Homework 6 Solutions

14.1 (e) Determine the convergence of $\sum \frac{\cos^2 n}{n^2}$ and justify your answer.

Solution. Since $|\cos^2 n| = |\cos n|^2 \leq 1$, we have $|\frac{\cos^2 n}{n^2}| \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, by comparison test, $\sum \frac{\cos^2 n}{n^2}$ also converges. □

14.2 (a) Determine the convergence of $\sum \frac{n-1}{n^2}$ and justify your answer.

Solution. The series is divergent. We observe that $\frac{n-1}{n^2}$ is approximately equal to $\frac{1}{n}$, and we know that $\sum \frac{1}{n}$ diverges. In order to show that $\sum \frac{n-1}{n^2}$ diverges, we compare $\frac{n-1}{n^2}$ with $\frac{1}{2n}$. Note that $\frac{n-1}{n^2} \geq \frac{1}{2n}$ is equivalent to that $2n(n-1) \geq n^2$, i.e., $n^2 \geq 2n$. So if $n \geq 2$, then $\frac{n-1}{n^2} \geq \frac{1}{2n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{2n}$ also diverges. By comparison test, $\sum \frac{n-1}{n^2}$ also diverges.

There is another way to show that the series diverges. We know that $\sum \frac{1}{n^2}$ converges. If $\sum \frac{n-1}{n^2}$ also converges, then $\sum(\frac{1}{n^2} + \frac{n-1}{n^2}) = \sum \frac{1}{n}$ would converge, which is a contradiction. □

14.4 (c) Determine the convergence of $\sum \frac{n!}{n^n}$ and justify your answer. Hint: You may use the limit $(1 + \frac{1}{n})^n \to e \approx 2.71828$.

Proof. We use the ratio test. We compute

$$|\frac{a_{n+1}}{a_n}| = \frac{\frac{(n+1)!}{(n+1)^{n+1}}\ \frac{n!}{n^n}}{=} \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1.$$

Since $\lim |\frac{a_{n+1}}{a_n}| < 1$, by ratio test, the series converges. □

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges. Hint: You may use the fact that if $0 \leq a < 1$ and $p > 1$, then $a^p \leq a$.

Proof. Since $\sum a_n$ converges, we have $a_n \to 0$. So there is $N \in \mathbb{N}$ such that for $n > N$, $|a_n - 0| < 1$. Since $a_n \geq 0$, we get $0 \leq a_n < 1$ for $n > N$. Since $p > 1$, we have $0 \leq a_n^p \leq a_n$ for $n > N$. Since $\sum a_n$ converges, by comparison test, $\sum a_n^p$ also converges. □

14.13 (b) Prove $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Hint: Use $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

(c) Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. Hint: Note $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$.

(d) Use (c) to calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Proof. (b) We first calculate the partial sum sequence. For any $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} (\frac{1}{n} - \frac{1}{n+1}) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{N} - \frac{1}{N+1}) = 1 - \frac{1}{N+1}.$$

Note that there are many cancelations. Since $\frac{1}{n+1} \to 0$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N\to\infty} (1 - \frac{1}{N+1}) = 1 - 0 = 1.$$

(c) Using that $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$, we can calculate the partial sum sequence:

$$\sum_{n=1}^{N} \frac{n-1}{2^{n+1}} = \sum_{n=1}^{N} \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right) = \left(\frac{1}{2^1} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \cdots + \left(\frac{N}{2^N} - \frac{N+1}{2^{N+1}}\right) = \frac{1}{2} - \frac{N+1}{2^{N+1}}.$$

We now show that $\frac{N}{2^N} \to 0$. This actually follows from ratio test. Note that $\frac{N+1}{2^{N+1}} / \frac{N}{2^N} = \frac{N+1}{2N} \to \frac{1}{2}$. By ratio test, $\sum \frac{N}{2^N}$ converges, which then implies that $\frac{N}{2^N} \to 0$. Thus,

$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{n-1}{2^{n+1}} = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{N+1}{2^{N+1}}\right) = \frac{1}{2}.$$

(d) From (c), we get $\sum_{n=1}^{\infty} \frac{n-1}{2^n} = 1$. We learned in class that for $r \in \mathbb{R}$ with $|r| < 1$, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, which implies that $\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} - 1 = \frac{r}{1-r}$. Taking $r = \frac{1}{2}$, we get $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. So

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \frac{n-1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + 2 = 2.$$

□