Homework 12 (due on 11/25)

- We will have the second midterm exam on November 18. The due date of this homework is postponed to November 25.
- Read Sections 26, 30, 31 for the next week.

29.2 Prove $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

29.3 Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$. Hint: Apply Theorem 29.3.

(b) Show $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$. Hint: Apply Theorems 29.3 and 29.8.

29.4 Let f and g be differentiable functions on an open interval I. Suppose a, b in I satisfy $a < b$ and $f(a) = f(b) = 0$. Show $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$. Hint: Consider $h(x) = f(x)e^{g(x)}$.

29.7 (a) Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.
Hint: Apply Corollary 29.4 to f' to conclude that f' is a constant a. Then apply Corollary 29.4 again to $g(x) := f(x) - ax$.

(b) Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim. Hint: Apply Corollary 29.4 three times.

29.9 Show $ex \leq e^x$ for all $x \in R$.
Hint: Let $f(x) = e^x - ex$. Observe the value of f(1) and the signs of $f'(x)$.

29.13 Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$.

29.16 Use Theorem 29.9 to obtain the derivative of the inverse $g = \mathrm{Tan}^{-1} = \arctan$ of f where $f(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

29.18 Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$.

(a) Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0)$, $s_2 = f(s_1)$, etc. Prove (s_n) is a convergent sequence.
Hint: To show that (s_n) is Cauchy, first show $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$.

(b) Show f has a fixed point, i.e., $f(s) = s$ for some s in $\mathbb{R}$, and such point is unique.