Currently, I am working on constructing SLE_κ loops for $\kappa \in (0, 8)$. It is a σ-finite measure on the loops in $\widehat{\mathbb{C}}$ that start and end at a given point, say 0. I expect that the SLE_κ loop satisfies the following properties. First, it satisfies the domain Markov property, i.e., given any stopping time, and conditional on the curve before the stopping time and the event that the curve is not finished at that time, the rest of the curve is a regular chordal SLE_κ curve growing in the remaining domain. Second, it satisfies reversibility, i.e., the reversal of an SLE_κ loop has the same law as itself. Last, the SLE_κ loop should be invariant under space-time shift. This means that, if the SLE_κ loop γ is parameterized by natural parametrization, i.e., the d-dimensional Minkowski content measure, and if it is extended to be defined on $\mathbb{R}$ periodically, then for any deterministic number $t_0 \in \mathbb{R}$, the new curve $\gamma_{t_0}(t) := \gamma(t_0 + t) - \gamma(t_0)$, $t \in \mathbb{R}$, has the same law as γ. The construction will use the results in [29], and the method may also be used to construct SLE_κ bubbles: a σ-finite measure on the loops in a simply connected domain that starts and ends at a boundary point, which satisfies domain Markov property and reversibility.

Another project I am working on now is to extend the results of [28, 31] with my Graduate students. We plan to get sharp bounds for the probability that a *radial* SLE visits a number of discs as well as prove the existence of multi-point Green's function for radial SLE. We also plan to prove the existence of multi-point boundary Green's function, where the $z_1, \dots, z_n$ lie on the boundary of the domain, and the exponent $d - 2$ is replaced by $-\alpha = 1 - \frac{8}{\kappa}$. Moreover, we want to study the properties of the multi-point Green's functions, e.g., how the function behaves when two points merge together.

The reversibility of chordal SLE and whole-plane SLE for $\kappa \in (4, 8]$ were recently proved in [9, 10]. Their papers used couplings of SLE with Gaussian free field. I am still looking for a proof, which only uses traditional SLE techniques. Since it was proved in [29] that for any $\kappa \in (0, 8)$, chordal SLE_κ can be constructed using two-sided radial SLE_κ, we may transform the reversibility of chordal SLE_κ into the reversibility of two-sided radial SLE_κ. Using radial Loewner equation, we find that the reversibility of two-sided radial SLE_κ is equivalent to constructing a pair of real valued stochastic processes which commute with each other. I have not been able to construct these processes, but there seems to be some hope.

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