Two-curve Green's function for 2-SLE: the interior case

Dapeng Zhan*

Michigan State University

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Abstract

A 2-SLE$_\kappa$ ($\kappa \in (0,8)$) is a pair of random curves (η_1, η_2) in a simply connected domain D connecting two pairs of boundary points such that conditioning on any curve, the other is a chordal SLE$_\kappa$ curve in a complement domain. In this paper we prove that for any $z_0 \in D$, the limit $\lim_{r\to 0^+} r^{-\alpha_0}\mathbb{P}[\operatorname{dist}(z_0,\eta_j) < r, j = 1,2]$, where $\alpha_0 = \frac{(12-\kappa)(\kappa+4)}{8\kappa}$, exists. Such limit is called a two-curve Green's function. We find the convergence rate and the exact formula of the Green's function in terms of a hypergeometric function up to a multiplicative constant. For $\kappa \in (4,8)$, we also prove the convergence of $\lim_{r\to 0^+} r^{-\alpha_0}\mathbb{P}[\operatorname{dist}(z_0,\eta_1\cap\eta_2) < r]$, whose limit is a constant times the previous Green's function. To derive these results, we work on two-time-parameter stochastic processes, and use orthogonal polynomials to derive the transition density of a two-dimensional diffusion process that satisfies some system of SDE.

1 Introduction

The Schramm-Loewner evolution (SLE), first introduced by Oded Schramm in 1999 ([24]), is a one-parameter ($\kappa \in (0,\infty)$) family of measures on non-self-crossing curves, which has received a lot of attention over the past two decades. It has been shown that, modulo time parametrization, many discrete random paths on grids have SLE with different parameters as their scaling limits. We refer the reader to Lawler's textbook [8] for basic properties of SLE.

One of the most important functions associated to SLE is the Green's function, which can be roughly defined as the scaling limit of the probability that an SLE curve hits a small disc around an interior or boundary point of its domain. The existence of chordal SLE Green's function for an interior point was given in [5], where conformal radius was used instead of Euclidean distance. The existence of the original one-point Green's function (using Euclidean distance) was proved later in [9]. The existence of boundary point Green's function for chordal SLE was given in [7]. Other related works include the Green's function for radial SLE ([1]), multipoint

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Green's function for chordal SLE ([11, 7, 9] for 2-point, [19] for n-point), and Green's function for $\mathrm{SLE}_\kappa(\rho)$ and hSLE ([12]).

A 2-SLE_κ (also called bi-chordal SLE_κ) is a pair of random curves in a simply connected domain connecting two pairs of boundary points, which satisfy that, when any one curve is given, the conditional law of the other curve is that of a chordal SLE_κ curve in one complement domain of the first curve. It is a special case of multiple N-SLE_κ (when $N = 2$) studied in [4], and exists for all $\kappa \in (0, 8)$ and any admissible link pattern. A 2-SLE arises naturally as a scaling limit of some lattice model with alternating boundary conditions ([26, 3]), as interacting flow lines in imaginary geometry ([15, 14]), and as two exploration curves of a CLE ([16, 17]).

Suppose (η_1, η_2) is a 2-SLE_κ in a simply connected domain D, and $z_0 \in \overline{D}$. Then the probability that both η_1 and η_2 visit a small disc centered at z_0 with radius ε tends to 0 as $\varepsilon \to 0$. It is expected that this probability decays like some power of ε, and the rescaled probability tends to a nontrivial limit, which is called the two-curve Green's function for this 2-SLE_κ. A similar object considered in [12] is the rescaled probability that either γ_1 or γ_2 gets close to a given interior point. Their Green's function is a sum of two one-curve Green's functions for the 2-SLE_κ, and is different from the one considered here. In this paper we focus on the interior point case, i.e., $z_0 \in D$. In the subsequent paper [28], we will work on the boundary point case, which uses a similar approach.

Below is our first main theorem, which holds for all $\kappa \in (0, 8)$.

Theorem 1.1. *Let $\kappa \in (0, 8)$. Let*

$$\alpha_0 = \frac{(12-\kappa)(\kappa+4)}{8\kappa} > 0. \tag{1.1}$$

Let F be the hypergeometric function ${}_2F_1(\frac{4}{\kappa}, 1-\frac{4}{\kappa}; \frac{8}{\kappa}, \cdot)$, which is known to be positive on $[0, 1]$. Let D be a simply connected domain with four distinct boundary points (prime ends) a_1, b_1, a_2, b_2 such that a_1 and a_2 together separate b_1 from b_2 on ∂D. Let $(\widehat{\eta}_1, \widehat{\eta}_2)$ be a 2-SLE_κ in D with link pattern $(a_1, b_1; a_2, b_2)$. Let $z_0 \in D$, and f_{z_0} be the conformal map from D onto $\mathbb{D}$ such that $f_{z_0}(z_0) = 0$ and $f_{z_0}'(0) > 0$. Let

$$G_{D;a_1,b_1;a_2,b_2}(z_0) := 4^{1-\frac{12}{\kappa}} |f'(z_0)|^{\alpha_0} \prod_{j=1}^{2} |f_{z_0}(a_j) - f_{z_0}(b_j)|^{\frac{8}{\kappa}-1} \prod_{x\in\{a,b\}} |f_{z_0}(x_1) - f_{z_0}(x_2)|^{\frac{4}{\kappa}} \times$$

$$\times F\Big(\frac{|f_{z_0}(a_1) - f_{z_0}(b_2)||f_{z_0}(a_2) - f_{z_0}(b_1)|}{|f_{z_0}(a_1) - f_{z_0}(a_2)||f_{z_0}(b_1) - f_{z_0}(b_2)|}\Big)^{-1}.$$

Let $\beta_0 = \frac{2+\frac{\kappa}{8}}{3+\frac{\kappa}{8}}$. Let $R = \operatorname{dist}(z_0, \partial D)$. Then there is a constant $C_0 > 0$ depending only on κ such that

$$\mathbb{P}[\operatorname{dist}(z_0, \widehat{\eta}_j) < r, j = 1, 2] = C_0 G_{D;a_1,b_1;a_2,b_2}(z_0) r^{\alpha_0} \Big(1 + O\Big(\Big(\frac{r}{R}\Big)^{\beta_0}\Big)\Big), \quad \text{as } r \to 0^+. \tag{1.2}$$

Here the implicit constants in the O symbol depend only on κ. In particular, it implies that there is a constant $C_0' > 0$ depending only on κ such that

$$\mathbb{P}[\operatorname{dist}(z_0, \widehat{\eta}_j) < r, j = 1,2] \le C_0' \Big(\frac{r}{R}\Big)^{\alpha_0}, \quad \forall r > 0. \tag{1.3}$$

Below is our second main theorem, which makes sense only for $\kappa \in (4,8)$.

Theorem 1.2. *Let $\kappa \in (4,8)$. We adopt the notation in the last theorem. Then there is a constant $C_1 > 0$ depending only on κ such that*

$$\mathbb{P}[\operatorname{dist}(z_0, \widehat{\eta}_1 \cap \widehat{\eta}_2) < r] = C_1 G_{D;a_1,b_1;a_2,b_2}(z_0) r^{\alpha_0} \Big(1 + O\Big(\frac{r}{R}\Big)^{\beta_0}\Big), \quad \textit{as } r \to 0^+.$$

Similar theorems also hold in the case that z_0 lies on the boundary, assuming that ∂D is smooth near z_0 ([28]), where the exponent α_0 is replaced by another exponent: $\frac{2}{\kappa}(12 - \kappa)$. Following the approach of [9], we expect that the second theorem above may be used to prove the existence of the Minkowski content of $\eta_1 \cap \eta_2$ of dimension $2 - \alpha_0$, which is the Hausdorff dimension of the double points of SLE$_\kappa$ ([18, Theorem 1.1]). This is closely related to the existence of Minkowski content of double points of a single SLE$_\kappa$ curve.

Definition 1.3. We call $G_{D;a_1,b_1;a_2,b_2}$ in Theorem 1.1 the two-curve Green's function for 2-SLE$_\kappa$ in D with link pattern $(a_1, b_1; a_2, b_2)$.

Remark 1.4. It is easy to derive the following properties of the two-curve Green's function.

(i) Using Koebe's 1/4 Theorem and the boundedness of F on $[0,1]$, we see that there is a constant $C > 0$ depending only on κ such that

$$G_{D;a_1,b_1;a_2,b_2}(z_0) \le C \operatorname{dist}(z_0, \partial D)^{-\alpha_0}. \tag{1.4}$$

(ii) For a_1, b_1, a_2, b_2 in the definition, there is another admissible link pattern, which is $(a_1, b_2; a_2, b_1)$. It is easy to see that $\frac{G_{D;a_1,b_2;a_2,b_1}(z_0)}{G_{D;a_1,b_1;a_2,b_2}(z_0)}$ does not depend on z_0, but only on the cross-ratio of a_1, b_1, a_2, b_2 in D.

The approach of the main theorems is somehow similar to that of the Green's function for a single chordal SLE$_\kappa$, where one parametrizes the curve according to the conformal radius viewed from the marked point and obtains an invariant measure on a process of harmonic measures. Here is how it goes for the setting here. By conformal invariance, we may assume that D is the unit disc $\mathbb{D} = \{|z| < 1\}$ and z_0 is the center 0. We may further reduce it to the case that b_1 and b_2 are opposite points on the circle, i.e., $b_1 + b_2 = 0$, by growing a part of η_1 or η_2 and mapping the remaining domain back to $\mathbb{D}$. In this special case, we choose to grow η_1 and η_2 simultaneously with random speeds so that at any time t, (i) the conformal radius of the remaining domain viewed from 0 is e^{-t}; and (ii) the harmonic measure in the remaining domain viewed from 0 of any boundary arc bounded by b_1 and b_2 is $1/2$. The process is stopped when either η_1 or η_2 finishes its journey, or the two curves together disconnect 0 from b_1 or b_2.

It turns out that the speeds of the curves are determined by the unparametrized curves. By Koebe's 1/4 theorem, the assumption on the conformal radius implies that at any time t that happens before the process ends, the minimum of $\operatorname{dist}(0,\eta_1[0,t])$ and $\operatorname{dist}(0,\eta_2[0,t])$ is comparable to e^{-t}. By Beurling's estimate and the harmonic measure assumption, we know that $\operatorname{dist}(0,\eta_1[0,t])$ is comparable with $\operatorname{dist}(0,\eta_2[0,t])$. Thus, both $\operatorname{dist}(0,\eta_1[0,t])$ and $\operatorname{dist}(0,\eta_2[0,t])$ are comparable to e^{-t}, if the time t happens before the lifetime of the process.

At any time t, conditionally on the event that the process does not end at time t, if we map the remaining domain back to $\mathbb{D}$ and fix 0, then the images of b_1 and b_2, say $b_1(t)$ and $b_2(t)$ are still opposite points on $\partial\mathbb{D}$. The tips of η_1 and η_2 are then mapped to two other points on $\partial\mathbb{D}$, say $a_1(t),a_2(t)$, that are separated by $b_1(t)$ and $b_2(t)$. The conditional joint law of the images of the remaining parts of η_1 and η_2 is then a function of $a_1(t),a_2(t),b_1(t),b_2(t)$ with rotation invariance. This means that the conditional probability that the images of the remaining parts of η_1 and η_2 both visit a small disc centered at 0 is a function of $\arg(a_j(t)/b_j(t))$, $j=1,2$.

The above observation motivates us to study the growth of the two-dimensional Markov process $(Z_1(t),Z_2(t))$ in $(0,\pi)^2$, where $Z_j(t)=\arg(a_j(t)/b_j(t))$. Using a framework of two-parameter martingales, we are able to show that (Z_1,Z_2) is a semi-martingale, and derive the system of SDEs for them. Then we follow the approach of [30, Appendix B] and use orthogonal polynomials to derive the explicit transition density for this Markov process. Using the transition density, we find that (Z_1,Z_2) has a quasi-invariant measure, say $\mu^\#_*$ on $(0,\pi)^2$, which means that if we start (Z_1,Z_2) from a random point with law $\mu^\#_*$, then for any deterministic $t>0$, the probability that the process survives at time t is $e^{-\alpha_0 t}$, and the law of $(Z_1(t),Z_2(t))$ conditional on this event is still $\mu^\#_*$. Furthermore, if we start (Z_1,Z_2) from any deterministic point, then the conditional distribution of $(Z_1(t),Z_2(t))$ approaches exponentially to $\mu^\#_*$. With this quasi-invariant measure in hand, the remaining part of the proofs of the main theorems are finished by using Koebe's distortion theorem.

The rest of the paper is organized as follows. In Sections 2.1 and 2.2, we review Loewner equations, SLE, 2-SLE, and hypergeometric SLE. In Section 2.3 we develop a framework on stochastic processes that depend on two time parameters. In Section 3 we describe the interaction between two radial Loewner chains, whose chordal counterpart appeared earlier in the works on the reversibility and duality of SLE ([33, 32]). The essential new stuff starts from Section 4, in which we grow the two curves in a 2-SLE simultaneously as described above and derive the SDEs for the process $(Z_1(t),Z_2(t))$. In Section 5 we derive the transition density and quasi-invariant density of this process. In the last section, we finish the proofs.

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2 Preliminary

2.1 Loewner equations, SLE and 2-SLE

In this subsection, we recall the definitions of Loewner equations, SLE and 2-SLE. Let $\mathbb{H}$ denote the upper half-plane $\{z\in\mathbb{C}:\operatorname{Im} z>0\}$. Let $\mathbb{D}$ and $\mathbb{T}$ denote the unit disc $\{z\in\mathbb{C}:|z|<1\}$ and its boundary, respectively.

We will extensively use radial Loewner equation in the paper. For the definition, we start with hulls in $\mathbb{D}$. A set $K\subset\mathbb{D}$ is called a $\mathbb{D}$-hull if $\mathbb{D}\setminus K$ is a simply connected domain that contains 0. For a $\mathbb{D}$-hull K, there is a unique conformal map g_K from $\mathbb{D}\setminus K$ onto $\mathbb{D}$ such that $g_K(0)=0$ and $g_K'(0)>0$. By Schwarz Lemma, $g_K'(0)\ge 1$, and the equality holds only when $K=\emptyset$. By Schwarz reflection principle, we may view g_K as a conformal map from $\mathbb{C}\setminus K^{\rm doub}$ onto $\mathbb{C}\setminus S_K$, where $K^{\rm doub}$ is the union of the closure of K and the reflection of K about $\mathbb{T}$, i.e., $\{1/\overline{z}:z\in K\}$, and S_K is a compact subset of $\mathbb{T}$. Let $\operatorname{dcap}(K):=\log(g_K'(0))\ge 0$ be called the $\mathbb{D}$-capacity of K. If $K_1\subset K_2$ are two $\mathbb{D}$-hulls, then we define $K_2/K_1:=g_{K_1}(K_2\setminus K_1)$, which is also a $\mathbb{D}$-hull, and satisfies $\operatorname{dcap}(K_2/K_1)=\operatorname{dcap}(K_2)-\operatorname{dcap}(K_1)$.

Let $\widehat{w}\in C([0,T),\mathbb{R})$ for some $T\in(0,\infty]$. The radial Loewner equation driven by $\widehat{w}$ is

$$\partial_t g_t(z)=g_t(z)\cdot\frac{e^{i\widehat{w}(t)}+g_t(z)}{e^{i\widehat{w}(t)}-g_t(z)},\quad 0\le t<T;\quad g_0(z)=z.$$

For each $t\in[0,T)$, let K_t be the set of $z\in\mathbb{D}$ such that the solution $g_\cdot(z)$ blows up before or at t (so that g_t is well defined on $\mathbb{D}\setminus K_t$). Then we call g_t and K_t the radial Loewner maps and hulls, respectively, driven by $\widehat{w}$. It turns out that, for each t, K_t is a $\mathbb{D}$-hull with $\operatorname{dcap}(K_t)=t$, and $g_{K_t}=g_t$. If for every $t\in[0,T)$, g_t^{-1} as a conformal map from $\mathbb{D}$ onto $\mathbb{D}\setminus K_t$ extends continuously to $\overline{\mathbb{D}}$, and $\eta(t):=g_t^{-1}(e^{i\widehat{w}(t)})$, $0\le t<T$, is continuous in t, then we say that η is a radial Loewner curve driven by $\widehat{w}$. Such η may not exist in general; when it exists, the hulls (K_t) are generated by η in the sense that for every t, $\mathbb{D}\setminus K_t$ is the connected component of $\mathbb{D}\setminus\eta([0,t])$ that contain 0.

Let $\widehat{w}$ be as above. Let u be a continuous and strictly increasing function defined on $[0,T)$ with $u(0)=0$. Suppose that the two families g^u_t and K^u_t, $0\le t<T$, satisfy that $g^u_{u^{-1}(t)}$ and $K^u_{u^{-1}(t)}$, $0\le t<u(T)$, are radial Loewner maps and hulls, respectively, driven by $\widehat{w}\circ u^{-1}$. Then we say that g^u_t and K^u_t, $0\le t<T$, are radial Loewner maps and hulls, respectively, driven by $\widehat{w}$ with speed du. If u is absolutely continuous, then we say that the speed is u'.

The following lemma is well known, and has appeared in the literature in different forms.

Lemma 2.1. *Suppose K_t, $0\le t<T$, are radial Loewner hulls driven by some $\widehat{w}\in C([0,T),\mathbb{R})$. Let L be a $\mathbb{D}$-hull such that $\overline{L}\cap\overline{K_t}=\emptyset$ for all $t\in[0,T)$. Then for any $t\in[0,T)$, $g_{K_t}(L)$ is a $\mathbb{D}$-hull that has positive distance from $e^{i\widehat{w}(t)}$, so that $g_{g_{K_t}(L)}$ is analytic at $e^{i\widehat{w}(t)}$; and*

$g_L(K_t)$, $0 \le t < T$, are radial Loewner hulls driven by some $\widehat{w}^L \in C([0,T),\mathbb{R})$ with speed $|g'_{g_{K_t}(L)}(e^{i\widehat{w}(t)})|^2$, where $\widehat{w}^L$ satisfies $e^{i\widehat{w}^L(t)} = g_{g_{K_t}(L)}(e^{i\widehat{w}(t)})$, $0 \le t < T$.

It will be useful to work on the covering radial Loewner equation. Let e^i denote the covering map $z \mapsto e^{iz}$ from $\mathbb{H}$ onto $\mathbb{D} \setminus \{0\}$. Let $\cot_2$ denote the function $\cot(\cdot/2)$. The covering radial Loewner equation driven by $\widehat{w} \in C([0,T),\mathbb{R})$ is

$$\partial_t \widetilde{g}_t(z) = \cot_2(\widetilde{g}_t(z) - \widehat{w}(t)), \quad g_0(z) = z.$$

For each $t \in [0,T)$, let $\widetilde{K}_t$ denote the set of of $z \in \mathbb{H}$ such that the solution $\widetilde{g}_\cdot(z)$ blows up before or at t. We call $\widetilde{g}_t$ and $\widetilde{K}_t$, $0 \le t < T$, the covering radial Loewner maps and hulls, respectively, driven by $\widehat{w}$. It turns out that $\widetilde{K}_t$ has period 2π, $\widetilde{g}_t$ maps $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$ with $\widetilde{g}_t(z+2\pi) = \widetilde{g}_t(z) + 2\pi$; and if (g_t) and (K_t) are the radial Loewner maps and hulls driven by $\widehat{w}$, then $\widetilde{K}_t = (e^i)^{-1}(K_t)$ and $e^i \circ \widetilde{g}_t = g_t \circ e^i$. If u is a continuous and strictly increasing function on $[0,T)$, we may similarly define covering radial Loewner maps $\widetilde{g}^u_t$ and hulls $\widetilde{K}^u_t$ with speed du driven by $\widehat{w}$.

If $\widehat{w}(t) = \sqrt{\kappa} B(t)$, $0 \le t < \infty$, where $\kappa > 0$ and $B(t)$ is a standard Brownian motion, then the radial Lowner curve η driven by $\widehat{w}$ is known to exist, and is called a radial SLE_κ curve in $\mathbb{D}$ from 1 to 0. What will be used in this paper is a generalization of radial SLE_κ: radial $\text{SLE}(\kappa;\underline{\rho})$, whose growth is affected by one or more force points lying on the boundary or the interior. For the generality needed here, we assume that all force points lie on the boundary and are distinct from the initial point of the curve. We start with the definition of radial $\text{SLE}(\kappa;\underline{\rho})$ in $\mathbb{D}$. Let $\rho_1,\dots,\rho_n \in \mathbb{R}$. Let $e^{iw}, e^{iv_1},\dots,e^{iv_n}$ be distinct points on $\mathbb{T}$. Let $B(t)$ be a standard Brownian motion. Suppose that $\widehat{w}(t)$ and $\widehat{v}_j(t)$, $1 \le j \le n$, $0 \le t < T$, solve the following system of SDEs with the maximal solution interval:

$$d\widehat{w}(t) = \sqrt{\kappa} dB(t) + \sum_{j=1}^n \frac{\rho_j}{2} \cot_2(\widehat{w}(t) - \widehat{v}_j(t))dt, \quad \widehat{w}(0) = w;$$

$$d\widehat{v}_j(t) = \cot_2(\widehat{v}_j(t) - \widehat{w}(t)), \quad \widehat{v}_j(0) = v_j, \quad 1 \le j \le n.$$

Then we call the radial Loewner curve driven by $\widehat{w}$ the $\text{SLE}(\kappa;\rho_1,\dots,\rho_n)$ curve in $\mathbb{D}$ started from e^{iw} aimed at 0 with force points $e^{iv_1},\dots,e^{iv_n}$. The covering radial Loewner maps implicitly appear in the definition: if $\widetilde{g}_t$ are covering radial Loewner maps, then $\widehat{v}_j(t) = \widetilde{g}_t(v_j)$.

Although we say that η is aimed at 0, it often happens that η does not end at 0. A radial $\text{SLE}(\kappa;\underline{\rho})$ curve in a general simply connected domain D started from a boundary point aimed at an interior point with force points on the boundary is defined by a conformal map from $\mathbb{D}$ onto D. The targeted interior point actually acts as another force point with force value $\kappa - 6 - \sum_{j=1}^n \rho_j$ (cf. [25]).

At the end of this subsection, we briefly recall chordal Loewner equation, chordal SLE_κ, and 2-SLE_κ. Let $\widehat{w} \in C([0,T),\mathbb{R})$ for some $T \in (0,\infty]$. The chordal Loewner equation driven by $\widehat{w}$ is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \widehat{w}(t)}, \quad 0 \le t < T; \quad g_0(z) = z.$$

For each $t \in [0,T)$, let K_t be the set of $z \in \mathbb{H}$ such that the solution $g_\cdot(z)$ blows up before or at t (so that g_t is well defined on $\mathbb{H} \setminus K_t$). Then we call g_t and K_t the chordal Loewner maps and hulls, respectively, driven by $\widehat{w}$. It turns out that, for each t, K_t is a bounded and relatively closed subset of $\mathbb{H}$, and g_t maps $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$. If for every $t \in [0,T)$, g_t^{-1} as a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus K_t$ extends continuously to $\overline{\mathbb{H}}$, and $\eta(t) := g_t^{-1}(\widehat{w}(t))$, $0 \le t < T$, is continuous in t, then we say that η is a chordal Loewner curve driven by $\widehat{w}$.

If $\widehat{w}(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$, where $\kappa > 0$ and $B(t)$ is a standard Brownian motion, then the chordal Lowner curve η driven by $\widehat{w}$ is known to exist, and is called a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞. In fact, we have $\eta(0) = \widehat{w}(0) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$ ([21]). If D is a simply connected domain with two distinct marked boundary points (prime ends) a and b, the chordal SLE_κ curve in D from a to b is defined to be the conformal image of a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞ under a conformal map from $(\mathbb{H}; 0, \infty)$ onto $(D; a, b)$.

For any $\kappa > 0$, both radial SLE_κ and chordal SLE_κ satisfy conformal invariance and Domain Markov Property (DMP). The DMP means that if η is a radial (resp. chordal) SLE_κ curve in D from a to b, and T is a stopping time, then conditionally on the part of η before T and the event that η does not reach b at time T, the part of η after T is a radial (resp. chordal) SLE_κ curve from $\eta(T)$ to b in one connected component of $D \setminus \eta([0,T])$. If $\kappa \in (0,8)$, chordal SLE_κ satisfies reversibility: the time-reversal of a chordal SLE_κ curve in D from a to b is a chordal SLE_κ curve in D from b to a, up to a time-change ([33, 13]).

Let D be a simply connected domain with distinct boundary points a_1, b_1, a_2, b_2 such that a_1 and a_2 together separate b_1 from b_2 on ∂D (and vice versa). Let $\kappa \in (0,8)$. A 2-SLE_κ in D with link pattern $(a_1, b_1; a_2, b_2)$ is a pair of random curves (η_1, η_2) in $\overline{D}$ such that for $j = 1, 2$, η_j connects a_j with b_j, and conditionally on η_{3-j}, η_j is a chordal SLE_κ curve in the connected component of $D \setminus \eta_{3-j}$ whose boundary contains a_j and b_j. Because of reversibility, we do not need to specify the orientation of η_1 and η_2. If we want to emphasize the orientation, then we use an arrow like $a_1 \to b_1$ in the link pattern. The existence of 2-SLE_κ was proved in [4] for $\kappa \in (0,4]$ using Brownian loop measure and in [15, 13] for $\kappa \in (4,8)$ using flow line theory. The uniqueness of 2-SLE_κ (for a fixed domain and link pattern) was proved in [14] (for $\kappa \in (0,4]$) and [16] (for $\kappa \in (4,8)$) using an ergodicity argument.

Using the DMP for chordal SLE_κ, it is easy to derive the following DMP for 2-SLE_κ: If (η_1, η_2) is a 2-SLE_κ in D with link pattern $(a_1 \to b_1; a_2, b_2)$, and if T is a stopping time for η_1, then conditionally on the part of η_1 before T and the event that η_1 neither reaches b_1 or disconnects b_1 from a_2, b_2 at time T, the rest part of η_1 and the complete η_2 form a 2-SLE_κ with link pattern $(\eta_1(T) \to b_1; a_2, b_2)$ in the connected component of $D \setminus \eta_1([0,T])$ whose boundary contains b_1, a_2, b_2. We will have a stronger DMP later in Lemma 6.1.

2.2 Hypergeometric SLE

We now review the hypergeometric SLE defined earlier in [31] (called intermediate $\mathrm{SLE}_\kappa(\rho)$ there) and [23]. Let $\kappa \in (0,8)$. Let F be the hypergeometric function ${}_2F_1(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}; \frac{8}{\kappa}, \cdot)$ in

Theorem 1.1. Such F is the solution of

$$x(1-x)F''(x)+\Big[\frac{8}{\kappa}-2x\Big]F'(x)-\frac{4}{\kappa}\Big(1-\frac{4}{\kappa}\Big)F(x)=0. \tag{2.1}$$

Since $\frac{8}{\kappa}>(1-\frac{4}{\kappa})+\frac{4}{\kappa}$, F extends continuously to 1 with $F(1)=\frac{\Gamma(\frac{8}{\kappa})\Gamma(\frac{8}{\kappa}-1)}{\Gamma(\frac{4}{\kappa})\Gamma(\frac{12}{\kappa}-1)}>0$. Using (2.1) one can prove that F is positive on $[0,1]$. Then we let $G(x)=\kappa x\frac{F'(x)}{F(x)}$, $\widetilde{F}(x)=x^{\frac{2}{\kappa}}F(x)$, and $\widetilde{G}(x)=\kappa x\frac{\widetilde{F}'(x)}{\widetilde{F}(x)}=G(x)+2$.

Definition 2.2. Let $0,v_1,v_2\in\mathbb{R}$ be such that $0<v_1<v_2$ or $0>v_1>v_2$. Let $B(t)$ and $B'(t)$ be two independent standard real Brownian motion. Suppose $\widehat{w},\widehat{v}_1,\widehat{v}_2\in C([0,\infty),\mathbb{R})$ satisfy the following properties. There is $T\in(0,\infty]$ such that $\widehat{w}(t)$ and $\widehat{v}_j(t)$, $0\le t<T$, $j=1,2$, are continuous random process such that they together solve the following SDE with the maximal solution interval and respective initial values 0 and v_j, $j=1,2$:

$$\begin{aligned}
d\widehat{w}(t)=&\sqrt{\kappa}dB(t)+\Big(\frac{1}{\widehat{w}(t)-\widehat{v}_1(t)}-\frac{1}{\widehat{w}(t)-\widehat{v}_2(t)}\Big)\widetilde{G}\Big(\frac{\widehat{w}(t)-\widehat{v}_1(t)}{\widehat{w}(t)-\widehat{v}_2(t)}\Big)dt;\\
d\widehat{v}_j(t)=&\frac{2dt}{\widehat{v}_j(t)-\widehat{w}(t)},\quad j=1,2.
\end{aligned}$$

Moreover, if $T<\infty$, then $\widehat{w}(T+t)=\widehat{w}(T)+\sqrt{\kappa}B'(t)$, $0\le t<\infty$. Then the chordal Loewner curve driven by $\widehat{w}$ is called a full hSLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞ with force points v_1,v_2.

If f maps $\mathbb{H}$ conformally onto a simply connected domain D, then the f-image of a full hSLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞ with force points v_1,v_2 is called a full hSLE$_\kappa$ curve in D from $f(0)$ to $f(\infty)$ with force points $f(v_1),f(v_2)$.

Remark 2.3. In the definition of full hSLE$_\kappa$ in $\mathbb{H}$, if $\kappa\in(0,4]$, then a.s. $T=\infty$, and we do not need the B' in the definition. If $\kappa\in(4,8)$, then a.s. $T<\infty$; and $\eta(t)$ tends to some point on $\mathbb{R}$ between ∞ and v_2. The assumption that $\widehat{w}(T+t)=\widehat{w}(T)+\sqrt{\kappa}B'(t)$, $0\le t<\infty$, means that given the part of η up to T, the rest of η is a chordal SLE$_\kappa$ curve from $\eta(T)$ to ∞ in the remaining domain. In both cases, a full hSLE$_\kappa$ curve always ends at its target.

We now describe hSLE using radial Loewner equation. Let $w_0,v_1,v_2,w_\infty\in\mathbb{R}$ be such that $w_0>v_1>v_2>w_\infty>w_0-2\pi$ or $w_0<v_1<v_2<w_\infty<w_0+2\pi$. Let $B(t)$ be a standard Brownian motion. Let $\widehat{w}_0(t)$, $\widehat{w}_\infty(t)$, and $\widehat{v}_j(t)$, $j=1,2$, $0\le t<T$, be the solution of the SDEs:

$$\begin{aligned}
d\widehat{w}_0(t)&=\sqrt{\kappa}dB(t)+\frac{\kappa-6}{2}\cot_2(\widehat{w}_0(t)-\widehat{w}_\infty(t))dt+\\
&\quad+\frac{1}{2}(\cot_2(\widehat{w}_0(t)-\widehat{v}_1(t))-\cot_2(\widehat{w}_0(t)-\widehat{v}_2(t)))\widetilde{G}(R(t))dt,\\
R(t)&=\frac{\sin_2(\widehat{w}_0(t)-\widehat{v}_1(t))\sin_2(\widehat{v}_2(t)-\widehat{w}_\infty(t))}{\sin_2(\widehat{w}_0(t)-\widehat{v}_2(t))\sin_2(\widehat{v}_1(t)-\widehat{w}_\infty(t))}\\
d\widehat{w}_\infty(t)&=\cot_2(\widehat{w}_\infty(t)-\widehat{w}_0(t))dt,\\
d\widehat{v}_j(t)&=\cot_2(\widehat{v}_j(t)-\widehat{w}_0(t))dt,\quad j=1,2,
\end{aligned}$$

with initial values w_0, w_∞, and v_j, $j=1,2$, respectively, such that $[0,T)$ is the maximal solution interval. Then we call the radial Loewner curve driven by $\widehat{w}_0$ a radial hSLE$_\kappa$ curve in $\mathbb{D}$ from e^{iw_0} to e^{iw_∞} with force points e^{iv_1}, e^{iv_2}, viewed from 0.

Proposition 2.4. *Let w_0, w_∞, v_1, v_2 be as above. Suppose $\eta(t)$, $0 \le t < T'$, is a full hSLE$_\kappa$ curve in $\mathbb{D}$ from e^{iw_0} to e^{iw_∞} with force points e^{iv_1}, e^{iv_2}. Let T be the first time that η separates 0 from any of $e^{iw_0}, e^{iv_1}, e^{iv_2}$. If such time does not exist, then we set $T = T'$. Then up to a time-change, $\eta(t)$, $0 \le t < T$, is a radial hSLE$_\kappa$ curve in $\mathbb{D}$ from e^{iw_0} to e^{iw_∞} with force points e^{iv_1}, e^{iv_2}, viewed from 0.*

Proof. This follows from the standard argument as in [25]. $\square$

One important property of hSLE is its connection with 2-SLE. If (η_1, η_2) is a 2-SLE$_\kappa$ in D with link pattern $(a_1 \to b_1; a_2 \to b_2)$, then for $j = 1, 2$, η_j is a full hSLE$_\kappa$ curve in D from a_j to b_j with force points b_{3-j} and a_{3-j} (see e.g., [26, Proposition 6.10]). The two curves η_1 and η_2 commute with each other in the following sense: if we run one curve, say η_{3-j} up to a stopping time T before reaching b_{3-j} or separating b_{3-j} from b_j or a_j, and condition on this part of η_{3-j}, then the whole η_j is a full hSLE$_\kappa$ curve from a_j to b_j in the remaining domain with force points $\eta_{3-j}(T)$ and b_{3-j}. This easily follows from the DMP of 2-SLE.

2.3 Two-parameter Stochastic Processes

We work on a measurable space $(\Omega, \mathcal{F})$. Let $\mathcal{Q}$ denote the first quadrant $[0,\infty)^2$ with partial order $\le$ such that $\underline{t} = (t_1, t_2) \le (s_1, s_2) = \underline{s}$ iff $t_1 \le s_1$ and $t_2 \le s_2$. It has a minimal element $\underline{0} = (0,0)$. We write $\underline{t} < \underline{s}$ if $t_1 < s_1$ and $t_2 < s_2$. Moreover, we define $\underline{t} \wedge \underline{s} = (t_1 \wedge s_1, t_2 \wedge s_2)$. Given $\underline{t}, \underline{s} \in \mathcal{Q}$, we define $[\underline{t}, \underline{s}] = \{\underline{r} \in \mathcal{Q} : \underline{t} \le \underline{r} \le \underline{s}\}$. For example, $[\underline{0}, \underline{t} \wedge \underline{s}] = [\underline{0}, \underline{t}] \cap [\underline{0}, \underline{s}]$. For $n \in \mathbb{N}$, we define $t^{\lfloor n \rfloor} = \frac{\lfloor 2^n t_1 \rfloor}{2^n}$ and $t^{\lceil n \rceil} = \frac{\lceil 2^n t_1 \rceil}{2^n}$ for $t \in [0,\infty)$, and $\underline{t}^{\lfloor n \rfloor} = (t_1^{\lfloor n \rfloor}, t_2^{\lfloor n \rfloor})$ and $\underline{t}^{\lceil n \rceil} = (t_1^{\lceil n \rceil}, t_2^{\lceil n \rceil})$ for $\underline{t} = (t_1, t_2) \in \mathcal{Q}$. Note that $\underline{t}^{\lfloor n \rfloor}, \underline{t}^{\lceil n \rceil} \in \mathcal{Q}$ and $\underline{t}^{\lfloor n \rfloor} \le \underline{t} \le \underline{t}^{\lceil n \rceil}$.

Definition 2.5. A family of sub-σ-fields $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$ of $\mathcal{F}$ is called a $\mathcal{Q}$-indexed filtration if $\mathcal{F}_{\underline{t}} \subset \mathcal{F}_{\underline{s}}$ whenever $\underline{t} \le \underline{s}$. A family of random variables $(X(\underline{t}))_{\underline{t} \in \mathcal{Q}}$ defined on $(\Omega, \mathcal{F})$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-adapted process if for any $\underline{t} \in \mathcal{Q}$, $X(\underline{t})$ is $\mathcal{F}_{\underline{t}}$-measurable. It is called continuous if $\underline{t} \mapsto X(\underline{t})$ is sample-wise continuous.

Definition 2.6. A random map $\underline{T} : \Omega \to \mathcal{Q}$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping time if for any deterministic $\underline{t} \in \mathcal{Q}$, $\{\underline{T} \le \underline{t}\} \in \mathcal{F}_{\underline{t}}$. Here we do not allow that $\underline{T}$ takes value infinity. For such $\underline{T}$, we define a new σ-field $\mathcal{F}_{\underline{T}}$ by

$$\mathcal{F}_{\underline{T}} = \{A \in \mathcal{F} : A \cap \{\underline{T} \le \underline{t}\} \in \mathcal{F}_{\underline{t}}, \quad \forall \underline{t} \in \mathcal{Q}\}.$$

The stopping time $\underline{T}$ is called bounded if there is a deterministic $\underline{t} \in \mathcal{Q}$ such that $\underline{T} \le \underline{t}$.

Note that any deterministic number $\underline{t} \in \mathcal{Q}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping time, and the $\mathcal{F}_{\underline{t}}$ defined by considering $\underline{t}$ as a stopping time agrees with the $\mathcal{F}_{\underline{t}}$ as in the filtration.

Lemma 2.7. *Let* $\underline{T}$ *and* $\underline{S}$ *be two* $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$*-stopping times. Then (i)* $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{S}}$*; (ii) if* $\underline{S}$ *is a deterministic time* $\underline{s}\in\mathcal{Q}$*, then* $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{T}}$*; and (iii) if* f *is an* $\mathcal{F}_{\underline{T}}$*-measurable function, then* $\mathbf{1}_{\{\underline{T}\le\underline{S}\}}f$ *is* $\mathcal{F}_{\underline{S}}$*-measurable. In particular, if* $\underline{T}\le\underline{S}$*, then* $\mathcal{F}_{\underline{T}}\subset\mathcal{F}_{\underline{S}}$*.*

Proof. (i) Let $\underline{t}=(t_1,t_2)\in\mathcal{Q}$. We have

$$\{\underline{T}\le\underline{S}\}\cap\{\underline{S}\le\underline{t}\}=\{\underline{T}\le\underline{t}\}\cap\{\underline{S}\le\underline{t}\}\cap\{\underline{T}\not\le\underline{S}\}^c$$

$$=\{\underline{S}\le\underline{t}\}\cap\{\underline{T}\le\underline{t}\}\cap[(\{S_1<T_1\le t_1\}\cap\{T_2\vee S_2\le t_2\})\cup(\{T_1\vee S_1\le t_1\}\cap\{S_2<T_2\le t_2\})].$$

Since $\underline{T}$ and $\underline{S}$ are stopping times, $\{\underline{S}\le\underline{t}\}\cap\{\underline{T}\le\underline{t}\}\in\mathcal{F}_{\underline{t}}$. Since we may write

$$\{S_1<T_1\le t_1\}\cap\{T_2\vee S_2\le t_2\}=\bigcup_{t_1'\in\mathbb{Q}\cap(0,t_1)}\{S_1\le t_1'<T_1\le t_1\}\cap\{T_2\vee S_2\le t_2\}$$

$$=\bigcup_{t_1'\in\mathbb{Q}\cap(0,t_1)}\{\underline{T}\le\underline{t}\}\cap\{\underline{S}\le(t_1',t_2)\}\cap\{\underline{T}\le(t_1',t_2)\}^c,$$

we get $\{S_1<T_1\le t_1\}\cap\{T_2\vee S_2\le t_2\}\in\mathcal{F}_{\underline{t}}$. Similarly, $\{T_1\vee S_1\le t_1\}\cap\{S_2<T_2\le t_2\}\in\mathcal{F}_{\underline{t}}$. Combining, we get $\{\underline{T}\le\underline{S}\}\cap\{\underline{S}\le\underline{t}\}\in\mathcal{F}_{\underline{t}}$. Thus, $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{S}}$.

(ii) If $\underline{S}=\underline{s}$ for some $\underline{s}\in\mathcal{Q}$, then $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{T}}$ because for any $\underline{t}\in\mathcal{Q}$,

$$\{\underline{T}\le\underline{S}\}\cap\{\underline{T}\le\underline{t}\}=\{\underline{T}\le\underline{s}\wedge\underline{t}\}\in\mathcal{F}_{\underline{s}\wedge\underline{t}}\subset\mathcal{F}_{\underline{t}}.$$

(iii) By monotone convergence, it suffices to consider the case that $f=\mathbf{1}_A$, where $A\in\mathcal{F}_{\underline{T}}$. Then for any $\underline{t}\in\mathcal{Q}$,

$$A\cap\{\underline{T}\le\underline{S}\}\cap\{\underline{S}\le\underline{t}\}=(A\cap\{\underline{T}\le\underline{t}\})\cap(\{\underline{T}\le\underline{S}\}\cap\{\underline{S}\le\underline{t}\})\in\mathcal{F}_{\underline{t}}.$$

So $A\cap\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{S}}$, which implies that $\mathbf{1}_{\{\underline{T}\le\underline{S}\}}f=\mathbf{1}_{A\cap\{\underline{T}\le\underline{S}\}}$ is $\mathcal{F}_{\underline{S}}$-measurable. ☐

Remark 2.8. In general, we do not have $\{\underline{T}\le\underline{S}\}\in\mathcal{F}_{\underline{T}}$ unless $\underline{S}$ is deterministic or separable (see Definition 2.12 and Lemma 2.13).

Lemma 2.9. *Let* $(X_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$ *be a continuous* $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$*-adapted process. Let* $\underline{T}$ *be an* $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$*-stopping time. Then* $X_{\underline{T}}$ *is* $\mathcal{F}_{\underline{T}}$*-measurable.*

Proof. Since $\underline{T}^{\lfloor n\rfloor}\uparrow\underline{T}$, as $n\to\infty$, by the continuity of X, it suffices to show that for every $n\in\mathbb{N}$, $X_{\underline{T}^{\lfloor n\rfloor}}$ is $\mathcal{F}_{\underline{T}}$-measurable. For a fixed $n\in\mathbb{N}$, since $\underline{T}^{\lfloor n\rfloor}$ takes values in the countable set $(\frac{\mathbb{Z}}{2^n})^2$; and for every $\underline{t}\in(\frac{\mathbb{Z}}{2^n})^2$, by Lemma 2.7 (i,ii), $\{\underline{T}^{\lfloor n\rfloor}=\underline{t}\}=\{\underline{t}\le\underline{T}\}\cap\{\underline{T}<\underline{t}+(\frac{1}{2^n},\frac{1}{2^n})\}\in\mathcal{F}_{\underline{T}}$, it suffices to show that $X_{\underline{T}^{\lfloor n\rfloor}}$ restricted to $\{\underline{T}^{\lfloor n\rfloor}=\underline{t}\}$ is $\mathcal{F}_{\underline{T}}$-measurable. To see this, we may write $\mathbf{1}_{\{\underline{T}^{\lfloor n\rfloor}=\underline{t}\}}X_{\underline{T}^{\lfloor n\rfloor}}=\mathbf{1}_{\{\underline{T}^{\lfloor n\rfloor}=\underline{t}\}}\mathbf{1}_{\{\underline{t}\le\underline{T}\}}X_{\underline{t}}$. Since $X_{\underline{t}}$ is $\mathcal{F}_{\underline{t}}$-measurable, by Lemma 2.7, $\mathbf{1}_{\{\underline{t}\le\underline{T}\}}X_{\underline{t}}$ is $\mathcal{F}_{\underline{T}}$-measurable. So $\mathbf{1}_{\{\underline{T}^{\lfloor n\rfloor}=\underline{t}\}}X_{\underline{T}^{\lfloor n\rfloor}}$ is $\mathcal{F}_{\underline{T}}$-measurable, as desired. ☐

From now on, we fix a a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$, and let $\mathbb{E}$ denote the corresponding expectation.

Definition 2.10. An $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-adapted process $(X_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-martingale (w.r.t. $\mathbb{P}$) if every $X_{\underline{t}}$ is integrable, and for any $\underline{s} \le \underline{t} \in \mathcal{Q}$, $\mathbb{E}[X_{\underline{t}}|\mathcal{F}_{\underline{s}}] = X_{\underline{s}}$. If there is $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_{\underline{t}} = \mathbb{E}[X|\mathcal{F}_{\underline{t}}]$ for every $\underline{t} \in \mathcal{Q}$, then it is clear that $(X_{\underline{t}})$ is an $(\mathcal{F}_{\underline{t}})$-martingale. We call such $(X_{\underline{t}})$ an X-Doob martingale or simply a Doob martingale.

Lemma 2.11 (Optional Stopping Theorem)**.** *Let $(X_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$ be a continuous $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-martingale. The following are true. (i) If $(X_{\underline{t}})$ is an X-Doob martingale for some $X \in L^1$, then for any $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time $\underline{T}$, $X_T = \mathbb{E}[X|\mathcal{F}_{\underline{T}}]$. (ii) If $\underline{T} \le \underline{S}$ are two bounded $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping times, then $\mathbb{E}[X_{\underline{S}}|\mathcal{F}_{\underline{T}}] = X_{\underline{T}}$.*

Proof. (i) Assume that $(X_{\underline{t}})$ is an X-Doob martingale. First, we assume that $\underline{T}$ takes values in $(\frac{\mathbb{Z}}{2^n})^2$ for some $n \in \mathbb{N}$. Since $X_{\underline{T}}$ is $\mathcal{F}_{\underline{T}}$-measurable by Lemma 2.9, it suffices to show that, for any $A \in \mathcal{F}_{\underline{T}}$, $\mathbb{E}[\mathbf{1}_A X_{\underline{T}}] = \mathbb{E}[\mathbf{1}_A X]$. We now fix $A \in \mathcal{F}_{\underline{T}}$. For any $\underline{t} \in \mathcal{Q} \cap (\frac{\mathbb{Z}}{2^n})^2$, since $A \cap \{\underline{T} = \underline{t}\} \in \mathcal{F}_{\underline{t}}$, using $\mathbb{E}[X|\mathcal{F}_{\underline{t}}] = X_{\underline{t}}$, we get $\mathbb{E}[\mathbf{1}_{A\cap\{\underline{T}=\underline{t}\}} X_{\underline{t}}] = \mathbb{E}[\mathbf{1}_{A\cap\{\underline{T}=\underline{t}\}} X]$. Summing up over $\underline{t} \in \mathcal{Q} \cap (\frac{\mathbb{Z}}{2^n})^2$, we get $\mathbb{E}[\mathbf{1}_A X_{\underline{T}}] = \mathbb{E}[\mathbf{1}_A X]$ in this special case.

Now we consider the general case. Note that for every $n \in \mathbb{N}$, $\underline{T}^{\lceil n\rceil}$ takes values in $(\frac{\mathbb{Z}}{2^n})^2$, and is a stopping time because for any $\underline{t} \in \mathcal{Q}$, $\{\underline{T}^{\lceil n\rceil} \le \underline{t}\} = \{T \le \underline{t}^{\lfloor n\rfloor}\} \in \mathcal{F}_{\underline{t}^{\lfloor n\rfloor}} \subset \mathcal{F}_{\underline{t}}$. Applying the special case to $\underline{T}^{\lceil n\rceil}$, we get $\mathbb{E}[X|\mathcal{F}_{\underline{T}^{\lceil n\rceil}}] = X_{\underline{T}^{\lceil n\rceil}}$. Since $\underline{T}^{\lceil n\rceil} \downarrow T$ as $n \to \infty$. By the continuity of X, we have $X_{\underline{T}^{\lceil n\rceil}} \to X_{\underline{T}}$. Since $\mathcal{F}_{\underline{T}} \subset \mathcal{F}_{\underline{T}^{\lceil n\rceil}}$ by Lemma 2.7, a standard argument involving uniform integrability shows that $\mathbb{E}[X|\mathcal{F}_{\underline{T}}] = X_{\underline{T}}$.

(ii) First assume that $\underline{S}$ is a constant $\underline{s} \in \mathbb{N}^2$. Then $\underline{S} \ge \underline{T}^{\lceil n\rceil}$ for all $n \in \mathbb{N}$. Using the same argument as in (i) with $X_{\underline{S}}$ in place of X, we get $\mathbb{E}[X_{\underline{S}}|\mathcal{F}_{\underline{T}}] = X_{\underline{T}}$.

Finally, we consider the general case. Since $\underline{S}$ is bounded, there is $\underline{r} \in \mathcal{Q} \cap \mathbb{N}^2$ such that $\underline{T} \le \underline{S} \le \underline{r}$. Let $A \in \mathcal{F}_{\underline{T}} \subset \mathcal{F}_{\underline{S}}$. From the special case of (ii), we get $\mathbb{E}[\mathbf{1}_A X_{\underline{S}}] = \mathbb{E}[\mathbf{1}_A X_{\underline{r}}] = \mathbb{E}[\mathbf{1}_A X_{\underline{T}}]$, which implies that $\mathbb{E}[X_{\underline{S}}|\mathcal{F}_{\underline{T}}] = X_{\underline{T}}$. $\square$

Definition 2.12. Suppose that there are two filtrations $(\mathcal{F}^1_{t_1})$ and $(\mathcal{F}^2_{t_2})$ such that $\mathcal{F}_{(t_1,t_2)} = \mathcal{F}^1_{t_1} \vee \mathcal{F}^2_{t_2}$, $(t_1, t_2) \in \mathcal{Q}$. Then we say that $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$ is a separable filtration generated by $(\mathcal{F}^1_{t_1})$ and $(\mathcal{F}^2_{t_2})$. For such a separable filtration, if T_j is a finite $(\mathcal{F}^j_t)$-stopping time, $j = 1, 2$, then (T_1, T_2) is called a separable $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time (w.r.t. $(\mathcal{F}^1_{t_1})$ and $(\mathcal{F}^2_{t_2})$).

Lemma 2.13. *Let $\underline{T}$ and $\underline{S}$ be two stopping times w.r.t. a separable filtration $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$. If $\underline{S}$ is separable, then $\{\underline{T} \le \underline{S}\} \in \mathcal{F}_{\underline{T}}$.*

Proof. We have $\{\underline{T} \le \underline{S}\} \in \mathcal{F}_{\underline{T}}$ because for any $\underline{t} \in \mathcal{Q}$,

$$\{\underline{T} \le \underline{S}\} \cap \{\underline{T} \le \underline{t}\} = \{\underline{T} \le \underline{S} \wedge \underline{t}\} \in \mathcal{F}_{\underline{S}\wedge\underline{t}} \subset \mathcal{F}_{\underline{t}}.$$

Here we use Lemma 2.7 (i,ii) and the fact that $\underline{S} \wedge \underline{t}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time (and so $\mathcal{F}_{\underline{S}\wedge\underline{t}}$ is well defined), which follows from the assumption that $\underline{S}$ is separable. $\square$

Definition 2.14. A relatively open subset $\mathcal{R}$ of $\mathcal{Q}$ is called a history complete region, or simply an HC region, if for any $\underline{t} \in \mathcal{R}$, we have $[\underline{0}, \underline{t}] \subset \mathcal{R}$. Given an HC region $\mathcal{R}$, we may define two functions $T^{\mathcal{R}}_1, T^{\mathcal{R}}_2 : [0, \infty) \to [0, \infty]$ such that

$$[0, T^{\mathcal{R}}_1(t_2)) = \{s_1 \ge 0 : (s_1, t_2) \in \mathcal{R}\}, \quad [0, T^{\mathcal{R}}_2(t_1)) = \{s_2 \ge 0 : (t_1, s_2) \in \mathcal{R}\}, \quad t_1, t_2 \ge 0.$$

A map $\mathcal{D}$ from Ω into the space of HC regions is called an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping region if for any $\underline{t}\in\mathcal{Q}$, $\{\omega\in\Omega:\underline{t}\in\mathcal{D}(\omega)\}\in\mathcal{F}_{\underline{t}}$. A random function $X(\underline{t})$ with a random domain $\mathcal{D}$ is called an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-adapted HC process if $\mathcal{D}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping region, and for every $\underline{t}\in\mathcal{Q}$, $X_{\underline{t}}$ restricted to $\{\underline{t}\in\mathcal{D}\}$ is $\mathcal{F}_{\underline{t}}$-measurable.

3 Ensemble of Two Radial Loewner Chains

3.1 Deterministic ensemble

Let $w_1,w_2,v_1,v_2\in\mathbb{R}$ be such that $w_1>v_1>w_2>v_2>w_2-2\pi$. For $j=1,2$, let $\widehat{w}_j\in C([0,S_j),\mathbb{R})$ be a radial Loewner driving function with $\widehat{w}_j(0)=w_j$. Suppose $\widehat{w}_j$ generates radial Loewner hulls $K_j(t)$, radial Loewner maps $g_j(t,\cdot)$, covering Loewner hulls $\widetilde{K}_j(t)$ and covering radial Loewner maps $\widetilde{g}_j(t,\cdot)$, $0\le t<S_j$. Let $\mathcal{D}$ denote the set of $(t_1,t_2)\in[0,S_1)\times[0,S_2)$ such that $\overline{K_1(t_1)}\cap\overline{K_2(t_2)}=\emptyset$ and $e^{iv_1},e^{iv_2}\not\in\overline{K_1(t_1)}\cup\overline{K_2(t_2)}$. Then $\mathcal{D}$ is an HC region as in Definition 2.14, and we may define functions $T_1^{\mathcal{D}}$ and $T_2^{\mathcal{D}}$. For $(t_1,t_2)\in\mathcal{D}$, let $K(t_1,t_2)=K_1(t_1)\cup K_2(t_2)$. Then $K(t_1,t_2)$ is also an $\mathbb{D}$-hull. Let $g((t_1,t_2),\cdot)=g_{K(t_1,t_2)}$, and $\mathrm{m}(t_1,t_2)=\mathrm{dcap}(K(t_1,t_2))$. For $(t_1,t_2)\in\mathcal{D}$ and $j\ne k\in\{1,2\}$, let $K_{j,t_k}(t_j)=g_k(t_k,K_j(t_j))$, and $g_{j,t_k}(t_j,\cdot)=g_{K_{j,t_k}(t_j)}$. Then we have

$$g_{1,t_2}(t_1,\cdot)\circ g_2(t_2,\cdot)=g((t_1,t_2),\cdot)=g_{2,t_1}(t_2,\cdot)\circ g_1(t_1,\cdot). \tag{3.1}$$

Let $\widetilde{K}(t_1,t_2),\widetilde{K}_{j,t_k}(t_j)\subset\mathbb{H}$ be the pre-images of $K(t_1,t_2),K_{j,t_k}(t_j)$, respectively, under the map e^i. Let $\widetilde{g}((t_1,t_2),\cdot)$, $(t_1,t_2)\in\mathcal{D}$, be the unique family of maps, such that $\widetilde{g}((t_1,t_2),z)$ is joint continuous in t_1,t_2,z; $\widetilde{g}((0,0),\cdot)=\mathrm{id}$; and for each $(t_1,t_2)\in\mathcal{D}$, $\widetilde{g}((t_1,t_2),\cdot):\mathbb{H}\backslash\widetilde{K}(t_1,t_2)\overset{\mathrm{Conf}}{\twoheadrightarrow}\mathbb{H}$, and $e^i\circ\widetilde{g}((t_1,t_2),\cdot)=g((t_1,t_2),\cdot)\circ e^i$. Define $\widetilde{g}_{1,t_2}(t_1,\cdot)$ and $\widetilde{g}_{2,t_1}(t_2,\cdot)$, $(t_1,t_2)\in\mathcal{D}$, similarly. Using (3.1) we get

$$\widetilde{g}_{1,t_2}(t_1,\cdot)\circ\widetilde{g}_2(t_2,\cdot)=\widetilde{g}((t_1,t_2),\cdot)=\widetilde{g}_{2,t_1}(t_2,\cdot)\circ\widetilde{g}_1(t_1,\cdot). \tag{3.2}$$

Note also that $\widetilde{g}((t_1,0),\cdot)=\widetilde{g}_1(t_1,\cdot)$ and $\widetilde{g}((0,t_2),\cdot)=\widetilde{g}_2(t_2,\cdot)$. So $\widetilde{g}_{1,t_2}(0,\cdot)$ (resp. $\widetilde{g}_{2,t_1}(0,\cdot)$) is an identity if $(0,t_2)\in\mathcal{D}$ (resp. $(t_1,0)\in\mathcal{D}$). Let $(t_1,t_2)\in\mathcal{D}$. From the assumption on e^{iv_1},e^{iv_2}, $g((t_1,t_2),\cdot)$ extends conformally to neighborhoods of e^{iv_1} and e^{iv_2}. Thus, $\widetilde{g}((t_1,t_2),\cdot)$ extends conformally to neighborhoods of v_1 and v_2. Then we define real valued functions

$$V_j(t_1,t_2)=\widetilde{g}((t_1,t_2),v_j),\quad V_{j,1}(t_1,t_2)=\widetilde{g}'((t_1,t_2),v_j),\quad (t_1,t_2)\in\mathcal{D}. \tag{3.3}$$

Here and below the prime means the partial derivative w.r.t. the last variable. Fix $j\ne k\in\{1,2\}$. From Lemma 2.1 we know that $g_{k,t_j}(t_k,\cdot)$ extends conformally to a neighborhood of $e^{i\widehat{w}_j(t_j)}$. Thus, $\widetilde{g}_{k,t_j}(t_k,\cdot)$ extends conformally to a neighborhood of $\widehat{w}_j(t_j)$. Now we define

$$W_j(t_1,t_2)=\widetilde{g}_{k,t_j}(t_k,\widehat{w}_j(t_j)),\quad W_{j,h}(t_1,t_2)=\widetilde{g}^{(h)}_{k,t_j}(t_k,\widehat{w}_j(t_j)),\quad (t_1,t_2)\in\mathcal{D}. \tag{3.4}$$

Here and below the superscript (h) means the h-th partial derivative w.r.t. the last variable. We then have $W_1>V_1>W_2>V_2>W_1-2\pi$. For a function X defined on $\mathcal{D}$, $k\in\{1,2\}$, and

$t_k \ge 0$, we let $X^{(k,t_k)}$ be the function defined on $[0, T_j^{\mathcal{D}}(t_k))$ obtained from X by fixing the k-th variable to be t_k. Since $\widetilde{g}_{k,t_j}(0,\cdot)$ are identity maps, we get

$$W_{j,1}^{(k,0)} \equiv 1, \quad W_{j,2}^{(k,0)} = W_{j,3}^{(k,0)} \equiv 0, \quad j \ne k \in \{1,2\}. \tag{3.5}$$

Using Lemma 2.1 we know that, for any $t_k \ge 0$, $K_{j,t_k}(t_j)$ and $\widetilde{g}_{j,t_k}(t_j,\cdot)$, $0 \le t_j < T_j^{\mathcal{D}}(t_k)$, are radial Loewner hulls and covering radial Loewner maps, respectively, driven by $W_j^{(k,t_k)}$ with speed $|W_{j,1}^{(k,t_k)}|^2$. This means that

$$\partial_j \,\mathrm{m} = \partial_j(\mathrm{dcap}(K_k(t_k)) + \mathrm{dcap}(K_{j,t_k}(t_j))) = W_{j,1}^2 \partial t_j; \tag{3.6}$$

$$\partial_j \widetilde{g}_{j,t_k}(t_j,z) = W_{j,1}(t_1,t_2)^2 \cot_2(\widetilde{g}_{j,t_k}(t_j,z) - W_j(t_1,t_2))\partial t_j. \tag{3.7}$$

Plugging $z = \widehat{w}_k(t_k)$ and $z = \widetilde{g}_k(t_k, v_s)$, respectively, into (3.7), we get

$$\partial_j W_k = W_{j,1}^2 \cot_2(W_k - W_j)\partial t_j, \quad \partial_j V_s = W_{j,1}^2 \cot_2(V_s - W_j)\partial t_j, \quad s = 1,2. \tag{3.8}$$

Differentiating (3.7) w.r.t. z, we get

$$\frac{\partial_j \widetilde{g}_{j,t_k}'(t_j,z)}{\widetilde{g}_{j,t_k}'(t_j,z)} = W_{j,1}(t_1,t_2)^2 \cot_2'(\widetilde{g}_{j,t_k}(t_j,z) - W_j(t_1,t_2))\partial t_j. \tag{3.9}$$

Plugging $z = \widehat{w}_k(t_k)$ and $z = \widetilde{g}_k(t_k, v_s)$, respectively, into (3.9), we get

$$\frac{\partial_j W_{k,1}}{W_{k,1}} = W_{j,1}^2 \cot_2'(W_k - W_j)\partial t_j, \quad \frac{\partial_j V_{s,1}}{V_{s,1}} = W_{j,1}^2 \cot_2'(V_s - W_j)\partial t_j, \quad s = 1,2. \tag{3.10}$$

Since $\cot_2'(W_k - W_j) < 0$, we see that $W_{k,1}$ is decreasing in t_j and stays positive. From (3.5) we see that $W_{k,1} \in (0,1]$. Since $\mathrm{m}(t_1,0) = t_1$ and $\mathrm{m}(0,t_2) = t_2$, from (3.6), we get

$$t_1 \vee t_2 \le \mathrm{m}(t_1,t_2) \le t_1 + t_2, \quad (t_1,t_2) \in \mathcal{D}. \tag{3.11}$$

Let $Sg := (\frac{g''}{g'})' - \frac{1}{2}(\frac{g''}{g'})^2$ denote the Schwarzian derivative of g, and let

$$W_{k,S} = S\widetilde{g}_{j,t_k}(t_j, \widehat{w}_k(t_k)) = \frac{W_{k,3}}{W_{k,1}} - \frac{3}{2}\Big(\frac{W_{k,2}}{W_{k,1}}\Big)^2. \tag{3.12}$$

Differentiating (3.9) w.r.t. z, we get

$$\partial_j \Big(\frac{\widetilde{g}_{j,t_k}''(t_j,z)}{\widetilde{g}_{j,t_k}'(t_j,z)}\Big) = W_{j,1}(t_1,t_2)^2 \cot_2''(\widetilde{g}_{j,t_k}(t_j,z) - W_j(t_1,t_2))\widetilde{g}_{j,t_k}'(t_j,z)\partial t_j.$$

Further differentiating this equation w.r.t. z and plugging $z = \widehat{w}_k(t_k)$, we get

$$\partial_j W_{K,S} = W_{j,1}^2 W_{k,1}^2 \cot_2'''(W_k - W_j)\partial t_j. \tag{3.13}$$

Differentiating (3.2) w.r.t. t_j and using (3.7), we get

$$\partial_{t_j}\widetilde{g}_{k,t_j}(t_k,\widetilde{g}_j(t_j,z))+\widetilde{g}'_{k,t_j}(t_k,\widetilde{g}_j(t_j,z))\cot_2(\widetilde{g}_j(t_j,z)-\widehat{w}_j(t_j))$$
$$=W_{j,1}(t_1,t_2)^2\cot_2(\widetilde{g}_{j,t_k}(t_j,\widetilde{g}_k(t_k,z))-W_j(t_1,t_2)).$$

Let $\widehat{z}=\widetilde{g}_j(t_j,z)$. Using (3.2) we get

$$\partial_{t_j}\widetilde{g}_{k,t_j}(t_k,\widehat{z})=\widetilde{g}'_{k,t_j}(t_k,\widehat{w}_j(t_j))^2\cot_2(\widetilde{g}_{k,t_j}(t_k,\widehat{z})-\widetilde{g}_{k,t_j}(t_k,\widehat{w}_j(t_j)))-\widetilde{g}'_{k,t_j}(t_k,\widehat{z})\cot_2(\widehat{z}-\widehat{w}_j(t_j)). \tag{3.14}$$

Sending $\widehat{z}\to\widehat{w}_j(t_j)$, we get

$$\partial_{t_j}\widetilde{g}_{k,t_j}(t_k,\widehat{z})|_{\widehat{z}=\widehat{w}_j(t_j)}=-3\widetilde{g}''_{k,t_j}(t_k,\widehat{w}_j(t_j))=-3W_{j,2}; \tag{3.15}$$

Differentiating (3.14) w.r.t. $\widehat{z}$ and then sending $\widehat{z}\to\widehat{w}_j(t_j)$, we get

$$\frac{\partial_{t_j}\widetilde{g}'_{k,t_j}(t_k,\widehat{z})|_{\widehat{z}=\widehat{w}_j(t_j)}}{\widetilde{g}'_{k,t_j}(t_k,\widehat{z})|_{\widehat{z}=\widehat{w}_j(t_j)}}=\frac{1}{2}\Big(\frac{W_{j,2}}{W_{j,1}}\Big)^2-\frac{4}{3}\frac{W_{j,3}}{W_{j,1}}-\frac{1}{6}(W_{j,1}^2-1). \tag{3.16}$$

Finally, suppose that $\widehat{w}_1$ and $\widehat{w}_2$ generate radial Loewner curves η_1 and η_2, respectively, and for any $j\neq k\in(1,2)$, and any $t_k\in[0,S_k)$, the radial Loewner process driven by $W_j^{(k,t_k)}$ with speed $|W_{j,1}^{(k,t_k)}|^2$ generates a radial Loewner curve η_{j,t_k}. Then we have $\eta_j(t_j)=g_k(t_k,\cdot)^{-1}(\eta_{j,t_k}(t_j))$, $0\le t_j<T_j^{\mathcal{D}}(t_k)$, where $g_k(t_k,\cdot)^{-1}$ is understood as the continuous extension of the original $g_k(t_k,\cdot)^{-1}$ from $\mathbb{D}$ to $\overline{\mathbb{D}}$.

3.2 Two-variable local martingales

We use the setup in the previous subsection. We view $(\widehat{w}_1(t))_{0\le t<S_1}$ and $(\widehat{w}_2(t))_{0\le t<S_2}$ as elements in $\Sigma:=\bigcup_{0<T\le\infty}C([0,T),\mathbb{R})$. The space Σ and a filtration $(\mathcal{F}_t)_{t\ge0}$ were defined in [29, Section 2]. Here is a brief review. For $f\in\Sigma$, let T_f be such that $[0,T_f)$ is the domain of f. For $0\le t<\infty$, the $\mathcal{F}_t$ is the σ-algebra on Σ generated by the values of the function at the times before t. More precisely, $\mathcal{F}_t$ is the σ-algebra generated by

$$\{f\in\Sigma:s<T_f,f(s)\in U\},\quad 0\le s\le t,U\in\mathcal{B}(\mathbb{R}).$$

Now we introduce randomness. Fix $\kappa\in(0,8)$ throughout. The boundary scaling exponent b and central charge c are defined by

$$\mathrm{b}=\frac{6-\kappa}{2\kappa},\quad \mathrm{c}=\frac{(3\kappa-8)(6-\kappa)}{2\kappa}. \tag{3.17}$$

We use $\cot_2,\tan_2,\sin_2,\cos_2$ to denote the functions $\cot(\cdot/2),\tan(\cdot/2),\sin(\cdot/2),\cos(\cdot/2)$, respectively. For $j=1,2$, we let $\mathbb{P}_B^j$ denote the law of $w_j+\sqrt{\kappa}B(t)$, $0\le t<\infty$, where $B(t)$ is a standard Brownian motion, which is a probability measure on $(\Sigma,\mathcal{F}_\infty)$. For $j=1,2$, let $\mathbb{P}_4^j$ denote the law of the radial Loewner driving function with initial value w_j for the radial

$\mathrm{SLE}_\kappa(2,2,2)$ curve in $\mathbb{D}$ started from e^{iw_j} aimed at 0 with force points $e^{iv_1}, e^{iv_2}, e^{iw_{3-j}}$. For $j=1,2$, let $\mathbb{P}_h^j$ denote the law of the radial Loewner driving function with initial value w_j for the radial hSLE_κ curve in $\mathbb{D}$ from e^{iw_j} to e^{iv_j} with force points $e^{iv_{3-j}}, e^{iw_{3-j}}$, viewed from 0.

From now on, when there is no ambiguity, we will not try to distinguish the law of a driving function and the law of the radial Loewner curve that it generates. We will mainly work on the product measurable space, and use the notation in Section 2.3. We naturally have the following product measures: $\mathbb{P}_{iB} := \mathbb{P}_B^1 \times \mathbb{P}_B^2$, $\mathbb{P}_{i4} := \mathbb{P}_4^1 \times \mathbb{P}_4^2$, and $\mathbb{P}_{ih} := \mathbb{P}_h^1 \times \mathbb{P}_h^2$. We use $\mathbb{E}_{iB}, \mathbb{E}_{i4}, \mathbb{E}_{ih}$ to denote the corresponding expectations, respectively.

Now suppose $(\widehat{\eta}_1, \widehat{\eta}_2)$ is a 2-SLE_κ in $\mathbb{D}$ with link pattern $(e^{iw_1} \to e^{iv_1}; e^{iw_2} \to e^{iv_2})$. Then $\widehat{\eta}_j$ is a full hSLE_κ curve in $\mathbb{D}$ from e^{iw_j} to e^{iv_j} with force points $e^{iv_{3-j}}$ and $e^{iw_{3-j}}$. For $j=1,2$, let η_j be the part of $\widehat{\eta}_j$ from w_j up to its lifetime or the time that it separates 0 from any of $e^{iv_j}, e^{iw_{3-j}}, e^{iv_{3-j}}$, if the later time exists. Then we may parametrize η_j using radial capacity, and get a radial Loewner curve. By Proposition 2.4, η_j is a radial hSLE_κ curve in $\mathbb{D}$ from e^{iw_j} to e^{iv_j} with force points $e^{iv_{3-j}}, e^{iw_{3-j}}$, viewed from 0. We use $\mathbb{P}_2$ to denote the joint law of the radial driving functions for η_1 and η_2. Such measure $\mathbb{P}_2$ is a coupling of $\mathbb{P}_h^1$ and $\mathbb{P}_h^2$, but is different from the product measure $\mathbb{P}_{ih}$. Instead, η_1 and η_2 that jointly follow the law $\mathbb{P}_2$ commute with each other in the following sense: for any $j \in \{1,2\}$, conditionally on the part of η_{3-j} up to a stopping time τ before its lifetime, if g maps the remaining domain conformally onto $\mathbb{D}$ with $g(0)=0$ and $g'(0)>0$, then the g-image of η_j up to the time that η_j hits $\eta_{3-j}[0,\tau]$ is a radial hSLE_κ curve in $\mathbb{D}$ from $g(e^{iw_j})$ to $g(e^{iv_j})$ with force points $g(e^{iv_{3-j}})$ and $g(\eta_{3-j}(\tau))$, viewed from 0. The measure $\mathbb{P}_2$ depends on the points w_1, v_1, w_2, v_2. When we want to emphasize the dependence, we use the symbol $\mathbb{P}_2^{w_1,v_1;w_2,v_2}$.

For $j=1,2$, let $(\mathcal{F}_t^j)$ be the filtration generated by the j-th function as described at the beginning of this subsection. Let $(\mathcal{F}_{\underline{t}})$ be the separable $\mathcal{Q}$-indexed filtration generated by $(\mathcal{F}_t^1)$ and $(\mathcal{F}_t^2)$. Then $\mathcal{D}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping region, and for $j \ne k \in \{1,2\}$ and $h=1,2,3$, $\widetilde{g}_j(t_j,\cdot)$, $w_j(t_j)$, $\widetilde{g}((t_1,t_2),\cdot)$, $\widetilde{g}_{j,t_k}(t_j,\cdot)$, $W_j(t_1,t_2)$, $W_{j,h}(t_1,t_2)$, $W_{j,S}(t_1,t_2)$, $V_j(t_1,t_2)$, $V_{j,1}(t_1,t_2)$, defined for $(t_1,t_2)\in\mathcal{D}$, are all continuous $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-adapted HC processes.

Let $B_1(t)$ and $B_2(t)$ be two independent standard Brownian motions. Suppose $\widehat{w}_j(t) = w_j + \sqrt{\kappa}B_j(t)$, $0 \le t < \infty$. Fix $j \ne k \in \{1,2\}$. Let $\mathcal{F}_{t_j}^{(k,\infty)}$ denote the σ-algebra $\mathcal{F}_{t_j}^j \vee \mathcal{F}_\infty^k$. Then we get a filtration $(\mathcal{F}_{t_j}^{(k,\infty)})_{t_j\ge 0}$. Since $(\widehat{w}_j)$ is independent of $(\mathcal{F}_\infty^k)$, it is a rescaled $(\mathcal{F}_{t_j}^{(k,\infty)})_{t_j\ge0}$-Brownian motion started from w_j. Fix an $(\mathcal{F}_\infty^k)$-measurable finite time τ_k. From now on, we will repeatedly use Itô's formula, where the variable t_k is fixed to be τ_k, the variable t_j ranges in $[0, T_j^{\mathcal{D}}(\tau_k))$, and all SDE are $(\mathcal{F}_{t_j}^{(k,\infty)})_{t_j\ge0}$-adapted. Recall that $X^{(k,\tau_k)}$ is the function obtained from a two-variable function X by fixing the k-th variable to be τ_k. Using (3.4,3.15), we get

$$dW_j^{(k,\tau_k)}(t_j) = W_{j,1}^{(k,\tau_k)}(t_j)d\widehat{w}_j(t_j) + \Big(\frac{\kappa}{2}-3\Big)W_{j,2}^{(k,\tau_k)}dt_j.$$

To make the symbols less heavy, we will omit the superscripts (k,τ_k) and the variables (t_j), and use the symbols ∂_j, $\partial\widehat{w}_j$ and ∂t_j to emphasize the role of t_j. The above SDE then becomes

$$\partial_j W_j = W_{j,1}\partial\widehat{w}_j + \Big(\frac{\kappa}{2}-3\Big)W_{j,2}\partial t_j.$$

Combining this with (3.8), we get (for $s=1,2$)

$$\begin{aligned}\partial_j(W_j-W_k)&=W_{j,1}\partial\widehat{w}_j+\Big(\frac{\kappa}{2}-3\Big)W_{j,2}\partial t_j+W_{j,1}^2\cot_2(W_j-W_k)\partial t_j;\\ \partial_j(W_j-V_s)&=W_{j,1}\partial\widehat{w}_j+\Big(\frac{\kappa}{2}-3\Big)W_{j,2}\partial t_j+W_{j,1}^2\cot_2(W_j-V_s)\partial t_j;\\ \partial_j(W_k-V_s)&=-W_{j,1}^2\cot_2(W_j-W_k)\partial t_j+W_{j,1}^2\cot_2(W_j-V_s)\partial t_j;\\ \partial_j(V_j-V_k)&=-W_{j,1}^2\cot_2(W_j-V_j)\partial t_j+W_{j,1}^2\cot_2(W_j-V_k)\partial t_j.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{\partial_j\sin_2(W_j-W_k)}{\sin_2(W_j-W_k)}=&\frac{1}{2}\cot_2(W_j-W_k)W_{j,1}\partial\widehat{w}_j+\frac{1}{2}W_{j,1}^2\cot_2^2(W_j-W_k)\partial t_j\\ &+\frac{1}{2}\cot_2(W_j-W_k)\Big(\frac{\kappa}{2}-3\Big)W_{j,2}\partial t_j-\frac{\kappa}{8}W_{j,1}^2\partial t_j; \qquad (3.18)\\ \frac{\partial_j\sin_2(W_j-V_s)}{\sin_2(W_j-V_s)}=&\frac{1}{2}\cot_2(W_j-V_s)W_{j,1}\partial\widehat{w}_j+\frac{1}{2}W_{j,1}^2\cot_2^2(W_j-V_s)\partial t_j\\ &+\frac{1}{2}\cot_2(W_j-V_s)\Big(\frac{\kappa}{2}-3\Big)W_{j,2}\partial t_j-\frac{\kappa}{8}W_{j,1}^2\partial t_j; \qquad (3.19)\\ \frac{\partial_j\sin_2(W_k-V_s)}{\sin_2(W_k-V_s)}=&-\frac{1}{2}W_{j,1}^2[1+\cot_2(W_j-W_k)\cot_2(W_j-V_s)]\partial t_j; \qquad (3.20)\\ \frac{\partial_j\sin_2(V_j-V_k)}{\sin_2(V_j-V_k)}=&-\frac{1}{2}W_{j,1}^2[1+\cot_2(W_j-V_j)\cot_2(W_j-V_k)]\partial t_j. \qquad (3.21)\end{aligned}$$

Using (3.16), we get

$$\frac{\partial_j W_{j,1}}{W_{j,1}}=\frac{W_{j,2}}{W_{j,1}}\partial\widehat{w}_j+\frac{1}{2}\Big(\frac{W_{j,2}}{W_{j,1}}\Big)^2\partial t_j+\Big(\frac{\kappa}{2}-\frac{4}{3}\Big)\frac{W_{j,3}}{W_{j,1}}\partial t_j-\frac{1}{6}(W_{j,1}^2-1)\partial t_j.$$

Recall the $W_{j,S}$ defined by (3.12) and the b,c defined by (3.17). The above SDE implies that

$$\frac{\partial_j W_{j,1}^{\mathrm{b}}}{W_{j,1}^{\mathrm{b}}}=\mathrm{b}\,\frac{W_{j,2}}{W_{j,1}}\partial\widehat{w}_j+\frac{\mathrm{c}}{6}W_{j,S}\partial t_j-\frac{\mathrm{b}}{6}(W_{j,1}^2-1)\partial t_j. \qquad (3.22)$$

Define a positive continuous function $M_{iB\to c4}$ on $\mathcal{D}$ by

$$M_{iB\to c4}:=e^{\frac{60}{8\kappa}\mathrm{m}+\frac{\mathrm{b}}{6}(\mathrm{m}-t_1-t_2)}[W_{1,1}W_{2,1}V_{1,1}V_{2,1}]^{\mathrm{b}}\Big[\prod_{j=1}^2\sin_2(W_j-V_j)\prod_{X,Y\in\{W,V\}}\sin_2(X_1-Y_2)\Big]^{\frac{2}{\kappa}}\times$$

$$\times\exp\Big(-\frac{\mathrm{c}}{6}\int_0^{t_1}\int_0^{t_2}W_{1,1}^2W_{2,1}^2\cot_2'''(W_1-W_2)ds_1ds_2\Big). \qquad (3.23)$$

Combing (3.6,3.10,3.13,3.18-3.22), we get

$$\frac{\partial_j M_{iB\to c4}}{M_{iB\to c4}}=\mathrm{b}\,\frac{W_{j,2}}{W_{j,1}}\partial\widehat{w}_j+\frac{1}{\kappa}\sum_{X\in\{W_k,V_1,V_2\}}\cot_2(W_j-X)W_{j,1}\partial\widehat{w}_j. \qquad (3.24)$$

This means that $M^{(k,\tau_k)}_{iB\to c4}$ is an $(\mathcal{F}^j_{t_j}\vee\mathcal{F}^k_{\infty})_{t_j\ge 0}$-local martingale up to $T^{\mathcal{D}}_j(\tau_k)$.

Let $F(x)$ be the hypergeometric function ${}_2F_1\Big(\frac{4}{\kappa},1-\frac{4}{\kappa};\frac{8}{\kappa};x\Big)$ as before. Recall that $G(x)=\kappa x\frac{F'(x)}{F(x)}$, $\widetilde{F}(x)=x^{\frac{2}{\kappa}}F(x)$, and $\widetilde{G}(x)=\kappa x\frac{\widetilde{F}'(x)}{\widetilde{F}(x)}=G(x)+2$. From (2.1) we get

$$\frac{\kappa}{8}x^2\frac{\widetilde{F}''(x)}{\widetilde{F}(x)}=\Big[\Big(\frac{1}{4}-\frac{1}{\kappa}\Big)\frac{x}{1-x}-\frac{1}{2\kappa}\Big]\widetilde{G}(x)+\frac{1}{4}\Big(\frac{6}{\kappa}-1\Big)=0.$$

Recall that W_1,V_1,W_2,V_2 are real valued functions defined on $\mathcal{D}$ that satisfy $W_1>V_1>W_2>V_2>W_1-2\pi$. Define the functions R and Φ_j on $\mathcal{D}$ by

$$R=\frac{\sin_2(W_1-V_2)\sin_2(V_1-W_2)}{\sin_2(W_1-W_2)\sin_2(V_1-V_2)}=-\frac{\sin_2(W_j-V_k)\sin_2(W_k-V_j)}{\sin_2(W_j-W_k)\sin_2(V_j-V_k)}\in(0,1).$$

$$\Phi_j=\cot_2(W_j-V_k)-\cot_2(W_j-W_k)=\frac{-\sin_2(W_k-V_k)}{\sin_2(W_j-V_k)\sin_2(W_j-W_k)}.$$

Note that R equals the cross-ratio $[e^{iW_1},e^{iV_1};e^{iV_2},e^{iW_2}]$. Using an identity of cross-ratio, we get

$$1-R=\frac{\sin_2(W_j-V_j)\sin_2(W_k-V_k)}{\sin_2(W_j-W_k)\sin_2(V_j-V_k)}.$$

Thus,

$$\frac{R\Phi_j}{1-R}=\frac{\sin_2(W_k-V_j)}{\sin_2(W_j-V_j)\sin_2(W_j-W_k)}=\cot_2(W_j-W_k)-\cot_2(W_j-V_j).$$

Using (3.18-3.21), we get

$$\begin{aligned}\frac{\partial_j R}{R}=&\frac{1}{2}W_{j,1}\Phi_j\partial\widehat{w}_j+\frac{1}{2}[\cot_2(W_j-W_k)+\cot_2(W_j-V_k)]W_{j,1}^2\Phi_j\partial t_j+\frac{1}{2}\Big(\frac{\kappa}{2}-3\Big)W_{j,2}\Phi_j\partial t_j\\&+\frac{1}{2}\cot_2(W_j-V_j)W_{j,1}^2\Phi_j\partial t_j-\frac{\kappa}{4}\cot_2(W_j-W_k)W_{j,1}^2\Phi_j\partial t_j.\end{aligned}\tag{3.25}$$

Combining the above formulas in this paragraph and using a tedious but straightforward computation, we get

$$\begin{aligned}\frac{\partial_j\widetilde{F}(R)}{\widetilde{F}(R)}=&\frac{1}{2\kappa}\widetilde{G}(R)W_{j,1}\Phi_j\partial\widehat{w}_j+\frac{1}{2\kappa}\Big(\frac{\kappa}{2}-3\Big)\widetilde{G}(R)W_{j,2}\Phi_j\partial t_j\\&+\frac{1}{4}\Big(\frac{6}{\kappa}-1\Big)\cot_2(W_j-V_j)\widetilde{G}(R)W_{j,1}^2\Phi_j\partial t_j+\frac{1}{4}\Big(\frac{6}{\kappa}-1\Big)W_{j,1}^2\Phi_j^2\partial t_j.\end{aligned}\tag{3.26}$$

Define another positive continuous function $M_{iB\to ch}$ on $\mathcal{D}$ by

$$M_{iB\to ch}:=e^{\frac{(\kappa-6)(\kappa-2)}{8\kappa}\mathrm{m}+\frac{\mathrm{b}}{6}(\mathrm{m}-t_1-t_2)}\widetilde{F}(R)[W_{1,1}W_{2,1}V_{1,1}V_{2,1}]^{\mathrm{b}}\Big[\prod_{j=1}^{2}\sin_2(W_j-V_j)\Big]^{-2\,\mathrm{b}}\times$$

$$\times \exp\Big(-\frac{\mathrm{c}}{6}\int_0^{t_1}\int_0^{t_2} W_{1,1}^2 W_{2,1}^2 \cot_2'''(W_1-W_2)ds_1ds_2\Big). \tag{3.27}$$

Combining (3.6,3.10,3.13,3.19,3.20,3.22,3.26), we get

$$\frac{\partial_j M_{iB\to ch}}{M_{iB\to ch}} = \mathrm{b}\,\frac{W_{j,2}}{W_{j,1}}\partial\widehat{w}_j + \frac{1}{2\kappa}\widetilde{G}(R)W_{j,1}\Phi_j\partial\widehat{w}_j - \mathrm{b}\cot_2(W_j-V_j)W_{j,1}\partial\widehat{w}_j. \tag{3.28}$$

This means that $M_{iB\to ch}^{(k,\tau_k)}$ is an $(\mathcal{F}^j_{t_j}\vee\mathcal{F}^k_\infty)_{t_j\ge 0}$-local martingale up to $T_j^{\mathcal{D}}(\tau_k)$.

3.3 Localization and Radon-Nikodym derivatives

For $j=1,2$, let Ξ_j denote the space of simple crosscuts of $\mathbb{D}$ that separate w_j from v_1, v_2, w_{3-j}, and 0. For $j=1,2$ and $\xi_j\in\Xi_j$, let $\tau^j_{\xi_j}$ be the first time that η_j hits the closure of ξ_j. If such time does not exist, then $\tau^j_{\xi_j}$ is defined to be the lifetime of η_j. We see that $\tau^j_{\xi_j}$ is bounded above by the $\mathbb{D}$-capacity of the $\mathbb{D}$-hull generated by ξ_j, and so is finite.

Let $\Xi=\{(\xi_1,\xi_2)\in\Xi_1\times\Xi_2,\operatorname{dist}(\xi_1,\xi_2)>0\}$. For $\underline{\xi}=(\xi_1,\xi_2)\in\Xi$, let $\tau_{\underline{\xi}}=(\tau^1_{\xi_1},\tau^2_{\xi_2})$. We may choose a countable set $\Xi^*\subset\Xi$ such that for every $\underline{\xi}=(\xi_1,\xi_2)\in\Xi$ there is $(\xi_1^*,\xi_2^*)\in\Xi^*$ such that ξ_j is enclosed by ξ_j^*, $j=1,2$.

Lemma 3.1. *For any* $\underline{\xi}\in\Xi$, $|\log(M_{iB\to c4})|$ *and* $|\log(M_{iB\to ch})|$ *are uniformly bounded on* $[\underline{0},\tau_{\underline{\xi}}]$ *by constants depending only on* $\kappa,\underline{\xi}$.

Proof. Fix $\underline{\xi}=(\xi_1,\xi_2)\in\Xi$. Throughout this proof, a constant depends only on $\kappa,\underline{\xi}$; by saying that a function is uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$, we mean that it is bounded by a constant on $[\underline{0},\tau_{\underline{\xi}}]$. It suffices to show that m, $|\log(W_{j,1})|$, $|\log(V_{j,1})|$, $|\log\sin_2(X_1-Y_2)|$, $X,Y\in\{W,V\}$, $|\log\sin_2(W_1-V_1)|$, $|\log\sin_2(W_2-V_2)|$, $|\log(\widetilde{F}(R))|$, and $|\int_0^{t_1}\int_0^{t_2}W_{1,1}^2W_{2,1}^2\cot_2'''(W_1-W_2)ds_1ds_2|$ are all uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$.

Let $K_{\underline{\xi}}$ be the $\mathbb{D}$-hull generated by $\xi_1\cup\xi_2$. Then $0\le t_1,t_2\le \mathrm{m}$ are uniformly bounded by the constant $\operatorname{dcap}(K_{\underline{\xi}})$ on $[\underline{0},\tau_{\underline{\xi}}]$. Note that $\mathbb{T}\setminus\overline{K_{\underline{\xi}}}$ is a disjoint union of two arcs, each of which contains one of e^{iv_s}, $s=1,2$. Denote the arcs I_1 and I_2 such that $e^{iv_s}\in I_s$, $j=1,2$. Each I_s is divided by e^{iv_s} into two open subarcs, which are denoted by $I_{s,1}$ and $I_{s,2}$ such that $I_{s,j}$ shares one endpoint with ξ_j, $j=1,2$. Let the positive constant $c_{s,j}$ be the harmonic measure in $\mathbb{D}\setminus K_{\underline{\xi}}$ viewed from 0 of the arc $I_{s,j}$. For any $\underline{t}=(t_1,t_2)\in[\underline{0},\tau_{\underline{\xi}}]$, the harmonic measure in $\mathbb{D}\setminus K_{(t_1,t_2)}$ viewed from 0 of the counterclockwise oriented arc from e^{iv_1} to the clockwise most point of $\eta_1([0,t])\cap\mathbb{T}$ is bounded from below by $c_{1,1}$. Thus, $W_1-V_1\ge c_{1,1}*2\pi$ on $[\underline{0},\tau_{\underline{\xi}}]$. Similarly, $V_1-W_2\ge c_{1,2}*2\pi$, $W_2-V_2\ge c_{2,2}*2\pi$, and $V_2+2\pi-W_1\ge c_{2,1}*2\pi$ on $[\underline{0},\tau_{\underline{\xi}}]$. Let $S=\{W_1-V_1,W_2-V_2,W_1-V_2,W_1-W_2,V_1-W_2,V_1-V_2\}$. Then we see that $\sin_2(Z)$, $Z\in S$, are all bounded below by positive constants on $[\underline{0},\tau_{\underline{\xi}}]$. So we get the uniform boundedness of $|\log\sin_2(Z)|$, $|\cot_2(Z)|$, $|\cot_2'(Z)|$, and $|\cot_2'''(Z)|$ on $[\underline{0},\tau_{\underline{\xi}}]$. Since $0<W_{j,1}\le 1$ and t_1,t_2 are uniformly bounded, we get the uniform boundedness of $|\int_0^{t_1}\int_0^{t_2}W_{1,1}^2W_{2,1}^2\cot_2'''(W_1-W_2)ds_1ds_2|$ on $[\underline{0},\tau_{\underline{\xi}}]$. From (3.10) and that $W_{k,1}|_{t_j=0}\equiv 1$ and $V_{j,1}(0,0)=1$ we conclude that $\log(W_{j,1})$ and $\log(V_{j,1})$, $j=1,2$, are uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$. From the definition of R we know

that $\log(R)$ is uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$. Since $\widetilde{F}(R)=R^{2/\kappa}F(R)$, and F is positive and continuous on $[0,1]$, we see that $\log(\widetilde{F}(R))$ is also uniformly bounded on $[\underline{0},\tau_{\underline{\xi}}]$. $\square$

Corollary 3.2. *For any* $s\in\{4,h\}$ *and* $\underline{\xi}\in\Xi$, $(M_{iB\to cs}(\underline{t}\wedge\tau_{\underline{\xi}}))_{\underline{t}\in\mathcal{Q}}$ *is an* $(\mathcal{F}_{\underline{t}})$-$M_{iB\to cs}(\tau_{\underline{\xi}})$-*Doob martingale w.r.t.* $\mathbb{P}_{iB}$.

Proof. Let $s\in\{4,h\}$ and $\underline{\xi}=(\xi_1,\xi_2)\in\Xi$. We need to show that, for any $\underline{t}=(t_1,t_2)\in\mathcal{Q}$,

$$\mathbb{E}_{iB}[M_{iB\to cs}(\tau_{\underline{\xi}})|\mathcal{F}_{\underline{t}}]=M_{iB\to cs}(\underline{t}\wedge\tau_{\underline{\xi}}). \tag{3.29}$$

From (3.24,3.28) we know that $M_{iB\to cs}(\tau^1_{\xi_1},t_2)$, $0\le t_2<T^{\mathcal{D}}_2(\tau^1_{\xi_1})$, is an $(\mathcal{F}^{(1,\infty)}_{t_2})_{t_2\ge 0}$-local martingale. By the previous lemma, $M_{iB\to cs}(\tau^1_{\xi_1},\cdot)$ is uniformly bounded on $[0,\tau^2_{\xi_2}]$. From the assumption on (ξ_1,ξ_2), we see that $\tau^2_{\xi_2}<T^{\mathcal{D}}_2(\tau^1_{\xi_1})$. So $M_{iB\to cs}(\tau^1_{\xi_1},\cdot\wedge\tau^2_{\xi_2})$ is an $(\mathcal{F}^{(1,\infty)}_{t_2})_{t_2\ge 0}$-$M_{iB\to cs}(\tau^1_{\xi_1},\tau^2_{\xi_2})$-Doob-martingale. This means that

$$\mathbb{E}_{iB}[M_{iB\to cs}(\tau^1_{\xi_1},\tau^2_{\xi_2})|\mathcal{F}^1_\infty\vee\mathcal{F}^2_{t_2}]=M_{iB\to cs}(\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2}). \tag{3.30}$$

A similar argument using $M_{iB\to cs}(\cdot,t_2\wedge\tau^2_{\xi_2})$ in place of $M_{iB\to cs}(\tau^1_{\xi_1},\cdot)$ implies that

$$\mathbb{E}_{iB}[M_{iB\to cs}(\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2})|\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_\infty]=M_{iB\to cs}(t_1\wedge\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2}). \tag{3.31}$$

Since

$$M_{iB\to cs}(t_1\wedge\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2})\in\mathcal{F}_{(t_1\wedge\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2})}\subset\mathcal{F}_{(t_1,t_2)}=\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_{t_2}\subset\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_\infty,$$

(3.31) implies that

$$\mathbb{E}_{iB}[M_{iB\to cs}(\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2})|\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_{t_2}]=M_{iB\to cs}(t_1\wedge\tau^1_{\xi_1},t_2\wedge\tau^2_{\xi_2}). \tag{3.32}$$

Combining (3.30,3.32) and using $\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_{t_2}\subset\mathcal{F}^1_\infty\vee\mathcal{F}^2_{t_2}$, we get (3.29). $\square$

The above corollary implies in particular that for any $s\in\{4,h\}$ and $\underline{\xi}\in\Xi$, we may define a probability measure $\mathbb{P}^{\underline{\xi}}_{cs}$ by $\frac{d\mathbb{P}^{\underline{\xi}}_{cs}}{d\mathbb{P}_{iB}}=\frac{M_{iB\to cs}(\tau_{\underline{\xi}})}{M_{iB\to cs}(\underline{0})}$. Suppose $(\widehat{w}_1,\widehat{w}_2)$ follows the law $\mathbb{P}^{\underline{\xi}}_{cs}$. We now describe the behavior of the radial Loewner curves η_1 and η_2 driven by $\widehat{w}_1$ and $\widehat{w}_2$, respectively. Fix $j\ne k\in\{1,2\}$. Let τ_k be an $(\mathcal{F}^k_{t_k})$-stopping time such that $\tau_k\le\tau^k_{\xi_k}$. From Lemma 2.11 and Corollary 3.2, for any $t_j\ge 0$,

$$\frac{d\mathbb{P}^{\underline{\xi}}_{cs}|\mathcal{F}^j_{t_j}\vee\mathcal{F}^k_{\tau_k}}{d\mathbb{P}_{iB}|\mathcal{F}^j_{t_j}\vee\mathcal{F}^k_{\tau_k}}=\frac{M^{(k,\tau_k)}_{iB\to cs}(t_j\wedge\tau^j_{\xi_j})}{M^{(k,\tau_k)}_{iB\to cs}(0)}.$$

From Girsanov Theorem and (3.24,3.28), we see that, under $\mathbb{P}^{\underline{\xi}}_{c4}$ and $\mathbb{P}^{\underline{\xi}}_{ch}$, $\widehat{w}_j$ respectively satisfies the following two SDEs up to $\tau^j_{\xi_j}$:

$$\begin{aligned}
\partial \widehat{w}_j =& \sqrt{\kappa}\partial B^4_{j,\tau_k} + \kappa\,\mathrm{b}\,\frac{W^{(k,\tau_k)}_{j,2}}{W^{(k,\tau_k)}_{j,1}}\partial t_j + \sum_{X\in\{W_k,V_1,V_2\}} \cot_2(W^{(k,\tau_k)}_j - X^{(k,\tau_k)})W^{(k,\tau_k)}_{j,1}\partial t_j,\\
\partial \widehat{w}_j =& \sqrt{\kappa}\partial B^h_{j,\tau_k} + \kappa\,\mathrm{b}\,\frac{W^{(k,\tau_k)}_{j,2}}{W^{(k,\tau_k)}_{j,1}}\partial t_j + \frac{1}{2}\widetilde{G}(R^{(k,\tau_k)})W^{(k,\tau_k)}_{j,1}\Phi^{(k,\tau_k)}_j\partial t_j\\
&- \kappa\,\mathrm{b}\cot_2(W^{(k,\tau_k)}_j - V^{(k,\tau_k)}_j)W^{(k,\tau_k)}_{j,1}\partial t_j,
\end{aligned}$$

where $B^s_{j,\tau_k}(t_j)$ is a standard $(\mathcal{F}^j_{t_j} \vee \mathcal{F}^k_{\tau_k})_{t_j\ge 0}$-Brownian motion under $\mathbb{P}^{\underline{\xi}}_{cs}$, $s \in \{4, h\}$. Using (3.4,3.15) we get the SDE satisfied by $W^{(k,\tau_k)}_j$ under $\mathbb{P}^{\underline{\xi}}_{c4}$ and $\mathbb{P}^{\underline{\xi}}_{ch}$, respectively, up to $\tau^j_{\xi_j}$:

$$\begin{aligned}
\partial W^{(k,\tau_k)}_j =& \sqrt{\kappa}W^{(k,\tau_k)}_{j,1}\partial B^4_{j,\tau_k} + \sum_{X\in\{W_k,V_1,V_2\}} \cot_2(W^{(k,\tau_k)}_j - X^{(k,\tau_k)})(W^{(k,\tau_k)}_{j,1})^2\partial t_j,\\
\partial W^{(k,\tau_k)}_j =& \sqrt{\kappa}W^{(k,\tau_k)}_{j,1}\partial B^h_{j,\tau_k} + \frac{1}{2}\widetilde{G}(R^{(k,\tau_k)})\Phi^{(k,\tau_k)}_j(W^{(k,\tau_k)}_{j,1})^2\partial t_j\\
&- \kappa\,\mathrm{b}\cot_2(W^{(k,\tau_k)}_j - V^{(k,\tau_k)}_j)(W^{(k,\tau_k)}_{j,1})^2\partial t_j.
\end{aligned}$$

Recall the ODE (3.8) satisfied by W_k and V_s, $s = 1, 2$. This implies that, under $\mathbb{P}^{\underline{\xi}}_{c4}$, conditionally on $\mathcal{F}^k_{\tau_k}$, $\eta_{j,\tau_k}(t_j) = g_k(\tau_k, \eta_j(t_j))$ is a radial SLE$_\kappa(2, 2, 2)$ curve with speed $(W^{(k,\tau_k)}_{j,1})^2$ started from $e^i(W^{(k,\tau_k)}_j(0)) = g_k(\tau_k, e^{i\widehat{w}_j(0)})$ with force points $e^i(W^{(k,\tau_k)}_k(0)) = e^{i\widehat{w}_k(0)} = g_k(\tau_k, \eta_k(\tau_k))$, $e^i(V^{(k,\tau_k)}_j(0)) = g_k(\tau_k, e^{iv_j})$ and $e^i(V^{(k,\tau_k)}_k(0)) = g_k(\tau_k, e^{iv_k})$, up to $\tau^j_{\xi_j}$; and under $\mathbb{P}^{\underline{\xi}}_{ch}$, conditionally on $\mathcal{F}^k_{\tau_k}$, $g_k(\tau_k, \eta_j(t_j))$ is a radial hSLE$_\kappa$ curve in $\mathbb{D}$ from $g_k(\tau_k, e^{i\widehat{w}_j(0)})$ to $g_k(\tau_k, e^{iv_j})$ with force points $g_k(\tau_k, e^{iv_k})$ and $g_k(\tau_k, \eta_k(\tau_k))$, up to $\tau^j_{\xi_j}$, viewed from 0. In particular, taking $\tau_k = 0$, we see that the j-th marginal measure of $\mathbb{P}^{\underline{\xi}}_{cs}$ restricted to $\mathcal{F}^j_{\tau^j_{\xi_j}}$ agrees with $\mathbb{P}^j_s$ restricted to $\mathcal{F}^j_{\tau^j_{\xi_j}}$. This means that the radial Loewner curves driven by $\widehat{w}_1$ and $\widehat{w}_2$, which jointly follow the law $\mathbb{P}^{\underline{\xi}}_{c4}$ (resp. $\mathbb{P}^{\underline{\xi}}_{ch}$), respectively stopped at $\tau^1_{\xi_1}$ and $\tau^2_{\xi_2}$, are two radial SLE$_\kappa(2, 2, 2)$ (resp. radial hSLE$_\kappa$) curves that locally commute with each other in the sense of [2]. Recall that $\mathbb{P}_2$ is the joint law of the radial Loewner driving functions for a 2-SLE$_\kappa$ in $\mathbb{D}$ with link pattern $(e^{iw_1} \to e^{iv_1}; e^{iw_2} \to e^{iv_2})$ up to certain separation times. Because of the commutation relation between the two curves in a 2-SLE$_\kappa$, we find that $\mathbb{P}^{\underline{\xi}}_{ch}|\mathcal{F}_{\underline{\xi}} = \mathbb{P}_2|\mathcal{F}_{\underline{\xi}}$.

Using the stochastic coupling technique developed and used in [33, 32] we may construct a probability measure $\mathbb{P}_{c4}$ on $\Sigma \times \Sigma$ such that for any $\underline{\xi} \in \Xi$, $\mathbb{P}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}} = \mathbb{P}^{\underline{\xi}}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}}$. Here is a brief review of the stochastic coupling technique for the setup here. From (3.24,3.28) and Girsanov

Theorem we know that $M^{(k,0)}_{iB\to c4}(\tau^j_{\xi_j})/M_{iB\to c4}(0,0)$ is the Radon-Nikodym derivative of $\mathbb{P}^j_4|\mathcal{F}^j_{\tau^j_{\xi_j}}$ against $\mathbb{P}^j_B|\mathcal{F}^j_{\tau^j_{\xi_j}}$. Define $M_{i4\to c4}$ on $\mathcal{D}$ by

$$M_{i4\to c4}(t_1,t_2)=\frac{M_{iB\to c4}(t_1,t_2)M_{iB\to c4}(0,0)}{M_{iB\to c4}(t_1,0)M_{iB\to c4}(0,t_2)}.$$

Then $M_{i4\to c4}(t_1,t_2)=1$ if $t_1\cdot t_2=0$; and under the probability measure $\mathbb{P}_{i4}=\mathbb{P}^1_4\times\mathbb{P}^2_4$, for any finite $(\mathcal{F}^k_{t_k})$-stopping time τ_k, $M^{(k,\tau_k)}_{i4\to c4}(t_j)$ is a local martingale. From Lemma 3.1 we know that, for any $\underline{\xi}\in\Xi$, $|\log M_{i4\to c4}|$ is bounded on $[\underline{0},\tau_{\underline{\xi}}]$. Let $(\underline{\xi}^k)_{k\in\mathbb{N}}$ be an enumeration of Ξ^*. From [33, Theorem 6.1] we know that, for any $n\in\mathbb{N}$, there is a uniformly bounded $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-Doob-martingale $M^{(n)}_{i4\to c4}$ defined on $[0,\infty]\times[0,\infty]$ such that $M^{(n)}_{i4\to c4}(t_1,t_2)=1$ if $t_1\cdot t_2=0$, and for any $1\le k\le n$, $M^{(n)}_{i4\to c4}$ agrees with $M_{i4\to c4}$ on $[\underline{0},\tau_{\underline{\xi}^k}]$. We may then define a sequence of probability measures $\mathbb{P}^{(n)}_{c4}$, $n\in\mathbb{N}$, by $d\mathbb{P}^{(n)}_{c4}=M^{(n)}_{i4\to c4}(\infty,\infty)d\mathbb{P}_{i4}$. Then every $\mathbb{P}^{(n)}_{c4}$ is a coupling of $\mathbb{P}^1_4$ and $\mathbb{P}^2_4$, and for $1\le k\le n$, $\frac{d\mathbb{P}^{(n)}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}^k}}}{d\mathbb{P}_{iB}|\mathcal{F}_{\tau_{\underline{\xi}^k}}}=\frac{M_{iB\to c4}(\tau_{\underline{\xi}^k})}{M_{iB\to c4}(\underline{0})}$. By a tightness argument, $(\mathbb{P}^{(n)}_{c4})$ contains a weakly convergent subsequence. Let $\mathbb{P}_{c4}$ denote any subsequential limit. Then for any $\underline{\xi}\in\Xi^*$, $\frac{d\mathbb{P}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}}}{d\mathbb{P}_{iB}|\mathcal{F}_{\tau_{\underline{\xi}}}}=\frac{M_{iB\to c4}(\tau_{\underline{\xi}})}{M_{iB\to c4}(\underline{0})}$. Since for every $\underline{\xi}\in\Xi$, there is $\underline{\xi}^*\in\Xi^*$ such that $\tau_{\underline{\xi}}\le\tau_{\underline{\xi}^*}$, by the martingale property of $M_{iB\to c4}(\cdot\wedge\tau_{\underline{\xi}^*})$, we get $\mathbb{P}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}}=\mathbb{P}^{\underline{\xi}}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}}$, as desired.

We may use the same idea to construct $\mathbb{P}_{ch}$. It satisfies $\mathbb{P}_{ch}|\mathcal{F}_{\tau_{\underline{\xi}}}=\mathbb{P}^{\underline{\xi}}_{ch}|\mathcal{F}_{\tau_{\underline{\xi}}}=\mathbb{P}_2|\mathcal{F}_{\tau_{\underline{\xi}}}$ for any $\xi\in\Xi$. At this moment we do not have a proof showing that $\mathbb{P}_{ch}=\mathbb{P}_2$, and we do not need this result. We now have the following lemma.

Lemma 3.3. *For any $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time $\underline{T}$,*

$$\frac{d\mathbb{P}_{c4}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\in\mathcal{D}\}}{d\mathbb{P}_{iB}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\in\mathcal{D}\}}=\frac{M_{iB\to c4}(\underline{T})}{M_{iB\to c4}(\underline{0})},\qquad\frac{d\mathbb{P}_{2}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\in\mathcal{D}\}}{d\mathbb{P}_{iB}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\in\mathcal{D}\}}=\frac{M_{iB\to ch}(\underline{T})}{M_{iB\to ch}(\underline{0})}.$$

Proof. We first work on $\mathbb{P}_{c4}$. We have $\{\underline{T}\in\mathcal{D}\}=\bigcup_{\underline{\xi}\in\Xi^*}\{\underline{T}\le\tau_{\underline{\xi}}\}$. Since by Lemma 2.13, $\{\underline{T}\le\tau_{\underline{\xi}}\}\in\mathcal{F}_{\underline{T}}$, it suffices to show that, for any $\underline{\xi}\in\Xi^*$,

$$\frac{d\mathbb{P}_{c4}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\le\tau_{\underline{\xi}}\}}{d\mathbb{P}_{iB}|\mathcal{F}_{\underline{T}}\cap\{\underline{T}\le\tau_{\underline{\xi}}\}}=\frac{M_{iB\to c4}(\underline{T})}{M_{iB\to c4}(\underline{0})}.\tag{3.33}$$

By Lemma 2.11 and Corollary 3.2, we see that $\mathbb{E}_{iB}[M_{iB\to c4}(\tau_{\underline{\xi}})|\mathcal{F}_{\underline{T}}]=M_{iB\to c4}(\underline{T}\wedge\tau_{\underline{\xi}})$. Let $A\in\mathcal{F}_{\underline{T}}$ with $A\subset\{\underline{T}\le\tau_{\underline{\xi}}\}$. Then we have $\mathbb{E}_{iB}[\mathbf{1}_AM_{iB\to c4}(\tau_{\underline{\xi}})]=\mathbb{E}_{iB}[\mathbf{1}_AM_{iB\to c4}(\underline{T})]$. Since $\frac{d\mathbb{P}_{c4}|\mathcal{F}_{\tau_{\underline{\xi}}}}{d\mathbb{P}_{iB}|\mathcal{F}_{\tau_{\underline{\xi}}}}=\frac{M_{iB\to c4}(\tau_{\underline{\xi}})}{M_{iB\to c4}(\underline{0})}$, and $A\subset\mathcal{F}_{\tau_{\underline{\xi}}}$ by Lemma 2.7, we get

$$M_{iB\to c4}(\underline{0})\mathbb{P}_{c4}[A]=\mathbb{E}_{iB}[\mathbf{1}_AM_{iB\to c4}(\tau_{\underline{\xi}})]=\mathbb{E}_{iB}[\mathbf{1}_AM_{iB\to c4}(\underline{T})].$$

Since this holds for any $A \in \mathcal{F}_{\underline{T}}$ with $A \subset \{\underline{T} \le \tau_{\underline{\xi}}\}$, we get (3.33) as desired.

A similar argument shows that (3.33) holds with $c4$ replaced by ch. Since $\mathcal{F}_{\underline{T}} \cap \{\underline{T} \le \tau_{\underline{\xi}}\} \subset \mathcal{F}_{\underline{\xi}}$ and $\mathbb{P}_{ch}$ agrees with $\mathbb{P}_2$ on $\mathcal{F}_{\underline{\xi}}$, we find that (3.33) holds with $M_{iB\to c4}$ replaced by $M_{iB\to ch}$ and $\mathbb{P}_{c4}$ replaced by $\mathbb{P}_2$. So we obtain the second equality. $\square$

We need the following lemma about the lifetime of a radial $\mathrm{SLE}_\kappa(\underline{\rho})$ curve.

Lemma 3.4. *Let $\kappa > 0$, $n \in \mathbb{N}$. Suppose $\underline{\rho} = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$ satisfies $\rho_1, \rho_n \ge \frac{\kappa}{2} - 2$ and $\rho_k \ge 0$, $1 \le k \le n$. Let $e^{iw}, e^{iv_1}, \dots, e^{iv_n}$ be distinct points on $\mathbb{T}$ such that $w > v_1 > \cdots > v_n > w - 2\pi$. Let $\eta(t)$, $0 \le t < T$, be a radial $SLE_\kappa(\underline{\rho})$ curve in $\mathbb{D}$ started from e^{iw} aimed at 0 with force points $e^{iv_1}, \dots, e^{iv_n}$. Then a.s. $T = \infty$, 0 is a subsequential limit of $\eta(t)$ as $t \to \infty$, and η does not hit the arc $J := \{e^{i\theta} : v_1 \ge \theta \ge v_n\}$.*

Proof. Let $\widehat{w}(t)$ and $\widehat{v}_j(t)$, $1 \le j \le n$, $0 \le t < T$, be the solutions of the system of SDE used to define this radial $\mathrm{SLE}_\kappa(\underline{\rho})$ curve. For any $t \in [0, T)$, we have $\widehat{w}(t) > \widehat{v}_1(t) > \cdots \widehat{v}_n(t) > \widehat{w}(t) - 2\pi$. If $T < \infty$, then one of the following events $E^0_{n'}, E^{2\pi}_{n'}$, $1 \le n' \le n$, must happen:

$$E^0_{n'} = \{\lim_{t\to T^-} \widehat{w}(t) - \widehat{v}_j(t) = 0, 1 \le j \le n'\} \cap \{\lim_{t\to T^-} \widehat{w}(t) - \widehat{v}_j(t) \in (0, 2\pi), n' + 1 \le j \le n\},$$

$$E^{2\pi}_k = \{\lim_{t\to T^-} \widehat{w}(t) - \widehat{v}_j(t) = 2\pi, n' \le j \le n\} \cap \{\lim_{t\to T^-} \widehat{w}(t) - \widehat{v}_j(t) \in (0, 2\pi), 1 \le j \le n' - 1\}.$$

To prove that $\mathbb{P}[T < \infty] = 0$, it suffices to show that $\mathbb{P}[E^0_{n'}] = \mathbb{P}[E^{2\pi}_{n'}] = 0$ for $1 \le n' \le n$. By symmetry, we only need to consider $E^0_{n'}$, $1 \le n' \le n$. If $\mathbb{P}[E^0_{n'}] > 0$, using Girsanov Theorem, we see that for a radial $\mathrm{SLE}_\kappa(\rho_1, \dots, \rho_{n'})$ process in $\mathbb{D}$ from e^{iw} to 0 with force points $e^{iv_1}, \dots, e^{iv_{n'}}$, there is a positive probability that the lifetime T is finite and $\lim_{t\to T^-} \widehat{w}(t) - \widehat{v}_j(t) = 0$, $1 \le j \le n'$. For this new process, $X_{n'}(t) := \widehat{w}(t) - \widehat{v}_{n'}(t)$ satisfies the SDE:

$$dX_{n'}(t) = \sqrt{\kappa} dB(t) + \sum_{j=1}^{n'} \frac{\rho_j}{2} \cot_2(\widehat{w}_1(t) - \widehat{v}_j(t)) dt + \cot_2(X_k(t)) dt.$$

Since $\cot_2(\widehat{w}_1(t) - \widehat{v}_j(t)) > \cot_2(X(t))$ and $\rho_j \ge 0$ for $1 \le j \le k-1$, the process $X_{n'}$ stochastically dominates the process Y, which satisfies the SDE: $dY(t) = \sqrt{\kappa} dB(t) + (1 + \frac{\sigma}{2}) \cot_2(Y(t)) dt$, where $\sigma = \sum_{j=1}^{n'} \rho_j \ge \rho_1 \ge \frac{\kappa}{2} - 2$. It is easy to see that $\frac{1}{2} Y_k(\frac{4}{\kappa} t)$ is a radial Bessel process of dimension $\delta = 1 + \frac{2}{\kappa}(2 + \sigma) \ge 2$, which a.s. does not tend to 0 at any finite time (cf. [6, Appendix A],[30, Appendix B]). So the probability that the $X_{n'}(t)$ for the new process tends to 0 at a finite time is also 0, which implies that the probability of the $E^0_{n'}$ for the original process is 0. Thus, a.s. $T = \infty$. By Koebe's 1/4 Theorem, we see that 0 is a subsequential limit of η as $t \to \infty$. If η hits the arc J, then when it happens, η separates 0 from either e^{iv_1} or e^{iv_n}, and the process stops at this time. Since a.s. $T = \infty$, such hitting a.s. can not happen. $\square$

Now we consider two radial Loewner curves η_1 and η, whose driving functions jointly follow $\mathbb{P}_{c4}$. From Lemma 3.4 (applied to $\kappa \in (0, 8)$ and $\rho_1 = \rho_2 = \rho_3 = 2$) we know that the the lifetimes of η_1 and η_2 are both a.s. ∞. Fix $\tau_2 < \infty$. Conditional on $\mathcal{F}^2_{\tau_2}$, $g_2(\tau_2, \eta_1(t_1))$, $t_1 \ge 0$, is a radial $\mathrm{SLE}_\kappa(2,2,2)$ curve in $\mathbb{D}$ started from $g_2(\tau_2, e^{iw_1})$ with force points $g_2(\tau_2, \eta_2(\tau_2))$, $g_2(\tau_2, e^{iv_1})$ and $g_2(\tau_2, e^{iv_2})$, up to the lifetime of η_1 or the first time that η_1 hits $\eta_2[0,\tau_2]$. If η_1 hits $\eta_2[0,\tau_2]$, then it means that $g_2(\tau_2, \eta_1(t_1))$ hits the boundary arc of $\mathbb{D}$ with end points $g_2(\tau_2, e^{iv_1})$ and $g_2(\tau_2, e^{iv_2})$ that contains $g_2(\tau_2, \eta_2(\tau_2))$, which is impossible by Lemma 3.4. Thus, the whole η_1 does not intersect $\eta_2[0,\tau_2]$. From Lemma 3.4 we also know that η_1 a.s does not intersect the boundary arc of $\mathbb{D}$ with end points e^{iv_1} and e^{iv_2} that contains the initial point of η_2: e^{iw_2}. From the definition of $\mathcal{D}$, we have $\mathbb{P}_{c4}$-a.s. $T_1^{\mathcal{D}}(\tau_2) = \infty$. Since this holds for any deterministic $\tau_2 < \infty$, and the lifetime of η_2 is a.s. ∞, we get the following lemma.

Lemma 3.5. *$\mathbb{P}_{c4}$-a.s. $\mathcal{D} = \mathcal{Q} = [0,\infty)^2$.*

Let $M_{2\to c4} = \frac{M_{iB\to c4}}{M_{iB\to ch}}$ and $M_{c4\to 2} = M_{2\to c4}^{-1}$. From Lemma 3.3 we see that, for any $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time $\underline{T}$,

$$\frac{d\mathbb{P}_2|\mathcal{F}_{\underline{T}} \cap \{\underline{T} \in \mathcal{D}\}}{d\mathbb{P}_{c4}|\mathcal{F}_{\underline{T}} \cap \{\underline{T} \in \mathcal{D}\}} = \frac{M_{c4\to 2}(\underline{T})}{M_{c4\to 2}(\underline{0})}. \tag{3.34}$$

Let $G(w_1, v_1; w_2, v_2)$ be defined by

$$G(w_1, v_1; w_2, v_2) = |\sin_2(w_1 - v_1)\sin_2(w_2 - v_2)|^{\frac{8}{\kappa}-1} |\sin_2(w_1 - w_2)\sin_2(v_1 - v_2)|^{\frac{4}{\kappa}} \times$$
$$\times F\Big(\Big|\frac{\sin_2(w_1 - v_2)\sin_2(v_1 - w_2)}{\sin_2(w_1 - w_2)\sin_2(v_1 - v_2)}\Big|\Big)^{-1}. \tag{3.35}$$

Then with α_0 defined by (1.1), we have

$$M_{2\to c4} = e^{\alpha_0 \cdot \mathrm{m}} G(W_1, V_1; W_2, V_2). \tag{3.36}$$

From Lemma 3.5 and (3.34) we see that for any $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time $\underline{T}$,

$$\mathbb{E}_2[\mathbf{1}_{\{\underline{T}\in\mathcal{D}\}} e^{\alpha_0 \cdot \mathrm{m}} G(W_1, V_1; W_2, V_2)|_{\underline{t}=\underline{T}}] = G(w_1, v_1; w_2, v_2). \tag{3.37}$$

4 A Time Curve in the Time Region

In the last section we have derived many random processes with two time parameters defined on the time region $\mathcal{D}$. We will now define a curve in $\mathcal{D}$ so that we can obtain one-parameter random processes from those two-parameter random processes.

Throughout this section, we suppose $v_1 - v_2 = \pi$. Let $\theta = V_1 - V_2 \in (0, 2\pi)$. Then $\theta(0,0) = \pi$. We are going to get a continuous and strictly increasing curve $\underline{u} : [0, T^u) \to \mathcal{D}$ with $\underline{u}(0) = \underline{0}$ such that $\theta(\underline{u}(t)) = \pi$ and $\mathrm{m}(\underline{u}(t)) = t$ for any $t \in [0, T^u)$, and the curve can not be further extended with this property. Note that

$$\partial_j \theta = W_{j,1}^2(\cot_2(V_1 - W_j) - \cot_2(V_2 - W_j))\partial t_j = \frac{-W_{j,1}^2 \sin_2(\theta)}{\sin_2(W_j - V_1)\sin_2(W_j - V_2)} \partial t_j. \tag{4.1}$$

So $\partial_1 \theta < 0$ and $\partial_2 \theta > 0$. Thus, $\theta(t, 0) < \pi$ for $t > 0$; and $\theta(0, t) > \pi$ for $t > 0$. Let

$$S_1 = \{t_1 \ge 0 : \exists t_2 > 0 \text{ such that } (t_1, t_2) \in \mathcal{D} \text{ and } \theta(t_1, t_2) > \pi\}.$$

Suppose $t_1 \in S_1$, and $t_2 > 0$ is such that $(t_1, t_2) \in \mathcal{D}$ and $\theta(t_1, t_2) > \pi$. Then for any $t_1' \in [0, t_1)$, $(t_1', t_2) \in \mathcal{D}$ and $\theta(t_1', t_2) > \theta(t_1, t_2) > \pi$, which implies that $t_1' \in S_1$. On the other hand, since $\mathcal{D}$ is relatively open in $\mathbb{R}_+^2$, by the continuity of θ, we can find $t_1'' > t_1$ such that $(t_1'', t_2) \in \mathcal{D}$ and $\theta(t_1'', t_2) > \pi$, which implies that $t_1'' \in S_1$. So $S_1 = [0, T_1^u)$ for some $T_1^u \in (0, \infty]$. For every $t_1 \geq T_1^u$ and any $t_2 \geq 0$ such that $(t_1, t_2) \in \mathcal{D}$, we must have $\theta(t_1, t_2) < \pi$. For $t_1 \in [0, T_1^u)$, applying the intermediate value theorem to $\theta(t_1, \cdot)$ and using the strict monotonicity of θ in t_2, we conclude that there is a unique $t_2 \geq 0$ such that $(t_1, t_2) \in \mathcal{D}$ and $\theta(t_1, t_2) = \pi$. Let $u_{1\to 2}$ denote the map $[0, T_1^u) \ni t_1 \mapsto t_2$. Since θ is strictly decreasing in t_1 and strictly increasing in t_2, $u_{1\to 2}$ is strictly increasing. A symmetric argument shows that there exists $T_2^u \in (0, \infty]$ such that for any $t_2 \geq T_2^u$ and any $t_1 \geq 0$ such that $(t_1, t_2) \in \mathcal{D}$, we have $\theta(t_1, t_2) > \pi$; for any $t_2 \in [0, T_2^u)$, there is a unique $t_1 \geq 0$ such that $(t_1, t_2) \in \mathcal{D}$ and $\theta(t_1, t_2) = \pi$; and the map $u_{2\to 1} : [0, T_2^u) \ni t_2 \mapsto t_1$ is strictly increasing. Thus, $u_{1\to 2}$ maps $[0, T_1^u)$ onto $[0, T_2^u)$, and $u_{2\to 1}$ is its inverse. Moreover, both $u_{1\to 2}$ and $u_{2\to 1}$ are continuous. Since m is continuous and strictly increasing in both t_1 and t_2, we see that the map $[0, T_1^u) \ni t_1 \mapsto \mathrm{m}(t_1, u_{1\to 2}(t_1))$ is continuous and strictly increasing. Since $u_{1\to 2}(0) = 0$ and $\mathrm{m}(0, 0) = 0$, the range of $\mathrm{m}(t_1, u_{1\to 2}(t_1))$ is $[0, T^u)$ for some $T^u \in (0, \infty]$. Let u_1 denote the inverse of this map, and let $u_2 = u_{1\to 2} \circ u_1$. Then for $j = 1, 2$, u_j is a continuous and strictly increasing function that maps $[0, T^u)$ onto $[0, T_j^u)$; and $\underline{u} := (u_1, u_2) : [0, T^u) \to \mathcal{D}$ is a strictly increasing curve that satisfies $\theta(u_1(t), u_2(t)) = \pi$ and $\mathrm{m}(u_1(t), u_2(t)) = t$ for any $0 \leq t < T^u$, and $\lim_{t \to T^-} \underline{u}(t) = (T_1^u, T_2^u)$. We see that (T_1^u, T_2^u) does not belong to $\mathcal{D}$ because if it does then $\theta(T_1^u, T_2^u) = \pi$, which contradicts the statement that for every $t_1 \geq T_1^u$ and any $t_2 \geq 0$ such that $(t_1, t_2) \in \mathcal{D}$, we have $\theta(t_1, t_2) < \pi$. Since m is increasing in t_1 and t_2, we get $u_1(t) = \mathrm{m}(u_1(t), 0) \leq \mathrm{m}(u_1(t), u_2(t)) = t$. Similarly, $u_2(t) \leq t$.

For any function X defined on $\mathcal{D}$, we define $X^u(t) = X(\underline{u}(t))$, $0 \leq t < T^u$. For example, if $X = \widehat{w}_j$, $j = 1, 2$, then $\widehat{w}_j^u(t) = \widehat{w}_j(u_j(t))$. Let $Z_j = W_j - V_j > 0$, $j = 1, 2$. Then $Z_2^u \in (0, \pi)$ because $Z_2^u < V_1^u - V_2^u = \pi$, and $Z_1^u \in (0, \pi)$ because $Z_1^u < V_2^u + 2\pi - V_1^u = \pi$. From (4.1) and that $\theta^u \equiv \pi$, we get

$$0 = \frac{-2(W_{1,1}^u)^2}{\sin(Z_1^u(t))} u_1'(t) + \frac{2(W_{2,1}^u)^2}{\sin(Z_2^u(t))} u_2'(t).$$

From $\mathrm{m}(u_1(t), u_2(t)) = t$ and (3.6) we get

$$1 = (W_{1,1}^u(t))^2 u_1'(t) + (W_{2,1}^u(t))^2 u_2'(t).$$

Combining, we get

$$(W_{j,1}^u)^2 u_j' = \frac{\sin(Z_j^u)}{\sin(Z_1^u) + \sin(Z_2^u)}, \quad j = 1, 2. \tag{4.2}$$

So far u_1 and u_2 are defined on $[0, T^u)$. If $T^u < \infty$, we extend u_1 and u_2 to $[0, \infty)$ such that for $t \geq T^u$, $u_j(t) = T_j^u$, $j = 1, 2$. From $u_j(t) \leq t$ we get $T_j^u \leq T^u < \infty$, $j = 1, 2$. Thus, the extended u_1 and u_2 are finite and continuous. Below is a lemma on the extended $\underline{u}$.

Lemma 4.1. *For any $t \in [0, \infty)$, $\underline{u}(t) = (u_1(t), u_2(t))$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping time.*

Proof. Fix $t \geq 0$ and $\underline{s} = (s_1, s_2) \in \mathcal{Q}$. We need to show that $\{\underline{u}(t) \leq \underline{s}\} \in \mathcal{F}_{\underline{s}}$. For this purpose, we consider three events. Let A_1 denote the event that the curve $\underline{u} \cap \mathcal{D}$ intersects $\{s_1\} \times [0, s_2)$; and let A_2 denote the event that the curve $\underline{u} \cap \mathcal{D}$ intersects $[0, s_1) \times \{s_2\}$. Then $A_1 \cap A_2 = \emptyset$,

$$A_1 = \bigcup_{t_2 \in [0,s_2) \cap \mathbb{Q}} \{(s_1, t_2) \in \mathcal{D}, \theta(s_1, t_2) > \pi\} \in \mathcal{F}_{\underline{s}},$$

and similarly $A_2 \in \mathcal{F}_{\underline{s}}$. Here we used the fact that $\mathcal{D}$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping region and θ is $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-adapted. Let $A_0 = (A_1 \cup A_2)^c \in \mathcal{F}_{\underline{s}}$. We have

$$\begin{aligned}
&\{\underline{u}(t) \leq \underline{s}\} \cap A_0 = A_0 \cap (\{\underline{s} \notin \mathcal{D}\} \cup \{\underline{s} \in \mathcal{D}, \theta(\underline{s}) = \pi, \mathrm{m}(\underline{s}) \geq t\});\\
&\{\underline{u}(t) \leq \underline{s}\} \cap A_1 = \bigcap_{n \in \mathbb{N}} \bigcup_{r_2 < t_2 \in [0,s_2) \cap \mathbb{Q}} \{(s_1, t_2) \in \mathcal{D}, \theta(s_1, t_2) > \pi > \theta(s_1, r_2), \mathrm{m}(s_1, r_2) > t - \frac{1}{n}\};\\
&\{\underline{u}(t) \leq \underline{s}\} \cap A_2 = \bigcap_{n \in \mathbb{N}} \bigcup_{r_1 < t_1 \in [0,s_1) \cap \mathbb{Q}} \{(t_1, s_2) \in \mathcal{D}, \theta(t_1, s_2) < \pi < \theta(r_1, s_2), \mathrm{m}(r_1, s_2) > t - \frac{1}{n}\}.
\end{aligned}$$

Since the events on the righthand side are all $\mathcal{F}_{\underline{s}}$-measurable, so is $\{\underline{u}(t) \leq \underline{s}\}$, as desired. □

We now get a new filtration $(\mathcal{F}^u_t := \mathcal{F}_{\underline{u}(t)})_{t \geq 0}$ by Lemma 2.7 since $\underline{u}$ is non-decreasing. For $\underline{\xi} = (\xi_1, \xi_2) \in \Xi$, let $\tau^u_{\underline{\xi}}$ denote the first $t \geq 0$ such that $u_1(t) = \tau^1_{\xi_1}$ or $u_2(t) = \tau^2_{\xi_2}$, whichever comes first. Note that such time exists and is finite because $[\underline{0}, \tau_{\underline{\xi}}] \subset \mathcal{D}$.

Lemma 4.2. *For $\underline{\xi} \in \Xi$, $\underline{u}(\tau^u_{\underline{\xi}})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping time, $\tau^u_{\underline{\xi}}$ is an $(\mathcal{F}^u_t)_{t \geq 0}$-stopping time, and for any $t \geq 0$, $\underline{u}(t \wedge \tau^u_{\underline{\xi}})$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$-stopping time.*

Proof. Let $\underline{\xi} \in \Xi$. Note that for any $\underline{t} = (t_1, t_2) \in \mathcal{Q}$, by Lemmas 2.7 and 2.9,

$$\{\underline{u}(\tau^u_{\underline{\xi}}) \leq \underline{t}\} \cap \{u_1(\tau^u_{\underline{\xi}}) = \tau^1_{\xi_1}\} = \{\tau^1_{\xi_1} \leq t_1\} \cap \{\theta(\tau^1_{\xi_1}, \tau^2_{\xi_2} \wedge t_2) \geq \pi\} \in \mathcal{F}_{\underline{t}}.$$

Similarly, $\{\underline{u}(\tau^u_{\underline{\xi}}) \leq \underline{t}\} \cap \{u_2(\tau^u_{\underline{\xi}}) = \tau^2_{\xi_1}\} \in \mathcal{F}_{\underline{t}}$. Since either $u_1(\tau^u_{\underline{\xi}}) = \tau^1_{\xi_1}$ or $u_2(\tau^u_{\underline{\xi}}) = \tau^2_{\xi_2}$, we get $\{\underline{u}(\tau^u_{\underline{\xi}}) \leq \underline{t}\} \in \mathcal{F}_{\underline{t}}$. Thus, $\underline{u}(\tau^u_{\underline{\xi}})$ is an $(\mathcal{F}_{\underline{t}})$-stopping time.

To prove that $\tau^u_{\underline{\xi}}$ is an $(\mathcal{F}^u_t)_{t \geq 0}$-stopping time, it suffices to show that, for any $t \geq 0$ and $\underline{s} \in \mathcal{Q}$, $\{\tau^u_{\underline{\xi}} \leq t\} \cap \{\underline{u}(t) \leq \underline{s}\} \in \mathcal{F}_{\underline{s}}$. We may choose a sequence $\underline{\xi}^n = (\xi^n_1, \xi^n_2)_{n \in \mathbb{N}}$ in Ξ, which approximates $\underline{\xi}$ such that $\{\tau^u_{\underline{\xi}} \leq t\} = \bigcap_{n=1}^\infty \{\tau^u_{\underline{\xi}^n} < t\}$. Then it suffices to show that, for any $n \in \mathbb{N}$, $\{\tau^u_{\underline{\xi}^n} < t\} \cap \{\underline{u}(t) \leq \underline{s}\} \in \mathcal{F}_{\underline{s}}$. Since $\underline{u}$ is strictly increasing on $[0, \tau^u_{\underline{\xi}^n} + \varepsilon)$,

$$\{\tau^u_{\underline{\xi}^n} < t\} \cap \{\underline{u}(t) \leq \underline{s}\} = \{\underline{u}(\tau^u_{\underline{\xi}^n}) < \underline{u}(t) \leq \underline{s}\} = \bigcup_{\underline{r} \in \mathbb{Q}^2 \cap [\underline{0}, \underline{s}]} \Big(\{\underline{u}(\tau^u_{\underline{\xi}^n}) \leq \underline{r}\} \cap \{\underline{r} < \underline{u}(t) \leq \underline{s}\}\Big).$$

Since $\underline{u}(\tau^u_{\underline{\xi}})$ and $\underline{u}(t)$ are $(\mathcal{F}_{\underline{t}})$-stopping times, the events in the union all belong to $\mathcal{F}_{\underline{s}}$. So $\{\tau^u_{\underline{\xi}^n} < t\} \cap \{\underline{u}(t) \leq \underline{s}\} \in \mathcal{F}_{\underline{s}}$, as desired.

Let $t \ge 0$ and $\underline{s} \in \mathcal{Q}$. Note that

$$\{\underline{u}(t \wedge \tau^u_{\underline{\xi}}) \le \underline{s}\} = (\{t < \tau^u_{\underline{\xi}}\} \cap \{\underline{u}(t) \le \underline{s}\}) \cup (\{\tau^u_{\underline{\xi}} \le t\} \cap \{\underline{u}(\tau^u_{\underline{\xi}}) \le \underline{s}\}).$$

The first event $\{t < \tau^u_{\underline{\xi}}\} \cap \{\underline{u}(t) \le \underline{s}\}$ belongs to $\mathcal{F}_{\underline{s}}$ because from that $\tau^u_{\underline{\xi}}$ is an $(\mathcal{F}^u_t)$-stopping time we know $\{t < \tau^u_{\underline{\xi}}\} \in \mathcal{F}^u_t = \mathcal{F}_{\underline{u}(t)}$. The other event $\{\tau^u_{\underline{\xi}} \le t\} \cap \{\underline{u}(\tau^u_{\underline{\xi}}) \le \underline{s}\}$ equals

$$\bigcap_{n\in\mathbb{N}} (\{\tau^u_{\underline{\xi}^n} < t\} \cap \{\underline{u}(\tau^u_{\underline{\xi}^n}) < \underline{s}\}) = \bigcap_{n\in\mathbb{N}} \bigcup_{\underline{r}\in\mathbb{Q}^2\cap[\underline{0},\underline{s})} (\{\underline{r} \in \mathcal{D}, \mathrm{m}(\underline{r}) < t\} \cap \{\underline{u}(\tau^u_{\underline{\xi}^n}) \le \underline{r}\}),$$

where we used that $\tau^u_{\underline{\xi}^n} = \mathrm{m}(\underline{u}(\tau^u_{\underline{\xi}^n}))$. The event on the RHS of the above displayed formula belongs to $\mathcal{F}_{\underline{s}}$ because m is $(\mathcal{F}_{\underline{t}})$-adapted and $\underline{u}(\tau^u_{\underline{\xi}^n})$ is an $(\mathcal{F}_{\underline{t}})$-stopping time. Thus, the event $\{\tau^u_{\underline{\xi}} \le t\} \cap \{\underline{u}(\tau^u_{\underline{\xi}}) \le \underline{s}\}$ also belongs to $\mathcal{F}_{\underline{s}}$. Then we get $\{\underline{u}(t \wedge \tau^u_{\underline{\xi}}) \le \underline{s}\} \in \mathcal{F}_{\underline{s}}$, as desired. □

Since under $\mathbb{P}_{iB}$, for $j = 1, 2$, $\widehat{w}_j(t_j) = w_j + \sqrt{\kappa} B_j(t_j)$, $t_j \ge 0$, where $(B_1(t_1))$ and $(B_2(t_2))$ are independent standard Brownian motions, we get five $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-martingales under $\mathbb{P}_{iB}$: $\widehat{w}_j(t_j)$, $\widehat{w}_j(t_j)^2 - \kappa t_j$, $j = 1, 2$, and $\widehat{w}_1(t_1)\widehat{w}_2(t_2)$. Using Lemmas 2.11 and 4.1 and the facts that $u_1(t), u_2(t) \le t$, we conclude that $\widehat{w}^u_j(t)$, $\widehat{w}^u_j(t)^2 - \kappa u_j(t)$, $j = 1, 2$, and $\widehat{w}^u_1(t)\widehat{w}^u_2(t)$ are all $(\mathcal{F}^u_t)$-martingales under $\mathbb{P}_{iB}$. So we get quadratic variations and co-variation for $\widehat{w}^u_j$, $j = 1, 2$:

$$\langle \widehat{w}^u_j \rangle_t = \kappa u_j(t), \quad j = 1, 2; \quad \langle \widehat{w}^u_1, \widehat{w}^u_2 \rangle_t \equiv 0. \tag{4.3}$$

Fix $\underline{\xi} = (\xi_1, \xi_2) \in \Xi$. From Lemmas 3.2 and 2.11 we know that $M_{iB\to cs}(u_1(t) \wedge \tau^1_{\xi_1}, u_2(t) \wedge \tau^2_{\xi_2})$, $t \ge 0$, $s \in \{4, h\}$, is an $(\mathcal{F}^u_t)_{t\ge0}$-martingale. Since $\tau^u_{\underline{\xi}}$ is an $(\mathcal{F}^u_t)_{t\ge0}$-stopping time, we see that

$$M_{iB\to cs}(u_1(t\wedge\tau^u_{\underline{\xi}})\wedge\tau^1_{\xi_1}, u_2(t\wedge\tau^u_{\underline{\xi}})\wedge\tau^2_{\xi_2}) = M_{iB\to cs}(u_1(t\wedge\tau^u_{\underline{\xi}}), u_2(t\wedge\tau^u_{\underline{\xi}})) = M^u_{iB\to cs}(t\wedge\tau^u_{\underline{\xi}}), \quad t \ge 0,$$

is an $(\mathcal{F}^u_t)_{t\ge0}$-martingale. Since $[0, T^u) = \bigcup_{\xi\in\Xi^*} [0, \tau^u_{\underline{\xi}}]$ and Ξ^* is countable, we conclude that $M^u_{iB\to cs}(t)$, $0 \le t < T^u$, is an $(\mathcal{F}^u_t)_{t\ge0}$-local martingale.

We now compute the SDE for $M^u_{iB\to c4}(t)$, $0 \le t < T^u$, in terms of $\widehat{w}^u_1$ and $\widehat{w}^u_2$. Using (3.23) we may express $M^u_{iB\to c4}$ as a product of several factors. Among these factors, $(W^u_{1,1})^{\mathrm{b}}$, $(W^u_{2,1})^{\mathrm{b}}$, $\sin_2(W^u_1 - W^u_2)^{\frac{2}{\kappa}}$, and $\sin_2(W^u_j - V^u_s)^{\frac{2}{\kappa}}$, $j, s \in \{1, 2\}$, contribute the martingale part of $M^u_{iB\to c4}$; and other factors are differentiable in t. For $j \ne k \in \{1, 2\}$, using (3.8,3.15,3.16) we get the $(\mathcal{F}^u_t)$-adapted SDEs:

$$dW^u_j = W^u_{j,1} d\widehat{w}^u_j + \Big(\frac{\kappa}{2} - 3\Big) W_{j,2} u'_j dt + \cot_2(W^u_j - W^u_k)(W^u_{k,1})^2 u'_k dt, \tag{4.4}$$

$$\frac{dW^u_{j,1}}{W^u_{j,1}} = \frac{W^u_{j,2}}{W^u_{j,1}} d\widehat{w}^u_j + \text{drift terms},$$

which imply that, for $s = 1, 2$,

$$\frac{d\sin_2(W^u_j - V^u_s)^{\frac{2}{\kappa}}}{\sin_2(W^u_j - V^u_s)^{\frac{2}{\kappa}}} = \frac{1}{\kappa}\cot_2(W^u_j - V^u_s) W^u_{j,1} d\widehat{w}^u_j + \text{drift terms},$$

$$\frac{d\sin_2(W_1^u - W_2^u)^{\frac{2}{\kappa}}}{\sin_2(W_1^u - W_2^u)^{\frac{2}{\kappa}}} = \frac{1}{\kappa}\cot_2(W_1^u - W_2^u)[W_{1,1}^u\,d\widehat{w}_1^u - W_{2,1}^u\,d\widehat{w}_2^u] + \text{drift terms},$$

$$\frac{d(W_{j,1}^u)^{\mathrm{b}}}{(W_{j,1}^u)^{\mathrm{b}}} = \mathrm{b}\,\frac{W_{j,2}^u}{W_{j,1}^u}\,d\widehat{w}_j^u + \text{drift terms}.$$

Since we already know that $\widehat{w}_1^u(t)$, $\widehat{w}_2^u(t)$, $M_{iB\to c4}^u(t)$, $0 \le t < T^u$, are $(\mathcal{F}_t^u)_{t\ge 0}$-local martingales, we get

$$\frac{dM_{iB\to c4}^u}{M_{iB\to c4}^u} = \sum_{j=1}^{2}\Big[\mathrm{b}\,\frac{W_{j,2}^u}{W_{j,1}^u} + \sum_{X\in\{W_{3-j},V_1,V_2\}}\frac{1}{\kappa}\cot_2(W_j^u - X^u)W_{j,1}^u\Big]\,d\widehat{w}_j^u. \tag{4.5}$$

One may also compute (4.5) directly, and conclude that $M_{iB\to c4}^u(t)$ is an $(\mathcal{F}_t^u)$-local martingale.

From Lemmas 3.3 and 4.2 we know that, for any $\underline{\xi} \in \Xi$ and $t \ge 0$,

$$\frac{d\mathbb{P}_{c4}|\mathcal{F}_{\underline{u}(t\wedge\tau_{\underline{\xi}}^u)}}{d\mathbb{P}_{iB}|\mathcal{F}_{\underline{u}(t\wedge\tau_{\underline{\xi}}^u)}} = \frac{M_{iB\to c4}^u(t\wedge\tau_{\underline{\xi}}^u)}{M_{iB\to c4}^u(0)}. \tag{4.6}$$

We will use a Girsanov argument to derive the SDEs for $\widehat{w}_j^u$, $j = 1,2$, under $\mathbb{P}_{c4}$.

Lemma 4.3. *Under $\mathbb{P}_{c4}$, there are two independent standard Brownian motions $B_j^u(t)$, $j = 1,2$, such that $\widehat{w}_j^u$ satisfies the SDE*

$$d\widehat{w}_j^u = \sqrt{\kappa u_j'}dB_j^u + \Big[\kappa\,\mathrm{b}\,\frac{W_{j,2}^u}{W_{j,1}^u} + \sum_{X\in\{W_{3-j},V_1,V_2\}}\cot_2(W_j^u - X^u)W_{j,1}^u\Big]u_j'dt, \quad 0 \le t < \infty.$$

Proof. For $j = 1,2$, define a process $\widetilde{w}_j^u$, which has initial value w_j, and satisfies the SDE

$$d\widetilde{w}_j^u = d\widehat{w}_j^u - \Big[\kappa\,\mathrm{b}\,\frac{W_{j,2}^u}{W_{j,1}^u} + \sum_{X\in\{W_{3-j},V_1,V_2\}}\cot_2(W_j^u - X^u)W_{j,1}^u\Big]u_j'dt. \tag{4.7}$$

From (4.5) we know that $\widetilde{w}_j^u(t)M_{iB\to c4}^u(t)$, $0 \le t < T^u$, is an $(\mathcal{F}_t^u)$-local martingale under $\mathbb{P}_{iB}$.

We claim that, for any $j \in \{1,2\}$ and $\underline{\xi} \in \Xi$, $|\widetilde{w}_j^u|$ is bounded on $[0,\tau_{\underline{\xi}}^u]$ by a constant depending only on $\kappa, \underline{\xi}, w_1, v_1, w_2, v_2$. The proof is similar to that of Lemma 3.1. We may write $\widetilde{w}_j^u(t) = \widehat{w}_j^u(t) + A_j(t)$ using (4.7). From that proof of Lemma 3.1 we know that $|\log(W_{j,1}^u)|$, $|W_{j,2}^u|$, $\cot_2(W_j^u - W_{3-j}^u)$, $W_j^u - V_s^u$, $s = 1,2$, are all uniformly bounded on $[0,\tau_{\underline{\xi}}^u]$. Since $\tau_{\underline{\xi}}^u = \mathrm{m}(\underline{u}(\tau_{\underline{\xi}}^u))$ and $K(\underline{u}(\tau_{\underline{\xi}}^u))$ is contained in the $\mathbb{D}$-hull generated by $\xi_1\cup\xi_2$, $\tau_{\underline{\xi}}^u$ is also uniformly bounded. From (4.2) we know that u_j' is uniformly bounded on $[0,\tau_{\underline{\xi}}^u]$. The above argument shows that $|A_j|$ is uniformly bounded on $[0,\tau_{\underline{\xi}}^u]$. In order to prove the uniform boundedness of $|\widehat{w}_j^u|$ on $[0,\tau_{\underline{\xi}}^u]$, it suffices to show that $|\widehat{w}_j|$ is uniformly bounded on $[0,\tau_{\xi_j}^j]$. For $\widehat{w}_2(t)$, we have

$$\widetilde{g}_2(t_2,v_1) > \widehat{w}_2(t_2) > \widetilde{g}_2(t_2,v_2), \quad 0 \le t_2 \le \tau_{\xi_2}^2. \tag{4.8}$$

Since $\partial_{t_2}\widetilde{g}_2(t_2,v_s)=\cot_2(\widetilde{g}_2(t_2,v_s)-\widehat{w}_2(t_2))$, by the uniform boundedness of $|\cot_2(\widetilde{g}_2(t_2,v_s)-\widehat{w}_2(t_2))|=|\cot_2(V_s(0,t_2)-W_2(0,t_2))|$ and t_2 on $[0,\tau^2_{\xi_2}]$, we see that $|\widetilde{g}_2(t_2,v_s)|$ is uniformly bounded on $[0,\tau^2_{\xi_2}]$ for $s=1,2$. Using (4.8) we get the uniform boundedness of $\widehat{w}_2(t_2)$ on $[0,\tau^j_{\xi_j}]$. The argument for $\widehat{w}_1(t_1)$ is similar except that we use $\widetilde{g}_1(t_1,v_2+2\pi)>\widehat{w}_1(t)>\widetilde{g}_1(t_1,v_1)$. So the claim is proved.

From Lemma 3.1 and the above claim, we see that, for any $j\in\{1,2\}$ and $\underline{\xi}\in\Xi$, $\widetilde{w}^u_j(t\wedge\tau^u_{\underline{\xi}})M^u_{iB\to c4}(t\wedge\tau^u_{\underline{\xi}})$, $t\ge 0$, is an $(\mathcal{F}^u_t)$-martingale under $\mathbb{P}_{iB}$. Since this process is $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})$-adapted, and $\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}\subset\mathcal{F}_{\underline{u}(t)}=\mathcal{F}^u_t$, we see that it is an $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})$-martingale. From (4.6) we see that $\widetilde{w}^u_j(t\wedge\tau^u_{\underline{\xi}})$, $t\ge 0$, is an $(\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})})$-martingale under $\mathbb{P}_{c4}$. We now show that $(\widetilde{w}^u_j(t\wedge\tau^u_{\underline{\xi}}))$ is an $(\mathcal{F}^u_t)$-martingale under $\mathbb{P}_{c4}$. To check this, we need to show that for any $t\ge s\ge 0$ and $A\in\mathcal{F}^u_s$,

$$\mathbb{E}_{c4}[\mathbf{1}_A\widetilde{w}^u_j(t\wedge\tau^u_{\underline{\xi}})]=\mathbb{E}_{c4}[\mathbf{1}_A\widetilde{w}^u_j(s\wedge\tau^u_{\underline{\xi}})]. \tag{4.9}$$

Write $A=A_1\cup A_2$, where $A_1=A\cap\{\tau^u_{\underline{\xi}}<s\}$ and $A_2=A\cap\{\tau^u_{\underline{\xi}}\ge s\}$. Since $t\wedge\tau^u_{\underline{\xi}}=s\wedge\tau^u_{\underline{\xi}}$ on A_1, (4.9) holds with A_1 in place of A. From Lemma 2.7, $A_2=A\cap\{\underline{u}(s)\le\underline{u}(s\wedge\tau^u_{\underline{\xi}})\}\in\mathcal{F}_{\underline{u}(s\wedge\tau^u_{\underline{\xi}})}$. So (4.9) also holds with A_2 in place of A. Combining, we get (4.9), as desired. Thus, $\widetilde{w}^u_j(t\wedge\tau^u_{\underline{\xi}})$, $t\ge 0$, is an $(\mathcal{F}^u_t)$-martingale under $\mathbb{P}_{c4}$. From Lemma 3.5 we know that $\mathbb{P}_{c4}$-a.s. $T^u=\infty$. Since $T^u=\sup_{\xi\in\Xi^*}\tau^u_{\underline{\xi}}$, we see that $\widetilde{w}^u_j(t)$, $0\le t<\infty$, is an $(\mathcal{F}^u_t)$-local martingale under $\mathbb{P}_{c4}$.

From (4.3) we know that, under $\mathbb{P}_{iB}$,

$$\langle\widetilde{w}^u_j(\cdot\wedge\tau^u_{\underline{\xi}})\rangle_t=\kappa u_j(t\wedge\tau^u_{\underline{\xi}}),\quad j=1,2;\quad \langle\widetilde{w}^u_1(\cdot\wedge\tau^u_{\underline{\xi}}),\widetilde{w}^u_2(\cdot\wedge\tau^u_{\underline{\xi}})\rangle_t\equiv 0 \tag{4.10}$$

Since $\mathbb{P}_{c4}\ll\mathbb{P}_{iB}$ on $\mathcal{F}_{\underline{u}(t\wedge\tau^u_{\underline{\xi}})}$ for any $t\ge 0$, we also have (4.10) under $\mathbb{P}_{c4}$. Since $T^u=\sup_{\xi\in\Xi^*}\tau^u_{\underline{\xi}}$, we conclude that, under $\mathbb{P}_{c4}$,

$$\langle\widetilde{w}^u_j\rangle_t=\kappa u_j(t),\quad j=1,2;\quad \langle\widetilde{w}^u_1,\widetilde{w}^u_2\rangle_t\equiv 0,\quad 0\le t<T^u=\infty. \tag{4.11}$$

Since $(\widetilde{w}^u_j)$, $j=1,2$, are $(\mathcal{F}^u_t)$-local martingales under $\mathbb{P}_{c4}$, we see that there are two independent standard Brownian motions $B^u_j(t)$, $j=1,2$, under $\mathbb{P}_{c4}$, such that $d\widetilde{w}^u_j(t)=\sqrt{\kappa u_j'(t)}dB^u_j(t)$, $0\le t<\infty$. Using (4.7) we then complete the proof. $\square$

Recall that $Z_j=W_j-V_j$, $j=1,2$. Since $W_1>V_1>W_2>V_2>W_1-2\pi$, and $\theta^u=V^u_1-V^u_2=\pi$, we have $Z^u_j\in(0,\pi)$, $j=1,2$. Let $k=3-j$. Using (3.8) we get

$$dV^u_j=-\cot_2(W^u_j-V^u_j)(W^u_{j,1})^2u_j'dt-\cot_2(W^u_k-V^u_j)(W^u_{k,1})^2u_k'dt.$$

Combining this formula with (4.2,4.4), and that $V^u_j-V^u_k=\pm\pi$, we get

$$dZ^u_j=\sqrt{\frac{\kappa\sin(Z^u_j)}{\sin(Z^u_1)+\sin(Z^u_2)}}dB^u_j+\frac{4\cos(Z^u_j)}{\sin(Z^u_1)+\sin(Z^u_2)}\,dt,\quad 0\le t<\infty,\quad j=1,2. \tag{4.12}$$

5 Transition Density

In this section, we are going to find out the transition density of the process (Z_1^u, Z_2^u) that satisfies (4.12). Define $B_+^u(t)$ and $B_-^u(t)$ such that

$$B_\pm^u(t) = \int_0^t \sqrt{\frac{\sin(Z_1^u(s))}{\sin(Z_1^u(s)) + \sin(Z_2^u(s))}}\, dB_1^u(s) \pm \int_0^t \sqrt{\frac{\sin(Z_2^u(s))}{\sin(Z_1^u(s)) + \sin(Z_2^u(s))}}\, dB_2^u(s).$$

Then both $B_+^u(t)$ and $B_-^u(t)$ are standard $(\mathcal{F}_t^u)$-Brownian motions, and their quadratic covariation satisfies

$$d\langle B_+^u, B_-^u\rangle_t = \cot_2(Z_1^u + Z_2^u)\tan_2(Z_1^u - Z_2^u)dt. \tag{5.1}$$

Let $Z_\pm^u = (Z_1^u \pm Z_2^u)/2$. Then $Z_+^u \in (0, \pi)$, $Z_-^u \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and they satisfy the SDEs

$$dZ_+^u = \frac{\sqrt{\kappa}}{2} dB_+^u + 2\cot(Z_+^u)dt, \quad 0 \le t < \infty.$$

$$dZ_-^u = \frac{\sqrt{\kappa}}{2} dB_-^u - 2\tan(Z_-^u)dt, \quad 0 \le t < \infty.$$

We are going to follow the argument in [30, Appendix B] to derive the transition density of (Z_+^u, Z_-^u). Let $X = \cos(Z_+^u)$ and $Y = \sin(Z_-^u)$. Then $X, Y \in (-1, 1)$, and satisfy the SDEs

$$dX = -\frac{\sqrt{\kappa}}{2}\sqrt{1 - X^2} dB_+^u - \Big(2 + \frac{\kappa}{8}\Big)X dt. \tag{5.2}$$

$$dY = +\frac{\sqrt{\kappa}}{2}\sqrt{1 - Y^2} dB_-^u - \Big(2 + \frac{\kappa}{8}\Big)Y dt. \tag{5.3}$$

From (5.1) we have

$$d\langle X, Y\rangle_t = -\frac{\kappa}{4} XY dt. \tag{5.4}$$

Since $X(t)^2 + Y(t)^2 = 1 - \sin(Z_1^u(t))\sin(Z_2^u(t)) < 1$, we see that $(X(t), Y(t)) \in \mathbb{D}$ for all $t \ge 0$. We will find out the transition density $p_t((x, y), (x^*, y^*))$ for the joint process (X, Y).

First, we assume that the transition density $p_t((x, y), (x^*, y^*))$ for (X, Y) exists, and make some observations. For any fixed $(x^*, y^*) \in \mathbb{D}$ and $t_0 > 0$, the process $M_t^{(x^*,y^*)} := p(t_0 - t, (X(t), Y(t)), (x^*, y^*))$, $0 \le t < t_0$, is a martingale. Assuming further that p is smooth in (t, x, y), then we get a PDE:

$$-\partial_t p + \mathcal{L}p = 0, \tag{5.5}$$

where $\mathcal{L}$ is the second order differential operator:

$$\mathcal{L} := \frac{\kappa}{8}(1 - x^2)\partial_x^2 + \frac{\kappa}{8}(1 - y^2)\partial_y^2 - \frac{\kappa}{4}xy\partial_x\partial_y - (2 + \frac{\kappa}{8})x\partial_x - (2 + \frac{\kappa}{8})y\partial_y.$$

We will derive the eigenvectors and eigenvalues of $\mathcal{L}$. Note that for integers $n, m \ge 0$,

$$\mathcal{L}(x^n y^m) = -\frac{\kappa}{8}(n + m)(n + m + \frac{16}{\kappa})x^n y^m + \frac{\kappa}{8}n(n - 1)x^{n-2}y^m + \frac{\kappa}{8}m(m - 1)x^n y^{m-2}.$$

Define

$$\lambda_n = -\frac{\kappa}{8} n(n+\frac{16}{\kappa}), \quad n \in \mathbb{N} \cup \{0\}. \tag{5.6}$$

Then $\mathcal{L}(x^n y^m)$ equals to $\lambda_{n+m} x^n y^m$ plus a polynomial in x, y of degree less than $n+m$. Hence, for each $n, m \in \mathbb{N} \cup \{0\}$, there is a polynomial $P_{(n,m)}(x, y)$ of degree $n+m$, which can be written as $x^n y^m$ plus a polynomial of degree less than $n+m$, such that

$$\mathcal{L} P_{(n,m)} = \lambda_{n+m} P_{(n,m)}.$$

Let $\Psi(x, y) = (1 - x^2 - y^2)^{\frac{8}{\kappa}-1}$, and define the inner product

$$\langle f, g \rangle_\Psi := \int\int_{\mathbb{D}} f(x, y) g(x, y) \Psi(x, y) dx dy.$$

Since $\Psi \equiv 0$ on $\mathbb{T}$, direct calculation shows that for smooth functions f and g on $\overline{\mathbb{D}}$,

$$\langle \mathcal{L} f, g \rangle_\Psi = \langle f, \mathcal{L} g \rangle_\Psi.$$

So $P_{(n,m)}$ is orthogonal to $P_{(n',m')}$ w.r.t. $\langle \cdot \rangle_\Psi$ if $n + m \neq n' + m'$. Thus, we may construct a sequence of polynomials $v_{(n,s)}$, $n = 0, 1, 2, \ldots$, $s = 0, 1, \ldots, n$, such that $v_{(n,s)}$ is a polynomial in x, y of degree n, $\mathcal{L} v_{(n,s)} = \lambda_n v_{(n,s)}$, and $\{v_{(n,s)}\}$ form an orthonormal basis w.r.t. $\langle \cdot \rangle_\Psi$. Here every $v_{(n,s)}$ is a linear combination of $P_{(j,k)}$ over $j, k \in \mathbb{N} \cup \{0\}$ such that $j + k = n$. On the other hand, if a sequence of polynomials $v_{(n,s)}$, $n = 0, 1, 2, \ldots$, $s = 0, 1, \ldots, n$, form an orthonormal basis w.r.t. $\langle \cdot \rangle_\Psi$, and each $v_{(n,s)}$ has degree n, then $\mathcal{L} v_{(n,s)} = \lambda_n v_{(n,s)}$. This is because $v_{(n,s)}$ is orthogonal to all polynomials of degree less than n, and so it must be a linear combination of $P_{(j,k)}$ over $j, k \in \mathbb{N} \cup \{0\}$ such that $j + k = n$. From [27, Section 1.2.2], we may choose $v_{(n,s)}$ such that for each $n \geq 0$, $v_{(n,0)}, v_{(n,1)}, \ldots, v_{(n,n)}$ are given by

$$v_{n,j,1} = h_{n,j,1} P_j^{(\frac{8}{\kappa}-1, n-2j)}(2r^2 - 1) r^{n-2j} \cos((n-2j)\theta), \quad 0 \leq 2j \leq n,$$

$$v_{n,j,2} = h_{n,j,2} P_j^{(\frac{8}{\kappa}-1, n-2j)}(2r^2 - 1) r^{n-2j} \sin((n-2j)\theta), \quad 0 \leq 2j \leq n-1,$$

where $P_j^{(\frac{8}{\kappa}-1, n-2j)}$ are Jacobi polynomials of index $(\frac{8}{\kappa} - 1, n - 2j)$, (r, θ) is the polar coordinate of (x, y): $x = r \cos\theta$ and $y = r \sin\theta$, and $h_{n,j,i} > 0$ are normalization constants. Using the polar integration and Formula ([20, Table 18.3.1])

$$\int_{-1}^{1} P_j^{(\alpha,\beta)}(x)^2 (1-x)^\alpha (1+x)^\beta dx = \frac{2^{\alpha+\beta+1} \Gamma(j+\alpha+1) \Gamma(j+\beta+1)}{j!(2j+\alpha+\beta+1) \Gamma(j+\alpha+\beta+1)} \tag{5.7}$$

with $\alpha = \frac{8}{\kappa} - 1$ and $\beta = n - 2j$, we compute

$$h_{n,j} := h_{n,j,1} = h_{n,j,2} = \sqrt{\frac{1 + \mathbf{1}_{n \neq 2j}}{\pi} \cdot \frac{j!(n+\frac{8}{\kappa}) \Gamma(n-j+\frac{8}{\kappa})}{\Gamma(j+\frac{8}{\kappa}) \Gamma(n-j+1)}}. \tag{5.8}$$

Using the super norm of $P_j^{(\alpha,\beta)}$ ([20, 18.14.1,18.14.2]):

$$\|P_j^{(\alpha,\beta)}\|_\infty = \frac{\Gamma(\max\{\alpha,\beta\}+j+1)}{j!\Gamma(\max\{\alpha,\beta\}+1)}, \quad \text{if } \max\{\alpha,\beta\} \ge -1/2,\ \min\{\alpha,\beta\} > -1, \tag{5.9}$$

we get

$$\|v_{n,j,1}\|_\infty = \|v_{n,j,2}\|_\infty = h_{n,j} \max\Big\{\frac{\Gamma(\frac{8}{\kappa}+j)}{j!\Gamma(\frac{8}{\kappa})}, \frac{\Gamma(n-j+1)}{j!\Gamma(n-2j+1)}\Big\}. \tag{5.10}$$

For $t>0$, $(x,y),(x^*,y^*)\in\mathbb{D}$, we define

$$p_t((x,y),(x^*,y^*)) = \sum_{n=0}^\infty \sum_{s=0}^n \Psi(x^*,y^*) v_{(n,s)}(x,y) v_{(n,s)}(x^*,y^*) e^{\lambda_n t}. \tag{5.11}$$

Let $p_\infty(x^*,y^*)$ be the term for $n=s=0$. Since $\lambda_0=0$ and $P_0^{\alpha,\beta}\equiv 1$, we have

$$p_\infty(x^*,y^*) = \frac{8}{\pi\kappa}\Psi(x^*,y^*) = \frac{8}{\pi\kappa}(1-(x^*)^2-(y^*)^2)^{\frac{8}{\kappa}-1}. \tag{5.12}$$

Lemma 5.1. *For any $t_0>0$, the series in (5.11) converges uniformly on $[t_0,\infty)\times\overline{\mathbb{D}}\times\overline{\mathbb{D}}$, and there is $C_{t_0}\in(0,\infty)$ depending only on κ and t_0 such that*

$$|p_t((x,y),(x^*,y^*)) - p_\infty(x^*,y^*)| \le C_{t_0} e^{-(2+\frac{\kappa}{8})t} p_\infty(x^*,y^*), \quad t\ge t_0, \quad (x,y),(x^*,y^*)\in\overline{\mathbb{D}}.$$

Moreover, for any $t>0$ and $(x^*,y^*)\in\overline{\mathbb{D}}$,

$$p_\infty(x^*,y^*) = \int\!\!\int_{\mathbb{D}} p_\infty(x,y) p_t((x,y),(x^*,y^*)) dxdy. \tag{5.13}$$

Proof. The uniform convergence of the series in (5.11) and the first formula follows from Stirling's formula, (5.8,5.10), and the facts that $\lambda_1 = -(2+\frac{\kappa}{8}) > \lambda_n$ for any $n>1$ and $\lambda_n \asymp -\frac{\kappa}{8}n^2$ for big n. Formula (5.13) follows from the orthogonality of $v_{n,s}$ w.r.t. $\langle\cdot,\cdot\rangle_\Psi$ and the uniform convergence of the series in (5.11). $\square$

Lemma 5.2. *Under $\mathbb{P}_{c4}$, $p_t((x,y),(x^*,y^*))$ is the transition density for $(X(t),Y(t))$ that satisfies (5.2,5.3,5.4), and p_∞ is the invariant density.*

Proof. Fix $(x,y)\in\mathbb{D}$. Let $(X(t),Y(t))$, $t\ge 0$, be the process that satisfies (5.2,5.3,5.4) with initial value (x,y). Fix $t_0>0$. For the first statement, it suffices to show that, for any $f\in C(\overline{\mathbb{D}},\mathbb{R})$,

$$\mathbb{E}_{c4}[f(X_{t_0},Y_{t_0})] = \int\!\!\int_{\mathbb{D}} p_{t_0}((x,y),(x^*,y^*)) f(x^*,y^*) dx^* dy^*. \tag{5.14}$$

Since $\mathcal{L}v_{(n,s)} = \lambda_n v_{(n,s)}$, every function $v_{(n,s)}(x,y)e^{\lambda_n t}$ solves (5.5). Let f be a polynomial in x, y. Let $a_{(n,s)} = \langle f, v_{(n,s)}\rangle_\Psi$. Then $f(x,y) = \sum_{n=0}^\infty \sum_{s=0}^n a_{(n,s)}v_{(n,s)}(x,y)$, where all but finitely many $a_{(n,s)}$ are not zero. Define

$$f(t,(x,y)) = \sum_{n=0}^\infty \sum_{s=0}^n a_{(n,s)}v_{(n,s)}(x,y)e^{\lambda_n t} = \int\int_{\mathbb{D}} p_{t_0}((x,y),(x^*,y^*))f(x^*,y^*)dx^*dy^*.$$

Then $f(t,(x,y))$ solves (5.5) since it is a linear combination of $v_{(n,s)}(x,y)e^{\lambda_n t}$. Let $(X(t),Y(t))$ be a stochastic process in $\mathbb{D}$, which solves (5.2,5.3,5.4) with initial value (x,y). Fix $t_0 > 0$ and define $M_t = f(t_0 - t,(X(t),Y(t)))$, $0 \le t \le t_0$. By Itô's formula, (M_t) is a bounded martingale w.r.t. $\mathbb{P}_4$, which implies that $\mathbb{E}_{c4}[f(X(t_0),Y(t_0))] = \mathbb{E}_{c4}[M_{t_0}] = M_0 = f(t_0,(x,y))$. So we get (5.14) for a polynomial f. Formula (5.14) for a general $f \in C(\overline{\mathbb{D}},\mathbb{R})$ follows from Stone-Weierstrass theorem. The statement on p_∞ follows immediately from (5.13). □

Corollary 5.3. *Under $\mathbb{P}_{c4}$, the transition density for (Z_1^u, Z_2^u) that satisfies (4.12) is*

$$p_t^Z((z_1,z_2),(z_1^*,z_2^*)) = p_t((\cos_2(z_1+z_2),\sin_2(z_1-z_2)),(\cos_2(z_1^*+z_2^*),\sin_2(z_1^*-z_2^*)))\frac{\sin z_1^* + \sin z_2^*}{4},$$

and the invariant density is

$$p_\infty^Z(z_1^*,z_2^*) = p_\infty(\cos_2(z_1^*+z_2^*),\sin_2(z_1^*-z_2^*))\frac{\sin z_1^* + \sin z_2^*}{4}.$$

Proof. This follows from the above lemma and the fact that $X(t) := \cos_2(Z_1^u(t)+Z_2^u(t))$ and $Y(t) := \sin_2(Z_1^u(t) - Z_2^u(t))$ satisfy (5.2,5.3,5.4). □

Next, we will derive the transition density $\widetilde{p}_t^Z((z_1,z_2),(z_1^*,z_2^*))$ under $\mathbb{P}_2$ for (Z_1^u,Z_2^u). Now B_1^u and B_2^u are not standard Brownian motions under $\mathbb{P}_2$, and we no longer have $\mathbb{P}_2$-a.s. $T^u = \infty$. In fact, we will see that $\mathbb{P}_2$-a.s. $T^u < \infty$. By saying that $\widetilde{p}_t^Z((z_1,z_2),(z_1^*,z_2^*))$ is the transition density for (Z_1^u,Z_2^u) under $\mathbb{P}_2$, we mean that, for any $t > 0$ and $(z_1,z_2) \in (0,\pi)^2$, if (Z_1^u,Z_2^u) starts from (z_1,z_2), then for any bounded measurable function f on $(0,\pi)^2$, we have

$$\mathbb{E}_2[\mathbf{1}_{\{T^u>t\}}f(Z_1^u(t),Z_2^u(t))] = \int_0^\pi\int_0^\pi \widetilde{p}_t^Z((z_1,z_2),(z_1^*,z_2^*))f(z_1^*,z_2^*)dz_2^*dz_1^*.$$

In particular, we have $\mathbb{P}_2[T^u > t] = \int_0^\pi\int_0^\pi \widetilde{p}_t^Z((z_1,z_2),(z_1^*,z_2^*))dz_2^*dz_1^*$.

From (3.34) we know that, for any $t \ge 0$,

$$\frac{d\mathbb{P}_2|\mathcal{F}_t^u \cap \{T^u > t\}}{d\mathbb{P}_{c4}|\mathcal{F}_t^u \cap \{T^u > t\}} = \frac{M_{c4\to 2}^u(t)}{M_{c4\to 2}^u(0)}.$$

Let $G^u(z_1,z_2)$ be a function defined for $z_1,z_2 \in (0,\pi)$ such that

$$G^u(z_1,z_2) = [\sin_2 z_1 \sin_2 z_2]^{\frac{8}{\kappa}-1}\cos_2(z_1-z_2)^{\frac{4}{\kappa}}F\Big(\frac{\cos_2 z_1 \cos_2 z_2}{\cos_2(z_1-z_2)}\Big)^{-1}.$$

From (3.35,3.36) we get $G(W_1^u, V_1^u; W_2^u, V_2^u) = G^u(Z_1^u, Z_2^u)$ and

$$M^u_{c4\to 2}(t) = e^{-\alpha_0 t} G^u(Z_1^u(t), Z_2^u(t))^{-1}.$$

Recall that $\mathbb{P}_{c4}$-a.s. $T^u = \infty$. So we obtain the following lemma.

Lemma 5.4. *Under* $\mathbb{P}_2$, $(Z_1^u(t), Z_2^u(t))$, $0 \le t < T^u$, *is a Markov process with transition density*

$$\widetilde{p}^Z_t((z_1,z_2),(z_1^*,z_2^*)) := e^{-\alpha_0 t} p^Z_t((z_1,z_2),(z_1^*,z_2^*)) \frac{G^u(z_1,z_2)}{G^u(z_1^*,z_2^*)}.$$

Note that for some explicit constant $C \in (0,\infty)$ depending on κ,

$$\frac{p^Z_\infty(z_1,z_2)}{G^u(z_1,z_2)} = C[\cos_2 z_1 \cos_2 z_2]^{\frac{8}{\kappa}-1} \sin_2(z_1+z_2) \cos_2(z_1-z_2)^{1-\frac{4}{\kappa}} F\Big(\frac{\cos_2 z_1 \cos_2 z_2}{\cos_2(z_1-z_2)}\Big).$$

So $\frac{p^Z_\infty(z_1,z_2)}{G^u(z_1,z_2)}$ extends to a continuous function on $[0,\pi]^2$, which vanishes at the corners. We may normalize it to get a probability density, i.e., we define

$$\mathcal{Z} = \int_0^\pi \int_0^\pi \frac{p^Z_\infty(z_1,z_2)}{G^u(z_1,z_2)} dz_1 dz_2 \in (0,\infty), \tag{5.15}$$

$$\widetilde{p}^Z_\infty(z_1,z_2) = \frac{1}{\mathcal{Z}} \frac{p^Z_\infty(z_1,z_2)}{G^u(z_1,z_2)}, \quad z_1, z_2 \in [0,\pi]. \tag{5.16}$$

From now on, if a quantity Q depends on $t \in (0,\infty)$ and other variables $\underline{x}$, and f is a positive function on $(0,\infty)$, we write Q as $O(f(t))$, if for any $t_0 > 0$ there is $C_{t_0} \in (0,\infty)$ depending only on κ and t_0 such that for any $t \ge t_0$ and any $\underline{x}$, $|Q(t,\underline{x})| \le Cf(t)$.

Lemma 5.5. *(i) For any* $t > 0$ *and* $z_1^*, z_2^* \in [0,\pi]$,

$$\int_0^\pi \int_0^\pi \widetilde{p}^Z_\infty(z_1,z_2) \widetilde{p}^Z_t((z_1,z_2),(z_1^*,z_2^*)) dz_1 dz_2 = \widetilde{p}^Z_\infty(z_1^*,z_2^*) e^{-\alpha_0 t}. \tag{5.17}$$

This means, under the law $\mathbb{P}_2$, *if the process* (Z_1^u, Z_2^u) *starts from a random point* $(z_1,z_2) \in (0,\pi)^2$ *with density* $\widetilde{p}^Z_\infty$, *then for any deterministic* $t \ge 0$, *the density of* $(Z_1^u(t), Z_2^u(t))$ *at time* t *is* $e^{-\alpha_0 t}\widetilde{p}^Z_\infty$.

(ii) For any $(z_1,z_2) \in (0,\pi)^2$ *and a process* (Z_1^u, Z_2^u) *started from* (z_1,z_2), *we have*

$$\mathbb{P}_2[T^u > t] = \mathcal{Z} G^u(z_1,z_2) e^{-\alpha_0 t}(1 + O(e^{-(2+\frac{\kappa}{8})t})); \tag{5.18}$$

$$\widetilde{p}^Z_t((z_1,z_2),(z_1^*,z_2^*)) = \mathbb{P}_2[T^u > t] \widetilde{p}^Z_\infty(z_1^*,z_2^*)(1 + O(e^{-(2+\frac{\kappa}{8})t})). \tag{5.19}$$

Here we emphasize that the implicit constants in the O *symbols do not depend on* (z_1,z_2).

Proof. Part (i) follows easily from (5.13). For part (ii), suppose (Z_1^u, Z_2^u) starts from (z_1, z_2). Using Lemmas 5.1, 5.4 and formulas (5.15,5.16), we get

$$\mathbb{P}_2[T^u > t] = \int_0^\pi \int_0^\pi \widetilde{p}_t^Z((z_1, z_2), (z_1^*, z_2^*)) dz_1^* dz_2^*$$

$$= \int_0^\pi \int_0^\pi e^{-\alpha_0 t} p_t^Z((z_1, z_2), (z_1^*, z_2^*)) \frac{G^u(z_1, z_2)}{G^u(z_1^*, z_2^*)} dz_1^* dz_2^*$$

$$= \int_0^\pi \int_0^\pi e^{-\alpha_0 t} p_\infty^Z(z_1^*, z_2^*)(1 + O(e^{-(2+\frac{\kappa}{8})t})) \frac{G^u(z_1, z_2)}{G^u(z_1^*, z_2^*)} dz_1^* dz_2^*$$

$$= \mathcal{Z} G^u(z_1, z_2) e^{-\alpha_0 t}(1 + O(e^{-(2+\frac{\kappa}{8})t})),$$

which is (5.18); and

$$\widetilde{p}_t^Z((z_1, z_2), (z_1^*, z_2^*)) = e^{-\alpha_0 t} p_t^Z((z_1, z_2), (z_1^*, z_2^*)) \frac{G^u(z_1, z_2)}{G^u(z_1^*, z_2^*)}$$

$$= e^{-\alpha_0 t} p_\infty^Z(z_1^*, z_2^*)(1 + O(e^{-(2+\frac{\kappa}{8})t})) \frac{G^u(z_1, z_2)}{G^u(z_1^*, z_2^*)}$$

$$= e^{-\alpha_0 t} \mathcal{Z} \widetilde{p}_\infty^Z(z_1^*, z_2^*)(1 + O(e^{-(2+\frac{\kappa}{8})t})) G^u(z_1, z_2),$$

which together with (5.18) implies (5.19). □

6 Proofs of Main Theorems

We will prove the main theorems of the paper in this section. We will need the Domain Markov Property for 2-SLE in the following form.

Lemma 6.1. *Let (η_1, η_2) be a 2-SLE$_\kappa$ in a simply connected domain D with link pattern $(a_1 \to b_1; a_2 \to b_2)$. Suppose, for $j = 1, 2$, η_j is parametrized by the chordal capacity viewed from b_j (determined by a conformal map from D onto $\mathbb{H}$ that takes b_j to ∞). Note that the lifetime of the parametrized η_1 and η_2 are both ∞. Let $(\mathcal{F}_t^j)_{t \ge 0}$ be the filtration generated by η_j, $j = 1, 2$, which together generate a separable $\mathcal{Q}$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t} \in \mathcal{Q}}$. Let $\underline{T} = (T_1, T_2)$ be an $(\mathcal{F}_{\underline{t}})$-stopping time. Let $D_{\underline{T}}^j$ denote the connected component of $D \setminus (\eta_1([0, T_1]) \cup \eta_2([0, T_2]))$ whose boundary contains b_j, $j = 1, 2$ Then*

- *(i) Conditioning on $\mathcal{F}_{\underline{T}}$ and the event that $D_{\underline{T}}^1 = D_{\underline{T}}^2 =: D_{\underline{T}}$ and $\eta_1(T_1) \ne \eta_2(T_2)$, $\eta_1|_{[T_1,\infty]}$ and $\eta_2|_{[T_2,\infty]}$) form a 2-SLE$_\kappa$ in $D_{\underline{T}}$ with link pattern $(\eta_1(T_1) \to b_1; \eta_2(T_2) \to b_2)$.*
- *(ii) Conditioning on $\mathcal{F}_{\underline{T}}$ and the event that $D_{\underline{T}}^1 \ne D_{\underline{T}}^2$, $\eta_j|_{[T_j,\infty]}$ is a chordal SLE$_\kappa$ curve in $D_{\underline{T}}^j$ from $\eta_j(T_j)$ to b_j, $j = 1, 2$, and $\eta_1|_{[T_1,\infty]}$ and $\eta_2|_{[T_2,\infty]}$ are independent.*

Proof. By the property of 2-SLE$_\kappa$, conditioning on $\mathcal{F}^2_\infty$, η_1 is a chordal SLE$_\kappa$ curve from a_1 to b_1 in a connected component of $D\setminus\eta_2$. Let $\mathcal{F}^{(2,\infty)}_{t_1}=\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_\infty$. Then we get a filtration $(\mathcal{F}^{(2,\infty)}_{t_1})_{t_1\ge 0}$. Since for any $t_1\ge 0$, $\{T_1\le t_1\}=\bigcup_{n\in\mathbb{N}}\{\underline{T}\le(t_1,n)\}$, we see that T_1 is an $(\mathcal{F}^{(2,\infty)}_{t_1})$-stopping time. If $A\in\mathcal{F}_{\underline{T}}$, then from $A\cap\{T_1\le t_1\}=\bigcup_{n\in\mathbb{N}}A\cap\{\underline{T}\le(t_1,n)\}$, $t_1\ge 0$, we see that $A\in\mathcal{F}^{(2,\infty)}_{T_1}$. Thus, $\mathcal{F}_{\underline{T}}\vee\mathcal{F}^2_\infty\subset\mathcal{F}^{(2,\infty)}_{T_1}$.

By the DMP of chordal SLE$_\kappa$, conditioning on $\mathcal{F}^{(2,\infty)}_{T_1}$, $\eta_1|_{[T_1,\infty]}$ has the law of a chordal SLE$_\kappa$ curve from $\eta_1(T_1)$ to b_1 in a connected component of $D\setminus(\eta_1([0,T_1])\cup\eta_2)$, which is denoted by $D_{T_1,\infty}$. Note that the triple $(D_{T_1,\infty};\eta_1(T_1),b_1)$ is measurable w.r.t. $\mathcal{F}_{\underline{T}}\vee\mathcal{F}^2_\infty$. Since $\mathcal{F}_{\underline{T}}\vee\mathcal{F}^2_\infty\subset\mathcal{F}^{(2,\infty)}_{T_1}$, we conclude that, conditioning on $\mathcal{F}_{\underline{T}}\vee\mathcal{F}^2_\infty$, $\eta_1|_{[T_1,\infty]}$ also has the law of a chordal SLE$_\kappa$ in $D_{T_1,\infty}$ from $\eta_1(T_1)$ to b_1. Since $\mathcal{F}_{\underline{T}}\vee\mathcal{F}^2_\infty$ agrees with the σ-algebra generated by $\mathcal{F}_{\underline{T}}$ and $\eta_2|_{[T_2,\infty]}$, we can say that, conditioning first on $\mathcal{F}_{\underline{T}}$ and then on $\eta_2|_{[T_2,\infty]}$, $\eta_1|_{[T_1,\infty]}$ has the law of a chordal SLE$_\kappa$ in $D_{T_1,\infty}$ from $\eta_1(T_1)$ to b_1. Similarly, conditioning first on $\mathcal{F}_{\underline{T}}$ and then on $\eta_1|_{[T_1,\infty]}$, $\eta_2|_{[T_2,\infty]}$ has the law of a chordal SLE$_\kappa$ in D_{∞,T_2}. On the event that $D^1_{\underline{T}}\ne D^2_{\underline{T}}$, D_{∞,T_j} does not depend on $\eta_{3-j}([T_{3-j},\infty])$, so $\eta_1|_{[T_1,\infty]}$ and $\eta_2|_{[T_2,\infty]}$ are conditionally independent given $\mathcal{F}_{\underline{T}}$ on this event. This is (ii). On the event that $D^1_{\underline{T}}=D^2_{\underline{T}}=:D_{\underline{T}}$ and $\eta_1(T_1)\ne\eta_2(T_2)$, the conditional joint law of $\eta_1|_{[T_1,\infty]}$ and $\eta_2|_{[T_2,\infty]}$ given $\mathcal{F}_{\underline{T}}$ agrees with that of the 2-SLE$_\kappa$ in $D_{\underline{T}}$ with link pattern $(\eta_1(T_1)\to b_1;\eta_2(T_2)\to b_2)$. So we get (i). $\square$

Proof of Theorem 1.1. We first work on (1.2). By Koebe's distortion theorem, it suffices to prove the theorem for $D=\mathbb{D}$ and $z_0=0$. By symmetry, we may assume that $a_j=e^{iw_j}$ and $b_j=e^{iv_j}$, $j=1,2$, and $w_1>v_1>w_2>v_2>w_1-2\pi$. We use $p(w_1,v_1,w_2,v_2;r)$ to denote the probability that both $\widehat{\eta}_1$ and $\widehat{\eta}_2$ have distance less than r from 0. Since $G_{\mathbb{D};e^{iw_1},e^{iv_1};e^{iw_2},e^{iv_2}}(0)$ agrees with the $G(w_1,v_1;w_2,v_2)$ defined by (3.35), it suffices to show that, for some constant $C_0\in(0,\infty)$,

$$p(w_1,v_1,w_2,v_2;r)=C_0G(w_1,v_1;w_2,v_2)r^{\alpha_0}(1+O(r^{\beta_0})),\quad\text{as } r\to 0^+. \tag{6.1}$$

For $j=1,2$, suppose $\widehat{\eta}_j$ is oriented from a_j to b_j, and let η_j be the part of $\widehat{\eta}_j$ from a_j up to b_j or the first time that $\widehat{\eta}_j$ separates 0 from any of b_j,a_{3-j},b_{3-j} if such time exists. Then we may parametrize η_1 and η_2 by the radial capacity viewed from 0 such that they are radial Loewner curves with lefetime $\widetilde{T}_1$ and $\widetilde{T}_2$ driven by some functions $\widehat{w}_1$ and $\widehat{w}_2$ with initial values w_1 and w_2, respectively. Then the law of $(\widehat{w}_1,\widehat{w}_2)$ is $\mathbb{P}_2^{w_1,v_1;w_2,v_2}$ as defined in Section 3.2.

We use the symbols in Section 3. Now we write $K_{\underline{t}}$ and $g_{\underline{t}}$ for $K(t_1,t_2)$ and $g((t_1,t_2),\cdot)$, and let $D_{\underline{t}}=\mathbb{D}\setminus K_{\underline{t}}$. Recall that $g_{\underline{t}}$ maps $D_{\underline{t}}$ conformally onto $\mathbb{D}$, fixes 0, and has derivative $e^{\mathrm{m}(\underline{t})}$ at 0. Moreover, $g_{\underline{t}}(\eta_j(t_j))=e^{iW_j(\underline{t})}$ and $g_{\underline{t}}(e^{iv_j})=e^{iV_j(\underline{t})}$, $j=1,2$. Suppose $\underline{T}=(T_1,T_2)$ is an $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$-stopping time such that $T_j<\widetilde{T}_j$, $j=1,2$. Then $\underline{T}$ corresponds to a stopping time w.r.t. the $\mathcal{Q}$-indexed filtration generated by $\widehat{\eta}_1,\widehat{\eta}_2$, which are parametrized by chordal capacities viewed from b_1,b_2, respectively. By Lemma 6.1, conditionally on $\mathcal{F}_{\underline{T}}$ and the event that $\underline{T}\in\mathcal{D}$, the $g_{\underline{T}}$-images of the parts of $\widehat{\eta}_j$ from $\eta_j(u_j(t_0))$ to e^{iv_j}, $j=1,2$, together form a 2-SLE$_\kappa$ in $\mathbb{D}$ with link pattern $(e^{iW_1(\underline{T})}\to e^{iV_1(\underline{T})};e^{iW_2(\underline{T})}\to e^{iV_2(\underline{T})})$.

Suppose that $v_1-v_2=\pi$ so that the time curve $\underline{u}$ in Section 4 can be defined, and we may use the results and symbols there. Fix $r\in(0,1/4)$. Suppose that it happens that $\operatorname{dist}(0,\widehat{\eta}_j)<r$,

$j=1,2$. Then the parts of $\widehat{\eta}_1$ and $\widehat{\eta}_2$ up to their respective hitting times at $\{|z|=r\}$ do not intersect. Because if they did intersect, then they together could disconnect e^{iv_1} and e^{iv_2} in $\mathbb{D}$, and the rest parts of $\widehat{\eta}_1$ and $\widehat{\eta}_2$ would grow in different domains, and could not both visit the disc $\{|z|<r\}$, which is a contradiction. Thus, for $j=1,2$, the above part of $\widehat{\eta}_j$ does not disconnect 0 from any of $e^{iv_j}, e^{iw_{3-j}}, e^{iv_{3-j}}$, and so belongs to η_j. Let τ_j be the first hitting time of η_j at $\{|z|=r\}$, $j=1,2$. Then $(\tau_1,\tau_2)\in\mathcal{D}$. By Koebe's 1/4 theorem, we get $\tau_j\ge -\log(4r)$, $j=1,2$. Recall that $\theta(\tau_1,0)<\pi<\theta(0,\tau_2)$. So there is $\underline{s}=(s_1,s_2)\in\{(\tau_1,t_2):0\le t_2\le\tau_2\}\cup\{(t_1,\tau_2):0\le t_1\le\tau_1\}$ such that $\theta(\underline{s})=\pi$. This implies that $\mathrm{m}(s_1,s_2)<T^u$ and $s_j=u_j(\mathrm{m}(s_1,s_2))$, $j=1,2$. Using (3.11) we get $T^u>s_1\vee s_2\ge -\log(4r)$.

Now fix $t_0\in[0,-\log(4r)]$. Then $\{\mathrm{dist}(0,\widehat{\eta}_j)<r, j=1,2\}\subset\{T^u>t_0\}$. When $T^u>t_0$ happens, since $u_j(t_0)\le t_0$, by Koebe's 1/4 theorem, $\mathrm{dist}(0,\eta_j[0,u_j(t_0)])\ge r$, $j=1,2$. Thus, $\mathrm{dist}(0,\widehat{\eta}_j)<r$, $j=1,2$, if and only if $T^u>t_0$ and the parts of $\widehat{\eta}_j$ after $\eta_j(u_j(t_0))$, $j=1,2$, both visit the disc $\{|z|<r\}$. Suppose $T^u>t_0$ does happen. Let $R_1<R_2\in(0,1)$ be such that $\frac{e^{-t_0}R_1}{(1-R_1)^2}=\frac{e^{-t_0}R_2}{(1+R_2)^2}=r$. Since $(g^u_{t_0})'(0)=e^{\mathrm{m}(\underline{u}(t_0))}=e^{t_0}$ and $r\le\frac14 e^{-t_0}$, by Koebe's distortion theorem, $\{|z|<r\}\subset D_{\underline{u}(t_0)}$, and

$$\{|z|<R_1\}\subset g^u_{t_0}(\{|z|<r\})\subset\{|z|<R_2\}. \tag{6.2}$$

By rotation symmetry, there is a function $p(z_1,z_2;r)$ such that $p(w_1,v_1,w_2,v_2;r)=p(w_1-v_1,w_2-v_2;r)$ if $v_1-v_2=\pi$. From the conditional joint law of the $g_{\underline{u}(t_0)}$-images of the parts of $\widehat{\eta}_j$ after $\eta_j(u_j(t_0))$, $j=1,2$, given $\mathcal{F}^u_{t_0}$, and the facts that $V^u_1(t_0)-V^u_2(t_0)=\pi$ and $Z^u_j=W^u_j-V^u_j$, $j=1,2$, we get

$$p(Z^u_1(t_0),Z^u_2(t_0);R_1)\le\mathbb{P}[\mathrm{dist}(0,\widehat{\eta}_j)<r,j=1,2|\mathcal{F}^u_{t_0},T^u>t_0]\le p(Z^u_1(t_0),Z^u_2(t_0);R_2). \tag{6.3}$$

We will first find the asymptotic of $p(r):=\int_0^\pi\int_0^\pi p(z_1,z_2;r)\widetilde{p}^Z_\infty(z_1,z_2)dz_1dz_2$ as $r\to0^+$. Such $p(r)$ is the probability that the two curves $\widehat{\eta}_1$ and $\widehat{\eta}_2$ in a 2-SLE$_\kappa$ in $\mathbb{D}$ with link pattern $(e^{iz_1}\to1;-e^{iz_2}\to-1)$ both get within distance r from 0, where z_1,z_2 are random numbers in $(0,\pi)$ with joint density $\widetilde{p}^Z_\infty$. From (5.18) we know that, $\mathbb{P}[T^u>t_0]=e^{-\alpha_0t_0}$, and conditioning on $T^u>t_0$, $(Z^u_1(t_0),Z^u_2(t_0))$ also has joint density $\widetilde{p}^Z_\infty$.

Suppose $0<t<T^u$. Let $d_j=\mathrm{dist}(0,\eta_j([0,u_j(t)]))$. Since $\mathrm{m}(u_1(t),u_2(t))=t$, by Schwarz Lemma, we have $d_1\wedge d_2\le e^{-t}$. By symmetry we may assume that $d_1\le d_2$. Since $\theta(u_1(t),u_2(t))=\pi$, we know that the harmonic measure of the union of $\eta_2([0,u_2(t)])$ and the subarc of $\mathbb{T}$ between e^{iv_1} and e^{iv_1} that contains e^{iw_2} in $\mathbb{D}\setminus K(u_1(t),u_2(t))$ viewed from 0 is exactly 1/2. Using Beurling estimate, we get $1/2\le 2(d_1/d_2)^{1/2}$, which implies that $d_2\le 16d_1$. Since $d_1\le e^{-t}$, we get $d_1,d_2\le 16e^{-t}$, and so $\mathrm{dist}(0,\eta_j)\le d_j\le16e^{-t}$, $j=1,2$. This means that $p(r)\ge\mathbb{P}[T^u>t]=e^{-\alpha_0t}>0$ if $r>16e^{-t}$. So p is positive. By (6.3) we get

$$e^{-\alpha_0t_0}p(R_1)\le p(r)\le e^{-\alpha_0t_0}p(R_2),\quad\text{if } \frac{e^{-t_0}R_1}{(1-R_1)^2}=\frac{e^{-t_0}R_2}{(1+R_2)^2}=r.$$

Let $q(r)=r^{-\alpha_0}p(r)$. Suppose $r,R\in(0,1)$ satisfy that $r<\frac{R}{(1+R)^2}$. By choosing $t_0>0$ such that $e^{t_0}=\frac{R/r}{(1+R)^2}$, we conclude from the above formula that $p(r)\le e^{-\alpha_0t_0}p(R)=(\frac{R/r}{(1+R)^2})^{-\alpha_0}p(R)$.

Thus, $q(r) \le (1+R)^{2\alpha_0} q(R)$. Similarly, by choosing $t_0 > 0$ such that $e^{t_0} = \frac{R/r}{(1-R)^2}$, we get $q(r) \ge (1-R)^{2\alpha_0} q(R)$. So we have

$$(1-R)^{2\alpha_0} q(R) \le q(r) \le (1+R)^{2\alpha_0} q(R), \quad \text{if } r < \frac{R}{(1+R)^2}. \tag{6.4}$$

Thus, $\lim_{r\to 0^+} \log(q(r))$ converges to a finite number, which implies that $\lim_{r\to 0^+} q(r)$ converges to a finite positive number. Let L denote the limit. Fixing $R \in (0,1)$ and sending $r \to 0^+$ in (6.4), we get $L(1+R)^{-2\alpha_0} \le q(R) \le L(1-R)^{-2\alpha_0}$. So $p(r) = Lr^{\alpha_0}(1+O(r))$ as $r \to 0$ for some $L \in (0,\infty)$.

Next, we find the asymptotic of $p(z_1,z_2;r)$ as $r \to 0^+$ for any $z_1, z_2 \in (0,\pi)$. From Lemma 5.5 we know that, for any $t_0 > 0$, $\mathbb{P}[T^u > t_0] = \mathcal{Z} G^u(z_1,z_2) e^{-\alpha_0 t_0}(1+O(e^{\lambda_1 t_0}))$, and conditionally on $\mathcal{F}^u_{t_0}$ and $T^u > t_0$, the joint density of $(Z^u_1(t_0), Z^u_2(t_0))$ is $\widetilde{p}^{\,Z}_\infty(z_1^*, z_2^*)(1+O(e^{\lambda_1 t_0}))$, where $\lambda_1 = -2 - \frac{\kappa}{8}$. Fix $r \in (0,1/4)$ and choose $t_0 > 0$ such that $t_0 < -\log(4r)$. We now still have (6.2). Note that $R_j = e^{t_0} r(1+O(e^{t_0} r))$, $j = 1,2$, if $e^{t_0} r$ is small. From (6.3) we get

$$\begin{aligned} p(z_1,z_2;r) &= \mathcal{Z} G^u(z_1,z_2) e^{-\alpha_0 t_0}(1+O(e^{\lambda_1 t_0})) p(e^{t_0} r(1+O(e^{t_0} r))) \\ &= \mathcal{Z} L G^u(z_1,z_2) e^{-\alpha_0 t_0}[e^{t_0} r(1+O(e^{t_0} r))]^{\alpha_0}(1+O(e^{\lambda_1 t_0}))(1+O(e^{t_0} r)) \\ &= \mathcal{Z} L G^u(z_1,z_2) r^{\alpha_0}(1+O(e^{\lambda_1 t_0}) + O(e^{t_0} r)). \end{aligned}$$

Since $\beta_0 = \frac{-\lambda_1}{1-\lambda_1}$, letting $C_0 = \mathcal{Z} L$ and choosing e^{t_0} such that $e^{t_0} = r^{\frac{-1}{1-\lambda_1}}$, we get

$$p(z_1,z_2;r) = C_0 G^u(z_1,z_2) r^{\alpha_0}(1+O(r^{\beta_0})).$$

This means that we obtain (6.1) in the case that $v_1 - v_2 = \pi$.

Finally, we consider the case that $\theta(0,0) = v_1 - v_2 \ne \pi$. First, suppose that $\theta(0,0) < \pi$. Recall that $\theta(t_1,t_2)$ is increasing in t_2. Let τ_2 be the first t_2 such that $(0,t_2) \in \mathcal{D}$ and $\theta(0,t_2) = \pi$, if such time exists; otherwise, let τ_2 be the lifetime $\widetilde{T}_2$ of η_2. Then τ_2 is an $(\mathcal{F}^2_t)$-stopping time. From (4.1) we know that $\partial_2 \theta(0,t_2) \ge 2\cot(\theta(0,t_2)/4)$, which implies that $\cos(\theta(0,t_2)/4) \le e^{-t/2}\cos(\theta(0,0)/4) < e^{-t/2}$. If $\log(2) < \widetilde{T}_2$, then $\cos(\theta(0,\log(2))/4) < 1/\sqrt{2}$, which implies that $\theta(0,\log(2)) > \pi$, and so $\tau_2 < \log(2)$. If $\log(2) \ge \widetilde{T}_2$, we then have $\tau_2 \le \widetilde{T}_2 \le \log(2)$. Thus, in both cases, τ_2 is bounded above by $\log(2)$, and we get an $(\mathcal{F}_{\underline{t}})$-stopping time $(0,\tau_2)$.

Moreover, if $(0,\tau_2) \notin \mathcal{D}$, then $\tau_2 = \widetilde{T}_2$, which means that the conformal radius of $\mathbb{D} \setminus \widehat{\eta}_2$ viewed from 0 is $e^{-\widetilde{T}_2} \ge 1/2$, and from Koebe's 1/4 theorem, we get $\mathrm{dist}(0,\widehat{\eta}_2) \ge 1/8$. Thus, if $\mathrm{dist}(0,\widehat{\eta}_2) < 1/8$, then $(0,\tau_2) \in \mathcal{D}$, and we get $V_1(0,\tau_2) - V_2(0,\tau_2) = \pi$. Conditional on $\mathcal{F}_{(0,\tau_2)}$ and the event that $(0,\tau_2) \in \mathcal{D}$, the $g_{(0,\tau_2)}$-image of $\widehat{\eta}_1$ and the part of $\widehat{\eta}_2$ after $\eta_2(\tau_2)$ form a 2-SLE$_\kappa$ in $\mathbb{D}$ with link pattern $(e^{iW_1(0,\tau_2)} \to e^{iV_1(0,\tau_2)}; e^{iW_2(0,\tau_2)} \to e^{iV_2(0,\tau_2)})$. Since $V_1(0,\tau_2) - V_2(0,\tau_2) = \pi$, by Koebe distortion theorem and the result in the case $v_1 - v_2 = \pi$, we get that, if $r < 1/8$,

$$p(w_1,v_1,w_2,v_2;r) = \mathbb{E}[\mathbf{1}_{\{(0,\tau_2)\in\mathcal{D}\}} p(Z_1(0,\tau_2), Z_2(0,\tau_2); e^{\tau_2} r(1+O(r)))]$$

$$= C_0\mathbb{E}[\mathbf{1}_{\{(0,\tau_2)\in\mathcal{D}\}}e^{\alpha_0\,\mathrm{m}(0,\tau_2)}G(W_1,V_1;W_2,V_2)|_{(0,\tau_2)}]r^{\alpha_0}(1+O(r^{\beta_0}))$$

$$= C_0G(w_1,v_1;w_2,v_2)r^{\alpha_0}(1+O(r^{\beta_0})),$$

where the last step follows from (3.37). The proof of the case that $\theta(0,0)>\pi$ is similar. So we have proved (6.1) in all cases, which implies (1.2).

Finally, from (1.2) we know that there are constants $\rho\in(0,1)$ and $C_1>0$ such that if $\frac{r}{R}<\rho$, then $\mathbb{P}[\mathrm{dist}(z_0,\widehat{\eta}_j)<r, j=1,2]\le C_1G_{D;a_1,b_1;a_2,b_2}(z_0)r^{\alpha_0}$. Using (1.4) we then get (1.3) in the case $\frac{r}{R}<\rho$. Since $\mathbb{P}[\mathrm{dist}(z_0,\widehat{\eta}_j)<r, j=1,2]\le 1$, we get (1.3) for all $r>0$. ☐

Proof of Theorem 1.2. The proof is almost the same as that of the previous theorem except that we need a new way to prove that $\mathbb{P}[\mathrm{dist}(z_0,\widehat{\eta}_1\cap\widehat{\eta}_2)<r]>0$ for all $r\in(0,R)$. To prove this, first note that from the previous theorem, the probability of the event E_r that both $\widehat{\eta}_1$ and $\widehat{\eta}_2$ visit the disc $\{|z-z_0|<r\}$ is positive, and when this event happens, the connected component of $D\backslash\widehat{\eta}_1$ whose boundary contains a_2,b_2, denoted by D_2, contains a part of the circle $\{|z-z_0|=r\}$ but not the whole circle. Thus, $\partial D_2\cap\{|z-z_0|<r\}$ is not empty. Since conditionally on $\widehat{\eta}_1$, $\widehat{\eta}_2$ is a chordal SLE_κ curve in D_2, and $\kappa\in(4,8)$, the conditional probability that $\widehat{\eta}_2$ intersects $\partial D_2\cap\{|z-z_0|<r\}$ given $\widehat{\eta}_1$ and E_r is positive, and when $\widehat{\eta}_2$ intersects $\partial D_2\cap\{|z-z_0|<r\}$, we have $\mathrm{dist}(z_0,\widehat{\eta}_1\cap\widehat{\eta}_2)<r$. So we get the desired positiveness. ☐

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