Convergent Sequences

Definition 1. A sequence of real numbers \((s_n)\) is said to converge to a real number \(s\) if

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ such that } n > N \text{ implies } |s_n - s| < \varepsilon.
\]

When this holds, we say that \((s_n)\) is a convergence sequence with \(s\) being its limit, and write \(s_n \to s\) or \(s = \lim_{n \to \infty} s_n\). If \((s_n)\) does not converge, then we say that \((s_n)\) is a divergent sequence.

We first show that one sequence \((s_n)\) can not have two different limits. Suppose \(s_n \to s\) and \(s_n \to t\). Let \(\varepsilon > 0\). Then \(\frac{\varepsilon}{2} > 0\). Since \(s_n \to s\), by definition there is \(N_1 \in \mathbb{N}\) such that for \(n > N_1\), \(|s_n - s| < \frac{\varepsilon}{2}\). Since \(s_n \to t\), by definition there is \(N_2 \in \mathbb{N}\) such that for \(n > N_2\), \(|s_n - t| < \frac{\varepsilon}{2}\). Here we use \(N_1\) and \(N_2\) in the two statements because the \(N\) coming from the two limits may not be the same. Let \(N = \max\{N_1, N_2\}\). If \(n > N\), then \(n > N_1\) and \(n > N_2\) both hold. So \(|s_n - s| < \frac{\varepsilon}{2}\) and \(|s_n - t| < \frac{\varepsilon}{2}\), which by triangle inequality imply that

\[
|s - t| \leq |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Now \(|s - t| < \varepsilon\) holds for every \(\varepsilon > 0\). We then conclude that \(|s - t| = 0\) (for otherwise \(|s - t| > 0\), we then get a contradiction by choosing \(\varepsilon = |s - t|\)). So \(s = t\), and the uniqueness holds.

We will use the following tools to check whether a sequence converges or diverges.

1. the definition
2. basic examples
3. limit theorems
4. boundedness and subsequences.

We have stated the definition. Now we consider some examples.

Example 1. Let \(s \in \mathbb{R}\). If \(s_n = s\) for all \(n\), i.e., \((s_n)\) is a constant sequence, then \(\lim s_n = s\).

Proof. For any given \(\varepsilon > 0\) we simply choose \(N = 1\). If \(n > N\), then \(|s_n - s| = 0 < \varepsilon\). \(\square\)

Example 2. We have \(\frac{1}{n} \to 0\).

Proof. Let \(\varepsilon > 0\). By Archimedean property, there is \(N \in \mathbb{N}\) such that \(\frac{1}{N} < \varepsilon\). If \(n > N\), then

\[
\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.
\]

\(\square\)

Example 3. The following two sequences are divergent
(i) \((s_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \ldots)\);
(ii) \((s_n) = (n) = (1, 2, 3, 4, 5, 6, \ldots)\).

**Proof.** (i) We use the notation of subsequence and statement that will be proved later. Suppose \(n_1 < n_2 < n_3 < \cdots\) is a strictly increasing sequence of indices, then \((s_{n_k})\) is a subsequence of \((s_n)\). We will prove a theorem, which asserts that, if \((s_n)\) converges to \(s\), then any subsequence of \((s_n)\) also converges to \(s\). The sequence \((s_n) = ((-1)^n)\) contains two constant sequences \((1, 1, \ldots)\) (with \(n_k = 2k\)) and \((-1, -1, -1, \ldots)\) (with \(n_k = 2k - 1\)), which converge to different limits. So the original \((s_n)\) cannot converge.

(ii) We use the following theorem. If \((s_n)\) is convergent, then it is a bounded sequence. In other words, the set \(\{s_n : n \in \mathbb{N}\}\) is bounded. So an unbounded sequence must diverge. Since for \(s_n = n, n \in \mathbb{N}\), the set \(\{s_n : n \in \mathbb{N}\} = \mathbb{N}\) is unbounded, the sequence \((n)\) is divergent. 

**Remark 1.** This example shows that we have two ways to prove that a sequence is divergent: (i) find two subsequences that converge to different limits; (ii) show that the sequence is unbounded. Note that the \((s_n)\) in (i) is bounded and divergent. The \((s_n)\) in (ii) is divergent, but \(\lim s_n\) actually exists, which is \(+\infty\), and its every subsequence also tends to \(+\infty\). We will define that limit later.

Now we state some limit theorems.

**Theorem 1** (Theorem 9.1). *Every convergent sequence is bounded.*

**Proof.** Let \((s_n)\) be a sequence that converges to \(s \in \mathbb{R}\). Applying the definition to \(\varepsilon = 1\), we see that there is \(N \in \mathbb{N}\) such that for any \(n > N\), \(|s_n - s| < 1\), which then implies that \(|s_n| \leq |s| + 1\). Let

\[ M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}. \]

The maximum exists since the set is finite. Then for any \(n \in \mathbb{N}\), \(|s_n| \leq M\) (consider the case \(n \leq N\) and \(n > N\) separately), i.e., \(-M \leq s_n \leq M\). So \(\{s_n : n \in \mathbb{N}\}\) is bounded. 

**Theorem 2** (Theorem 9.3). *If \((s_n)\) converges to \(s\) and \((t_n)\) converges to \(t\), then \((s_n + t_n)\) converges to \(s + t\).*

**Proof.** Let \(\varepsilon > 0\). Then \(\frac{\varepsilon}{2} > 0\). Since \(s_n \to s\), there is \(N_1 \in \mathbb{N}\) such that for any \(n > N_1\), \(|s_n - s| < \frac{\varepsilon}{2}\). Since \(t_n \to t\), there is \(N_2 \in \mathbb{N}\) such that for any \(n > N_2\), \(|t_n - t| < \frac{\varepsilon}{2}\). Let \(N = \max\{N_1, N_2\}\). If \(n > N\), then \(n > N_1\) and \(n > N_2\) both hold, and so \(|s_n - s| < \frac{\varepsilon}{2}\) and \(|t_n - t| < \frac{\varepsilon}{2}\), which together imply (by triangle inequality) that

\[ |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

So we have the desired convergence. 

**Theorem 3** (Theorem 9.4). *If \((s_n)\) converges to \(s\) and \((t_n)\) converges to \(t\), then \((s_n \cdot t_n)\) converges to \(s \cdot t\).*
Discussion. We need to bound $|s_nt_n - st|$ from above for big $n$. We write

$$s_nt_n - st = s_nt_n - s_nt + s_nt - st = s_n(t_n - t) + t(s_n - s).$$

By triangle inequality, we get

$$|s_nt_n - st| \leq |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|.$$

Since $t_n \to t$ and $s_n \to s$, we know that $|t_n - t|$ and $|s_n - s|$ can be arbitrarily small if we choose $n$ big enough. Thus, if $|s_n|$ and $|t|$ are not too big, then we can control the sum on the RHS (righthand side). In fact, the size of $|s_n|$ can be controlled because of Theorem 9.1.

**Proof.** Since $(s_n)$ is convergent, by Theorem 9.1, there is $M > 0$ such that $|s_n| \leq M$ for every $n$. We may choose $M$ big such that $M \geq |t|$. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M} > 0$. Since $s_n \to s$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2M}$. Since $t_n \to t$, there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2M}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and $n > N_2$ both hold, and so $|s_n - s| < \frac{\varepsilon}{2M}$ and $|t_n - t| < \frac{\varepsilon}{2M}$, which together with $|s_n| \leq M$ and $|t| \leq M$ imply that

$$|s_nt_n - st| \leq |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|$$

$$\leq M|t_n - t| + M|s_n - s| < M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon.$$

\[\square\]

**Corollary 1.** If $(s_n)$ converges to $s$, $k \in \mathbb{R}$, and $m \in \mathbb{N}$, then $(ks_n)$ converges to $ks$ and $s_n^m$ converges to $s^m$.

**Proof.** For the sequence $(ks_n)$, we apply Theorem 9.4 to the sequence $(t_n)$ with $t_n = k$ for all $n$. For the sequence $(s_n^m)$ we use induction. In the induction step, note that $s_{n+1}^m = s_n^m s_n^1$ and apply Theorem 9.4 to $t_n = s_n^m$.

\[\square\]

**Corollary 2.** If $(s_n)$ converges to $s$ and $(t_n)$ converges to $t$, then $(s_n - t_n)$ converges to $s - t$.

**Proof.** We write $s_n + t_n = s_n + (-1)t_n$ and apply Theorem 9.3 and the previous corollary. \[\square\]

From this corollary we see that $s_n \to s$ iff $s_n - s \to 0$. By the Theorem below, the latter statement is equivalent to that $|s_n - s| \to 0$.

**Theorem 4.** (a) Suppose two sequences $(s_n)$ and $(t_n)$ satisfy that $t_n \to 0$ and $|s_n| \leq |t_n|$ for all but finitely many $n$. Then $s_n \to 0$.

(b) For any sequence $(s_n)$, $s_n \to 0$ if and only if $|s_n| \to 0$.

**Proof.** (a) Let $N_0 \in \mathbb{N}$ be such that $|s_n| \leq |t_n|$ for $n > N_0$. Let $\varepsilon > 0$. Since $t_n \to 0$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|t_n - 0| < \varepsilon$. Let $N = \max\{N_0, N_1\}$. For $n > N$, $|s_n| \leq |t_n|$ and $|t_n - 0| < \varepsilon$, which imply that $|s_n - 0| = |s_n| \leq |t_n| = |t_n - 0| < \varepsilon$.

(b) From (a) we know that if $|s_n| = |t_n|$ for all $n$, then $s_n \to 0$ iff $t_n \to 0$. We then apply this result to $t_n = |s_n|$ and use that $||s_n|| = |s_n|$. \[\square\]
Lemma 1 (Lemma 9.5). If \((s_n)\) converges to \(s\) such that \(s \neq 0\) and \(s_n \neq 0\) for all \(n\), then \((1/s_n)\) converges to \(1/s\).

**Discussion.** We need to bound \(|1/s_n - 1/s|\) from above for big \(n\). We write

\[
\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s_n s} \right| = \frac{|s_n - s|}{|s_n||s|}.
\]

Since \(s_n \to s\), \(|s_n - s|\) can be arbitrarily small if we choose \(n\) big enough. Thus, if \(|s_n|\) and \(|s|\) are not too close to 0, then we can control the size of the RHS. This means that we need a positive lower bound of the set \(\{|s_1|, |s_2|, \ldots\}\).

**Proof.** Since \(s \neq 0\), we have \(|s| > 0\). Since \(s_n \to s\), applying the definition to \(\varepsilon = |s|/2\), we get \(N \in \mathbb{N}\) such that for \(n > N\), \(|s_n - s| < |s|/2\), which then implies by triangle inequality that \(|s_n| \geq |s| - |s_n - s| > |s| - |s|/2 = |s|/2\). Let \(m = \min\{|s_1|, |s_2|, \ldots, |s_N|, |s|/2\}\). Then \(m\) exists and is positive since the set is a finite set of positive numbers.

Let \(\varepsilon > 0\). Then \(m|s|\varepsilon > 0\). Since \(s_n \to s\), there is \(N' \in \mathbb{N}\) such that \(n > N'\) implies that \(|s_n - s| < m|s|\varepsilon\), which together with \(|s_n| \geq m|s|\), for all \(n\) implies that

\[
\left| \frac{1}{s_n} - \frac{1}{s} \right| \leq \frac{|s_n - s|}{m|s|} < \frac{m|s|\varepsilon}{m|s|} = \varepsilon.
\]

\(\square\)

Theorem 5 (Theorem 9.6). Suppose \((s_n)\) converges to \(s\) and \((t_n)\) converges to \(t\). If \(s \neq 0\) and \(s_n \neq 0\) for all \(n\), then \((s_n/t_n)\) converges to \(t/s\).

**Proof.** By Lemma 9.5, \((1/s_n)\) converges to \(1/s\). Applying Theorem 9.4 to the sequences \((1/s_n)\) and \((t_n)\), we get the conclusion. \(\square\)

Example 4. Derive \(\lim 3n + 1 \over n-1\) and \(\lim 4 + 3n/n^3 - 6\)

**Solution.** We write

\[
\frac{3n + 1}{n-4} = \frac{3 + 1/n}{7 + (-4) \times 1/n}, \quad \frac{4n^3 + 3n}{n^3 - 6} = \frac{4 + 3 \times (1/n)^2}{1 + (-6) \times 1/n}.
\]

We have shown that \(\lim 1/n = 0\). So (i) \(\lim (3 + 1/n) = 3 + 0 = 3\) and \(\lim (7 + (-4) \times 1/n) = 7 + (-4) \times 0 = 7\), which imply that \(\lim 3n + 1 = \lim (3 + 1/n) / \lim (7 + (-4) \times 1/n) = 3/7\); (ii) \(\lim (4 + 3 \times (1/n)^2) = 4 + 3 \times 0^2 = 4\) and \(\lim (1 + (-6) \times 1/n) = 1 + (-6) \times 0 = 1\), which imply that \(\lim 4n^3 + 3n / n^3 - 6 = \lim (4 + 3 \times (1/n)^2) / \lim (1 + (-6) \times 1/n) = 4\).

\(\square\)

We now state some theorems about the relation between limits and orders.

**Theorem 6** (Exercise 8.9). (a) If \((s_n)\) converges to \(s\), and there is \(N_0 \in \mathbb{N}\) such that \(s_n \geq 0\) for all \(n > N_0\), then \(s \geq 0\).

(b) Suppose \((s_n)\) converges to \(s\) and \((t_n)\) converges to \(t\). If there \(N_0 \in \mathbb{N}\) such that \(s_n \leq t_n\) for all \(n > N_0\), then \(s \leq t\).
Proof. (a) We prove by contradiction. Suppose \( s < 0 \). Let \( \varepsilon = |s| = -s > 0 \). Since \( s_n \to s \), there is \( N \in \mathbb{N} \) such that for \( n > N \), \( |s_n - s| < \varepsilon \), which implies that \( s_n < s + \varepsilon = 0 \). Let \( n = \max\{N, N_0\} + 1 \). Then \( n > N_0 \) and \( n > N \). From \( n > N_0 \) we get \( s_n \geq 0 \); from \( n > N \) we get \( s_n < 0 \). This is the contradiction.

(b) Applying (i) to the sequence \((t_n - s_n)\) we conclude that its limit \( t - s \) is nonnegative. \( \square \)

For \( x \in [0, \infty) \) and \( n \in \mathbb{N} \), the power root \( x^{1/n} \) is defined as the unique \( y \in [0, \infty) \) such that \( y^n = x \). The uniqueness of such \( y \) follows from the fact that if \( 0 \leq y_1 < y_2 \), then \( y_1^n < y_2^n \). The existence follows from the “Intermediate Value Theorem” for continuous function \( f(x) = x^n \), which will be stated and proved later. We now just accept the existence of \( x^{1/n} \) for any \( x \in [0, \infty) \), although in the case that \( n \) is an odd number, we can also define \( x^{1/n} \) for \( x < 0 \).

When \( n = 2 \), \( x^{1/2} \) is often written as \( \sqrt{x} \). We have the following theorem.

**Theorem 7** (Example 5). Suppose \((s_n)\) converges to \( s \) and \( s_n \geq 0 \) for all \( n \). Then \((\sqrt{s_n})\) converges to \( \sqrt{s} \).

**Discussion** We want to bound \( |\sqrt{s_n} - \sqrt{s}| \) from above for big \( n \). It is useful to note the equality

\[
(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s}) = (\sqrt{s_n})^2 - (\sqrt{s})^2 = s_n - s.
\]

Taking absolute value, we get

\[
|\sqrt{s_n} - \sqrt{s}| \cdot |\sqrt{s_n} + \sqrt{s}| = |s_n - s|.
\]

If \( \sqrt{s} > 0 \), then

\[
|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}}.
\]

**Proof.** By Theorem 6, \( s \geq 0 \). First suppose \( s > 0 \). Then \( \sqrt{s} > 0 \). Let \( \varepsilon > 0 \). Then \( \sqrt{s\varepsilon} > 0 \). Since \( s_n \to s \), there is \( N \in \mathbb{N} \) such that for \( n > N \), \( |s_n - s| < \sqrt{s\varepsilon} \), which implies that

\[
|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s\varepsilon}}{\sqrt{s}} = \varepsilon.
\]

We leave the proof in the case \( s = 0 \) as an exercise. Note that for \( x \geq 0 \), \( \sqrt{x} < \varepsilon \) iff \( x^2 < \varepsilon \). \( \square \)