1.3 Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers $n$.

**Solution.** We prove by induction. The base case $1^3 = 1^2$ is true. Suppose the statement holds for $n$. So

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n + 1)}{2}\right)^2,$$

where the second “=” follows from $1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$. We now prove the statement for $n + 1$. By the displayed formula,

$$1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 = \left(\frac{n(n + 1)}{2}\right)^2 + (n + 1)^3 = \frac{1}{4}(n + 1)^2(n^2 + 4(n + 1))$$

$$= \frac{1}{4}(n + 1)^2(n + 2)^2 = \left(\frac{(n + 1)(n + 2)}{2}\right)^2 = (1 + 2 + \cdots + n + (n + 1))^2.$$

Thus the induction step is also proven, and the claim is true.

1.8 The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) $P_m$ is true, (ii) $P_{n+1}$ is true whenever $P_n$ is true and $n \geq m$.

(a) Prove $n^2 > n + 1$ for all integers $n \geq 2$.

(b) Prove $n! > n^2$ for all integers $n \geq 4$. [Recall $n! = n(n-1)\cdots 2 \cdot 1$; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.]

**Solution.** (a) The base case $2^2 > 2 + 1$ is true. Suppose the statement holds for some $n \geq 2$. We now prove the statement for $n + 1$. We have

$$(n + 1)^2 = n^2 + 2n + 1 > n^2 + 1 > (n + 1) + 1.$$  

So the induction step is proven, and the claim is true.

(b) The base case $4! > 4^2$ is true because $4! = 24$ and $4^2 = 16$. Suppose the statement holds for some $n \geq 4$. We now prove the statement for $n + 1$. We have

$$(n + 1)! = (n + 1)n! > (n + 1)n^2 > (n + 1)^2,$$

where the last step follows from (a). So the induction step is proven, and the claim is true.

1.9 (a) Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.
Proof. (a) We observe that $2^0 = 1, 0^2 = 0; 2^1 = 2, 1^2 = 1; 2^2 = 4, 2^2 = 4; 2^3 = 8, 3^2 = 9; 2^4 = 16, 4^2 = 16; 2^5 = 32, 5^2 = 25; 2^6 = 64, 6^2 = 36; 2^7 = 128, 7^2 = 49.$ For negative integers $n$, $2^n < 1$ and $n^2 ≥ 1$. So we conjecture that $2^n > n^2$ holds if and only if $n ∈ \{0, 1\}$ or $n ≥ 5$.

(b) We have excluded the case $n < 0$ and checked the case $n = 0, 1, 2, 3, 4$ one by one. We now show that $2^n > n^2$ for $n ≥ 5$ by induction. The base case $2^5 > 5^2$ is also checked above. Suppose the statement holds for some $n ≥ 5$. We now prove the statement for $n + 1$. Note $n^2 + 2n + 1 = (n + 1)^2$ implies $n^2 > 2n + 1$. So

$$2^{n+1} = 2 \cdot 2^n > 2n^2 = n^2 + 2n^2 > n^2 + 2n + 1 = (n + 1)^2.$$ 

So the induction step is proven, and the claim is true.

2.3 Show $\sqrt{2 + \sqrt{2}}$ is not a rational number.

Solution. We see that if $a = \sqrt{2 + \sqrt{2}}$, then we have $a^2 = 2 + \sqrt{2}, a^2 - 2 = \sqrt{2}$, and $(a^2 - 2)^2 = 2$. Expanding the formula, we see that $a$ is a solution of the equation

$$x^4 - 4x^2 + 2 = 0.$$ 

If $a ∈ Q$, then by a corollary of Rational Zeroes Theorem, $a$ is an integer that divides 2. So $a$ must be one of $1, -1, 2, -2$. Plugging these numbers into the equation, we see that none of them are roots. So $a$ can not be a rational number.

2.4 Show $\sqrt{5 - \sqrt{3}}$ is not a rational number.

Solution. We now use a slightly different approach. First we show that $b = \sqrt{3}$ is not a rational number. Note that $b$ solves the equation $x^2 - 3 = 0$. If $b ∈ Q$, then by a corollary of Rational Zeroes Theorem, $b$ is an integer that divides $-3$. So $b$ must be one of $1, -1, 3, -3$. Plugging these numbers into the equation, we see that none of them are roots. So $b$ can not be a rational number. Suppose now $a = \sqrt{5 - \sqrt{3}}$ is a rational number, say $\frac{c}{d}$. Then $5 - \sqrt{3} = a^2 = \frac{c^2}{d^2}$ is also a rational number, and so $\sqrt{3} = 5 - \frac{c^2}{d^2}$ is a rational number, which contradicts the first part of the proof. So $a$ is also not a rational number.

2.7 Show the following irrational-looking expressions are actually rational numbers:

(a) $\sqrt[3]{4 + 2\sqrt{3} - \sqrt{3}}$, (b) $\sqrt[4]{6 + 4\sqrt{2} - \sqrt{2}}$.

Solution. (a) We observe that $(1 + \sqrt{3})^2 = 1^2 + 2 \cdot 1 \cdot \sqrt{3} + (\sqrt{3})^2 = 1 + 2\sqrt{3} + 3 = 4 + 2\sqrt{3}$. So $\sqrt[3]{4 + 2\sqrt{3}} = 1 + \sqrt{3}$, and $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = 1$.

(b) We observe that $(2 + \sqrt{2})^2 = 2^2 + 2 \cdot 2 \cdot \sqrt{2} + (\sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}$. So $\sqrt{6 + 4\sqrt{2}} = 2 + \sqrt{2}$, and $\sqrt{6 + 4\sqrt{2} - \sqrt{2}} = 2$. 

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