MTH 320　　Section 004　　Midterm 2　　Sample

1. (a) [4pts] Let $f : \mathbb{R} \to \mathbb{R}$. What does it mean to say that f is continuous at x_0?

   (b) [6pts] A set S is said to be *dense* in $\mathbb{R}$ if every open interval contains a point in S. (For example, both the rationals and the irrationals are dense in $\mathbb{R}$.) Suppose S is dense in $\mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$, and $f(s) = g(s)$ for every $s \in S$. Prove that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

   Solution. (a) We say that f is continuous at x_0 if for any sequence (x_n) in $\mathbb{R}$ with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

   (b) Let $x \in \mathbb{R}$. Since S is dense, for any $n \in \mathbb{N}$, $(x - \frac{1}{n}, x + \frac{1}{n})$ contains an element in S. Let it be denoted by x_n. Then we get a sequence $(x_n)_{n\in\mathbb{N}}$ in S. Since $x - \frac{1}{n} < x_n < x + \frac{1}{n}$ for each n, by squeeze lemma we have $x_n \to x$. For each n, since $x_n \in S$, we have $f(x_n) = g(x_n)$. Since f and g are continuous at x, we have

   $$f(x) = \lim f(x_n) = \lim g(x_n) = g(x).$$

   This is true for every $x \in \mathbb{R}$. So $f(x) = g(x)$ for every $x \in \mathbb{R}$. $\square$

2. For each of the following, either give an example of a power series with the given properties, or prove that one cannot exist. The center does not have to be 0.

   (a) [3pts.] A power series with interval of convergence $(0, 2]$.

   (b) [4pts.] A power series which converges uniformly on its interval of convergence.

   (c) [3pts.] A power series with interval of convergence $(-2, 2)$.

   solution. (a) The series $\sum \frac{(-1)^n x^n}{n}$ has radius $R = 1$ because $|\frac{a_{n+1}}{a_n}| = \frac{n}{n+1} \to 1$. At $x = -1$, the series becomes $\sum \frac{1}{n}$, which converges. At $x = 1$, the series becomes $\sum \frac{(-1)^n}{n}$, which converges by the alternative series test. So the exact interval of convergence is $(-1, 1]$. Note that $(0, 2]$ is the translation of $(-1, 1]$ by 1 to the right. So $\sum \frac{(-1)^n (x-1)^n}{n}$ has interval of convergence $(0, 2]$.

   (b) The series $\sum \frac{x^n}{n^2}$ has radius $R = 1$ because $|\frac{a_{n+1}}{a_n}| = \frac{n^2}{(n+1)^2} \to 1$. We now show that the series converges uniformly on $[-1, 1]$, and so the interval of convergence is $[-1, 1]$. The uniform convergence follows from Weierstrass M-test because $|\frac{x^n}{n^2}| \leq \frac{1}{n^2}$ for all $x \in [-1, 1]$ and $n \in \mathbb{N}$, and $\sum \frac{1}{n^2}$ converges.

   (c) The series $\sum x^n$ has radius $R = 1$ because $|a_n|^{1/n} = 1 \to 1$. At $x = 1$ or $x = -1$, $|x^n| = 1 \not\to 0$, which implies that $x^n \not\to 0$. So $\sum x^n$ does not converge at 1 or -1. Then the interval of convergence of $\sum x^n$ is $(-1, 1)$. Note that $x \in (-2, 2)$ if and only if $x/2 \in (-1, 1)$. So the series $\sum \frac{x^n}{2^n} = \sum (x/2)^n$ has interval of convergence $(-2, 2)$. $\square$

3. (a) [4pts] Let $f : \mathbb{R} \to \mathbb{R}$. What does it mean to say that f is differentiable at x_0?

(b) [6pts] Prove that $f(x) = \cos(\sin(x^3) + e^{\frac{1}{x^2}})$ is differentiable on $\mathbb{R} \setminus \{0\}$, and compute $f'(x)$. Carefully justify each step.

Solution. (a) We say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite.

(b) Since x^3 and $\sin x$ are differentiable on $\mathbb{R}$ with $\frac{d}{dx}(x^3) = 3x^2$ and $\sin' x = \cos x$, by the chain rule, $\sin(x^3)$ is differentiable on $\mathbb{R}$ with

$$\frac{d}{dx}\sin(x^3) = \cos(x^3) \cdot 3x^2.$$

Since $\frac{1}{x^2} = x^{-2}$ is differentiable on $\mathbb{R} \setminus \{0\}$ with $\frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$, and e^x is differentiable on $\mathbb{R}$ with $\frac{d}{dx}e^x = e^x$, by the chain rule, $e^{\frac{1}{x^2}}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and $\frac{d}{dx}e^{\frac{1}{x^2}} = e^{\frac{1}{x^2}} \cdot \frac{-2}{x^3}$. By the sum rule, $\sin(x^3) + e^{\frac{1}{x^2}}$ is differentiable on $\mathbb{R} \setminus \{0\}$ with derivative $\cos(x^3) \cdot 3x^2 + e^{\frac{1}{x^2}} \cdot \frac{-2}{x^3}$. Since $\cos x$ is differentiable on $\mathbb{R}$ with $\cos' x = -\sin x$, by the chain rule, f is differentiable on $\mathbb{R} \setminus \{0\}$, and

$$f'(x) = -\sin(\sin(x^3) + e^{\frac{1}{x^2}}) \cdot (\cos(x^3) \cdot 3x^2 + e^{\frac{1}{x^2}} \cdot \frac{-2}{x^3}).$$

□

4. (a) [4pts] What is the Weierstrass M-test?

(b) [6pts] Suppose that a power series $\sum a_n x^n$ has radius of convergence $R > 0$. Let $0 < R_0 < R$. Prove that the series $\sum a_n x^n \cos(x^2)$ converges uniformly on $[-R_0, R_0]$ to a continuous function.

Solution. (a) Suppose (g_n) is a sequence of functions defined on S and (M_n) is a sequence of nonnegative real numbers such that $\sum M_n$ converges. If $|g_n(x)| \leq M_n$ for every $x \in S$ and $n \in \mathbb{N}$, then $\sum g_n$ converges uniformly on S.

(b) By the definition of R, we know that $\sum |a_n| x^n$ also have radius R. Since $|R_0| = R_0 < R$, we have that $\sum |a_n| R_0^n$ converges. Let $g_n(x) = a_n x^n \cos(x^2)$, $M_n = |a_n| R_0^n$ and $S = [-R_0, R_0]$. We have shown that $\sum M_n$ converges. Note that for any n and $x \in S$, $|g_n(x)| = |a_n x^n \cos(x^2)| \leq |a_n||x|^n \leq |a_n| R_0^n = M_n$. By Weierstrass M-test, $\sum a_n x^n \cos(x^2) = \sum g_n(x)$ converges uniformly on $S = [-R_0, R_0]$. Finally, since each g_n is continuous on S, the uniform limit of the series $\sum g_n$ should also be continuous on S.

□

5. Let (a_n) be a sequence of positive numbers such that $\lim a_n = 0$. (a) [5pts.] Give an example to show that $\sum a_n$ need not converge. (b) [5pts.] Prove that there exists a subsequence (a_{n_k}) of (a_n) such that $\sum a_{n_k}$ converges.

Solution. (a) Let $a_n = \frac{1}{n}$, $n \in \mathbb{N}$. We know that $\frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges.

(b) We will prove that there exist $1 \le n_1 < n_2 < \cdots$ such that $|a_{n_k}| \le \frac{1}{k^2}$ for each k. Then since $\sum \frac{1}{k^2}$ converges, by comparison test, $\sum a_{n_k}$ would converge. We construct those n_k's inductively. Since $a_n \to 0$, letting $\varepsilon = 1$, we find that there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|a_n - 0| < 1$. Taking $n_1 = N_1 + 1$. Then $|a_{n_1}| < \frac{1}{1^2}$. Suppose we have found $1 \le n_1 < \cdots < n_m$ such that $|a_{n_k}| \le \frac{1}{k^2}$ for all $1 \le k \le m$. Letting $\varepsilon = \frac{1}{(m+1)^2}$ and using $a_n \to 0$, we find that there is $N_{m+1} \in \mathbb{N}$ such that for $n > N_{m+1}$, $|a_n - 0| < \frac{1}{(m+1)^2}$. Let $n_{m+1} = \max\{n_m, N_{m+1}\} + 1$. Then $n_{m+1} > n_m$ and $n_{m+1} > N_{m+1}$. The latter implies that $|a_{n_{m+1}}| \le \frac{1}{(m+1)^2}$. So we now have $1 \le n_1 < \cdots < n_m < n_{m+1}$ such that $|a_{n_k}| \le \frac{1}{k^2}$ for all $1 \le k \le m+1$. By induction, we get the desired subsequence (a_{n_k}). $\square$