Quiz 2 Solutions

1 (a) (2 points) State the Comparison Test for Series.

(b) (3 points) Does the series $\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^2}$ converge? Why?

Solution. (a) For two series $\sum a_n$ and $\sum b_n$, if $|a_n| \leq b_n$ for each n, and $\sum b_n$ converges, then $\sum a_n$ also converges; if $a_n \geq b_n \geq 0$ for each n, and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

(b) Since $|\sin(n^3)| \leq 1$, we have $|\frac{\sin(n^3)}{n^2}| \leq \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. Taking $p = 2$, we see that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Since $|\frac{\sin(n^3)}{n^2}| \leq \frac{1}{n^2}$ for each n, by comparison test, $\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^2}$ also converges. □

2 (a) (2 points) State the Intermediate Value Theorem. Make sure to include all the hypotheses.

(b) (3 points) Prove there is some $x \in (0, \pi/2)$ for which $\pi \sin x + x = \pi$, justifying each step of your answer carefully.

Solution. (a) If f is continuous on $[a, b]$, and $y \in \mathbb{R}$ satisfies either $f(a) > y > f(b)$ or $f(a) < y < f(b)$, then there is $x \in (a, b)$ such that $f(x) = y$.

(b) Let $f(x) = \pi \sin x + x$. Since $\sin x$ and x are continuous on $\mathbb{R}$, f is continuous on $\mathbb{R}$, and so is continuous on $[0, \pi]$. We calculate $f(0) = 0 < \pi$ and $f(\pi/2) = \pi + \pi/2 > \pi$. By Intermediate Value Theorem, there is $x \in (0, \pi/2)$ such that $f(x) = \pi$, i.e., $\pi \sin x + x = \pi$. □