Homework 10 Solutions

23.1 For each of the following power series, find the radius of convergence and determine the exact interval of convergence. (a) $\sum n^2x^n$; (c) $\sum(\frac{2^n}{n^2})x^n$; (e) $\sum(\frac{2^n}{n!})x^n$; (g) $\sum(\frac{3^n}{n\cdot 4^n})x^n$.

Solution. (a) Here $a_n = n^2$. So $|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^2}{n^2} = (1+\frac{1}{n})^2 \to 1$. Thus, $\beta = 1$ and the radius $R = 1/\beta = 1$. It remains to check the convergence at $x = 1$ and $x = -1$. Since for $x = 1$ or -1, $|n^2x^n| = n^2 \to \infty$, we do not have $n^2x^n \to 0$. So $\sum n^2x^n$ diverges if $x = 1$ or -1. Thus, the interval of convergence I is $(-1, 1)$.

(c) Here $a_n = \frac{2^n}{n^2}$. So $|\frac{a_{n+1}}{a_n}| = \frac{2^{n+1}/(n+1)^2}{2^n/n^2} = \frac{2^{n+1}n^2}{2^n(n+1)^2} = \frac{2}{(1+1/n)^2} \to 2$. Thus, $\beta = 2$ and $R = 1/2$. It remains to check the convergence at $x = 1/2$ and $x = -1/2$. Since when $|x| = 1/2$, $|(\frac{2^n}{n^2})x^n| = \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges, by comparison test, $\sum(\frac{2^n}{n^2})x^n$ converges at $x = 1/2$ and $x = -1/2$. Thus, $I = [-1/2, 1/2]$.

(e) Here $a_n = \frac{2^n}{n!}$. So $|\frac{a_{n+1}}{a_n}| = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}n!}{2^n(n+1)!} = \frac{2}{n+1} \to 0$. Thus, $\beta = 0$ and $R = \infty$. Then we have $I = \mathbb{R}$.

(g) Here $a_n = \frac{3^n}{n\cdot 4^n}$. So $|\frac{a_{n+1}}{a_n}| = \frac{(3/4)^{n+1}/(n+1)}{(3/4)^n/n} = \frac{3/4n}{n+1} = \frac{3/4}{1+1/n} \to \frac{3}{4}$. Thus, $\beta = 3/4$ and $R = 4/3$. It remains to check the convergence at $x = 4/3$ and $x = -4/3$. When $x = 4/3$, the series becomes $\sum \frac{1}{n}$, which diverges. When $x = -4/3$, the series becomes $\sum \frac{(-1)^n}{n}$, which converges by alternative series test. So $I = [-4/3, 4/3)$. $\square$

23.4 For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.

(a) Find $\limsup |a_n|^{1/n}$, $\liminf |a_n|^{1/n}$, $\limsup |\frac{a_{n+1}}{a_n}|$ and $\liminf |\frac{a_{n+1}}{a_n}|$.

(b) Do the series $\sum a_n$ and $\sum(-1)^na_n$ converge? Explain briefly.

(c) Now consider the power series $\sum a_nx^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

Solution. (a) We have $|a_n|^{1/n} = \frac{4+2(-1)^n}{5} = \frac{6}{5}$ if n is even; $= \frac{2}{5}$ if n is odd. Thus, $\limsup |a_n|^{1/n} = \frac{6}{5}$ and $\liminf |a_n|^{1/n} = \frac{2}{5}$. If n is odd, $|\frac{a_{n+1}}{a_n}| = \frac{(6/5)^{n+1}}{(2/5)^n} = \frac{6}{5}\cdot 3^n \to \infty$; if n is even, $|\frac{a_{n+1}}{a_n}| = \frac{(2/5)^{n+1}}{(6/5)^n} = \frac{2}{5}\cdot(\frac{1}{3})^n \to 0$. So $\limsup |\frac{a_{n+1}}{a_n}| = \infty$ and $\liminf |\frac{a_{n+1}}{a_n}| = 0$.

(b) From (a) we know that the radius of the power series $\sum a_nx^n$ is $R = 1/\limsup |a_n|^{1/n} = 5/6$. Since $\sum a_n$ and $\sum(-1)^na_n$ are the power series at 1 and -1, and $|1| = |-1| > R$, they should both diverge.

(c) We have found in (b) that the radius is $5/6$. When $x = 5/6$ or $-5/6$, we have $a_nx^n = 1$ for even n. So we do not have $a_nx^n \to 0$. Then $\sum a_nx^n$ diverges at $x = 5/6$ or $-5/6$. Thus, the interval of convergence is $(-5/6, 5/6)$. $\square$

23.5 Consider a power series $\sum a_nx^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leq 1$.

(b) Prove that if $\limsup |a_n| > 0$, then $R \leq 1$. Hint: You may work out (b) first and use it to prove (a).

Proof. (b) We may find a subsequence (a_{n_k}) of (a_n) such that $\lim |a_{n_k}| = \limsup |a_n| > 0$. Since the limit is positive, we can find $x > 0$ and $N \in \mathbb{N}$ such that $|a_{n_k}| \geq x$ for $k > N$. Then we have $|a_{n_k}|^{1/n_k} \geq x^{1/n_k}$ for all $k > N$. From $x^{1/n_k} \to 1$ we then get $\beta = \limsup |a_n|^{1/n} \geq \limsup |a_{n_k}|^{1/n_k} \geq \lim x^{1/n_k} = 1$. Then $R = 1/\beta \leq 1$.

(a) If a_n are all integers and infinitely many of them are nonzero, then there are infinitely many n such that $|a_n| \geq 1$. Then we have $\limsup |a_n| \geq 1$. From (b) we conclude that $R \leq 1$. □

24.2 For $x \in [0, \infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution. (a) Since $\lim f_n(x) = \lim \frac{x}{n} = 0$ for all $x \in \mathbb{R}$, f is constant 0 on $\mathbb{R}$.

(b) We have $f_n \to f$ uniformly on $[0, 1]$ by Remark 24.4 because $\sup\{|f_n(x) - f(x)| : x \in [0, 1]\} = \sup\{\frac{x}{n} : x \in [0, 1]\} = \frac{1}{n} \to 0$.

(c) By Remark 24.4, we do not have $f_n \to f$ uniformly on $[0, \infty)$ because $\sup\{|f_n(x) - f(x)| : x \in [0, \infty)\} = \sup\{\frac{x}{n} : x \in [0, \infty)\} = \infty$ for each n. □

24.3 Repeat Exercise 24.2 for $f_n(x) = \frac{1}{1+x^n}$.

Solution. (a) If $x \in [0, 1)$, $x^n \to 0$; if $x = 1$, $x^n \to 1$; if $x > 1$, $x^n \to \infty$. Thus, $f(x) = \lim f_n(x) = \frac{1}{1+0} = 1$ on $[0, 1)$; $= \frac{1}{1+1} = \frac{1}{2}$ at 1; and $= \frac{1}{1+\infty} = 0$ on $(1, \infty)$.

(b) We do not have $f_n \to f$ uniformly on $[0, 1]$ because if the uniform convergence holds, then from the continuity of each f_n we could conclude from Theorem 24.3 that f is continuous. However, f is not continuous at 1. So the uniform convergence does not hold.

(c) For a similar reason, f_n does not converge to f uniformly on $[0, \infty)$. □

24.10 (a) Prove that if $f_n \to f$ uniformly on S and $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ and $g_n \to g$ uniformly on S, there are $N_f, N_g \in \mathbb{N}$ such that if $n > N_f$ then for every $x \in S$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$; and if $n > N_g$ then for every $x \in S$, $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$. Let $N = \max\{N_f, N_g\}$. If $n > N$, then for every $x \in S$,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then we conclude that $f_n + g_n \to f + g$ uniformly on S. $\square$

24.11 Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for all $x \in \mathbb{R}$.

(a) Observe $f_n \to f$ uniformly on R [obvious!] and $g_n \to g$ uniformly on R [almost obvious].

(b) Observe the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

Proof. (a) Since $\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = 0 \to 0$ and $\sup\{|g_n(x) - g(x)| : x \in \mathbb{R}\} = \frac{1}{n} \to 0$, we get $f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$.

(b) $f_n(x)g_n(x) = \frac{x}{n}$ and $f(x)g(x) = 0$. From Exercise 24.2, we know that $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. $\square$

24.12 Prove the assertion in Remark 24.4: A sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0.$$

Proof. First suppose $f_n \to f$ uniformly on S. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for any $n > N$, we have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$, which implies that $\sup_{x\in S} |f_n(x) - f(x)| \leq \varepsilon$. From this we see that $\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$. On the other hand, suppose $\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for $n > N$, $\sup_{x\in S} |f_n(x) - f(x)| < \varepsilon$, which then implies that $|f_(x) - f(x)| \leq \sup_{x\in S} |f_n(x) - f(x)| < \varepsilon$ for all $x \in S$. So $f_n \to f$ uniformly on S. $\square$