Homework 3 (due on 9/20)

We will frequently use the following facts about $\sup S$ and $\inf S$. They hold true no matter whether or not S is bounded above or bounded below. They have been proved in class. So you can use them directly in homework or exams. For your convenience, we now include a proof of each statement.

Let S be a nonempty subset of $\mathbb{R}$. Then we have the following statements.

(U1) For any $s \in S$, $s \le \sup S$. Proof: If S is bounded above, this holds because $\sup S$ is an upper bound of S. If S is not bounded above, this holds because $\sup S = +\infty$, and $+\infty > x$ for any $x \in \mathbb{R}$.

(U2) Let $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. If for any $s \in S$, $s \le a$, then $\sup S \le a$. Proof: If $a = +\infty$, then $\sup S \le a$ always holds because $\sup S$ is either $+\infty$ or a real number. If $a \in \mathbb{R}$, then the condition means that a is an upper bound of S. So S is bounded above, and $\sup S$ is the smallest upper bound of S. So we have $\sup S \le a$. Finally, a could not be $-\infty$ because $-\infty < x$ for any $x \in \mathbb{R}$. Remark: The converse is also true. If $\sup S \le a$, then for any $s \in S$, by (U1) $s \le \sup S \le a$.

(L1) For any $s \in S$, $s \ge \inf S$. Proof: If S is bounded below, this holds because $\inf S$ is a lower bound of S. If S is not bounded below, this holds because $\inf S = -\infty$, and $-\infty < x$ for any $x \in \mathbb{R}$.

(L2) Let $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. If for any $s \in S$, $s \ge a$, then $\inf S \ge a$. Proof: If $a = -\infty$, then $\inf S \ge a$ always holds because $\inf S$ is either $-\infty$ or a real number. If $a \in \mathbb{R}$, then the condition means that a is a lower bound of S. So S is bounded below, and $\inf S$ is the biggest lower bound of S. So we have $\inf S \ge a$. Finally, a could not be $+\infty$ because $+\infty > x$ for any $x \in \mathbb{R}$. Remark: The converse is also true. If $\inf S \ge a$, then for any $s \in S$, by (L1) $s \ge \inf S \ge a$.

4.7 & 5.6 Let S and T be nonempty subsets of $\mathbb{R}$.

(a) Prove if $S \subseteq T$, then $\inf T \le \inf S \le \sup S \le \sup T$.

(b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Note: S and T may or may not be bounded above or below. In part (b), do not assume $S \subseteq T$.

Solution. (a) For every $s \in S$, we have $s \in T$, and so by (U1), $s \le \sup T$. Since this inequality holds for every $s \in S$, by (U2) we have $\sup S \le \sup T$. For every $s \in S$, we have $s \in T$, and so by (L1), $s \ge \inf T$. Since this inequality holds for every $s \in S$, by (L2) we have $\inf S \ge \inf T$. Since S is nonempty, we may pick $s_0 \in S$. By (U1) and (L1), $\inf S \le s_0$ and $s_0 \le \sup S$. So $\inf S \le \sup S$.

(b) Since S and T are subsets of $S \cup T$, we get $\sup S, \sup T \leq \sup(S \cup T)$ by (a). So $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$. For any $s \in S$, $s \leq \sup S \leq \max\{\sup S, \sup T\}$; and for any $t \in T$, $t \leq \sup T \leq \max\{\sup S, \sup T\}$. Thus, for any $x \in S \cup T$, $x \leq \max\{\sup S, \sup T\}$. By (U2) we get $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$. So we have $\sup(S \cup T) = \max\{\sup S, \sup T\}$. $\square$

4.18 Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

(a) Observe S is bounded above and T is bounded below.

(b) Prove $\sup S \leq \inf T$.

(c) Give an example of such sets S and T where $S \cap T$ is nonempty.

(d) Give an example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set.

Solution. (a) By assumption, any $t \in T$ is an upper bound of S, and any $s \in S$ is a lower bound of T. Since S and T are not empty, we conclude that S is bounded above and T is bounded below.

(b) Fix $t \in T$. Since $s \leq t$ for every $s \in S$, by (U2) we get $\sup S \leq t$. Now $\sup S \leq t$ for every $t \in T$. By (L2) we get $\sup S \leq \inf T$.

(c) $S = (-\infty, 0)$ and $T = (0, \infty)$.

(c) $S = (-\infty, 0]$ and $T = [0, \infty)$. Note that $\sup S = 0 = \inf T$. $\square$

4.12 The elements of $\mathbb{R} \setminus \mathbb{Q}$ are called irrational numbers. Prove if $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$. Hint: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{R} \setminus \mathbb{Q}$.

Solution. We already know that $\sqrt{2}$ is an irrational number. We then conclude that for any $r \in \mathbb{Q}$, $r + \sqrt{2}$ is an irrational number. In fact, if $r + \sqrt{2} = r' \in \mathbb{Q}$, then since $\mathbb{Q}$ is a field, we get $\sqrt{2} = r' - r \in \mathbb{Q}$, a contradiction. By the denseness of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$, which implies $a < r + \sqrt{2} < b$. Then $x := r + \sqrt{2}$ is an irrational number that we need. $\square$

4.15 Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$. Compare Exercise 3.8.

Solution. Suppose $a \leq b$ does not hold. Then $a > b$, and so $a - b > 0$. By Archimedean Property, there is $n_0 \in \mathbb{N}$ such that $n_0(a - b) > 1$, which implies that $a - b > \frac{1}{n_0}$ and $a > b + \frac{1}{n_0}$. This contradicts that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$. So $a \leq b$. The assumption of this problem is weaker than that of Exercise 3.8. Instead of requiring that $a < b_1$ for any $b_1 > b$, we now only requiring that $a < b_1$ for any b_1 with the form $b_1 = b + \frac{1}{n}$, $n \in \mathbb{N}$. But we reach the same conclusion: $a \leq b$. $\square$

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

Solution. Let $S = \{r \in \mathbb{Q} : r < a\}$. Then for any $r \in S$, $r < a$. So by (U2), $\sup S \leq a$. If $\sup S \neq a$, then $\sup S < a$. By the denseness of $\mathbb{Q}$, there is $r \in \mathbb{Q}$ such that $\sup S < r < a$. Since $r < a$, we get $r \in S$. By (U1) we have $r \leq \sup S$, which contradicts that $\sup S < r$. So $\sup S = a$. $\square$

8.1 (a) Prove $\lim \frac{(-1)^n}{n} = 0$.

Solution. We have two ways to prove the equality. One way is to use the definition. Let $\varepsilon > 0$. By Archimedean Property, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then for any $n > N$, $|\frac{(-1)^n}{n} - 0| = |\frac{(-1)^n}{n}| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. So we get $\lim \frac{(-1)^n}{n} = 0$.

Another way is to use the limit theorem, which says that $s_n \to 0$ if and only if $|s_n| \to 0$. Since $|\frac{(-1)^n}{n}| = \frac{1}{n}$, and we proved $\frac{1}{n} \to 0$ in class, we get $\frac{(-1)^n}{n} \to 0$. $\square$

8.2 (e) Determine the limits of the sequence $s_n = \frac{1}{n}\sin n$, and then prove your claim.

Solution. The limit is 0. We have two ways to prove this claim. One way is to use the definition. Let $\varepsilon > 0$. By Archimedean Property, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then for any $n > N$, $|\frac{1}{n}\sin n - 0| = |\frac{1}{n}\sin n| = \frac{|\sin n|}{n} \leq \frac{1}{n} < \frac{1}{N} < \varepsilon$. Here we use that $|\sin x| \leq 1$. So we get $\lim \frac{1}{n}\sin n = 0$.

Another way is to use the squeeze lemma. Since $|\sin n| \leq 1$, we get $-\frac{1}{n} \leq s_n \leq \frac{1}{n}$. We have $\frac{1}{n} \to 0$. By a limit theorem, $-\frac{1}{n} \to -0 = 0$. By squeeze lemma, we then get $s_n \to 0$ as well. $\square$

8.3 Let (s_n) be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

Discussion. For a fixed $\varepsilon > 0$, we need a threshold N such that for $n > N$, $|\sqrt{s_n} - 0| < \varepsilon$. Since s_n and $\sqrt{s_n}$ are nonnegative, this is equivalent to that $|s_n| < \varepsilon^2$, which is possible for big enough n since $\lim s_n = 0$. $\square$

Solution. Fix $\varepsilon > 0$. Since $\lim s_n = 0$, by definition there is $N \in \mathbb{N}$ such that for $n > N$, $|s_n - 0| < \varepsilon^2$, which then implies that $|\sqrt{s_n} - 0| < \varepsilon$. So $\lim \sqrt{s_n} = 0$. $\square$

9.1 (c) Using the limit Theorems 9.2 - 9.7, prove the following. Justify all steps.

$$\lim \frac{17n^5 + 73n^4 - 18n^2 + 3}{23n^5 + 13n^3} = \frac{17}{23}.$$

Solution. By multiplying n^{-5} to both the enumerator and the denominator, we rewrite $\frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3}$ as

$$\frac{17+73n^{-1}-18n^{-3}+3n^{-5}}{23+13n^{-2}}.$$

Since $\lim n^{-1}=0$, we get $n^{-2}, n^{-3}, n^{-5} \to 0$. So the enumerator converges to $17+73\cdot 0-18\cdot 0+3\cdot 0=17$, and the denominator converges to $23+13\cdot 0=23$. Since both the enumerator and the denominator converge, and the latter limit is not zero, the fractal converges to $\frac{17}{23}$. □

9.5 Let $t_1=1$ and $t_{n+1}=\frac{t_n^2+2}{2t_n}$ for $n\geq 1$. Assume (t_n) converges and find the limit.

Solution. The proof is based on the assumption that (t_n) converges. We do not try to prove the assumption here. We will also use the fact that, when $\lim t_n$ exists, the limit of the sequence (t_{n+1}) also exists and agrees with $\lim t_n$.

Let $L=\lim t_n$. Since $t_{n+1}=\frac{t_n^2+2}{2t_n}$, by limit theorems, $\lim t_{n+1}=\frac{L^2+2}{2L}$. Since (t_{n+1}) has the same limit as (t_n), we get the equality $L=\frac{L^2+2}{2L}$. Solving the equation, we get $2L^2=L^2+2$ and $L^2=2$. There are two solutions $L=\sqrt{2}$ and $L=-\sqrt{2}$. The $-\sqrt{2}$ can not be the limit of (t_n) because every term of (t_n) is positive (which can be proved easily by induction), and so $\lim t_n\geq 0$ (Exercise 8.9 (a)). Thus, $\lim t_n=\sqrt{2}$. □