Do not attempt to apply the limit theorems for finite limit we have learned before to infinite limits. Instead, we now derive some theorems for $\lim s_n = +\infty$ or $-\infty$, which will be useful.

Lemma 1 (Exercise 9.10 (b))**.** *$s_n \to -\infty$ if and only if $-s_n \to +\infty$.*

Proof. Suppose $s_n \to -\infty$. Let $M > 0$. Then $-M < 0$. So there is $N \in \mathbb{N}$ such that for $n > N$, $s_n < -M$, which implies that $-s_n > M$. So we get $s_n \to +\infty$. On the other hand, assume $s_n \to +\infty$. Let $M < 0$. Then $-M > 0$. So there is $N \in \mathbb{N}$ such that for $n > N$, $s_n > -M$, which implies that $-s_n < M$. So we get $-s_n \to -\infty$. □

Theorem 1 (Theorem 9.9)**.** *If $\lim s_n = +\infty$ and $\lim t_n = +\infty$ or $\lim t_n = a \in (0, \infty)$, then $\lim(s_n t_n) = +\infty$.*

Proof. If $t_n \to +\infty$, then there is $N_t \in \mathbb{N}$ such that for $n > N_t$, $t_n > 1$. If $t_n \to a \in (0, \infty)$, then by taking $\varepsilon = \frac{a}{2}$, we see that there is $N_t \in \mathbb{N}$ such that for $n > N_t$, $|t_n - t| < \frac{a}{2}$, which implies that $t_n > a - \frac{a}{2} = \frac{a}{2} > 0$. In either case, there are $b \in (0, \infty)$ and $N_t \in \mathbb{N}$ such that $t_n > b$ for $n > N_t$. Let $M > 0$. Since $s_n \to +\infty$, there is $N_s \in \mathbb{N}$ such that for $n > N_s$, $s_n > M/b$. Let $N = \max\{N_t, N_s\}$. If $n > N$, then $t_n > b$ and $s_n > M/b$, and so $s_n t_n > M$. Thus, $s_n t_n \to +\infty$. □

Corollary 1. *If $\lim s_n = -\infty$ and $\lim t_n = +\infty$ or $\lim t_n = a \in (0, +\infty)$, then $\lim(s_n t_n) = -\infty$. If $\lim s_n = +\infty$ and $\lim t_n = -\infty$ or $\lim t_n = a \in (-\infty, 0)$, then $\lim(s_n t_n) = -\infty$. If $\lim s_n = -\infty$ and $\lim t_n \in -\infty$ or $\lim t_n = a \in (-\infty, 0)$, then $\lim(s_n t_n) = +\infty$.*

Proof. This follows from Lemma 1 and Theorem 9.9. □

Theorem 2 (Exercise 9.11)**.** *(i) If $s_n \to +\infty$ and (t_n) is bounded below, then $s_n + t_n \to +\infty$. (ii) If $s_n \to -\infty$ and (t_n) is bounded above, then $s_n + t_n \to -\infty$. (iii) If $s_n \to +\infty$ and (t_n) converges or diverges to $+\infty$, then $s_n + t_n \to +\infty$. (iv) If $s_n \to -\infty$ and (t_n) converges or diverges to $-\infty$, then $s_n + t_n \to -\infty$.*

Proof. (i) There is $L \in \mathbb{R}$ such that $t_n > L$ for all n. Let $M > 0$. There is $M_s > 0$ such that $M_s > M - L$. Since $s_n \to +\infty$, there is $N \in \mathbb{N}$ such that $n > N$ implies that $s_n > M_s$, which in turn implies that $s_n + t_n > M_s + L > M$. (ii) is similar to (i). (iii) follows from (i) because when (t_n) converges or diverges to $+\infty$, it is bounded below. (iv) follows from (ii) in a similar way. □

Theorem 3 (Theorem 9.10)**.** *(i) For a positive sequence (s_n), $s_n \to +\infty$ if and only if $\frac{1}{s_n} \to 0$. (ii) For a negative sequence (s_n), $s_n \to -\infty$ if and only if $\frac{1}{s_n} \to 0$.*

Proof. (i) First suppose $s_n \to +\infty$. Let $\varepsilon > 0$. Then $\frac{1}{\varepsilon} > 0$. So there is $N \in \mathbb{N}$ such that for $n > N$, $s_n > \frac{1}{\varepsilon}$, which implies that $|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \varepsilon$. So we get $\frac{1}{s_n} \to 0$. Second, suppose $\frac{1}{s_n} \to 0$. Let $M > 0$. Then $\frac{1}{M} > 0$. So there is $N \in \mathbb{N}$ such that for $n > N$, $\frac{1}{s_n} = |\frac{1}{s_n} - 0| < \frac{1}{M}$, which implies that $s_n > M$. So we get $s_n \to +\infty$. (ii) follows from (i) and Lemma 1 □

Remark 1. If we do not assume that (s_n) is a positive sequence, we can still conclude that $1/s_n \to 0$ from $s_n \to +\infty$. This is because we have $s_n > 0$ for n big enough. Here $1/s_n$ may not be defined for finitely many n, but that does not affect the limit. Similarly, $s_n \to -\infty$ also implies that $1/s_n \to 0$. But $1/s_n \to 0$ does not imply $s_n \to +\infty$ or $s_n \to -\infty$. The sequence (s_n) may have alternative signs. Consider the example $s_n = (-1)^n n$.

Theorem 4. *If $s_n \to +\infty$, then for any $r \in \mathbb{Q}$ with $r > 0$, $s_n^r \to +\infty$.*

Proof. Let $M > 0$. Then $M^{1/r} > 0$. Since $s_n \to +\infty$, there is $N \in \mathbb{N}$ such that for $n > N$, $s_n > M^{1/r}$, which implies that $s_n^r > M$. □

Remark 2. We may understand the above propositions formally as

$$(\pm\infty) \times (\pm\infty) = +\infty, \quad (\pm\infty) \times (\mp\infty) = -\infty;$$

$$a(>0) \times (\pm\infty) = \pm\infty, \quad a(<0) \times (\pm\infty) = \mp\infty;$$

$$(\pm\infty) + (\pm\infty) = (\pm\infty), \quad a + (\pm\infty) = \pm\infty;$$

$$\frac{1}{\pm\infty} = 0^{\pm}, \quad \frac{1}{0^{\pm}} = \pm\infty, \quad (+\infty)^r = +\infty, \quad r > 0.$$

There are no results about $0 \cdot (\pm\infty)$ or $(+\infty) + (-\infty)$.

Example 1. We have $\lim n^2 = +\infty$, $\lim(-n) = -\infty$, $\lim 2^n = +\infty$, and $\lim(\sqrt{n} + 7) = +\infty$. To see this, recall that $\lim n = +\infty$. Using the product theorem $(+\infty) \times (+\infty) = +\infty$, we get $\lim n^2 = +\infty$. Using the theorem $(-1) \times (+\infty) = -\infty$, we get $\lim(-n) = -\infty$. Using the theorems $(+\infty)^{1/2} = +\infty$ and $(+\infty) + a = +\infty$, we get $\lim(\sqrt{n} + 7) = +\infty$. Finally, since $2^n > 0$ and $\frac{1}{2^n} = (\frac{1}{2})^n \to 0$ (because $0 < \frac{1}{2} < 1$), using the theorem $\frac{1}{0^+} = +\infty$, we get $\lim 2^n = +\infty$.

Example 2. Show $\frac{n^2+3}{n+1} = +\infty$.

Solution. Since $\frac{n^2+3}{n+1} > 0$ for all n, it suffices to show that $\frac{n+1}{n^2+3} \to 0$. This is the case because

$$\frac{n+1}{n^2+3} = \frac{1/n + 1/n^2}{1 + 3/n^2} \to \frac{0 + 0^2}{1 + 3 * 0^2} = 0.$$

□

Monotone Sequences

Definition 1. A sequence (s_n) is called an increasing sequence if $s_n \le s_{n+1}$ for all n, and is called a decreasing sequence if $s_n \ge s_{n+1}$ for all n. If (s_n) is increasing, then for any $n \le m$, $s_n \le s_m$. If (s_n) is decreasing, then for any $n \le m$, $s_n \ge s_m$. An increasing or decreasing sequence is called a monotone sequence.

Remark 3. An increasing sequence is bounded below: the first element is a lower bound. A decreasing sequence is bounded above: the first element is an upper bound.

Example 3. The sequence (n) is increasing. The sequences $(-n)$ and $(\frac{1}{n})$ are decreasing. The sequence $((-1)^n)$ is neither increasing or decreasing.

Theorem 5 (Theorems 10.2, 10.4, 10.5)**.** *(i) If* (s_n) *is increasing, then* $\lim s_n$ *exists and equals* $\sup\{s_n : n \in \mathbb{N}\}$*. If* (s_n) *is bounded above, then* (s_n) *converges.*

(ii) If (s_n) *is decreasing, then* $\lim s_n$ *exists and equals* $\inf\{s_n : n \in \mathbb{N}\}$*. If* (s_n) *is bounded below, then* (s_n) *converges.*

Proof. (i) Let $S = \{s_n : n \in \mathbb{N}\}$ and $s = \sup S$. Consider two cases. Case 1. S is bounded above. In this case $s \in \mathbb{R}$ is the smallest upper bound of S. Let $\varepsilon > 0$. Since s is the smallest upper bound of S, $s - \varepsilon$ is not an upper bound of S. Thus, S contains an element greater than $s - \varepsilon$. This means, for some $N \in \mathbb{N}$, we have $s_N > s - \varepsilon$. Since (s_n) is increasing, for any $n > N$, $s_n \geq s_N > s - \varepsilon$. On the other hand, $s_n \leq s$ for all $n \in \mathbb{N}$ since s is an upper bound of S. So for any $n > N$, $s - \varepsilon < s_n \leq s$, which implies that $|s_n - s| < \varepsilon$. Thus, (s_n) converges to s. Case 2. S is not bounded above. Then $s = +\infty$. Let $M > 0$. Since S is not bounded above, M is not an upper bound of S. So S contains an element greater than M, i.e., for some $N \in \mathbb{N}$, we have $s_N > M$. Since (s_n) is increasing, for any $n > N$, $s_n \geq s_N > M$. Thus, $s_n \to +\infty = s$.

(ii) This is similar to (i). We leave it as a homework problem. □