Homework 8 Solutions

17.2 Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$.

(a) Determine the following functions: $f + g$, fg, $f \circ g$, $g \circ f$. Be sure to specify their domains.

(b) Which of the functions f, g, $f + g$, fg, $f \circ g$, $g \circ f$ is continuous?

Solution. (a) Since $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$, $\text{dom}(f + g) = \text{dom}(fg) = \text{dom}(f \circ g) = \text{dom}(g \circ f) = \mathbb{R}$. We have

– $(f + g)(x) = 4 + x^2$ for $x \geq 0$; $= x^2$ for $x < 0$;

– $(fg)(x) = 4x^2$ for $x \geq 0$; $= 0$ for $x < 0$;

– $(f \circ g)(x) = 4$;

– $(g \circ f)(x) = 16$ for $x \geq 0$; $= 0$ for $x < 0$.

(b) $g, fg, f \circ g$ are continuous, $f, f + t, g \circ f$ are not continuous. □

17.3 Accept on faith that the following familiar functions are continuous on their domains: $\sin x$, $\cos x$, e^x, 2^x, $\log_e x$ for $x > 0$, x^p for $x > 0$ [p any real number]. Use these facts and theorems in this section to prove the following functions are also continuous. (b) $[\sin^2 x + \cos^6 x]^\pi$ (e) $\tan x$ for $x \neq$ odd multiple of $\frac{\pi}{2}$.

Solution. (b) From that $\sin x$ is continuous, we get $\sin^2 x = \sin x \cdot \sin x$ is continuous. From the continuity of $\cos x$ and x^6, we see that their composition $\cos^6 x$ is continuous. Combining the continuity of $\sin^2 x$ and $\cos^6 x$, we see that $\sin^2 x + \cos^6 x$ is continuous. Combining this fact with the continuity of x^π, we see that their composition $[\sin^2 x + \cos^6 x]^\pi$ is continuous.

(e) Since $\sin x$ and $\cos x$ are continuous, their ratio $\tan x = \frac{\sin x}{\cos x}$ is continuous for $x \in \mathbb{R}$ such that $\cos x \neq 0$. Since $\cos x = 0$ iff x is an odd multiple of $\frac{\pi}{2}$, we get the conclusion. □

17.10 Prove the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the $\varepsilon - \delta$ property in Theorem 17.2.

(a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;

(b) $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$, $x_0 = 0$;

(c) $\text{sgn}(x) = 1$ for $x > 0$, $\text{sgn}(x) = -1$ for $x < 0$, and $\text{sgn}(0) = 0$, $x_0 = 0$.

Proof. (a) Let $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then $x_n \to 0 = x_0$. Since $x_n > 0$, $f(x_n) = 1$ for all n. But $f(x_0) = 0$. So we do not have $f(x_n) \to f(x_0)$.

(b) We know $\sin(2n\pi + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$. Let $x_n = \frac{1}{2n\pi+\pi/2}$, $n \in \mathbb{N}$. Then $x_n \to 0 = x_0$, and $g(x_n) = \sin(2n\pi + \pi/2) = 1$ for all n. So $g(x_n) \to 1 \neq 0 = g(x_0)$.

(c) Let $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then $x_n \to 0 = x_0$. Since $x_n > 0$, $\text{sgn}(x_n) = 1$ for all n. But $\text{sgn}(x_0) = 0$. So we do not have $\text{sgn}(x_n) \to \text{sgn}(x_0)$. □

17.12 (a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$. Hint: Use part (a).

Proof. (a) Let $x_0 \in (a, b)$. Let $s = \min\{x_0 - a, b - x_0\} > 0$. If $|x - x_0| < s$, then $x \in (x_0 - s, x_0 + s) \subseteq (a, b)$. For each $n \in \mathbb{N}$, by the denseness of $\mathbb{Q}$, there is $r_n \in \mathbb{Q}$ lying in the interval $(x_0 - \frac{s}{n}, x_0 + \frac{s}{n}) \subseteq (x_0 - s, x_0 + s) \subseteq (a, b)$. Thus, (r_n) is a sequence in (a, b), and $|r_n - x_0| < \frac{s}{n}$ for each n. So we have $r_n \to x_0$. By the assumption, $f(r_n) = 0$ for each n. By the continuity of f at x_0, $f(x_0) = \lim f(r_n) = 0$.

(b) Let $h = f - g$. Then h is continuous on (a, b) and $h(r) = 0$ for each rational number r in (a, b). By (a) $h(x) = 0$ for all $x \in (a, b)$, which implies that $f(x) = g(x)$ for all $x \in (a, b)$. □

18.2 Reread the proof of Theorem 18.1 (a continuous function reaches max and min) with $[a, b]$ replaced by (a, b). Where does it break down? Discuss.

Solution. When we apply the Bolzano-Weierstrass Theorem, we get a convergent subsequence (x_{n_k}) in (a, b). The limit (x_{n_k}) may be a or b, at which f is not defined, and so we can not conclude that the sequence $(f(x_{n_k}))$ converges. □

18.6 Prove $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

Proof. Let $f(x) = x - \cos x$. Since x and $\cos x$ are continuous, f is continuous on $\mathbb{R}$. We calculate $f(0) = 0 - \cos 0 = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$. By Intermediate Value Theorem, there is $x \in (0, \frac{\pi}{2})$ such that $0 = f(x) = x - \cos x$, which implies that $x = \cos x$. □

18.9 Prove that a polynomial function f of odd degree has at least one real root.

Proof. Suppose f is of degree n and expressed by $f(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$. We may assume that the leading coefficient a_n is positive because otherwise $-f$ is also a polynomial of odd degree, whose leading coefficient is positive, and we may work on $-f$ (a root of $-f$ is also a root of f).

We calculate $\frac{f(m)}{m^n} = \sum_{k=0}^{n} a_k m^{k-n} = a_n + \sum_{k=0}^{n-1} a_k m^{k-n}$. For $0 \leq k \leq n-1$, we have $k - n < 0$, and so $m^{k-n} \to 0$ as $m \to \infty$. Thus,

$$\lim_{m\to\infty} \frac{f(m)}{m^n} = a_n + \sum_{k=0}^{n-1} a_k \lim_{m\to\infty} m^{k-n} = a_n > 0.$$

On the other hand, using the oddness of n, we get

$$\lim_{m\to\infty} \frac{f(-m)}{m^n} = (-1)^n a_n + \sum_{k=0}^{n-1} a_k \lim_{m\to\infty} (-1)^k m^{k-n} = -a_n < 0.$$

Thus, there are $m_1, m_2 \in \mathbb{N}$ such that $\frac{f(m_1)}{m_1^n} > 0$ and $\frac{f(-m_2)}{m_2^n} < 0$. Since $m_1^n, m_2^n > 0$, we get $f(m_1) > 0$ and $f(m_2) < 0$. Since f is continuous on $\mathbb{R}$, by Intermediate Value Theorem, there is $x \in (-m_2, m_1)$ such that $f(x) = 0$. Such x is a real root of f. $\square$

E1 Let $f(x) = 0$ for all $x \in \mathbb{Q}$ and $f(x) = 1$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that f is not continuous at any $x \in \mathbb{R}$. Hint: Use the denseness of $\mathbb{Q}$ (4.7) and the denseness of $\mathbb{R} \setminus \mathbb{Q}$ (Exercise 4.12).

Proof. Fix $x_0 \in \mathbb{R}$. We now show that f is not continuous at x_0. Case 1. $x_0 \in \mathbb{R}\setminus\mathbb{Q}$. Then $f(x_0) = 1$. By the denseness of $\mathbb{Q}$ in 4.7, for any $n \in \mathbb{N}$, there is $x_n \in \mathbb{Q} \cap (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then $|x_n - x_0| < \frac{1}{n}$ for all n, which implies that $x_n \to x_0$. Since $x_n \in \mathbb{Q}$, we have $f(x_n) = 0$ So we do not have $f(x_n) \to f(x_0)$, and so f is not continuous at x_0. Case 2. $x_0 \in \mathbb{Q}$. Then $f(x_0) = 0$. By the denseness of $\mathbb{R} \setminus \mathbb{Q}$ as in Exercise 4.12, for any $n \in \mathbb{N}$, there is $x_n \in (\mathbb{R} \setminus \mathbb{Q}) \cap (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then $|x_n - x_0| < \frac{1}{n}$ for all n, which implies that $x_n \to x_0$. Since $x_n \in \mathbb{R} \setminus \mathbb{Q}$, we have $f(x_n) = 1$ So we do not have $f(x_n) \to f(x_0)$, and so f is not continuous at x_0. $\square$