Convergent Sequences

Definition 1. A sequence of real numbers (s_n) is said to converge to a real number s if

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \text{such that } n > N \text{ implies } |s_n - s| < \varepsilon. \tag{1}$$

When this holds, we say that (s_n) is a convergence sequence with s being its limit, and write $s_n \to s$ or $s = \lim_{n\to\infty} s_n$. If (s_n) does not converge, then we say that (s_n) is a divergent sequence.

We first show that one sequence (s_n) can not have two different limits. Suppose $s_n \to s$ and $s_n \to t$. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$, by definition there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$. Since $s_n \to t$, by definition there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|s_n - t| < \frac{\varepsilon}{2}$. Here we use N_1 and N_2 in the two statements because the N coming from the two limits may not be the same. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and $n > N_2$ both hold. So $|s_n - s| < \frac{\varepsilon}{2}$ and $|s_n - t| < \frac{\varepsilon}{2}$, which by triangle inequality imply that

$$|s - t| \le |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now $|s-t| < \varepsilon$ holds for every $\varepsilon > 0$. We then conclude that $|s-t| = 0$ (for otherwise $|s-t| > 0$, we then get a contradiction by choosing $\varepsilon = |s-t|$). So $s = t$, and the uniqueness holds.

We will use the following tools to check whether a sequence converges or diverges.

1. the definition
2. basic examples
3. limit theorems
4. boundedness and subsequences.

We have stated the definition. Now we consider some examples.

Example 1. Let $s \in \mathbb{R}$. If $s_n = s$ for all n, i.e., (s_n) is a constant sequence, then $\lim s_n = s$.

Proof. For any given $\varepsilon > 0$ we simply choose $N = 1$. If $n > N$, then $|s_n - s| = 0 < \varepsilon$. □

Example 2. We have $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. By Archimedean property, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. If $n > N$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

□

Example 3. The following two sequences are divergent

(i) $(s_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \dots)$;

(ii) $(s_n) = (n) = (1, 2, 3, 4, 5, 6, \dots)$.

Proof. (i) We use the notation of subsequence and statement that will be proved later. Suppose $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of indices, then (s_{n_k}) is a subsequence of (s_n). We will prove a theorem, which asserts that, if (s_n) converges to s, then any subsequence of (s_n) also converges to s. The sequence $(s_n) = ((-1)^n)$ contains two constant sequences $(1, 1, 1, \dots)$ (with $n_k = 2k$) and $(-1, -1, -1, \dots)$ (with $n_k = 2k-1$), which converge to different limits. So the original (s_n) can not converge.

(ii) We use the following theorem. If (s_n) is convergent, then it is a bounded sequence. In other words, the set $\{s_n : n \in \mathbb{N}\}$ is bounded. So an unbounded sequence must diverge. Since for $s_n = n$, $n \in \mathbb{N}$, the set $\{s_n : n \in \mathbb{N}\} = \mathbb{N}$ is unbounded, the sequence (n) is divergent. □

Remark 1. This example shows that we have two ways to prove that a sequence is divergent: (i) find two subsequences that convergent to different limits; (ii) show that the sequence is unbounded. Note that the (s_n) in (i) is bounded and divergent. The (s_n) in (ii) is divergent, but $\lim s_n$ actually exists, which is $+\infty$, and its every subsequence also tends to $+\infty$. We will define that limit later.

Now we state some limit theorems.

Theorem 1 (Theorem 9.1)**.** *Every convergent sequence is bounded.*

Proof. Let (s_n) be a sequence that converges to $s \in \mathbb{R}$. Applying the definition to $\varepsilon = 1$, we see that there is $N \in \mathbb{N}$ such that for any $n > N$, $|s_n - s| < 1$, which then implies that $|s_n| \le |s| + 1$. Let

$$M = \max\{|s_1|, |s_2|, \dots, |s_N|, |s| + 1\}.$$

The maximum exists since the set is finite. Then for any $n \in \mathbb{N}$, $|s_n| \le M$ (consider the case $n \le N$ and $n > N$ separately), i.e., $-M \le s_n \le M$. So $\{s_n : n \in \mathbb{N}\}$ is bounded. □

Theorem 2 (Theorem 9.3)**.** *If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.*

Proof. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$. Since $t_n \to t$, there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and $n > N_2$ both hold, and so $|s_n - s| < \frac{\varepsilon}{2}$ and $|t_n - t| < \frac{\varepsilon}{2}$, which together imply (by triangle inequality) that

$$|(s_n + t_n) - (s + t)| \le |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we have the desired convergence. □

Theorem 3 (Theorem 9.4)**.** *If (s_n) converges to s and (t_n) converges to t, then $(s_n \cdot t_n)$ converges to $s \cdot t$.*

Discussion. We need to bound $|s_n t_n - st|$ from above for big n. We write

$$s_n t_n - st = s_n t_n - s_n t + s_n t - st = s_n(t_n - t) + t(s_n - s).$$

By triangle inequality, we get

$$|s_n t_n - st| \le |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|.$$

Since $t_n \to t$ and $s_n \to s$, we know that $|t_n - t|$ and $|s_n - s|$ can be arbitrarily small if we choose n big enough. Thus, if $|s_n|$ and $|t|$ are not too big, then we can control the sum on the RHS (righthand side). In fact, the size of $|s_n|$ can be controlled because of Theorem 9.1.

Proof. Since (s_n) is convergent, by Theorem 9.1, there is $M > 0$ such that $|s_n| \le M$ for every n. We may choose M big such that $M \ge |t|$. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M} > 0$. Since $s_n \to s$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2M}$. Since $t_n \to t$, there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2M}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1$ and $n > N_2$ both hold, and so $|s_n - s| < \frac{\varepsilon}{2M}$ and $|t_n - t| < \frac{\varepsilon}{2M}$, which together with $|s_n| \le M$ and $|t| \le M$ imply that

$$|s_n t_n - st| \le |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|$$

$$\le M|t_n - t| + M|s_n - s| < M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon.$$

□

Corollary 1. *If (s_n) converges to s, $k \in \mathbb{R}$, and $m \in \mathbb{N}$, then (ks_n) converges to ks and s_n^m converges to s^m.*

Proof. For the sequence (ks_n), we apply Theorem 9.4 to the sequence (t_n) with $t_n = k$ for all n. For the sequence (s_n^m) we use induction. In the induction step, note that $s_n^{m+1} = s_n * s_n^m$ and apply Theorem 9.4 to $t_n = s_n^m$ □

Corollary 2. *If (s_n) converges to s and (t_n) converges to t, then $(s_n - t_n)$ converges to $s - t$.*

Proof. We write $s_n + t_n = s_n + (-1)t_n$ and apply Theorem 9.3 and the previous corollary. □

From this corollary we see that $s_n \to s$ iff $s_n - s \to 0$. By the Theorem below, the latter statement is equivalent to that $|s_n - s| \to 0$.

Theorem 4. *(a) Suppose two sequences (s_n) and (t_n) satisfy that $t_n \to 0$ and $|s_n| \le |t_n|$ for all but finitely many n. Then $s_n \to 0$.*

(b) For any sequence (s_n), $s_n \to 0$ if and only if $|s_n| \to 0$.

Proof. (a) Let $N_0 \in \mathbb{N}$ be such that $|s_n| \le |t_n|$ for $n > N_0$. Let $\varepsilon > 0$. Since $t_n \to 0$, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|t_n - 0| < \varepsilon$. Let $N = \max\{N_0, N_1\}$. For $n > N$, $|s_n| \le |t_n|$ and $|t_n - 0| < \varepsilon$, which imply that $|s_n - 0| = |s_n| \le |t_n| = |t_n - 0| < \varepsilon$.

(b) From (a) we know that if $|s_n| = |t_n|$ for all n, then $s_n \to 0$ iff $t_n \to 0$. We then apply this result to $t_n = |s_n|$ and use that $||s_n|| = |s_n|$. □

Lemma 1 (Lemma 9.5)**.** *If (s_n) converges to s such that $s \neq 0$ and $s_n \neq 0$ for all n, then $(1/s_n)$ converges to $1/s$.*

Discussion. We need to bound $|1/s_n - 1/s|$ from above for big n. We write

$$\Big|\frac{1}{s_n} - \frac{1}{s}\Big| = \Big|\frac{s - s_n}{s_n s}\Big| = \frac{|s_n - s|}{|s_n||s|}.$$

Since $s_n \to s$, $|s_n - s|$ can be arbitrarily small if we choose n big enough. Thus, if $|s_n|$ and $|s|$ are not too close to 0, then we can control the size of the RHS. This means that we need a positive lower bound of the set $\{|s_1|, |s_2|, \dots\}$.

Proof. Since $s \neq 0$, we have $\frac{|s|}{2} > 0$. Since $s_n \to s$, applying the definition to $\varepsilon = \frac{|s|}{2}$, we get $N \in \mathbb{N}$ such that for $n > N$, $|s_n - s| < \frac{|s|}{2}$, which then implies by triangle inequality that $|s_n| \geq |s| - |s_n - s| > |s| - \frac{|s|}{2} = \frac{|s|}{2}$. Let $m = \min\{|s_1|, |s_2|, \dots, |s_N|, \frac{|s|}{2}\}$. Then m exists and is positive since the set is a finite set of positive numbers.

Let $\varepsilon > 0$. Then $m|s|\varepsilon > 0$. Since $s_n \to s$, there is $N' \in \mathbb{N}$ such that $n > N'$ implies that $|s_n - s| < m|s|\varepsilon$, which together with $|s_n| \geq m$ for all n implies that

$$\Big|\frac{1}{s_n} - \frac{1}{s}\Big| = \frac{|s_n - s|}{|s_n||s|} \leq \frac{|s_n - s|}{m|s|} < \frac{m|s|\varepsilon}{m|s|} = \varepsilon.$$

□

Theorem 5 (Theorem 9.6)**.** *Suppose (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.*

Proof. By Lemma 9.5, $(1/s_n)$ converges to $1/s$. Applying Theorem 9.4 to the sequences $(1/s_n)$ and (t_n), we get the conclusion. □

Example 4. Derive $\lim \frac{3n+1}{7n-4}$ and $\lim \frac{4n^3+3n}{n^3-6}$

Solution. We write

$$\frac{3n+1}{7n-4} = \frac{3 + 1/n}{7 + (-4) * 1/n}, \quad \frac{4n^3 + 3n}{n^3 - 6} = \frac{4 + 3 * (1/n)^2}{1 + (-6) * 1/n}.$$

We have shown that $\lim 1/n = 0$. So (i) $\lim(3 + 1/n) = 3 + 0 = 3$ and $\lim(7 + (-4) * 1/n) = 7 + (-4) * 0 = 7$, which imply that $\lim \frac{3n+1}{7n-4} = \lim(3 + 1/n)/\lim(7 + (-4) * 1/n) = 3/7$; (ii) $\lim(4 + 3 * (1/n)^2) = 4 + 3 * 0^2 = 4$ and $\lim(1 + (-6) * 1/n) = 1 + (-6) * 0 = 1$, which imply that $\lim \frac{4n^3+3n}{n^3-6} = \lim(4 + 3 * (1/n)^2)/\lim(1 + (-6) * 1/n) = 4$. □

We now state some theorems about the relation between limits and orders.

Theorem 6 (Exercise 8.9)**.** *(a) If (s_n) converges to s, and there is $N_0 \in \mathbb{N}$ such that $s_n \geq 0$ for all $n > N_0$, then $s \geq 0$.*

(b) Suppose (s_n) converges to s and (t_n) converges to t. If there $N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n > N_0$, then $s \leq t$.

Proof. (a) We prove by contradiction. Suppose $s < 0$. Let $\varepsilon = |s| = -s > 0$. Since $s_n \to s$, there is $N \in \mathbb{N}$ such that for $n > N$, $|s_n - s| < \varepsilon$, which implies that $s_n < s + \varepsilon = 0$. Let $n = \max\{N, N_0\} + 1$. Then $n > N_0$ and $n > N$. From $n > N_0$ we get $s_n \geq 0$; from $n > N$ we get $s_n < 0$. This is the contradiction.

(b) Applying (i) to the sequence $(t_n - s_n)$ we conclude that its limit $t - s$ is nonnegative. □

For $x \in [0, \infty)$ and $n \in \mathbb{N}$, the power root $x^{1/n}$ is defined as the unique $y \in [0, \infty)$ such that $y^n = x$. The uniqueness of such y follows from the fact that if $0 \leq y_1 < y_2$, then $y_1^n < y_2^n$. The existence follows from the "Intermediate Value Theorem" for continuous function $f(x) = x^n$, which will be stated and proved later. We now just accept the existence of $x^{1/n}$ for any $x \in [0, \infty)$. It is clear that $0 \leq x_1 < x_2$ implies that $0 \leq x_1^{1/n} < x_2^{1/n}$. We restrict our attention to $[0, \infty)$ although in the case that n is an odd number, we can also define $x^{1/n}$ for $x < 0$.

When $n = 2$, $x^{1/2}$ is often written as $\sqrt{x}$. We have the following theorem.

Theorem 7 (Example 5)**.** *Suppose (s_n) converges to s and $s_n \geq 0$ for all n. Then $(\sqrt{s}_n)$ converges to $\sqrt{s}$.*

Discussion We want to bound $|\sqrt{s_n} - \sqrt{s}|$ from above for big n. It is useful to note the equality

$$(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s}) = (\sqrt{s_n})^2 - (\sqrt{s})^2 = s_n - s.$$

Taking absolute value, we get

$$|\sqrt{s_n} - \sqrt{s}| \cdot |\sqrt{s_n} + \sqrt{s}| = |s_n - s|.$$

If $\sqrt{s} > 0$, then

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}}.$$

Proof. By Theorem 6, $s \geq 0$. First suppose $s > 0$. Then $\sqrt{s} > 0$. Let $\varepsilon > 0$. Then $\sqrt{s}\varepsilon > 0$. Since $s_n \to s$, there is $N \in \mathbb{N}$ such that for $n > N$, $|s_n - s| < \sqrt{s}\varepsilon$, which implies that

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\varepsilon}{\sqrt{s}} = \varepsilon.$$

We leave the proof in the case $s = 0$ as an exercise. Note that for $x \geq 0$, $\sqrt{x} < \varepsilon$ iff $x^2 < \varepsilon$. □