Homework 4 (due on 9/27)

- Read Sections 10 and 11 for the next week.

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Solution. (a) Let $M > 0$. Since $s_n \to +\infty$, there is $N_s \in \mathbb{N}$ such that $n > N_s$ implies that $s_n > M$. Let $N = \max\{N_0, N_s\}$. If $n > N$, then $t_n = s_n > M$. So $t_n \to +\infty$.

(b) Let $M < 0$. Since $s_n \to -\infty$, there is $N_s \in \mathbb{N}$ such that $n > N_s$ implies that $s_n < M$. Let $N = \max\{N_0, N_s\}$. If $n > N$, then $t_n = s_n < M$. So $t_n \to +\infty$.

(c) We have to consider different cases. Case 1. $\lim s_n$ and $\lim t_n$ are both finite. In this case we can apply a limit theorem to conclude that $\lim s_n \leq \lim t_n$. Case 2. $\lim s_n$ is not finite. There are two subcases. Case 2.1. $\lim s_n = -\infty$. Then $\lim s_n \leq \lim t_n$ always holds because $\lim t_n$ takes values in $\mathbb{R} \cup \{+\infty, -\infty\}$, and for any $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $-\infty \leq a$. Case 2.2. $\lim s_n = +\infty$. Then by (a) $\lim t_n = +\infty$, and so $\lim s_n \leq \lim t_n$ still holds. Case 3. $\lim t_n$ is not finite. There are two subcases. Case 3.1. $\lim t_n = +\infty$. Then $\lim s_n \leq \lim t_n$ always holds because $\lim s_n$ takes values in $\mathbb{R} \cup \{+\infty, -\infty\}$, and for any $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $a \leq +\infty$. Case 3.2. $\lim t_n = -\infty$. Then by (b) $\lim s_n = -\infty$, and so $\lim s_n \leq \lim t_n$ still holds. □

9.12 Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. Hint: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |sn| = +\infty$. Hint: Apply (a) to the sequence $t_n = \frac{1}{s_n}$; see Theorem 9.10.

Proof. (a) Since $L < 1$, we may choose $a \in (L, 1)$. Let $\varepsilon = a - L$. Since $|\frac{s_{n+1}}{s_n}| \to L$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then $-\varepsilon < |\frac{s_{n+1}}{s_n}| - L < \varepsilon$, which implies that $|\frac{s_{n+1}}{s_n}| < a$ and so $|s_{n+1}| < a|s_n|$. We now show that $|s_n| < a^{n-N}|s_N|$ for $n > N$ by induction. The basis case is when $n = N + 1$, $|s_{N+1}| < a|s_N|$. This is true by taking $n = N$ in $|s_{n+1}| < a|s_n|$. Suppose $|s_n| < a^{n-N}|s_N|$ for some $n > N$. Then $|s_{n+1}| < a|s_n| < a \cdot a^{n-N}|s_N| = a^{n+1-N}|s_N|$. So $|s_n| < a^{n-N}|s_N|$ holds for all $n > N$. Since $0 \leq |s_n| \leq a^{n-N}|s_N|$, and $a^{n-N}|s_N| \to 0$ because $0 < a < 1$, by squeeze lemma, we get $|s_n| \to 0$, which then implies that $s_n \to 0$.

(b) Let $t_n = \frac{1}{s_n}$. Then $\lim |\frac{t_{n+1}}{t_n}|$ exists and equals $\frac{1}{L}$ if $L < \infty$ and equals 0 if $L = +\infty$. In any case we have $\lim |\frac{t_{n+1}}{t_n}| < 1$. Applying (a) to (t_n), we get $t_n \to 0$, and so $|t_n| \to 0$. Since $|s_n| = \frac{1}{|t_n|}$ and $|s_n| > 0$ for all n, we get $|s_n| \to +\infty$. □

9.13 Show

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & \text{if } |a|<1 \\ 1, & \text{if } a=1 \\ +\infty, & \text{if } a>1 \\ \text{does not exist}, & \text{if } a\le -1 \end{cases}$$

Proof. If $|a|<1$, then by Theorem 9.7 (b), $a^n \to 0$. If $a=1$, then $a^n=1$ for all n, and so $a^n \to 1$ trivially. If $a>1$, then $|\frac{1}{a}| = \frac{1}{a} < 1$. Since $\frac{1}{a^n} = (\frac{1}{a})^n$, we get $\frac{1}{a^n} \to 0$. Since $a^n > 0$ for all n, we get $a^n \to +\infty$. If $a=-1$, we proved in class that $((-1)^n)$ has no limit. Finally, suppose $a<-1$. Then $|a|>1$. So by the previous case, $|a^n| = |a|^n \to +\infty$. Since $a<0$, (a^n) has alternative signs. We claim that the sequence (a^n) is neither bounded above nor bounded below. For this purpose, we show that for any $M \in (0,\infty)$, there are $n_1, n_2 \in \mathbb{N}$ such that $a^{n_1} > M$ and $a^{n_2} < -M$. Since $|a^n| \to +\infty$, there is $N \in \mathbb{N}$ such that for $n > N$, $|a^n| > M$. Since $a^n > 0$ for even n and $a^n < 0$ for odd n, if we choose an even number n_1 and an odd number n_2 with $n_1, n_2 > N$. Then $a^{n_1} = |a^{n_1}| > M$ and $a^{n_2} = -|a^{n_2}| < -M$. So the claim is proved. Now since any sequence (s_n) with a limit is either bounded above or bounded below, we conclude that (a^n) has no limit if $a<-1$. □

9.16 (a) Prove $\lim \frac{n^4+8n}{n^2+9} = +\infty$.

Proof. Since $\frac{n^4+8n}{n^2+9} > 0$ for all n, it suffices to show that $\lim \frac{n^2+9}{n^4+8n} = 0$. This is true because

$$\frac{n^2+9}{n^4+8n} = \frac{1/n^2+9/n^4}{1+8/n^3} \to \frac{0^2+9*0^4}{1+8*0^3} = 0.$$

□

9.18 (a) Verify $1+a+a^2+\cdots+a^n = \frac{1-a^{n+1}}{1-a}$ for $a \ne 1$.

(b) Find $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n)$ for $|a|<1$.

(d) What is $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n)$ for $a \ge 1$?

Proof. (a) We prove this by induction. The basis case is $1+a = \frac{1-a^2}{1-a}$, which is obvious. Suppose the statement holds for n. Then

$$1+a+a^2+\cdots+a^n+a^{n+1} = \frac{1-a^{n+1}}{1-a} + a^{n+1} = \frac{1-a^{n+1}}{1-a} + \frac{a^{n+1}(1-a)}{1-a}$$

$$= \frac{(1-a^{n+1})+(a^{n+1}-a^{n+2})}{1-a} = \frac{1-a^{n+2}}{1-a} = \frac{1-a^{(n+1)+1}}{1-a}.$$

So the statement is also true for $n+1$. Thus, it is true for all $n \in \mathbb{N}$.

(b) By (a), we need to calculate $\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$. Since $|a| < 1$, by Exercise 9.13, $a^{n+1} \to 0$. So $\frac{1-a^{n+1}}{1-a} \to \frac{1-0}{1-a} = \frac{1}{1-a}$.

(d) From $a \geq 1$ we get $a^n \geq 1^n$, and so $1 + a + a^2 + \cdots + a^n \geq n + 1$ for all n. Since $\lim(n + 1) = +\infty$, by Exercise 9.9 we get $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = +\infty$. □

10.7 Let S be a bounded nonempty subset of R such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Proof. Since S is bounded, $\sup S \in \mathbb{R}$. Since $\sup S$ is the least upper bound of S, for any $n \in \mathbb{N}$, $\sup S - \frac{1}{n}$ is not an upper bound of S, and so there is an element in S, denoted by s_n, which is greater than $\sup S - \frac{1}{n}$. Then we get a sequence (s_n) in S such that $s_n > \sup S - \frac{1}{n}$ for any n. Since $\sup S$ is an upper bound of S and $s_n \in S$, we also have $\sup S \geq s_n$ for all n. Applying Squeeze lemma to the inequalities $\sup S \geq s_n > \sup S - \frac{1}{n}$ we conclude that $s_n \to \sup S$. □

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3, and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence. Hint: Still use induction.

(d) Show $\lim s_n$ exists and find $\lim s_n$. Hint: $\lim s_{n+1} = \lim s_n$.

Solution. (a) $s_2 = \frac{1}{3}(1 + 1) = \frac{2}{3}$, $s_3 = \frac{1}{3}(\frac{2}{3} + 1) = \frac{5}{9}$, $s_4 = \frac{1}{3}(\frac{5}{9} + 1) = \frac{14}{27}$.

(b) The basis case is $s_1 = 1 > \frac{1}{2}$, which is obvious. Suppose $s_n > \frac{1}{2}$. Then $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$. So the induction step also holds. Thus, $s_n > \frac{1}{2}$ for all n.

(c) We still prove by induction. We need to show that $s_n \geq s_{n+1}$ for all n. The basis is $s_1 \geq s_2$, which is obvious since $s_1 = 1$ and $s_2 = \frac{2}{3}$. Suppose $s_n \geq s_{n+1}$. Then $s_{n+1} = \frac{1}{3}(s_n + 1) \geq \frac{1}{3}(s_{n+1} + 1) = s_{n+2}$. So the induction step also holds. Thus, $s_n \geq s_{n+1}$ for all n, and (s_n) is decreasing.

(d) Since (s_n) is decreasing and bounded below, it converges to a real number, say s. Since $s_{n+1} = \frac{1}{3}(s_n+1)$, by limit theorems, (s_{n+1}) converges to $\frac{1}{3}(s+1)$. Since $\lim s_{n+1} = \lim s_n$, we get $\frac{1}{3}(s + 1) = s$. Solving this equation we get $s = \frac{1}{2}$. Thus, $\lim s_n = \frac{1}{2}$. □

E1 Prove that if (s_n) is decreasing, then $\lim s_n$ exists and equals $\inf\{s_n : n \in \mathbb{N}\}$. If (s_n) is bounded below, then (s_n) converges.

Proof. Let $S = \{s_n : n \in \mathbb{N}\}$ and $s = \inf S$. Consider two cases. Case 1. S is bounded below. In this case $s \in \mathbb{R}$ is the biggest lower bound of S. Let $\varepsilon > 0$. Since s is the biggest lower bound of S, $s + \varepsilon$ is not a lower bound of S. Thus, S contains an element smaller than $s + \varepsilon$. This means, for some $N \in \mathbb{N}$, we have $s_N < s + \varepsilon$. Since (s_n) is decreasing, for any $n > N$, $s_n \leq s_N < s + \varepsilon$. On the other hand, $s_n \geq s$ for all $n \in \mathbb{N}$ since s is a lower bound of S. So for any $n > N$, $s + \varepsilon > s_n \geq s$, which implies that $|s_n - s| < \varepsilon$. Thus, (s_n) converges to s. Case 2. S is not bounded below. Then $s = -\infty$. Let $M < 0$. Since S is not bounded below, M is not a lower bound of S. So S contains an element less than M, i.e., for some $N \in \mathbb{N}$, we have $s_N < M$. Since (s_n) is decreasing, for any $n > N$, $s_n \leq s_N < M$. Thus, $s_n \to -\infty = s$. □