## Homework 9 (due on 11/1)

- Read Sections 20, 23, and 24 for the next week.
- We are going to have the second quiz on Monday, October 28. It covers the material we have learned up to Oct 18.

19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish. (a) $f(x) = x^{17}\sin x - e^x \cos 3x$ on $[0, \pi]$; (d) $f(x) = x^3$ on $\mathbb{R}$; (f) $f(x) = \sin\frac{1}{x^2}$ on $(0, 1]$; (g) $f(x) = x^2 \sin\frac{1}{x}$.

19.2 Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$ property in Definition 19.1. (c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$.

19.4 (a) Prove that if $f$ is uniformly continuous on a bounded set $S$, then $f$ is a bounded function on $S$. Hint: Assume not. Use Theorems 11.5 and 19.4.

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

19.5 Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems or Exercise 19.4(a). (a) $\tan x$ on $[0, \frac{\pi}{4}]$, (b) $\tan x$ on $[0, \frac{\pi}{2})$, (c) $\frac{1}{x}\sin^2 x$ on $(0, \pi]$, (d) $\frac{1}{x-3}$ on $(0, 3)$, (e) $\frac{1}{x-3}$ on $(3, \infty)$, (f) (d) $\frac{1}{x-3}$ on $[4, \infty)$.

19.8 (a) Use the Mean Value theorem to prove $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$; see the proof of Theorem 19.6.

(b) Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

20.6 Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to 0} f(x)$ *when they exist* for the function $f(x) = \frac{x^3}{|x|}$. Prove your assertions.

20.11 Find the following limits. (a) $\lim_{x\to a} \frac{x^2-a^2}{x-a}$; (b) $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$, $b > 0$; (c) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$. Hint: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$.

20.16 Suppose that the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.

(a) Show if $f_1(x) \le f_2(x)$ for all $x$ in some interval $(a, b)$, then $L_1 \le L_2$. Hint: You may use the results on limits of sequences: If $x_n \to L_1$ and $y_n \to L$, and $x_n \le y_n$ for each $n$, then $L_1 \le L_2$. See Exercises 8.9 and 9.9.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all $x$ in some interval $(a, b)$. Can you conclude that $L_1 < L_2$?

20.17 Show that if $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ and if $f_1(x) \le f_2(x) \le f_3(x)$ for all $x$ in some interval $(a, b)$, then $\lim_{x\to a^+} f_2(x) = L$. Hint: When $L \in \mathbb{R}$, use Exercise 8.5 (Squeeze Lemma for sequences, Theorem 1 in the lecture notes of Sep 16-18). When $L = +\infty$ or $-\infty$, use Exercise 9.9 (the first problem in Homework 4).