Homework 4 (due on 9/27)

- Read Sections 10 and 11 for the next week.

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Hint: For (c), if $\lim s_n$ or $\lim t_n$ are both finite, you can apply a limit theorem for finite limits in lecture notes. Otherwise you can use (a) and (b).

9.13 Show

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & \text{if } |a| < 1 \\ 1, & \text{if } a = 1 \\ +\infty, & \text{if } a > 1 \\ \text{does not exist}, & \text{if } a \leq -1 \end{cases}$$

Hint: For the last case, if $a = -1$, it discussed in class; if $a < -1$, you may show that (a^n) is neither bounded above nor bounded below.

9.16 (a) Prove $\lim \frac{n^4+8n}{n^2+9} = +\infty$.

9.18 (a) Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?

Hint: Use Exercises 9.13 and 9.9.

10.7 Let S be a bounded nonempty subset of R such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$. Hint: Use the fact that for any $n \in \mathbb{N}$, $\sup S - \frac{1}{n}$ is not an upper bound of S. Then use Squeeze lemma.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3, and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence. Hint: Still use induction.

(d) Show $\lim s_n$ exists and find $\lim s_n$. Hint: $\lim s_{n+1} = \lim s_n$.

E1 Prove that if (s_n) is decreasing, then $\lim s_n$ exists and equals $\inf\{s_n : n \in \mathbb{N}\}$. If (s_n) is bounded below, then (s_n) converges.