Homework 5 (due on 10/4)

- Read Sections 12 and 14 for the next week.
- The first midterm is to be held on October 7. It is an one hour exam. The class time will be extended for 10 minutes.

10.6 (a) Let (s_n) be a sequence such that $|s_{n+1} - s_n| \leq 2^{-n}$ for all $n \in \mathbb{N}$. Prove (s_n) is a Cauchy sequence and hence a convergent sequence. Hint: If $n > m$, write $s_n - s_m$ as $\sum_{k=m}^{n-1}(s_{k+1} - s_k)$. Then use triangle inequality, Exercise 9.18, and Theorem 9.7.

- Exercise 11.2, 11.3, 11.4 for sequences (b_n), (c_n), (s_n), (w_n). Do them as a single problem. You only need to provide answers. No rigorous proofs are needed.

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0,1]$. This means that every element in $\mathbb{Q} \cap (0,1]$ appears in the sequence exactly once.

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

Hint: Use Exercise 8.9 to give an upper bound and a lower bound of the subsequential limits. Use denseness of $\mathbb{Q}$ to show that any open subinterval of $(0,1]$ contains infinitely many elements in the sequence. Then apply Theorems 11.2 and 11.8.

11.8 Use Definition 10.6 and Exercise 5.4 to prove $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).

11.9 (a) Show the closed interval $[a,b]$ is a closed set.

(b) Is there a sequence (s_n) such that $(0,1)$ is its set of subsequential limits?

12.4 Show $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). Hint: First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply Exercise 9.9(c).

12.6 Let (s_n) be a bounded sequence, and let k be a nonnegative real number.

(a) Prove $\limsup(ks_n) = k \limsup s_n$.

(b) Do the same for $\liminf$. Hint: Use Exercise 11.8.

(c) What happens in (a) and (b) if $k < 0$?

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

E1 Let (s_n) be a sequence of real numbers and $x \in \mathbb{R}$. Prove that (a) if $\limsup s_n > x$, then there are infinitely many n such that $s_n > x$; (b) if there are infinitely many n such that $s_n \geq x$, then $\limsup s_n \geq x$.