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Duality of chordal SLE

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Abstract. We derive some geometric properties of chordal SLE$(\kappa;\vec{\rho})$ processes. Using these results and the method of coupling two SLE processes, we prove that the outer boundary of the final hull of a chordal SLE$(\kappa;\vec{\rho})$ process has the same distribution as the image of a chordal SLE$(\kappa';\vec{\rho}')$ trace, where $\kappa > 4$, $\kappa' = 16/\kappa$, and the forces $\vec{\rho}$ and $\vec{\rho}'$ are suitably chosen. We find that for $\kappa \ge 8$, the boundary of a standard chordal SLE(κ) hull stopped on swallowing a fixed $x \in \mathbb{R}\setminus\{0\}$ is the image of some SLE$(16/\kappa;\vec{\rho})$ trace started from x. Then we obtain a new proof of the fact that chordal SLE(κ) trace is not reversible for $\kappa > 8$. We also prove that the reversal of SLE$(4;\vec{\rho})$ trace has the same distribution as the time-change of some SLE$(4;\vec{\rho}')$ trace for certain values of $\vec{\rho}$ and $\vec{\rho}'$.

1 Introduction

The Schramm–Loewner evolution (SLE) has become a fast growing area in probability theory since 1999 [12]. SLE describes some random fractal curve, which is called an SLE trace, that grows in a plane domain. The behavior of the trace depends on a real parameter $\kappa > 0$. We write SLE(κ) to emphasize the parameter κ. If $\kappa \in (0, 4]$, the trace is a simple curve; if $\kappa > 4$, the trace is not simple; and if $\kappa \ge 8$, the trace is space-filling. For basic properties of SLE, see [6] and [11].

Many two-dimensional lattice models from statistical physics have been proved to have SLE as their scaling limits when the mesh of the grid tends to 0, e.g., the convergence of critical percolation on triangular lattice to SLE(6) [16], loop-erased random walk (LERW) to SLE(2) [9,18], uniform spanning tree (UST) Peano curve to SLE(8) [9], Gaussian free field contour line to SLE(4) [13], and some Ising models to SLE(3) and SLE(16/3) [15]. And there are some promising conjectures, e.g., the convergence of self-avoiding walk to SLE(8/3) [8], and double domino tilling to SLE(4) [11].

For $\kappa > 4$, people are also interested in the hulls that are generated by the SLE(κ) traces. Duplantier proposed a rough conjecture about the duality between SLE(κ) and SLE($16/\kappa$), which says that when $\kappa > 4$, the boundary of an SLE(κ) hull looks locally like an SLE($16/\kappa$) trace.

For $\kappa \leq 8$, the Hausdorff dimension of an SLE(κ) trace was proved to be $1 + \kappa/8$ [3]. If the duality conjecture is true, then we may conclude that for $\kappa > 4$, the Hausdorff dimension of the boundary of an SLE(κ) hull is $1 + 2/\kappa$.

For some parameter κ, the duality is already known. The duality between SLE(8) and SLE(2) follows from the convergence of UST and LERW to SLE(8) and SLE(2), respectively, and the Wilson's algorithm [17] that links UST with LERW. The duality between SLE(6) and SLE(8/3) follows from the conformal restriction property [8]. The duality between SLE(16/3) and SLE(3) follows from the convergence of Ising models.

In [4], J. Dubédat proposed some specific conjectures about the duality of SLE, one of which says that for $\kappa > 4$, the right boundary of the final hull of a chordal SLE($\kappa; \kappa - 4$) process started from $(0, 0^+)$ has the same law as a chordal SLE($\kappa'; \frac{1}{2}(\kappa' - 4)$) trace started from $(0, 0^-)$, where $\kappa' = 16/\kappa$. And he justified his conjecture by studying the distributions of the sets obtained by adding Brownian loop soups to SLE($\kappa; \kappa - 4$) and SLE($\kappa'; \frac{1}{2}(\kappa' - 4)$), respectively.

Recently, a new technique about constructing a coupling of two SLE processes that grow in the same domain was introduced in [19] to prove the reversibility of chordal SLE(κ) trace when $\kappa \in (0, 4]$. In this paper, we will use this technique to prove some specific versions of the duality conjecture, which are not exactly the same as those in [4]. For example, one of our results is that for $\kappa > 4$ and $\kappa' = 16/\kappa$, the right boundary of the final hull of a chordal SLE($\kappa; \kappa - 4$) process started from $(0, 0^+)$ has the same law as the image under the map $z \mapsto 1/\overline{z}$ of a chordal SLE($\kappa'; \frac{1}{2}(\kappa' - 4)$) trace started from $(0, 0^-)$. If the degenerate chordal SLE($\kappa'; \frac{1}{2}(\kappa' - 4)$) trace satisfies reversibility, which is Conjecture 1 of this paper, then Dubédat's conjecture is proved.

This paper is organized in the following way. In Sect. 2, we review the definitions of the chordal and strip (i.e., dipolar) Loewner equations and SLE($\kappa; \vec{\rho}$) processes. The conformal invariance of chordal and strip SLE($\kappa; \vec{\rho}$) processes are introduced. In Sect. 3, we study the tail behavior of a chordal or strip SLE($\kappa; \vec{\rho}$) trace when the force points and forces satisfy certain conditions. In Sect. 4, for $\kappa \geq 4 \geq \kappa' > 0$ with $\kappa\kappa' = 16$, some commutation result of a chordal SLE($\kappa; \vec{\rho}$) process with a chordal SLE($\kappa'; \vec{\rho}'$) process is described in terms of a two-dimensional martingale. This is closely related with J. Dubédat's work in [5]. Then the technique in [19] is applied to get a coupling of the above two SLE processes. In Sect. 5, we consider the coupling in the previous section with some special choices of force points and forces, and apply the geometry results from Sect. 3 to prove that in this coupling, the chordal SLE($\kappa'; \vec{\rho}'$) trace becomes the outer boundary of the chordal SLE($\kappa; \vec{\rho}$) hull, and so prove the duality conjecture.

Then we derive the equation of the boundary of a standard chordal SLE(κ) hull, $\kappa \geq 8$, at the time when a fixed $x \in \mathbb{R} \setminus \{0\}$ is swallowed. Then we give a new proof of the fact that for $\kappa > 8$, the chordal SLE(κ) trace is not reversible. This result was claimed in [11]. At the end, we derive the reversibility property of some chordal SLE($4; \vec{\rho}$) traces.

2 Preliminary

2.1 Chordal SLE. If H is a bounded and relatively closed subset of $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and $\mathbb{H} \setminus H$ is simply connected, then we call H a hull in $\mathbb{H}$ w.r.t. ∞. For such H, there is φ_H that maps $\mathbb{H} \setminus H$ conformally onto $\mathbb{H}$, and satisfies $\varphi_H(z) = z + \frac{c}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$, where $c = \operatorname{hcap}(H) \geq 0$ is called the capacity of H in $\mathbb{H}$ w.r.t. ∞. If $H_1 \subset H_2$ are hulls in $\mathbb{H}$ w.r.t. ∞, then $H_2/H_1 := \varphi_{H_1}(H_2 \setminus H_1)$ is also a hull in $\mathbb{H}$ w.r.t. ∞, and $\operatorname{hcap}(H_2/H_1) = \operatorname{hcap}(H_2) - \operatorname{hcap}(H_1)$. If $H_1 \subset H_2 \subset H_3$ are three hulls in $\mathbb{H}$ w.r.t. ∞, then $H_2/H_1 \subset H_3/H_1$ and $(H_3/H_1)/(H_2/H_1) = H_3/H_2$.

Proposition 2.1 *Suppose Ω is an open neighborhood of $x_0 \in \mathbb{R}$ in $\mathbb{H}$. Suppose W maps Ω conformally into $\mathbb{H}$ such that for some $r > 0$, if $z \in \Omega$ approaches $(x_0 - r, x_0 + r)$ then $W(z)$ approaches $\mathbb{R}$. So W extends conformally across $(x_0 - r, x_0 + r)$ by Schwarz reflection principle. Then for any $\varepsilon > 0$, there is some $\delta > 0$ such that if a hull H in $\mathbb{H}$ w.r.t. ∞ is contained in $\{z \in \mathbb{H} : |z - x_0| < \delta\}$, then $W(H)$ is also a hull in $\mathbb{H}$ w.r.t. ∞, and*

$$\left|\operatorname{hcap}(W(H)) - W'(x_0)^2 \operatorname{hcap}(H)\right| \leq \varepsilon |\operatorname{hcap}(H)|.$$

Proof. This is Lemma 2.8 in [7]. □

For a real interval I, we use $C(I)$ to denote the space of real continuous functions on I. For $T > 0$ and $\xi \in C([0, T))$, the chordal Loewner equation driven by ξ is

$$\partial_t \varphi(t, z) = \frac{2}{\varphi(t, z) - \xi(t)}, \quad \varphi(0, z) = z.$$

For $0 \leq t < T$, let $K(t)$ be the set of $z \in \mathbb{H}$ such that the solution $\varphi(s, z)$ blows up before or at time t. We call $K(t)$ and $\varphi(t, \cdot)$, $0 \leq t < T$, chordal Loewner hulls and maps, respectively, driven by ξ.

Definition 2.1 *We call $(K(t), 0 \leq t < T)$ a Loewner chain in $\mathbb{H}$ w.r.t. ∞, if each $K(t)$ is a hull in $\mathbb{H}$ w.r.t. ∞; $K(0) = \emptyset$; $K(s) \subsetneqq K(t)$ if $s < t$; and for each fixed $a \in (0, T)$ and compact $F \subset \mathbb{H} \setminus K(a)$, the extremal length* [1] *of the curves in $\mathbb{H} \setminus K(t + \varepsilon)$ that disconnect $K(t + \varepsilon) \setminus K(t)$ from F tends to 0 as $\varepsilon \to 0^+$, uniformly in $t \in [0, a]$.*

Proposition 2.2 *(a) Suppose $K(t)$ and $\varphi(t, \cdot)$, $0 \leq t < T$, are chordal Loewner hulls and maps, respectively, driven by $\xi \in C([0, T))$. Then*

$(K(t), 0 \le t < T)$ is a Loewner chain in $\mathbb{H}$ w.r.t. ∞, $\varphi_{K(t)} = \varphi(t,\cdot)$, and $\mathrm{hcap}(K(t)) = 2t$ *for any $0 \le t < T$. Moreover, for every $t \in [0, T)$,*

$$\{\xi(t)\} = \bigcap_{\varepsilon \in (0, T-t)} \overline{K(t+\varepsilon)/K(t)}.$$

(b) Let $(L(s), 0 \le s < S)$ be a Loewner chain in $\mathbb{H}$ w.r.t. ∞. Let $v(s) =$ $\mathrm{hcap}(L(s))/2$, $0 \le s < S$. Then v is a continuous (strictly) increasing function with $u(0) = 0$. Let $T = v(S)$ and $K(t) = L(v^{-1}(t))$, $0 \le t < T$. Then $K(t)$, $0 \le t < T$, are chordal Loewner hulls driven by some $\xi \in C([0, T))$.

Proof. This is almost the same as Theorem 2.6 in [7]. □

Let D be a domain and $K \subset D$. Let p_1 and p_2 be two boundary points or prime ends of D. We say that K does not separate p_1 from p_2 in D if there are neighborhoods U_1 and U_2 of p_1 and p_2, respectively, in D such that U_1 and U_2 lie in the same pathwise connected component of $D \setminus K$. In our definition, K may separates some p from itself. Let Q be a set of boundary points or prime ends of D. We say that K does not divide Q in D if for any $p_1, p_2 \in D$, K does not separate p_1 from p_2 in D.

Let $\varphi(t,\cdot)$ and $K(t)$ be as before. Let $x \in \mathbb{R}$. If at time t, $\varphi(t, x)$ does not blow up, then $K(t)$ does not separate x from ∞ in $\mathbb{H}$, and vice versa. In fact, we have a slightly stronger result: if $\varphi(s, x)$ blows up before or at $s = t \in [0, T)$, then $\bigcup_{s<t} K(s)$ also separates x from ∞ in $\mathbb{H}$. This follows from the property of a Loewner chain.

Let $B(t)$, $0 \le t < \infty$, be a (standard linear) Brownian motion. Let $\kappa \ge 0$. Then $K(t)$ and $\varphi(t,\cdot)$, $0 \le t < \infty$, driven by $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < \infty$, are called standard chordal SLE(κ) hulls and maps, respectively. It is known [11,9] that almost surely for any $t \in [0, \infty)$,

$$\beta(t) := \lim_{\mathbb{H} \ni z \to \xi(t)} \varphi(t,\cdot)^{-1}(z) \tag{2.1}$$

exists, and $\beta(t)$, $0 \le t < \infty$, is a continuous curve in $\overline{\mathbb{H}}$. Moreover, if $\kappa \in (0, 4]$ then β is a simple curve, which intersects $\mathbb{R}$ only at the initial point, and for any $t \ge 0$, $K(t) = \beta((0, t])$; if $\kappa > 4$ then β is not simple, and intersects $\mathbb{R}$ at infinitely many points; and in general, $\mathbb{H} \setminus K(t)$ is the unbounded component of $\mathbb{H} \setminus \beta((0, t])$ for any $t \ge 0$. Such β is called a standard chordal SLE(κ) trace.

If $(\xi(t))$ is a semi-martingale, and $d\langle \xi(t)\rangle = \kappa dt$ for some $\kappa > 0$, then from Girsanov theorem and the existence of standard chordal SLE(κ) trace, almost surely for any $t \in [0, T)$, $\beta(t)$ defined by (2.1) exists, and has the same property as a standard chordal SLE(κ) trace (depending on the value of κ) as described in the last paragraph.

Let $\kappa \ge 0$, $\rho_1, \ldots, \rho_N \in \mathbb{R}$, $x \in \mathbb{R}$, and $p_1, \ldots, p_N \in \widehat{\mathbb{R}} \setminus \{x\}$, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a circle. Let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, be the solutions to the SDE:

$$\begin{cases} d\xi(t) = \sqrt{\kappa} dB(t) + \sum_{k=1}^{N} \frac{\rho_k dt}{\xi(t) - p_k(t)} \\ dp_k(t) = \frac{2dt}{p_k(t) - \xi(t)}, \quad 1 \le k \le N, \end{cases} \tag{2.2}$$

with initial values $\xi(0) = x$ and $p_k(0) = p_k$, $1 \le k \le N$. If $\varphi(t, \cdot)$ are chordal Loewner maps driven by $\xi(t)$, then $p_k(t) = \varphi(t, p_k)$. Here if some $p_k = \infty$ then $p_k(t) = \infty$ and $\frac{\rho_k}{\xi(t) - p_k(t)} = 0$ for all $t \ge 0$, so p_k has no effect on the equation. Suppose $[0, T)$ is the maximal interval of the solution. Let $K(t)$, $0 \le t < T$, be chordal Loewner hulls driven by ξ. Then we call $K(t)$, $0 \le t < T$, a (full) chordal SLE$(\kappa; \rho_1, \dots, \rho_N)$ process started from $(x; p_1, \dots, p_N)$. Since $(\xi(t))$ is a semi-martingale, and $d\langle \xi(t) \rangle = \kappa dt$, so the chordal Loewner trace $\beta(t)$, $0 \le t < T$, driven by ξ exists, and is called a chordal SLE$(\kappa; \rho_1, \dots, \rho_N)$ trace started from $(x; p_1, \dots, p_N)$. If we let $\vec{\rho}$ and $\vec{p}$ to denote the vectors $(\rho_1, \dots, \rho_N)$ and $(p_1, \dots, p_N)$, then we may call $K(t)$ and $\beta(t)$, $0 \le t < T$, chordal SLE$(\kappa; \vec{\rho})$ process and trace, respectively, started from $(x; \vec{p})$. If $S \in (0, T]$ is a stopping time, then $K(t)$ and $\beta(t)$, $0 \le t < S$, are called partial chordal SLE$(\kappa; \vec{\rho})$ process and trace, respectively, started from $(x; \vec{p})$.

These p_k's and ρ_k's are called force points and forces, respectively. For $0 \le t < T$ and $1 \le k \le N$, $\varphi(t, p_k)$ does not blow up, so $K(t)$ does not divide $\{\infty, p_1, \dots, p_N\}$ in $\mathbb{H}$. If $T < \infty$ then there must exist some $p_k \in \mathbb{R}$ such that $\varphi(t, p_k) - \xi(t) \to 0$ as $t \to T$, so $\bigcup_{t<T} K(t)$ separates p_k from ∞ in $\mathbb{H}$. If $T = \infty$ then $\bigcup_{t<T} K(t)$ is unbounded, so $\bigcup_{t<T} K(t)$ separates ∞ from itself in $\mathbb{H}$. Thus in any case, $\bigcup_{t<T} K(t)$ divides $\{\infty, p_1, \dots, p_N\}$ in $\mathbb{H}$.

The chordal SLE$(\kappa; \vec{\rho})$ defined above are of generic cases. We now introduce degenerate SLE$(\kappa; \vec{\rho})$, where one of the force points takes value x^+ or x^-, or two of the force points take values x^+ and x^-, respectively, where $x \in \mathbb{R}$ is the initial point of the trace. Let $\kappa \ge 0$; $\rho_1, \dots, \rho_N \in \mathbb{R}$, and $\rho_1 \ge \kappa/2 - 2$; $p_1 = x^+$, $p_2, \dots, p_N \in \widehat{\mathbb{R}} \setminus \{x\}$. Let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, $0 < t < T$, be the maximal solution to (2.2) with initial values $\xi(0) = p_1(0) = x$, and $p_k(0) = p_k$, $1 \le k \le N$. Moreover, we require that $p_1(t) > \xi(t)$ for any $0 < t < T$. If $N = 1$, the existence of the solution follows from the Bessel process (see [8]). The condition $\rho_1 \ge \kappa/2 - 2$ is to guarantee that p_1 is not immediately swallowed after time 0. If $N \ge 2$, the existence of the solution follows from the above result and Girsanov Theorem. Then we obtain chordal SLE$(\kappa; \rho_1, \dots, \rho_N)$ process and trace started from $(x; x^+, p_2, \dots, p_N)$. If the condition $p_1(t) > \xi(t)$ is replaced by $p_1(t) < \xi(t)$, then we get chordal SLE$(\kappa; \rho_1, \dots, \rho_N)$ process and trace started from $(x; x^-, p_2, \dots, p_N)$. Now suppose $N \ge 2$, $\rho_1, \rho_2 \ge \kappa/2 - 2$, $p_1 = x^+$, and $p_2 = x^-$. Let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, $0 < t < T$, be the maximal solution to (2.2) with initial values $\xi(0) = p_1(0) = p_2(0) = x$, and $p_k(0) = p_k$, $1 \le k \le N$, such that $p_1(t) > \xi(t) > p_2(t)$ for all $0 < t < T$. Then we obtain chordal SLE$(\kappa; \rho_1, \dots, \rho_N)$ process and trace started from $(x; x^+, x^-, p_3, \dots, p_N)$. The existence of the solution to the equation follows from [13] and Girsanov theorem.

The force point x^+ or x^- is called a degenerate force point. Other force points are called generic force points. Let $\varphi(t,\cdot)$ be the chordal Loewner maps driven by ξ. Since for any generic force point p_j, we have $p_j(t) = \varphi(t, p_j)$, so it is reasonable to write $\varphi(t, p_j)$ for $p_j(t)$ in the case that p_j is a degenerate force point. Suppose ρ_j is the force associated with some degenerate force point p_j. If we allow that the process continues growing after p_j is swallowed, the condition that $\rho_j \ge \kappa/2 - 2$ may be weakened to $\rho_j > -2$ [8].

From the work in [14], we get the conformal invariance of chordal SLE$(\kappa;\vec{\rho})$ processes, which is the following lemma.

Lemma 2.1 *Suppose $\kappa \ge 0$ and $\vec{\rho} = (\rho_1, \dots, \rho_N)$ with $\sum_{m=1}^N \rho_m = \kappa - 6$. For $j = 1, 2$, let $K_j(t)$, $0 \le t < T_j$, be a generic or degenerate chordal* SLE$(\kappa;\vec{\rho})$ *process started from $(x_j;\vec{p}_j)$, where $\vec{p}_j = (p_{j,1}, \dots, p_{j,N})$, $j = 1, 2$. Suppose W is a conformal or conjugate conformal map from $\mathbb{H}$ onto $\mathbb{H}$ such that $W(x_1) = x_2$ and $W(p_{1,m}) = p_{2,m}$, $1 \le m \le N$. Let $p_{1,\infty} = W^{-1}(\infty)$ and $p_{2,\infty} = W(\infty)$. For $j = 1, 2$, let $S_j \in (0, T_j]$ be the largest number such that for $0 \le t < S_j$, $K_j(t)$ does not separate $p_{j,\infty}$ from ∞ in $\mathbb{H}$. Then $(W(K_1(t)), 0 \le t < S_1)$ has the same law as $(K_2(t), 0 \le t < S_2)$ up to a time-change. A similar result holds for the traces.*

Proof. Here we only consider the generic cases. The proof of the degenerate cases is similar. Let $Q_j = \{\infty, p_{j,1}, \dots, p_{j,N}, p_{j,\infty}\}$, $j = 1, 2$. Then $W(Q_1) = Q_2$, and S_j is the maximum number in $(0, T_j]$ such that for $0 \le t < S_j$, $K_j(t)$ does not divide Q_j in $\mathbb{H}$. For $0 \le t < S_1$, since $K_1(t)$ does not divide Q_1 in $\mathbb{H}$, so $W(K_1(t))$ does not divide Q_2 in $\mathbb{H}$. From Theorem 3 in [14], after a time-change, $(W(K_1(t)), 0 \le t < S_1)$ is a partial chordal SLE$(\kappa;\vec{\rho})$ process started from $(x_2;\vec{p}_2)$. We now suffice to show that this chordal Loewner chain can not be further extended without dividing Q_2 in $\mathbb{H}$. If this is not true, then $\bigcup_{0\le t<S_1} W(K_1(t))$ does not divide Q_2 in $\mathbb{H}$. So $\bigcup_{0\le t<S_1} K_1(t)$ does not divide Q_1 in $\mathbb{H}$, which contradicts the choice of S_1. □

Note that if $\kappa \in (0, 4]$ then $S_j = T_j$, $j = 1, 2$, so we conclude that $(W(K_1(t)), 0 \le t < T_1)$ has the same distribution as $(K_2(t), 0 \le t < T_2)$ up to a time-change. In general, by adding ∞ to be a force point with suitable value of force, we may always make the sum of forces equals to $\kappa - 6$, so the lemma can be applied.

2.2 Strip SLE. Strip SLE is studied independently in [20] and [2] (where it is called dipolar SLE). For $h > 0$, let $\mathbb{S}_h = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < h\}$ and $\mathbb{R}_h = ih + \mathbb{R}$. If H is a bounded closed subset of $\mathbb{S}_\pi$, $\mathbb{S}_\pi \setminus H$ is simply connected, and has $\mathbb{R}_\pi$ as a boundary arc, then we call H a hull in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$. For such H, there is a unique ψ_H that maps $\mathbb{S}_\pi \setminus H$ conformally onto $\mathbb{S}_\pi$, such that for some $c \ge 0$, $\psi_H(z) = z \pm c + o(1)$ as $z \to \pm\infty$ in $\mathbb{S}_\pi$. We call such c the capacity of H in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$, and denote it by scap(H).

For $\xi \in C([0, T))$, the strip Loewner equation driven by ξ is

$$\partial_t \psi(t, z) = \coth\left(\frac{\psi(t, z) - \xi(t)}{2}\right), \quad \psi(0, z) = z. \tag{2.3}$$

For $0 \le t < T$, let $L(t)$ be the set of $z \in \mathbb{S}_\pi$ such that the solution $\psi(s, z)$ blows up before or at time t. We call $L(t)$ and $\psi(t, \cdot)$, $0 \le t < T$, strip Loewner hulls and maps, respectively, driven by ξ. It turns out that $\psi(t, \cdot) = \psi_{L(t)}$ and $\operatorname{scap}(L(t)) = t$ for each t. From now on, we write $\coth_2(z)$, $\tanh_2(z)$, $\cosh_2(z)$, and $\sinh_2(z)$ for functions $\coth(z/2)$, $\tanh(z/2)$, $\cosh(z/2)$, and $\sinh(z/2)$, respectively.

Let $\kappa \ge 0$, $\rho_1, \dots, \rho_N \in \mathbb{R}$, $x \in \mathbb{R}$, and $p_1, \dots, p_N \in \mathbb{R} \cup \mathbb{R}_\pi \cup \{+\infty, -\infty\} \setminus \{x\}$. Let $B(t)$ be a Brownian motion. Let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, be the solutions to the SDE:

$$\begin{cases} d\xi(t) = \sqrt{\kappa} dB(t) + \sum_{k=1}^{N} \frac{\rho_k}{2} \coth_2(\xi(t) - p_k(t)) dt \\ dp_k(t) = \coth_2(p_k(t) - \xi(t)) dt, \quad 1 \le k \le N, \end{cases} \tag{2.4}$$

with initial values $\xi(0) = x$ and $p_k(0) = p_k$, $1 \le k \le N$. Here if some $p_k = \pm\infty$ then $p_k(t) = \pm\infty$ and $\coth_2(\xi(t) - p_k(t)) = \mp 1$ for all $t \ge 0$, so p_k has a constant effect on the equation. Suppose $[0, T)$ is the maximal interval of the solution. Let $L(t)$, $0 \le t < T$, be strip Loewner hulls driven by ξ. Then we call $L(t)$, $0 \le t < T$, a (full) strip $\mathrm{SLE}(\kappa; \vec{\rho})$ process started from $(x; \vec{p})$, where $\vec{\rho} = (\rho_1, \dots, \rho_N)$ and $\vec{p} = (p_1, \dots, p_N)$.

The following two lemmas show that strip $\mathrm{SLE}(\kappa; \vec{\rho})$ processes also satisfy conformal invariance, and are conformally equivalent to the corresponding chordal $\mathrm{SLE}(\kappa; \vec{\rho})$ processes. The proofs are similar to that of Lemma 2.1, and use the result of Sect. 4 in [14], so we omit the proofs.

Lemma 2.2 *Suppose $\kappa \ge 0$ and $\vec{\rho} = (\rho_1, \dots, \rho_N)$ with $\sum_{m=1}^{N} \rho_m = \kappa - 6$. For $j = 1, 2$, let $L_j(t)$, $0 \le t < T_j$, be a strip $\mathrm{SLE}(\kappa; \vec{\rho})$ process started from $(x_j; \vec{p}_j)$, where $\vec{p}_j = (p_{j,1}, \dots, p_{j,N})$. Suppose W is a conformal or conjugate conformal map from $\mathbb{S}_\pi$ onto $\mathbb{S}_\pi$ such that $W(x_1) = x_2$ and $W(p_{1,m}) = p_{2,m}$, $1 \le m \le N$. Let $I_1 = W^{-1}(\mathbb{R}_\pi)$ and $I_2 = W(\mathbb{R}_\pi)$. For $j = 1, 2$, let $S_j \in (0, T_j]$ be the largest number such that for $0 \le t < S_j$, $L_j(t)$ does not separate I_j from $\mathbb{R}_\pi$ in $\mathbb{S}_\pi$. Then $(W(L_1(t)), 0 \le t < S_1)$ has the same law as $(L_2(t), 0 \le t < S_2)$ up to a time-change.*

Lemma 2.3 *Suppose $\kappa \ge 0$ and $\vec{\rho} = (\rho_1, \dots, \rho_N)$ with $\sum_{m=1}^{N} \rho_m = \kappa - 6$. Let $K(t)$, $0 \le t < T$, be a chordal $\mathrm{SLE}(\kappa; \vec{\rho})$ process started from $(x; \vec{p})$, where $\vec{p} = (p_1, \dots, p_N)$. Let $L(t)$, $0 \le t < S$, be a strip $\mathrm{SLE}(\kappa; \vec{\rho})$ process started from $(y; \vec{q})$, where $\vec{q} = (q_1, \dots, q_N)$. Suppose W is a conformal or conjugate conformal map from $\mathbb{H}$ onto $\mathbb{S}_\pi$ such that $W(x) = y$ and $W(p_k) = q_k$, $1 \le k \le N$. Let $I = W^{-1}(\mathbb{R}_\pi)$ and $q_\infty = W(\infty)$. Let $T' \in (0, T]$ be the largest number such that for $0 \le t < T'$, $K(t)$ does not separate I from ∞ in $\mathbb{H}$. Let $S' \in (0, S]$ be the largest number such that for*

$0 \le t < S'$, $L(t)$ *does not separate* q_∞ *from* $\mathbb{R}_\pi$. *Then* $(W(K(t)), 0 \le t < T')$ *has the same law as* $(L(t), 0 \le t < S')$ *up to a time-change.*

As usual, if $\kappa \in [0, 4]$, then $S_j = T_j$, $j = 1, 2$, in Lemma 2.2, and $T' = T$ and $S' = S$ in Lemma 2.3. In general, for a strip SLE$(\kappa; \vec{\rho})$ process, by adding $+\infty$ and $-\infty$ to be force points with suitable values of forces, we may always make the sum of forces equals to $\kappa - 6$, so the above two lemmas can be applied. From Lemma 2.3, we have the existence of the strip SLE$(\kappa; \vec{\rho})$ trace, and the above two lemmas also hold for traces.

3 Geometric properties

Suppose $\beta(t)$, $0 \le t < T$, is a chordal SLE$(\kappa; \vec{\rho})$ trace. In this section, we will study the existence and property of the limit or subsequential limit of $\beta(t)$ as $t \to T$ in certain cases. The three lemmas in the last section will be frequently used.

3.1 Many force points. Let $\kappa > 0$, $\vec{p}_\pm = (p_{\pm 1}, \ldots, p_{\pm N_\pm})$, $\vec{\rho}_\pm = (\rho_{\pm 1}, \ldots, \rho_{\pm N_\pm})$, where $0 < p_1 < \cdots < p_{N_+}$, $0 > p_{-1} > \cdots > p_{-N-}$, and $\rho_{\pm j} \in \mathbb{R}$, $j = 1, \ldots, N_\pm$. Let $\beta(t)$, $0 \le t < T$, be a chordal SLE$(\kappa; \vec{\rho}_+, \vec{\rho}_-)$ trace started from $(0; \vec{p}_+, \vec{p}_-)$. Let $\varphi(t, \cdot)$ and $\xi(t)$, $0 \le t < T$, be the chordal Loewner maps and driving function, respectively, for the trace γ.

Theorem 3.1 *Suppose for any* $1 \le k \le N_\pm$, $\sum_{j=1}^{k} \rho_{\pm j} \ge \kappa/2 - 2$.

(i) Almost surely $T = \infty$, *so* ∞ *is a subsequential limit of* $\beta(t)$ *as* $t \to T$.
(ii) If in addition, $\kappa \in (0, 4]$, *then almost surely* $\overline{\beta((0, \infty))} \cap (\mathbb{R} \setminus \{0\}) = \emptyset$.

Proof. Let $\rho_\infty = \kappa - 6 - \sum_{j=1}^{N_+} \rho_j - \sum_{j=1}^{N_-} \rho_{-j}$. Let $\chi_0 = 0$. For $1 \le k \le N_\pm$, let $\chi_{\pm k} = \sum_{j=1}^{k} \rho_{\pm j} \ge \kappa/2 - 2$. Let $\chi^\pm_{\max} = \max\{\chi_{\pm k} : 1 \le k \le N_\pm\}$.

(i) If $T = \infty$, then the diameter of $\beta((0, t])$ tends to ∞ as $t \to \infty$, so ∞ is a subsequential limit of $\beta(t)$ as $t \to T$. So we suffice to prove that $T = \infty$ a.s. If $T < \infty$, then for $x = p_1$ or p_{-1}, $\varphi(t, x) - \xi(t) \to 0$ as $t \to T$, where $\xi(t)$ and $\varphi(t, \cdot)$ are the driving function and chordal Loewner map. For any $t \in [0, T)$, $\varphi(t, p_{-1}) < \xi(t) < \varphi(t, p_1)$, so $\partial_t \varphi(t, p_1) = 2/(\varphi(t, p_1) - \xi(t)) > 0$ and $\partial_t \varphi(t, p_{-1}) = 2/(\varphi(t, p_{-1}) - \xi(t)) < 0$. Thus $\varphi(t, p_1) - \varphi(t, p_{-1})$ increases. If $\varphi(t, p_1) - \xi(t) \to 0$, then $(\varphi(t, p_1) - \xi(t))/(\varphi(t, p_1) - \varphi(t, p_{-1})) \to 0$, so $(\varphi(t, p_1) - \xi(t))/(\xi(t) - \varphi(t, p_{-1})) \to 0$. Similarly, if $\xi(t) - \varphi(t, p_{-1}) \to 0$, then $(\xi(t) - \varphi(t, p_{-1}))/(\varphi(t, p_1) - \xi(t)) \to 0$. Thus if $T < \infty$, then $\ln(\xi(t) - \varphi(t, p_{-1})) - \ln(\varphi(t, p_1) - \xi(t))$ tends to $+\infty$ or $-\infty$ as $t \to T$.

Suppose W maps $\mathbb{H}$ conformally onto $\mathbb{S}_\pi$ such that $W(0) = 0$ and $W(p_{\pm 1}) = \pm\infty$. Let $q_\infty = W(\infty)$ and $q_{\pm j} = W(p_{\pm j})$, $1 \le j \le N_\pm$. Let $\gamma(t) = \beta(u^{-1}(t))$ for $0 \le t < S = u(T)$, where u is a continuous increasing function on $[0, T)$ such that $\mathrm{scap}(L(t)) = t$ for any t, and $L(t)$ is the hull in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$ generated by $\gamma((0,t])$. From Lemma 2.3, $\gamma(t)$, $0 \le t < S$, is a strip SLE$(\kappa; \rho_\infty, \vec{\rho}_+, \vec{\rho}_-)$ trace started from $(0; q_\infty, \vec{q}_+, \vec{q}_-)$, where $\vec{q}_\pm = (q_{\pm 1}, \dots, q_{\pm N_\pm})$. Since all q_j's are either $\pm\infty$ or lie on $\mathbb{R}_\pi$, which will never be swallowed, so $S = \infty$.

Let $\psi(t,\cdot)$ and $\eta(t)$, $0 \le t < \infty$, be the strip Loewner maps and driving function, respectively, for the trace γ. Let $X_\infty(t) = \operatorname{Re}\psi(t, q_\infty) - \eta(t)$ and $X_{\pm j}(t) = \operatorname{Re}\psi(t, q_{\pm j}) - \eta(t)$, $1 \le j \le N_\pm$. Then $X_{-2}(t) < \cdots < X_{-N_-}(t) < X_\infty(t) < X_{N_+}(t) < \cdots < X_2(t)$. And for some Brownian motion $B(t)$, $\eta(t)$ satisfies the SDE:

$$\begin{aligned} d\eta(t) = \sqrt{\kappa}dB(t) - \frac{\rho_\infty}{2}\tanh_2(X_\infty(t))dt \\ - \sum_{j=1}^{N_+}\frac{\rho_j}{2}\tanh_2(X_j(t))dt - \sum_{j=1}^{N_-}\frac{\rho_{-j}}{2}\tanh_2(X_{-j}(t))dt. \end{aligned}$$

For $0 \le t < T$, let $W_t = \psi(u(t),\cdot) \circ W \circ \varphi(t,\cdot)^{-1}$. Then W_t maps $\mathbb{H}$ conformally onto $\mathbb{S}_\pi$, $W_t(\xi(t)) = \eta(u(t))$, $W_t(\infty) = \psi(u(t), q_\infty)$, and $W_t(\varphi(t, p_{\pm 1})) = \pm\infty$. Thus

$$\begin{aligned} \ln(\xi(t) - \varphi(t, p_{-1})) - \ln(\varphi(t, p_1) - \xi(t)) &= \operatorname{Re}\psi(u(t), q_\infty) - \eta(u(t)) \\ &= X_\infty(u(t)). \end{aligned}$$

Thus if $T < \infty$ then $X_\infty(t)$ tends to $+\infty$ or $-\infty$ as $t \to \infty$. So now we suffice to show that a.s. $\limsup_{t\to\infty} X_\infty(t) = +\infty$ and $\liminf_{t\to\infty} X_\infty(t) = -\infty$. We will prove that a.s. $\limsup_{t\to\infty} X_\infty(t) = +\infty$. The other statement follows from symmetry.

Let $X_{N_+ +1}(t) = X_{-N_- -1}(t) = X_\infty(t)$. Then $X_\infty(t)$ satisfies the SDE:

$$\begin{aligned} dX_\infty(t) &= -\sqrt{\kappa}dB(t) + \left(\frac{\kappa}{2} - 2 - \frac{\chi_{N_+} + \chi_{-N_-}}{2}\right)\tanh_2(X_\infty(t))dt \\ &\quad + \sum_{j=1}^{N_+}\frac{\chi_j - \chi_{j-1}}{2}\tanh_2(X_j(t))dt \\ &\quad + \sum_{j=1}^{N_-}\frac{\chi_{-j} - \chi_{-j+1}}{2}\tanh_2(X_{-j}(t))dt \\ &= -\sqrt{\kappa}dB(t) + \left(\frac{\kappa}{2} - 2\right)\tanh_2(X_\infty(t))dt \\ &\quad + \sum_{j=1}^{N_+}\frac{\chi_j}{2}(\tanh_2(X_j(t)) - \tanh_2(X_{j+1}(t)))dt \\ &\quad + \sum_{j=1}^{N_-}\frac{\chi_{-j}}{2}(\tanh_2(X_{-j}(t)) - \tanh_2(X_{-j-1}(t)))dt. \end{aligned}$$

Note that for $1 \le j \le N_\pm$, $\pm(\tanh_2(X_{\pm j}(t)) - \tanh_2(X_{\pm j\pm 1}(t))) > 0$. Since $\kappa/2 - 2 \le \chi_j \le \chi^\pm_{\max}$ for $1 \le j \le N_\pm$, so for some adapted process $A(t) \ge 0$,

$$\begin{aligned} dX_\infty(t) &= -\sqrt{\kappa} dB(t) + A(t)dt + \left(\frac{\kappa}{2} - 2\right)\tanh_2(X_\infty(t))dt \\ &\quad + \left(\frac{\kappa}{4} - 1\right)\sum_{j=1}^{N_+}(\tanh_2(X_j(t)) - \tanh_2(X_{j+1}(t)))dt \\ &\quad + \frac{\chi^-_{\max}}{2}\sum_{j=1}^{N_-}(\tanh_2(X_{-j}(t)) - \tanh_2(X_{-j-1}(t)))dt \\ &= -\sqrt{\kappa} dB(t) + A(t)dt + \left(\frac{\kappa}{4} - 1 - \frac{\chi^-_{\max}}{2}\right)(1 + \tanh_2(X_\infty(t)))dt. \end{aligned}$$

Note that $\tanh_2(X_{\pm 1}(t)) = \pm 1$. Define f on $\mathbb{R}$ such that for any $x \in \mathbb{R}$, $f'(x) = (e^x + 1)^{\frac{2}{\kappa}(\chi^-_{\max} + 2 - \kappa/2)}$. Since $\chi^-_{\max} \ge \kappa/2 - 2$, so $f'(x) \ge 1$ for any $x \in \mathbb{R}$. Thus f maps $\mathbb{R}$ onto $\mathbb{R}$. Let $Y(t) = f(X_\infty(t))$, and $\widetilde{A}(t) = f'(X_\infty(t))A(t) \ge 0$. From Ito's formula, we have

$$dY(t) = -\sqrt{\kappa} f'(X_\infty(t))dB(t) + \widetilde{A}(t)dt.$$

Let $M(t) = Y(t) - \int_0^t \widetilde{A}(s)ds$. Then we have $Y(t) \ge M(t)$ and $dM(t) = -\sqrt{\kappa} f'(X_\infty(t))dB(t)$. Let $v(t) = \int_0^t \kappa f'(X_\infty(s))^2 ds$. Then v is a continuous increasing function on $[0, \infty)$, and maps $[0, \infty)$ onto $[0, \infty)$. And $M(v^{-1}(t))$, $0 \le t < \infty$, is a Brownian motion. Thus a.s. $\limsup_{t\to\infty} M(t) = +\infty$. Since $Y(t) \ge M(t)$ for any t, so a.s. $\limsup_{t\to\infty} Y(t) = +\infty$. Since $X_\infty(t) = f^{-1}(Y(t))$, so a.s. $\limsup_{t\to\infty} X_\infty(t) = +\infty$, as desired.

(ii) From symmetry, we suffice to show that a.s. $\overline{\beta((0, \infty))} \cap (-\infty, 0) = \emptyset$. Fix any $r_+ \in (-\infty, p_{-N_-}) \cap \mathbb{Q}$ and $r_- \in (p_{-1}, 0) \cap \mathbb{Q}$. We suffice to show that a.s. $\overline{\beta((0, \infty))} \cap (r_+, r_-) = \emptyset$. Choose W that maps $\mathbb{H}$ conformally onto $\mathbb{S}_\pi$ such that $W(0) = 0$ and $W(r_\pm) = \pm\infty$. Let $q_{\pm j} = W(p_{\pm j})$, $1 \le j \le N_\pm$, and $\vec{q}_\pm = (q_{\pm 1}, \dots, q_{\pm N_\pm})$. Let $q_\infty = W(\infty) \in (0, \infty)$. Let $\gamma(t) = \beta(u^{-1}(t))$ for $0 \le t < S = u(T)$, where u is a continuous increasing function on $[0, T)$ such that $\mathrm{scap}(\gamma((0, t])) = t$ for any $t \in [0, S)$. From Lemma 2.3, $\gamma(t)$, $0 \le t < S$, is a strip SLE$(\kappa; \rho_\infty, \vec{\rho}_+, \vec{\rho}_-)$ trace started from $(0; q_\infty, \vec{q}_+, \vec{q}_-)$.

Let $\psi(t, \cdot)$ and $\eta(t)$, $0 \le t < S$, be the strip Loewner maps and driving function, respectively, for the trace γ. Let $X_\infty(t) = \psi(t, q_\infty) - \eta(t)$, $q_{N_++1} = q_{-N_--1} = q_\infty$, and $X_{\pm j}(t) = \psi(t, q_{\pm j}) - \eta(t)$, $1 \le j \le N_\pm + 1$. Then there is a Brownian motion $B(t)$ such that $X_\infty(t)$ satisfies:

$$\begin{aligned} dX_\infty(t) &= -\sqrt{\kappa} dB(t) + \left(1 + \frac{\rho_\infty}{2}\right)\coth_2(X_\infty(t))dt \\ &\quad + \sum_{j=1}^{N_+}\frac{\rho_j}{2}\coth_2(X_j(t))dt + \sum_{j=1}^{N_-}\frac{\rho_{-j}}{2}\coth_2(X_{-j}(t))dt \end{aligned}$$

$$= -\sqrt{\kappa}dB(t) + \left(\frac{\kappa}{2} - 2\right)\coth_2(X_\infty(t))dt$$
$$+ \sum_{j=1}^{N_+} \frac{\chi_j}{2}(\coth_2(X_j(t)) - \coth_2(X_{j+1}(t)))dt$$
$$+ \sum_{j=1}^{N_-} \frac{\chi_{-j}}{2}(\coth_2(X_{-j}(t)) - \coth_2(X_{-j-1}(t)))dt.$$

Since $X_j(t)$, $1 \le j \le N_+ + 1$, lie on the boundary of $\mathbb{S}_\pi$ in the counterclockwise direction; and $X_{-j}(t)$, $1 \le j \le N_- + 1$, lie on the boundary of $\mathbb{S}_\pi$ in the clockwise direction, so we have $\pm(\coth_2(X_{\pm j}(t)) - \coth_2(X_{\pm(j+1)})) > 0$ for $1 \le j \le N_\pm$. Since $\chi_{-j} \ge \kappa/2 - 2$, $1 \le j \le N_-$, and $\chi_j \le \chi^+_{\max}$, $1 \le j \le N_+$, so for some adapted process $A_1(t) \ge 0$,

$$dX_\infty(t) = -\sqrt{\kappa}dB(t) - A_1(t)dt + \left(\frac{\kappa}{2} - 2\right)\coth_2(X_\infty(t))dt$$
$$+ \left(\frac{\kappa}{4} - 1\right)(\coth_2(X_{-1}(t)) - \coth_2(X_{-N_- -1}(t)))dt$$
$$+ \frac{\chi^+_{\max}}{2}(\coth_2(X_1(t)) - \coth_2(X_{N_+ +1}(t)))dt$$
$$= -\sqrt{\kappa}dB(t) - A_1(t)dt$$
$$+ \left(\frac{\kappa}{4} - 1\right)(\coth_2(X_{-1}(t)) + \coth_2(X_\infty(t)))dt$$
$$+ \frac{\chi^+_{\max}}{2}(\coth_2(X_1(t)) - \coth_2(X_\infty(t)))dt.$$

We have $\coth_2(X_{-1}(t)) + \coth_2(X_\infty(t)) > 0$ because $X_{-1}(t) \in \mathbb{R}_\pi$ and $X_\infty(t) \in (0, \infty)$. Since $\kappa \in (0, 4]$, so $\kappa/4 - 1 \le 0$. Thus for some adapted process $A_2(t) \ge A_1(t) \ge 0$,

$$dX_\infty(t) = -\sqrt{\kappa}dB(t) - A_2(t)dt + \frac{\chi^+_{\max}}{2}(\coth_2(X_1(t)) - \coth_2(X_\infty(t)))dt.$$

For $0 \le t < S$, since $X_\infty(t) > 0$, so

$$\sqrt{\kappa}B(t) \le \frac{\chi^+_{\max}}{2}\int_0^t (\coth_2(X_1(s)) - \coth_2(X_\infty(s)))ds.$$

Since $0 < X_1(s) < X_\infty(s)$ for $0 \le s < S$, so the integrand is positive. Thus if $\chi^+_{\max} \le 0$, then $B(t) \le 0$ for $0 \le t < S$. Now suppose $\chi^+_{\max} > 0$. Let $q_1(t) = \psi(t, q_1)$ and $q_\infty(t) = \psi(t, q_\infty)$. From the strip Loewner equation, for $0 \le t < S$,

$$\sqrt{\kappa}B(t) \le \frac{\chi^+_{\max}}{2}(q_1(s) - q_\infty(s))\Big|_{s=0}^{s=t}$$
$$\le -\frac{\chi^+_{\max}}{2}(q_1(s) - q_\infty(s))\Big|_{s=0} = \frac{\chi^+_{\max}}{2}(q_\infty - q_1),$$

where the second "$\le$" follows from the fact that $q_1(t) < q_\infty(t)$. Thus in any case, $B(t)$ is uniformly bounded above on $[0, S)$. So we have $S < \infty$ a.s.

For a hull H in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$, if $\mathrm{scap}(H) = s$ then the height of H is no more than $2\cos^{-1}(e^{-s/2})$, and the equality is attained when H is some vertical line segment. Now for $0 \le t < S$, $\mathrm{scap}(\gamma((0,t])) = t < S$, so the distance between $\gamma((0,t])$ and $\mathbb{R}_\pi$ is bigger than $\pi - 2\cos^{-1}(e^{-S/2})$. Since a.s. $S < \infty$, so $\gamma((0,S))$ is bounded away from $\mathbb{R}_\pi$. From the property of W and the definition of γ, we conclude that a.s. $\beta((0,\infty))$ is bounded away from (r_+, r_-). So we are done. □

Theorem 3.2 *Suppose $x \in \mathbb{R}$, $\kappa \in (0,4]$, $\rho_1, \rho_2 \ge \kappa/2 - 2$, and $\beta(t)$, $0 \le t < \infty$, is a chordal* $\mathrm{SLE}(\kappa; \rho_1, \rho_2)$ *trace started from* $(x; p_1, p_2)$.

(i) If $p_1 = x^-$ and $p_2 = x^+$, then a.s. $\lim_{t\to\infty} \beta(t) = \infty$.
(ii) If $p_1 \in (-\infty, x)$ and $p_2 \in (x, +\infty)$, then a.s. $\lim_{t\to\infty} \beta(t) = \infty$.

Proof. We may suppose $x = 0$. We first consider the case that $p_1 = x^- = 0^-$ and $p_2 = x^+ = 0^+$. Let Z denote the set of subsequential limits in $\overline{\mathbb{H}}$ of $\beta(t)$ as $t \to \infty$. We suffice to show that $Z = \emptyset$ a.s. From Lemma 2.1, for any $a > 0$, $a^2\beta(t)$, $0 \le t < \infty$, has the same distribution as $\beta(at)$, $0 \le t < \infty$, which implies that $a^2 Z$ has the same distribution as Z. Thus we suffice to show that a.s. $0 \notin Z$.

Let $\varphi(t,\cdot)$ and $\xi(t)$ be the chordal Loewner maps and driving function for the trace β. Choose W_t that maps $(\mathbb{H}; \xi(t), \varphi(t,0^+), \varphi(t,0^-))$ conformally onto $(\mathbb{S}_\pi; 0, +\infty, -\infty)$, and let $X_\infty(t) = \mathrm{Re}\, W_t(\infty)$. Then $X_\infty(t) = \ln(\varphi(t,0^+) - \xi(t)) - \ln(\xi(t) - \varphi(t,0^-))$. From the proof of Theorem 3.1 (i), we see that a.s. $\limsup X_\infty(t) = +\infty$ and $\liminf X_\infty(t) = -\infty$. Thus a.s. there is $t \ge 1$ such that $X_\infty(t) = 0$, i.e., $\varphi(t,0^+) - \xi(t) = \xi(t) - \varphi(t,0^-)$. Let T denote the first t with this property. So T is a finite stopping time.

Let $g(z) = (\varphi(T,z) - \xi(T))/(\varphi(T,0^+) - \xi(T))$ and $f = g^{-1}$. Then g maps $\mathbb{H} \setminus \beta((0,T])$ conformally onto $\mathbb{H}$, $g(\beta(T)) = 0$; and f extends continuously to $\mathbb{H} \cup \mathbb{R}$ such that $f^{-1}(0) = \{-1, 1\}$. Let $\gamma(t) = g(\beta(T+t))$, $t \ge 0$. Then after a time-change, $\gamma(t)$, $0 \le t < \infty$, has the same distribution as a chordal $\mathrm{SLE}(\kappa; \rho_1, \rho_2)$ trace started from $(0; -1, 1)$. From Theorem 3.1 (ii), $\gamma((0,\infty))$ is bounded away from $\{-1, 1\}$ a.s. Thus a.s. $\beta([T,\infty))$ is bounded away from 0, which implies that $0 \notin Z$. So we proved (i).

(ii) Suppose $p_1 \in (-\infty, x)$ and $p_2 \in (x, \infty)$. Let $r = (p_2 - x)/(x - p_1)$. Let $\beta_0(t)$ be a chordal $\mathrm{SLE}(\kappa; \rho_1, \rho_2)$ trace started from $(0; 0^-, 0^+)$. Let $\varphi(t,\cdot)$ and $\xi(t)$, $0 \le t < \infty$, be the chordal Loewner maps and driving function for the trace β_0. Let $X_\infty(t)$ be defined as in the last paragraph with β replaced by β_0. Then there is a.s. $t \ge 1$ such that $X_\infty(t) = \ln(r)$, i.e., $(\varphi(t,0^+) - \xi(t))/(\xi(t) - \varphi(t,0^-)) = r$. Let T_r denote this time. Since $(X_\infty(t))$ is recurrent, T is a finite stopping time. Let

$$g(z) = x + \frac{(p_2 - p_1)(\varphi(T_r, z) - \xi(T_r))}{\varphi(T_r, 0^+) - \varphi(T_r, 0^-)}.$$

Then we see that g maps $(\mathbb{H}\setminus\beta_0((0,T_r]);\beta_0(T_r),0^-,0^+)$ conformally onto $(\mathbb{H};x,p_1,p_2)$. So after a time-change, $(g(\beta_0(T_r+t)),0\le t<\infty)$, has the same distribution as $(\beta(t),0\le t<\infty)$. From (i), a.s. $\lim_{t\to\infty}\beta_0(t)=\infty$, so we have a.s. $\lim_{t\to\infty}\beta(t)=\infty$. □

Conjecture 1 (Reversibility) Suppose $\kappa\in(0,4)$, $\rho_-,\rho_+\ge\kappa/2-2$, and $\beta(t),0\le t<\infty$, is a chordal SLE$(\kappa;\rho_-,\rho_+)$ trace started from $(0;0^-,0^+)$. Let $W(z)=1/\overline{z}$. Then after a time-change, the reversal of $(W(\beta(t)))$ has the same distribution as $(\beta(t))$.

If $\kappa=0$, the conjecture is trivial because the trace is a half line. If $\rho_+=\rho_-=0$, i.e., β is a standard chordal SLE(κ) trace, the reversibility is known in [19]. If $\kappa=4$, the reversibility is a result of the convergence of discrete Gaussian free field contour line in [13]; and is also a special case of Theorem 5.5 in this paper. To prove this conjecture using the technique in [19] and this paper, one may need to know the conditional distribution of $\beta(t)$, $T_1\le t<T_2$, given its initial segment $\beta([0,T_1])$ and final segment $\beta([T_2,\infty))$, where T_1 is a stopping time, T_2 is a "backward" stopping time, and $T_1<T_2$. In the case that β is a standard chordal SLE(κ) trace, we find that $\beta(t),T_1\le t<T_2$, is a chordal SLE(κ) trace in $\mathbb{H}\setminus(\beta((0,T_1]\cup[T_2,\infty)))$ from $\beta_1(T_1)$ to $\beta_2(T_2)$, up to a time-change. If $\kappa=4$, we will see in the proof of Theorem 5.5 that after a time-change, $\beta(t)$, $T_1\le t<T_2$, is a generic SLE$(\kappa;\rho_-,\rho_+)$ trace in $\mathbb{H}\setminus(\beta((0,T_1]\cup[T_2,\infty)))$. In general, this conditional distribution may not be an SLE$(\kappa;\vec{\rho})$ trace.

3.2 Two force points. We now study a strip SLE process with two force points at ∞ and $-\infty$. Let $\kappa>0$ and $\rho_+,\rho_-\in\mathbb{R}$. Suppose $\beta(t),0\le t<T$, is a strip SLE$(\kappa;\rho_+,\rho_-)$ trace started from $(0;+\infty,-\infty)$. Let $\sigma=(\rho_--\rho_+)/2$. Then $T=\infty$ and the driving function is $\xi(t)=\sqrt{\kappa}B(t)+\sigma t$, $0\le t<\infty$, for some Brownian motion $B(t)$. Let $L(t)$ and $\psi(t,\cdot)$, $0\le t<\infty$, be the strip Loewner hulls and maps, respectively, driven by ξ.

We first consider the case that $|\sigma|<1$. Then $\xi(t)$ satisfies

$$|\xi(t)|\le A(\omega)+\sigma' t,\quad \forall t\ge 0, \tag{3.1}$$

where $\sigma':=(1+|\sigma|)/2<1$ and $A(\omega)>0$ is a random number.

Lemma 3.1 *If $|\sigma|<1$, then $L(\infty)$ is bounded.*

Proof. Let $\sigma''=(1+|\sigma'|)/2$. We may choose $R>0$ such that $\operatorname{Re}\coth_2(z)>\sigma''$ when $z\in\mathbb{S}_\pi$ and $\operatorname{Re}z\ge R$. From (3.1) there is $a=a(\omega)\ge R+1$ such that $R+1+\xi(t)-\sigma''t\le a$ for all $t\ge0$. Consider a point $z\in\mathbb{S}_\pi$ with $\operatorname{Re}z\ge a$. Suppose there is t such that $\operatorname{Re}\psi(t,z)-\xi(t)<R$. Since $\psi(0,z)=z$, so $\operatorname{Re}\psi(0,z)=\operatorname{Re}z\ge a>R$. Since $\xi(0)=0$, so $\operatorname{Re}\psi(0,z)-\xi(0)\ge a>R$. Thus there is a first t_0 such that $\operatorname{Re}\psi(t_0,z)-\xi(t_0)=R$. For $t\in[0,t_0]$, we have $\operatorname{Re}\psi(t,z)-\xi(t)\ge R$, and so

$$\partial_t\operatorname{Re}\psi(t,z)=\operatorname{Re}\coth_2(\psi(t_0,z)-\xi(t_0))\ge\sigma''.$$

Integrating the above inequality w.r.t. t from 0 to t_0, we get

$$R = \mathrm{Re}\,\psi(t_0, z) - \xi(t_0) \geq \mathrm{Re}\,\psi(0, z) + \sigma'' t_0 - \xi(t_0)$$
$$\geq a + \sigma'' t_0 - \xi(t_0) \geq R + 1,$$

where the last inequality uses the property of a. So we get a contradiction. Therefore $\mathrm{Re}\,\psi(t, z) - \xi(t) \geq R$ for all $t \geq 0$. So $\psi(t, z)$ will never blow up, which means that $z \notin L(t)$ for all $t \geq 0$, and so $z \notin L(\infty)$. Similarly, there is $a' = a'(\omega) > 0$ such that if $z \in \mathbb{S}_\pi$ and $\mathrm{Re}\, z \leq -a'$ then $z \notin L(\infty)$. Thus $L(\infty)$ is contained in $\{x + iy : -a' < x < a, 0 < y < \pi\}$, and so is bounded. □

Let

$$f_{\kappa,\sigma}(x) = \int_{-\infty}^{x} \exp(s/2)^{\frac{4\sigma}{\kappa}} \cosh_2(s)^{-\frac{4}{\kappa}} ds. \tag{3.2}$$

Since $|\sigma| < 1$, so $f_{\kappa,\sigma}$ maps $\mathbb{R}$ onto the interval $(0, A_{\kappa,\sigma})$ for some $A_{\kappa,\sigma} < \infty$.

Let

$$X_t(z) = \mathrm{Re}\,\psi(t, z) - \xi(t).$$

Now fix $z_0 = x_0 + \pi i \in \mathbb{R}_\pi$. Then $\psi(t, z_0) \in \mathbb{R}_\pi$ for all t. Let X_t denote $X_t(z_0)$ temporarily. Then $dX_t = \tanh_2(X_t)dt - d\xi(t)$. From Ito's formula, we have

$$df_{\kappa,\sigma}(X_t) = -\exp(X_t/2)^{\frac{4\sigma}{\kappa}} \cosh_2(X_t)^{-\frac{4}{\kappa}} \sqrt{\kappa} dB(t).$$

Thus $f_{\kappa,\sigma}(X_t)$ is a local martingale.

Let $u(0) = 0$ and $u'(t) = [\exp(X_s/2)^{\frac{4\sigma}{\kappa}} \cosh_2(X_s)^{-\frac{4}{\kappa}} \sqrt{\kappa}]^2$. Then u is a continuous increasing function. Let $T = u(\infty) \in (0, +\infty]$, and $v = u^{-1}$. Then $(f_{\kappa,\sigma}(X_{v(t)}), 0 \leq t < T)$ has the same distribution as $(B(t), 0 \leq t < T)$. Since $f_{\kappa,\sigma}(X_{v(t)})$ stays inside $(0, A_{\kappa,\sigma})$, so from the property of Brownian motion, we have a.s. $T < \infty$ and $\lim_{t\to T} f_{\kappa,\sigma}(X_{v(t)})$ exists. If $\lim_{t\to T} f_{\kappa,\sigma}(X_{v(t)})$ is neither 0 nor $A_{\kappa,\sigma}$, then $f_{\kappa,\sigma}(X_{v(t)})$ is uniformly bounded away from 0 and $A_{\kappa,\sigma}$ on $[0, T)$, so X_t is uniformly bounded on $[0, \infty)$, which implies that $u'(t)$ is uniformly bounded below, and so $T = u(\infty) = \infty$. Since $T < \infty$ a.s., so $\lim_{t\to T} f_{\kappa,\sigma}(X_{v(t)}) \in \{0, A_{\kappa,\sigma}\}$ a.s. Thus $\lim_{t\to\infty} X_t \in \{\pm\infty\}$ a.s. Moreover, the probability that $X_t \to +\infty$ is equal to $f_{\kappa,\sigma}(x_0)/A_{\kappa,\sigma}$ by the Markov property.

Define

$$J_+ = \inf\{x \in \mathbb{R} : \lim_{t\to\infty} X_t(x + \pi i) = +\infty\};$$
$$J_- = \sup\{x \in \mathbb{R} : \lim_{t\to\infty} X_t(x + \pi i) = -\infty\}.$$

Since $x_1 < x_2$ implies $\mathrm{Re}\,\psi(t, x_1 + \pi i) < \mathrm{Re}\,\psi(t, x_2 + \pi i)$ for all t, so we have $J_- \leq J_+$; and for $x < J_-$, $X_t(x + \pi i) \to -\infty$, for $x > J_+$, $X_t(x + \pi i) \to +\infty$ as $t \to \infty$. Hence $\mathbf{P}\{J_+ < x\} \leq f_{\kappa,\sigma}(x)/A_{\kappa,\sigma} \leq \mathbf{P}\{J_- \leq x\}$ for all $x \in \mathbb{R}$. Since $f_{\kappa,\sigma}$ is strictly increasing, so $J_- = J_+$ a.s. By discarding an event of probability 0, we may assume that $J_+ = J_-$, and let it be denoted by J. The density of J is $\exp(x/2)^{\frac{4\sigma}{\kappa}} \cosh_2(x)^{-\frac{4}{\kappa}} / A_{\kappa,\sigma}$.

Lemma 3.2 $\overline{L(\infty)} \cap \mathbb{R}_\pi = \{J + \pi i\}$.

Proof. If $J + \pi i \notin \overline{L(\infty)}$, then there are $b, c > 0$ such that $\text{dist}(x + \pi i, L(\infty)) > c$ for all $x \in [J - b, J + b]$. From the definition of J, $X_t(J \pm b + \pi i) \to \pm\infty$ as $t \to \infty$. Thus $\text{Re}\,\psi(t, J + b + \pi i) - \text{Re}\,\psi(t, J - b + \pi i) \to +\infty$ as $t \to \infty$. By mean value theorem, for each t, there is $x_t \in [J - b, J + b]$ such that $|\partial_z \psi(t, x_t + \pi i)| \to \infty$ as $t \to \infty$. From Koebe's 1/4 theorem, we conclude that $\text{dist}(x_t + \pi i, L(t)) \to 0$, which is a contradiction. Thus $J + \pi i \in \overline{L(\infty)}$.

Suppose $x_0 > J$. Then $X_t(x_0 + \pi i) \to +\infty$ as $t \to \infty$. Thus $\partial_t \psi(t, x_0 + \pi i) \to 1$ as $t \to \infty$. Recall that $0 < \sigma' < 1$, and $|\xi(t)| \le A(\omega) + \sigma' t$ for all $t \ge 0$. So there is $H > 0$ such that when $t \ge H$, $X_t(x_0 + \pi i) = \text{Re}\,\psi(t, x_0 + \pi i) - \xi(t) > \frac{1-\sigma'}{2} t$. So $X_t(x + \pi i) > \frac{1-\sigma'}{2} t$ for all $x \ge x_0$ and $t \ge H$.

Differentiate (2.3) w.r.t. z, then we get

$$\partial_t \partial_z \psi(t, z) = -1/2 \cdot \partial_z \psi(t, z) \cdot \sinh_2(\psi(t, z) - \xi(t))^{-2}.$$

Thus

$$\partial_t \ln |\partial_z \psi(t, z_0)| = \text{Re}\left(-1/2 \cdot \sinh_2(\psi(t, z_0) - \xi(t))^{-2}\right). \tag{3.3}$$

It follows that for all $x \ge x_0$,

$$\begin{aligned} |\partial_z \psi(t, x + \pi i)| &= \exp\left(\int_0^t \text{Re}\left(\frac{-1/2}{\sinh_2(\psi(s, x + \pi i) - \xi(s))^2}\right) ds\right) \\ &= \exp\left(\int_0^t \text{Re}\left(\frac{-1/2}{\sinh_2(X_s(x + \pi i) + \pi i)^2}\right) ds\right) \\ &= \exp\left(\int_0^t \frac{1/2}{\cosh_2(X_s(x + \pi i))^2} ds\right) \\ &\le \exp\left(\int_0^H \frac{ds}{2} + \int_H^\infty \frac{1}{2\cosh_2\left(\frac{1-\sigma'}{2} s\right)^2} ds\right) < +\infty. \end{aligned}$$

Then by Koebe's 1/4 theorem, for all $x \ge x_0$, $x + \pi i$ is bounded away from $L(\infty)$ uniformly. Thus $\overline{L(\infty)}$ is disjoint from $[x_0 + \pi i, +\infty)$ for all $x_0 > J$. So $\overline{L(\infty)}$ is disjoint from $(J + \pi i, +\infty)$. Similarly, $\overline{L(\infty)}$ is disjoint from $(-\infty, J + \pi i)$. Thus $\overline{L(\infty)}$ intersects $\mathbb{R}_\pi$ only at $J + \pi i$. □

Theorem 3.3 *If* $\kappa \in (0, 4]$ *and* $|\sigma| < 1$, *then a.s.* $\lim_{t \to \infty} \beta(t) \in \mathbb{R}_\pi$.

Proof. Let $Q = \bigcap_{0 \le t < \infty} \overline{\beta[t, \infty)}$. By Lemma 3.1, Q is nonempty and compact. Suppose $\widetilde{\xi}$ has the same law as ξ, and is independent of ξ. Let $\widetilde{\beta}(t)$ and $\widetilde{\psi}(t, \cdot)$, $0 \le t < \infty$, be the strip Loewner trace and maps driven by $\widetilde{\xi}$, respectively. Let $(\widetilde{\mathcal{F}}_t)$ be the filtration generated by $\widetilde{\xi}$. For $h \in (0, 1)$, let T_h be the first t such that $\text{Im}\,\widetilde{\beta}(t) = \pi - h$. From Lemma 3.2, T_h is a finite $(\widetilde{\mathcal{F}}_t)$-

stopping time. Let $\xi_*(t) = \widetilde{\xi}(t)$ for $0 \le t \le T_h$; $\xi_*(t) = \widetilde{\xi}(T_h) + \xi(t - T_h)$ for $t \ge T_h$. Then ξ_* has the same distribution as ξ. Let $\beta_*(t)$ be the strip Loewner trace driven by ξ_*. Then $\beta_*(t) = W_{T_h}(\beta(t - T_h))$ for $t \ge T_h$, where $W_{T_h}(z) := \widetilde{\psi}(T_h, \cdot)^{-1}(\widetilde{\xi}(T_h) + z)$. Since β_* has the same distribution as β, so $W_{T_h}(Q)$ has the same law as Q.

Let Λ_- denote the set of curves in $\mathbb{S}_\pi \setminus \widetilde{\beta}((0, T_h])$ that connecting $(-\infty, 0)$ with the union of $[0, \infty)$ and the righthand side of $\widetilde{\beta}((0, T_h])$. Let $p = \operatorname{Re} \widetilde{\beta}(T_h) + \pi i$, and $A = \{z \in \mathbb{S}_\pi : h < |z - p| < \pi\}$. Then every curve in Λ_- crosses A. Thus the extremal length [1] of Λ_- is at least $(\ln(\pi) - \ln(h))/\pi$. From the property of ψ_{T_h}, W_{T_h} maps $\mathbb{S}_\pi$ conformally onto $\mathbb{S}_\pi \setminus \widetilde{\beta}((0, T_h])$. There are $c_h < 0 < d_h$ such that $W_{T_h}((-\infty, c_h]) = (-\infty, 0]$ and $W_{T_h}([d_h, \infty)) = [0, \infty)$. Since $\kappa \in (0, 4]$, so $W_{T_h}((c_h, d_h)) = \beta((0, T_h]) \subset \mathbb{S}_\pi$. Moreover, $W_{T_h}(0) = \widetilde{\beta}(T_h)$, and W_{T_h} maps $[0, d_h)$ to the righthand side of $\widetilde{\beta}((0, T_h])$. From conformal invariance of extremal length, the extremal distance between $(-\infty, c_h)$ and $[0, \infty)$ in $\mathbb{S}_\pi$ is not less than $(\ln(\pi) - \ln(h))/\pi$. Thus $c_h \to -\infty$ uniformly as $h \to 0$. Similarly, $d_h \to +\infty$ uniformly as $h \to 0$.

For any $z \in \overline{\mathbb{S}}_\pi$, we have $\operatorname{Im} W_{T_h}(z) \ge \operatorname{Im} z$; and the strict inequality holds when $z \in \mathbb{S}_\pi$ or $z \in (c_h, d_h)$. Thus $\min\{\operatorname{Im} W_{T_h}(Q)\} \ge \min\{\operatorname{Im} Q\}$. Since $W_{T_h}(Q)$ has the same law as Q, so a.s. $\min\{\operatorname{Im} W_{T_h}(Q)\} = \min\{\operatorname{Im} Q\}$. Suppose now $Q \not\subset \mathbb{R}_\pi$ holds with a positive probability. Since Q is a bounded set, there is $R > 0$ such that $\mathbf{P}[\mathcal{E}_R] > 0$, where $\mathcal{E}_R$ denotes the event that $Q \subset \{z : |\operatorname{Re} z| < R\}$ and $Q \not\subset \mathbb{R}_\pi$ both hold. If h is small enough, we have $|c_h|, |d_h| > R$. Assume that $\mathcal{E}_R$ occurs. For any $z \in Q \setminus \mathbb{R}_\pi$, either $z \in \mathbb{S}_\pi$ or $z \in (c_h, d_h)$. In both cases, we have $\operatorname{Im} W_{T_h}(z) > \operatorname{Im} z$. Thus $\min\{\operatorname{Im} W_{T_h}(Q)\} > \min\{\operatorname{Im} Q\}$ on $\mathcal{E}_R$, which is a contradiction. Thus a.s. $Q \subset \mathbb{R}_\pi$. From Lemma 3.2, we have a.s. $Q = \{J + \pi i\}$, which means that $\lim_{t\to\infty} \beta(t) = J + \pi i$. □

Now we consider the case that $|\sigma| \ge 1$.

Theorem 3.4 *If* $\kappa \in (0, 4]$ *and* $\pm\sigma \ge 1$, *then almost surely* $\lim_{t\to\infty} \beta(t) = \pm\infty$.

Proof. Let $\sigma \ge 1$. Let $W(z) = e^z - 1$. Then W maps $(\mathbb{S}_\pi; 0, +\infty, -\infty)$ conformally onto $(\mathbb{H}; 0, \infty, -1)$. From Lemma 2.3, after a time-change, $W(\beta(t)), 0 \le t < \infty$, has the same distribution as a chordal SLE$(\kappa; \frac{\kappa}{2} - 3 + \sigma)$ trace started from $(0; -1)$, which is also a chordal SLE$(\kappa; \frac{\kappa}{2} - 3 + \sigma, 0)$ trace started from $(0; -1, 1)$. Since $\sigma \ge 1$, so $\frac{\kappa}{2} - 3 + \sigma \ge \frac{\kappa}{2} - 2$. Since $\kappa \in (0, 4]$, so $0 \ge \frac{\kappa}{2} - 2$. Thus from Theorem 3.2 (ii), a.s. $\lim_{t\to\infty} W(\beta(t)) = \infty$, which implies that $\lim_{t\to\infty} \beta(t) = +\infty$. The case $\sigma \le -1$ is similar. □

Remark. Theorems 3.3 and 3.4 should hold true in the case $\kappa > 4$. For example, the only part that the condition $\kappa \in (0, 4]$ is used in the proof of Theorem 3.3 is that $\operatorname{Im} W_{T_h}(x) > 0 = \operatorname{Im} x$ for $c_h < x < d_h$. If this is not true for any $\kappa > 4$, then we get some cut point of the hull that lies on the real line, which does not seem to be possible. If $\kappa > 4$ in Theorem 3.4, we can prove that if $\sigma \ge 1$ (resp. $\sigma \le -1$), then $L(\infty)$ is bounded from left (resp. right) and unbounded from right (resp. left), and $\overline{L(\infty)} \cap \mathbb{R}_\pi = \emptyset$.

3.3 Three or four force points. First, we consider a strip Loewner process with three force points. Let $\kappa > 0$ and $\rho_+ + \rho_- + \rho = \kappa - 6$. Suppose $\beta(t)$, $0 \le t < T$, is a strip SLE$(\kappa; \rho_+, \rho_-, \rho)$ trace started from $(0; +\infty, -\infty, p)$ for some $p \in \mathbb{R}_\pi$. Then $T = \infty$. Let $\bar{p} = \operatorname{Re} p$. Then the driving function $\xi(t)$, $0 \le t < \infty$, is the solution to the SDE:

$$\begin{cases} d\xi(t) = \sqrt{\kappa} dB(t) + \frac{\rho_- - \rho_+}{2} dt - \frac{\rho}{2} \tanh_2(\bar{p}(t) - \xi(t)) dt; \\ d\bar{p}(t) = \tanh_2(\bar{p}(t) - \xi(t)) dt. \end{cases} \tag{3.4}$$

Here $\bar{p}(t) \in \mathbb{R}$ and $\bar{p}(t) + \pi i = \psi(t, p)$ for any $t \ge 0$, where $\psi(t, \cdot)$, $0 \le t < \infty$, are strip Loewner maps driven by ξ. Let $X(t) = \bar{p}(t) - \xi(t)$. Then $X(t)$ satisfies the SDE:

$$dX(t) = -\sqrt{\kappa} dB(t) - \frac{\rho_- - \rho_+}{2} dt + \left(1 + \frac{\rho}{2}\right) \tanh_2(X(t)) dt. \tag{3.5}$$

Suppose f is a real valued function on $\mathbb{R}$, and for any $x \in \mathbb{R}$,

$$f'(x) = \exp(x/2)^{\frac{4}{\kappa} \cdot \frac{\rho_- - \rho_+}{2}} \cosh_2(x)^{-\frac{4}{\kappa}(1 + \frac{\rho}{2})}.$$

From Ito's formula, $f(X(t))$ is a local martingale.

Let $I = f(\mathbb{R})$. Recall that $\rho = \kappa - 6 - \rho_+ - \rho_-$. If $\rho_+ \ge \kappa/2 - 2$ and $\rho_- \ge \kappa/2 - 2$, then $I = \mathbb{R}$, so a.s. $\limsup X(t) = +\infty$ and $\liminf X(t) = -\infty$. If $\rho_+ < \kappa/2 - 2$ and $\rho_- \ge \kappa/2 - 2$, then $I = (a, \infty)$ for some $a \in \mathbb{R}$, so a.s. $\lim X(t) = -\infty$. If $\rho_+ \ge \kappa/2 - 2$ and $\rho_- < \kappa/2 - 2$, then $I = (-\infty, b)$ for some $b \in \mathbb{R}$, so a.s. $\lim X(t) = +\infty$. If $\rho_+ < \kappa/2 - 2$ and $\rho_- < \kappa/2 - 2$, then $I = (a, b)$ for some $a, b \in \mathbb{R}$, so with some probability $P \in (0, 1)$, $\lim X(t) = -\infty$; and with probability $1 - P$, $\lim X(t) = +\infty$.

Let $I_1 = [\kappa/2 - 2, \infty)$, $I_2 = (\kappa/2 - 4, \kappa/2 - 2)$, and $I_3 = (-\infty, \kappa/2 - 4]$. Let Case (jk) denote the case that $\rho_+ \in I_j$ and $\rho_- \in I_k$. We use $(p, +\infty)$ or $(-\infty, p)$ to denote the open subarc of $\mathbb{R}_\pi$ between p and $+\infty$ or between p and $-\infty$, respectively.

Theorem 3.5 *Suppose $\kappa \in (0, 4]$. In Case* (11)*, a.s. $\lim_{t\to\infty} \beta(t) = p$. In Case* (12)*, a.s. $\lim_{t\to\infty} \beta(t) \in (-\infty, p)$. In Case* (21)*, a.s. $\lim_{t\to\infty} \beta(t) \in (p, +\infty)$. In Case* (13)*, a.s. $\lim_{t\to\infty} \beta(t) = -\infty$. In Case* (31)*, a.s. $\lim_{t\to\infty} \beta(t) = +\infty$. In Case* (22)*, a.s. $\lim_{t\to\infty} \beta(t) \in (-\infty, p)$ or $\in (p, +\infty)$. In Case* (23)*, a.s. $\lim_{t\to\infty} \beta(t) = -\infty$ or $\in (p, +\infty)$. In Case* (32)*, a.s. $\lim_{t\to\infty} \beta(t) \in (-\infty, p)$ or $= +\infty$. In Case* (33)*, a.s. $\lim_{t\to\infty} \beta(t) = -\infty$ or $= +\infty$. And in each of the last four cases, both events happen with some positive probability.*

Proof. The result in Case (11) follows from Theorem 3.2 and Lemma 2.3. Now consider Case (12). We have a.s. $\lim X(t) = +\infty$. Let $Y(t) = X(t) + \sqrt{\kappa} B(t)$. From (3.5), a.s.

$$Y'(t) = -\frac{\rho_- - \rho_+}{2} + \left(1 + \frac{\rho}{2}\right) \tanh_2(X(t))$$
$$\to \frac{\rho + \rho_+ - \rho_-}{2} + 1 = \frac{\kappa}{2} - 2 - \rho_-$$

as $t \to \infty$. Thus a.s.

$$\lim_{t\to\infty} X(t)/t = \lim_{t\to\infty} Y(t)/t = \kappa/2 - 2 - \rho_- > 0. \tag{3.6}$$

From (3.4), we see that as $t \to \infty$, the SDE for $\xi(t)$ tends to $d\xi(t) = \sqrt{\kappa} dB(t) + \sigma dt$, where $\sigma := \frac{\rho_- - \rho_+}{2} - \rho/2 = \rho_- - (\kappa/2 - 3) \in (-1, 1)$. From Theorem 3.3, it is reasonable to guess that a.s. $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$. This will be rigorously proved below.

Let

$$a(t) = \frac{\rho/2}{\sqrt{\kappa}}(1 - \tanh_2(X(t))); \tag{3.7}$$

$$M(t) = \exp\left(-\int_0^t a(s)dB(s) - \frac{1}{2}\int_0^t a(s)^2 ds\right). \tag{3.8}$$

From (3.6), a.s. $\int_0^\infty a(t)^2 dt < \infty$, so a.s. $\lim_{t\to\infty} M(t) \in (0, \infty)$. From Ito's formula, $M(t)$ is a positive local martingale, and $dM(t)/M(t) = -a(t)dB(t)$. For $N \in \mathbb{N}$, let $T_N \in [0, \infty]$ be the largest number such that $M(t) \in (1/N, N)$ for $0 \le t < T_N$. Then T_N is a stopping time, $M(t \wedge T_N)$ is a bounded martingale, and $\mathbf{P}[\{T_N = \infty\}] \to 1$ as $N \to \infty$. Define $\mathbf{Q}$ such that $d\mathbf{Q} = M(T_N)d\mathbf{P}$, where $M(\infty) := \lim_{t\to\infty} M(t)$. Then $\mathbf{Q}$ is also a probability measure. For $t \ge 0$, let $\widetilde{B}(t) = B(t) + \int_0^t a(s)ds$. From (3.4), we have

$$\xi(t) = \xi(0) + \sqrt{\kappa}\widetilde{B}(t) + \sigma t.$$

From Girsanov theorem, $\widetilde{B}(t)$, $0 \le t < T_N$, is a partial $\mathbf{Q}$-Brownian motion. Since $\kappa \in (0, 4]$ and $|\sigma| < 1$, so from Theorem 3.3, $\mathbf{Q}$-a.s. $\lim_{t\to T_N} \beta(t) \in \mathbb{R}_\pi$ on $\{T_N = \infty\}$. Since $1/N \le d\mathbf{Q}/d\mathbf{P} \le N$, so $\mathbf{Q}$ is equivalent to $\mathbf{P}$. Thus ($\mathbf{P}$-)a.s. $\lim_{t\to T_N} \beta(t) \in \mathbb{R}_\pi$ on $\{T_N = \infty\}$. For any $\varepsilon > 0$, there is N such that $\mathbf{P}[\{T_N = \infty\}] > 1 - \varepsilon$. Thus with probability greater than $1 - \varepsilon$, $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$. Since $\varepsilon > 0$ is arbitrary, so a.s. $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$. Now for any $x \in \mathbb{R}$ and $x \ge \bar{p}$, $\psi(t, x + \pi i) \in \mathbb{R}_\pi$ and $\operatorname{Re}\psi(t, x + \pi i) \ge \operatorname{Re}\psi(t, \bar{p} + \pi i)$ for any $t \ge 0$. Thus $\operatorname{Re}\psi(t, x + \pi i) - \xi(t) \to \infty$ as $t \to \infty$. From an argument in the proof of Lemma 3.2, we have $\operatorname{dist}(x + \pi i, \beta((0, \infty))) > 0$. Thus $\lim_{t\to\infty} \beta(t) \notin [p, +\infty)$, so a.s. $\lim_{t\to\infty} \beta(t) \in (-\infty, p)$.

Now consider Case (13). The argument is similar to that in Case (12) except that now $\sigma = \rho_- - (\kappa/2 - 3) \le -1$, so from Theorem 3.4, we have a.s. $\lim_{t\to\infty}\beta(t) = -\infty$. Cases (21) and (31) are symmetric to the above two cases. In Case (22), a.s. $\lim_{t\to\infty} X(t) = +\infty$ or $= -\infty$. If $\lim_{t\to\infty} X(t) = +\infty$, then as $t \to \infty$, the SDE for $\xi(t)$ tends to $d\xi(t) = \sqrt{\kappa}dB(t) + \sigma dt$, where $\sigma = \rho_- - (\kappa/2 - 3) \in (-1, 1)$. Using the argument in Case (12), we get a.s. $\lim_{t\to\infty}\beta(t) \in (-\infty, p)$ whenever $\lim_{t\to\infty} X(t) = +\infty$. Similarly, a.s. $\lim_{t\to\infty}\beta(t) \in (p, +\infty)$ whenever $\lim_{t\to\infty} X(t) = -\infty$. The arguments in the other three cases are similar to that in Case (22). □

Next, we consider a strip Loewner process with four force points. Let $\kappa > 0$ and $\rho_+ + \rho_- + \rho_1 + \rho_2 = \kappa - 6$. Suppose $\beta(t)$, $0 \le t < T$, is a strip SLE$(\kappa; \rho_+, \rho_-, \rho_1, \rho_2)$ trace started from $(0; +\infty, -\infty, p_1, p_2)$ for some $p_1, p_2 \in \mathbb{R}$ with $p_1 > 0 > p_2$. Then the driving function $\xi(t)$, $0 \le t < T$, is the maximal solution to the SDE:

$$\begin{cases} d\xi(t) = \sqrt{\kappa}dB(t) + \frac{\rho_- - \rho_+}{2}dt - \sum_{j=1}^{2}\frac{\rho_j}{2}\coth_2(p_j(t) - \xi(t))dt; \\ dp_j(t) = \coth_2(p_j(t) - \xi(t))dt, \quad j = 1, 2. \end{cases} \tag{3.9}$$

Here $p_j(t) = \psi(t, p_j) \in \mathbb{R}$, $0 \le t < T$, $j = 1, 2$, where $\psi(t, \cdot)$, $0 \le t < T$, are strip Loewner maps driven by ξ.

Theorem 3.6 *Suppose* $\kappa \in (0, 4]$, $\rho_1, \rho_2 \ge \frac{\kappa-4}{2}$, $|(\rho_1+\rho_+) - (\rho_2+\rho_-)| < 2$, *and* $\min\{\rho_1, \rho_2\} \le 0$. *Then a.s.* $T = \infty$ *and* $\lim_{t\to\infty}\beta(t) \in \mathbb{R}_\pi$.

Proof. We only consider the case that $\rho_2 \le 0$. The case $\rho_1 \le 0$ is symmetric. Let $X_j(t) = p_j(t) - \xi(t)$, $j = 1, 2$. Then $X_1(t) > 0 > X_2(t)$, $0 \le t < T$. And we have

$$\begin{aligned} dX_1(t) = &-\sqrt{\kappa}dB(t) - \frac{\rho_- - \rho_+}{2}dt \\ &+ \left(1 + \frac{\rho_1}{2}\right)\coth_2(X_1(t))dt + \frac{\rho_2}{2}\coth_2(X_2(t))dt. \end{aligned}$$

Define f on $(0, \infty)$ such that for any $x > 0$,

$$f'(x) = \exp(x/2)^{\frac{4}{\kappa}\cdot\frac{\rho_- - \rho_+ + \rho_2}{2}}\sinh_2(x)^{-\frac{4}{\kappa}(1+\frac{\rho_1}{2})}.$$

Then for any $x > 0$,

$$\frac{\kappa}{2}f''(x) = f'(x)\left(\frac{\rho_- - \rho_+ + \rho_2}{2} - \left(1 + \frac{\rho_1}{2}\right)\coth_2(x)\right).$$

Let $Y(t) = f(X_1(t))$ for any $t \in [0, T)$. From Ito's formula, we have

$$dY(t) = -\sqrt{\kappa}f'(X_1(t))dB(t) + \frac{\rho_2}{2}f'(X_1(t))(1 + \coth_2(X_2(t)))dt.$$

From the conditions of ρ_j's, f maps $(0, \infty)$ onto $(-\infty, b)$ for some $b \in \mathbb{R}$. Since $\rho_2 \le 0$ and $X_2(t) < 0$, so the drift is non-negative. Thus a.s.

$\lim_{t\to T} Y(t) = b$, which implies that $\lim_{t\to T} X_1(t) = +\infty$. Let $Z(t) = X_1(t) + \sqrt{\kappa}B(t)$. Since $\coth_2(X_2(t)) < -1$ and $\rho_2 \le 0$, so if $T = \infty$, then as $t \to \infty$,

$$Z'(t) \ge \frac{\rho_+ - \rho_- - \rho_2}{2} + \left(1 + \frac{\rho_1}{2}\right)\coth_2(X_1(t)) \to 1 + \frac{\rho_+ + \rho_1 - \rho_- - \rho_2}{2}.$$

Then $\liminf_{t\to\infty} X_1(t)/t = \liminf_{t\to\infty} Z(t)/t \ge \sigma := 1 + (\rho_+ + \rho_1 - \rho_- - \rho_2)/2 > 0$.

Let $a(t)$ and $M(t)$ be defined by (3.7) and (3.8) except that ρ and $\tanh_2(X(t))$ in (3.7) are replaced by ρ_1 and $\coth_2(X_1(t))$, respectively. If $T = \infty$, since $\liminf_{t\to\infty} X_1(t)/t \ge \sigma > 0$, so a.s. $\lim_{t\to\infty} M(t) \in (0, \infty)$. This is clearly true if $T < \infty$ because $a(s)$ is bounded. Let $\widetilde{B}(t) = B(t) + \int_0^t a(s)ds$, $0 \le t < T$. From (3.9) we have

$$d\xi(t) = \sqrt{\kappa}d\widetilde{B}(t) + \frac{\rho_- - \rho_+ - \rho_1}{2}dt - \frac{\rho_2}{2}\coth_2(X_2(t))dt.$$

If under some probability measure $\mathbf{Q}$, $(\widetilde{B}(t))$ is a partial Brownian motion, then $\beta(t)$, $0 \le t < T$, is a partial strip $\mathrm{SLE}(\kappa; \rho'_+, \rho_-, \rho_2)$ process started from $(0; +\infty, -\infty, p_2)$, where $\rho'_+ = \rho_+ + \rho_1$. Since $\rho'_+ + \rho_- + \rho_2 = \kappa - 6$, $\rho'_+ \in (\kappa/2 - 4, \kappa/2 - 2)$ and $\rho_2 \ge \kappa/2 - 2$, so from Lemma 2.2 and Theorem 3.6, we have $\mathbf{Q}$-a.s. $\lim_{t\to T} \beta(t) \in \mathbb{R}_\pi \cup \mathbb{S}_\pi$. From the proof in Case (12) of Theorem 3.6, we have a.s. $\lim_{t\to T} \beta(t) \in \mathbb{R}_\pi \cup \mathbb{S}_\pi$. Since β is a full trace, it separates either p_1 or p_2 from $\mathbb{R}_\pi$ in $\mathbb{S}_\pi$, so $\lim_{t\to T} \beta(t) \in \mathbb{S}_\pi$ is not possible. Thus $\lim_{t\to T} \beta(t) \in \mathbb{R}_\pi$ a.s. This implies that $T = \lim_{t\to T} \mathrm{scap}(\beta((0, t])) = \infty$. □

4 Coupling of two SLE processes

Let $\kappa_1, \kappa_2 > 0$; $\kappa_1\kappa_2 = 16$; $\rho_{j,m} \in \mathbb{R}$, $1 \le m \le N$, $j = 1, 2$, $N \in \mathbb{N}$; $\rho_{2,m} = -\kappa_2\rho_{1,m}/4$, $1 \le m \le N$; $x_1, x_2, p_1, \dots, p_N \in \mathbb{R}$ are distinct points. Let $\vec{\rho}_j = (\rho_{j,1}, \dots, \rho_{j,N})$, $j = 1, 2$, and $\vec{p} = (p_1, \dots, p_N)$. Note that if $\kappa_1 = \kappa_2 = 4$, then $\vec{\rho}_1 + \vec{\rho}_2 = \vec{0}$; if $\kappa_1, \kappa_2 \ne 4$, then $\vec{\rho}_1/(\kappa_1 - 4) = \vec{\rho}_2/(\kappa_2 - 4)$. The goal of this section is to prove the following theorem.

Theorem 4.1 *There is a coupling of $K_1(t)$, $0 \le t < T_1$, and $K_2(t)$, $0 \le t < T_2$, such that (i) for $j = 1, 2$, $K_j(t)$, $0 \le t < T_j$, is a chordal* $\mathrm{SLE}(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ *process started from $(x_j; x_{3-j}, \vec{p})$; and (ii) for $j \ne k \in \{1, 2\}$, if $\bar{t}_k$ is an $(\mathcal{F}_t^k)$-stopping time with $\bar{t}_k < T_k$, then conditioned on $\mathcal{F}_{\bar{t}_k}^k$, $\varphi_k(\bar{t}_k, K_j(t))$, $0 \le t \le T_j(\bar{t}_k)$, has the same distribution as a time-change of a partial chordal* $\mathrm{SLE}(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ *process started from $(\varphi_k(\bar{t}_k, x_j); \xi_k(\bar{t}_k), \varphi_k(\bar{t}_k, \vec{p}))$, where $\varphi_k(t, \vec{p}) = (\varphi_k(t, p_1), \dots, \varphi_k(t, p_N))$,*

$\varphi_k(t,\cdot)=\varphi_{K_k(t)}$, $T_j(\bar t_k)\in(0,T_j]$ *is the largest number such that* $\overline{K_j(t)}\cap\overline{K_k(\bar t_k)}=\emptyset$ *for* $0\le t<T_j(\bar t_k)$*, and* $(\mathcal{F}^j_t)$ *is the filtration generated by* $(K_j(t))$*,* $j=1,2$*.*

In many cases we can prove that $\varphi_k(\bar t_k,K_j(t))$, $0\le t\le T_j(\bar t_k)$, has the same distribution as a time-change of a *full* chordal SLE$(\kappa_j;-\frac{\kappa_j}{2},\vec\rho_j)$ process started from $(\varphi_k(\bar t_k,x_j);\xi_k(\bar t_k),\varphi_k(\bar t_k,\vec p))$. From the property of $T_j(\bar t_k)$, $\bigcup_{0\le t<T_j(\bar t_k)}K_j(t)$ touches $K_k(\bar t_k)$, so $\bigcup_{0\le t<T_j(\bar t_k)}\varphi_k(\bar t_k,K_j(t))$ touches $\mathbb{R}$. So the chain can not be further extended while staying bounded away from the boundary. Thus if $\kappa_j\le 4$, it is a full process. Another case is when there is some force point p_k that lies between x_1 and x_2. Then $\bigcup_{0\le t<T_j}(\bar t_k)K_j(t)$ separates $\varphi_k(\bar t_k,p_k)$ from ∞. So again we get a full process.

4.1 Ensembles. Let's review the results in Sect. 3 of [19]. For $j=1,2$, let $K_j(t)$ and $\varphi_j(t,\cdot)$, $0\le t<S_j$, be chordal Loewner hulls and maps driven by $\xi_j\in C([0,S_j))$. Suppose $K_1(t_1)\cap K_2(t_2)=\emptyset$ for any $t_1\in[0,S_1)$ and $t_2\in[0,S_2)$. For $j\ne k\in\{1,2\}$, $t_0\in[0,S_k)$ and $t\in[0,S_j)$, let

$$K_{j,t_0}(t)=(K_j(t)\cup K_k(t_0))/K_k(t_0),\quad \varphi_{j,t_0}(t,\cdot)=\varphi_{K_{j,t_0}(t)}. \tag{4.1}$$

Then for any $t_1\in[0,S_1)$ and $t_2\in[0,S_2)$,

$$\varphi_{K_1(t_1)\cup K_2(t_2)}=\varphi_{1,t_2}(t_1,\cdot)\circ\varphi_2(t_2,\cdot)=\varphi_{2,t_1}(t_2,\cdot)\circ\varphi_1(t_1,\cdot). \tag{4.2}$$

We use ∂_1 and ∂_z to denote the partial derivatives of $\varphi_j(\cdot,\cdot)$ and $\varphi_{j,t_0}(\cdot,\cdot)$ w.r.t. the first (real) and second (complex) variables, respectively, inside the bracket; and use ∂_0 to denote the partial derivative of $\varphi_{j,t_0}(\cdot,\cdot)$ w.r.t. the subscript t_0. From (3.10)–(3.14) in Sect. 3 of [19], we have

$$\partial_0\varphi_{k,t}(s,\xi_j(t))=-3\partial_z^2\varphi_{k,t}(s,\xi_j(t)); \tag{4.3}$$

$$\frac{\partial_0\partial_z\varphi_{k,t}(s,\xi_j(t))}{\partial_z\varphi_{k,t}(s,\xi_j(t))}=\frac12\cdot\left(\frac{\partial_z^2\varphi_{k,t}(s,\xi_j(t))}{\partial_z\varphi_{k,t}(s,\xi_j(t))}\right)^2-\frac43\cdot\frac{\partial_z^3\varphi_{k,t}(s,\xi_j(t))}{\partial_z\varphi_{k,t}(s,\xi_j(t))}; \tag{4.4}$$

$$\partial_1\varphi_{j,t_0}(t,z)=\frac{2\partial_z\varphi_{k,t}(t_0,\xi_j(t))^2}{\varphi_{j,t_0}(t,z)-\varphi_{k,t}(t_0,\xi_j(t))}; \tag{4.5}$$

$$\frac{\partial_1\partial_z\varphi_{j,s}(t,z)}{\partial_z\varphi_{j,s}(t,z)}=\frac{-2\partial_z\varphi_{k,t}(s,\xi_j(t))^2}{(\varphi_{j,s}(t,z)-\varphi_{k,t}(s,\xi_j(t)))^2}; \tag{4.6}$$

$$\partial_1\left(\frac{\partial_z^2\varphi_{j,s}(t,z)}{\partial_z\varphi_{j,s}(t,z)}\right)=\frac{4\partial_z\varphi_{k,t}(s,\xi_j(t))^2\partial_z\varphi_{j,s}(t,z)}{(\varphi_{j,s}(t,z)-\varphi_{k,t}(s,\xi_j(t)))^3}; \tag{4.7}$$

$$\partial_1\partial_z\left(\frac{\partial_z^2\varphi_{j,s}(t,z)}{\partial_z\varphi_{j,s}(t,z)}\right)=\frac{4\partial_z\varphi_{k,t}(s,\xi_j(t))^2\partial_z^2\varphi_{j,s}(t,z)}{(\varphi_{j,s}(t,z)-\varphi_{k,t}(s,\xi_j(t)))^3}-\frac{12\partial_z\varphi_{k,t}(s,\xi_j(t))^2\partial_z\varphi_{j,s}(t,z)^2}{(\varphi_{j,s}(t,z)-\varphi_{k,t}(s,\xi_j(t)))^4}. \tag{4.8}$$

4.2 Martingales. Suppose $x_1, x_2, p_1, \dots, p_N$ are distinct points on $\mathbb{R}$. Let $\xi_j \in C([0, T_j))$, $j = 1, 2$, be two independent semi-martingales that satisfy $d\langle \xi_j(t)\rangle = \kappa_j dt$, where $\kappa_1, \kappa_2 > 0$. Let $\varphi(t, \cdot)$ and $K_j(t)$, $0 \le t < \infty$, be chordal Loewner maps and hulls driven by ξ_j, $j = 1, 2$. Let

$$\mathcal{D} := \Big\{(t_1, t_2) : \overline{K_1(t_1)} \cap \overline{K_2(t_2)} = \emptyset,\quad \varphi(t_j, p_m) \text{ does not blow up}, 1 \le m \le N, j = 1, 2\Big\}. \tag{4.9}$$

For $(t_1, t_2) \in \mathcal{D}$, $j \ne k \in \{1, 2\}$, and $h \in \mathbb{Z}_{\ge 0}$, define $A_{j,h}(t_1, t_2) = \partial_z^h \varphi_{k,t_j}(t_k, \xi_j(t_j))$. For $(t_1, t_2) \in \mathcal{D}$, $1 \le m \le N$, and $h \in \mathbb{Z}_{\ge 0}$, let $B_{m,h}(t_1, t_2) = \partial_z^h \varphi_{K_1(t_1)\cup K_2(t_2)}(p_m)$. For $j = 1$, $k = 2$, and $1 \le m \le N$, we have the following SDEs:

$$\partial_j A_{j,0} = A_{j,1}\partial \xi_j(t_j) + \left(\frac{\kappa_j}{2} - 3\right) A_{j,2}\partial t_j; \tag{4.10}$$

$$\frac{\partial_j A_{j,1}}{A_{j,1}} = \frac{A_{j,2}}{A_{j,1}}\partial \xi_j(t_j) + \left(\frac{1}{2}\cdot\frac{A_{j,2}^2}{A_{j,1}^2} + \left(\frac{\kappa_j}{2} - \frac{4}{3}\right)\cdot\frac{A_{j,3}}{A_{j,1}}\right)\partial t_j; \tag{4.11}$$

$$\partial_j A_{k,0} = \frac{2A_{j,1}^2}{A_{k,0} - A_{j,0}}\partial t_j, \qquad \frac{\partial_j A_{k,1}}{A_{k,1}} = \frac{-2A_{j,1}^2}{(A_{k,0} - A_{j,0})^2}\partial t_j; \tag{4.12}$$

$$\partial_j B_{m,0} = \frac{2A_{j,1}^2}{B_{m,0} - A_{j,0}}\partial t_j, \qquad \frac{\partial_j B_{m,1}}{B_{m,1}} = \frac{-2A_{j,1}^2}{(B_{m,0} - A_{j,0})^2}\partial t_j. \tag{4.13}$$

Here ∂_j means the partial derivative w.r.t. t_j. Note that (4.10) and (4.11) are (4.10) and (4.11) in [19]; (4.12) follows from (4.5) and (4.6) here; and (4.13) follows from (4.5), (4.6), and (4.2). By symmetry, (4.10)–(4.13) also hold for $j = 2$ and $k = 1$.

For $j \ne k \in \{1, 2\}$, let $E_{j,0} = A_{j,0} - A_{k,0} = -E_{k,0}$; $E_{j,m} = A_{j,0} - B_{m,0}$, $1 \le m \le N$; and $C_{m_1,m_2} = B_{m_1,0} - B_{m_2,0}$, $1 \le m_1 < m_2 \le N$. From (4.10), (4.12), and (4.13), for $0 \le m \le N$,

$$\frac{\partial_j E_{j,m}}{E_{j,m}} = \frac{A_{j,1}}{E_{j,m}}\partial \xi_j(t_j) + \left(\left(\frac{\kappa_j}{2} - 3\right)\cdot\frac{A_{j,2}}{E_{j,m}} + 2\frac{A_{j,1}^2}{E_{j,m}^2}\right)\partial t_j. \tag{4.14}$$

From (4.12) and (4.13), for $1 \le m \le N$ and $1 \le m_1 < m_2 \le N$

$$\frac{\partial_j E_{k,m}}{E_{k,m}} = \frac{-2A_{j,1}^2}{E_{j,0}E_{j,m}}\partial t_j, \qquad \frac{\partial_j C_{m_1,m_2}}{C_{m_1,m_2}} = \frac{-2A_{j,1}^2}{E_{j,m_1}E_{j,m_2}}\partial t_j. \tag{4.15}$$

Now suppose $\kappa_1\kappa_2 = 16$. For $j = 1, 2$, let

$$\alpha_j = \frac{6 - \kappa_j}{2\kappa_j}, \qquad \lambda_j = \frac{(8 - 3\kappa_j)(6 - \kappa_j)}{2\kappa_j}. \tag{4.16}$$

Then $\lambda_1 = \lambda_2$. Let it be denoted by λ. From (4.11) and (4.12), we have

$$\frac{\partial_j A_{j,1}^{\alpha_j}}{A_{j,1}^{\alpha_j}} = \frac{6-\kappa_j}{2\kappa_j} \cdot \frac{A_{j,2}}{A_{j,1}} \partial \xi_j(t_j) + \lambda\left(\frac{1}{4} \cdot \frac{A_{j,2}^2}{A_{j,1}^2} - \frac{1}{6} \cdot \frac{A_{j,3}}{A_{j,1}}\right) \partial t_j; \tag{4.17}$$

$$\frac{\partial_j A_{k,1}^{\alpha_k}}{A_{k,1}^{\alpha_k}} = -2\alpha_k \frac{A_{j,1}^2}{E_{j,0}^2} \partial t_j = -\frac{3\kappa_j - 8}{8} \frac{A_{j,1}^2}{E_{j,0}^2} \partial t_j. \tag{4.18}$$

Suppose $\vec{\rho}_j = (\rho_{j,1}, \ldots, \rho_{j,N}) \in \mathbb{R}^N$, $j = 1, 2$, and $\vec{\rho}_2 = -\frac{\kappa_2}{4} \vec{\rho}_1$. Let $\rho_{j,m}^* = \rho_{j,m}/\kappa_j$, $1 \le m \le N$, $j = 1, 2$. Then $\rho_{2,m}^* = -\kappa_1 \rho_{1,m}^*/4$ and $\rho_{1,m}^* = -\kappa_2 \rho_{2,m}^*/4$ for $1 \le m \le N$. From (4.14) and (4.15), for $j \ne k \in \{1, 2\}$ and $1 \le m \le N$, we have

$$\begin{aligned}\frac{\partial_j |E_{j,m}|^{\rho_{j,m}^*}}{|E_{j,m}|^{\rho_{j,m}^*}} &= \rho_{j,m}^* \frac{A_{j,1}}{E_{j,m}} \partial \xi_j(t_j) + \rho_{j,m}^* \cdot \frac{\kappa_j - 6}{2} \cdot \frac{A_{j,2}}{E_{j,m}} \partial t_j \\ &\quad + \left(\frac{\kappa_j}{2} \rho_{j,m}^* (\rho_{j,m}^* - 1) + 2\rho_{j,m}^*\right) \frac{A_{j,1}^2}{E_{j,m}^2} \partial t_j;\end{aligned} \tag{4.19}$$

$$\frac{\partial_j |E_{k,m}|^{\rho_{k,m}^*}}{|E_{k,m}|^{\rho_{k,m}^*}} = -2\rho_{k,m}^* \frac{A_{j,1}^2}{E_{j,0} E_{j,m}} \partial t_j = \frac{\kappa_j \rho_{j,m}^*}{2} \cdot \frac{A_{j,1}^2}{E_{j,0} E_{j,m}} \partial t_j. \tag{4.20}$$

Let $E = |E_{1,0}| = |E_{2,0}|$. From (4.14), for $j = 1, 2$,

$$\frac{\partial_j E^{-1/2}}{E^{-1/2}} = -\frac{1}{2} \cdot \frac{A_{j,1}}{E_{j,0}} \partial \xi_j(t_j) + \left(\frac{6-\kappa_j}{4} \cdot \frac{A_{j,2}}{E_{j,0}} + \frac{3\kappa_j - 8}{8} \frac{A_{j,1}^2}{E_{j,0}^2}\right) \partial t_j. \tag{4.21}$$

For $1 \le m \le N$, let

$$\gamma_m = \frac{\kappa_1}{4} \rho_{1,m}^* (\rho_{1,m}^* - 1) + \rho_{1,m}^* = \frac{\kappa_2}{4} \rho_{2,m}^* (\rho_{2,m}^* - 1) + \rho_{2,m}^*.$$

For $j = 1, 2$, from (4.13) we have

$$\frac{\partial_j B_{m,1}^{\gamma_m}}{B_{m,1}^{\gamma_m}} = -\left(\frac{\kappa_j}{2} \rho_{j,m}^* (\rho_{j,m}^* - 1) + 2\rho_{j,m}^*\right) \frac{A_{j,1}^2}{E_{j,m}^2} \partial t_j. \tag{4.22}$$

For $1 \le m_1 < m_2 \le N$, let

$$\delta_{m_1,m_2} = \frac{\kappa_1}{2} \rho_{1,m_1}^* \rho_{1,m_2}^* = \frac{\kappa_2}{2} \rho_{2,m_1}^* \rho_{2,m_2}^*.$$

From (4.15), for $j = 1, 2$,

$$\frac{\partial_j |C_{m_1,m_2}|^{\delta_{m_1,m_2}}}{|C_{m_1,m_2}|^{\delta_{m_1,m_2}}} = -\kappa_j \rho_{j,m_1}^* \rho_{j,m_2}^* \frac{A_{j,1}^2}{E_{j,m_1} E_{j,m_2}} \partial t_j. \tag{4.23}$$

For $(t_1,t_2)\in\mathcal{D}$, let

$$F(t_1,t_2)=\exp\bigg(\int_0^{t_2}\int_0^{t_1}\frac{2A_{1,1}(s_1,s_2)A_{2,1}(s_1,s_2)}{E(s_1,s_2)^2}ds_1ds_2\bigg). \tag{4.24}$$

From (4.15) in [19], for $j=1,2$,

$$\frac{\partial_j F^{-\lambda}}{F^{-\lambda}}=-\lambda\bigg(\frac14\cdot\frac{A_{j,2}^2}{A_{j,1}^2}-\frac16\cdot\frac{A_{j,3}}{A_{j,1}}\bigg)\partial t_j. \tag{4.25}$$

Let

$$\widetilde{M}=\frac{A_{1,1}^{\alpha_1}A_{2,1}^{\alpha_2}}{E^{1/2}}F^{-\lambda}\prod_{m=1}^N\bigg(B_{m,1}^{\gamma_m}\prod_{j=1}^2|E_{j,m}|^{\rho^*_{j,m}}\bigg)\prod_{1\le m_1<m_2\le N}|C_{m_1,m_2}|^{\delta_{m_1,m_2}}. \tag{4.26}$$

Now we compute the SDE for $\partial_j\widetilde{M}/\widetilde{M}$ in terms of $\partial\xi_j(t_j)$ and ∂t_j. The coefficient of the $\partial\xi_j(t_j)$ term should be the sum of the coefficients of the $\partial\xi_j(t_j)$ terms in (4.17)–(4.25). The SDEs in (4.17)–(4.25) that contain stochastic terms are (4.17), (4.19) and (4.21). So the sum is equal to

$$\frac{6-\kappa_j}{2\kappa_j}\cdot\frac{A_{j,2}}{A_{j,1}}-\frac12\cdot\frac{A_{j,1}}{E_{j,0}}+\sum_{m=1}^N\rho^*_{j,m}\frac{A_{j,1}}{E_{j,m}}. \tag{4.27}$$

The coefficient of the ∂t_j term equals to the sum of the coefficients of the ∂t_j terms in (4.17)–(4.25) plus the sum of the coefficients of the drift terms coming out of products. The drift term in the SDE for $\partial_j\widetilde{M}/\widetilde{M}$ contributed by the products of (4.17) and SDEs in (4.19) is

$$\kappa_j\cdot\sum_{m=1}^N\frac{6-\kappa_j}{2\kappa_j}\cdot\frac{A_{j,2}}{A_{j,1}}\cdot\rho^*_{j,m}\cdot\frac{A_{j,1}}{E_{j,m}}=-\sum_{m=1}^N\rho^*_{j,m}\cdot\frac{\kappa_j-6}{2}\cdot\frac{A_{j,2}}{E_{j,m}}. \tag{4.28}$$

The drift term contributed by the products of (4.21) and SDEs in (4.19) is

$$\kappa_j\cdot\sum_{m=1}^N\rho^*_{j,m}\cdot\frac{A_{j,1}}{E_{j,m}}\cdot\bigg(-\frac12\cdot\frac{A_{j,1}}{E_{j,0}}\bigg)=-\sum_{m=1}^N\frac{\kappa_j\rho^*_{j,m}}{2}\cdot\frac{A_{j,1}^2}{E_{j,0}E_{j,m}}. \tag{4.29}$$

The drift term contributed by the product of (4.17) and (4.21) is

$$\kappa_j\cdot\frac{6-\kappa_j}{2\kappa_j}\cdot\frac{A_{j,2}}{A_{j,1}}\cdot\bigg(-\frac12\cdot\frac{A_{j,1}}{E_{j,0}}\bigg)=-\frac{6-\kappa_j}{4}\cdot\frac{A_{j,2}}{E_{j,0}}. \tag{4.30}$$

The drift term contributed by the products of pairs of SDEs in (4.19) is

$$\kappa_j\cdot\sum_{1\le m_1<m_2\le N}\rho^*_{j,m_1}\cdot\frac{A_{j,1}}{E_{j,m_1}}\cdot\rho^*_{j,m_2}\cdot\frac{A_{j,1}}{E_{j,m_2}}=\sum_{1\le m_1<m_2\le N}\kappa_j\rho^*_{j,m_1}\rho^*_{j,m_2}\frac{A_{j,1}^2}{E_{j,m_1}E_{j,m_2}}. \tag{4.31}$$

The sum of the coefficients of the ∂t_j terms in (4.17)–(4.25) is equal to

$$\lambda\left(\frac{1}{4}\cdot\frac{A_{j,2}^2}{A_{j,1}^2}-\frac{1}{6}\cdot\frac{A_{j,3}}{A_{j,1}}\right)-\frac{3\kappa_j-8}{8}\frac{A_{j,1}^2}{E_{j,0}^2}+\sum_{m=1}^{N}\rho_{j,m}^*\cdot\frac{\kappa_j-6}{2}\cdot\frac{A_{j,2}}{E_{j,m}}$$
$$+\sum_{m=1}^{N}\left(\frac{\kappa_j}{2}\rho_{j,m}^*(\rho_{j,m}^*-1)+2\rho_{j,m}^*\right)\frac{A_{j,1}^2}{E_{j,m}^2}+\sum_{m=1}^{N}\frac{\kappa_j\rho_{j,m}^*}{2}\cdot\frac{A_{j,1}^2}{E_{j,0}E_{j,m}}$$
$$+\left(\frac{6-\kappa_j}{4}\cdot\frac{A_{j,2}}{E_{j,0}}+\frac{3\kappa_j-8}{8}\frac{A_{j,1}^2}{E_{j,0}^2}\right)-\sum_{m=1}^{N}\left(\frac{\kappa_j}{2}\rho_{j,m}^*(\rho_{j,m}^*-1)+2\rho_{j,m}^*\right)\frac{A_{j,1}^2}{E_{j,m}^2}$$
$$-\sum_{1\le m_1<m_2\le N}\kappa_j\rho_{j,m_1}^*\rho_{j,m_2}^*\frac{A_{j,1}^2}{E_{j,m_1}E_{j,m_2}}-\lambda\left(\frac{1}{4}\cdot\frac{A_{j,2}^2}{A_{j,1}^2}-\frac{1}{6}\cdot\frac{A_{j,3}}{A_{j,1}}\right)$$
$$=\sum_{m=1}^{N}\rho_{j,m}^*\cdot\frac{\kappa_j-6}{2}\cdot\frac{A_{j,2}}{E_{j,m}}+\sum_{m=1}^{N}\frac{\kappa_j\rho_{j,m}^*}{2}\cdot\frac{A_{j,1}^2}{E_{j,0}E_{j,m}}$$
$$+\frac{6-\kappa_j}{4}\cdot\frac{A_{j,2}}{E_{j,0}}-\sum_{1\le m_1<m_2\le N}\kappa_j\rho_{j,m_1}^*\rho_{j,m_2}^*\frac{A_{j,1}^2}{E_{j,m_1}E_{j,m_2}}. \tag{4.32}$$

From (4.28)–(4.32), the SDE for $\partial_j\widetilde{M}/\widetilde{M}$ has no ∂t_j terms. Thus from (4.27), for $j=1,2$, we have

$$\frac{\partial_j\widetilde{M}}{\widetilde{M}}=\left(\frac{6-\kappa_j}{2\kappa_j}\cdot\frac{A_{j,2}}{A_{j,1}}-\frac{1}{2}\cdot\frac{A_{j,1}}{E_{j,0}}+\sum_{m=1}^{N}\rho_{j,m}^*\frac{A_{j,1}}{E_{j,m}}\right)\partial\xi_j(t_j). \tag{4.33}$$

For $(t_1,t_2)\in\mathcal{D}$, let

$$M(t_1,t_2)=\frac{\widetilde{M}(t_1,t_2)\widetilde{M}(0,0)}{\widetilde{M}(t_1,0)\widetilde{M}(0,t_2)}. \tag{4.34}$$

Then $M(t_1,0)=M(0,t_2)=1$ for $t_j\in[0,T_j)$, $j=1,2$.

The process $\widetilde{M}$ turns out to be the local Radon–Nikodym derivative of the coupling measure in Theorem 4.1 w.r.t. the product measure of two standard chordal SLE(κ) processes. Fix $j\ne k\in\{1,2\}$. Such $\widetilde{M}$ must satisfy SDE (4.33). So there are factors $A_{j,1}^{\alpha_j}$, $\prod_m|E_{j,m}|^{\rho_{j,m}^*}$, and $E^{-1/2}$ in (4.26). Other factors in (4.26) make $\widetilde{M}$ a local martingale in t_j, for any fixed t_k. Moreover, if t_j is fixed, $\widetilde{M}$ should also be a local martingale in t_k. And we expect some symmetry between j and k in the definition of $\widetilde{M}$. This gives restrictions on the values of κ_j and $\rho_{j,m}$, $j=1,2$, $1\le m\le N$. The process M then becomes the local Radon–Nikodym derivative of the coupling measure in Theorem 4.1 w.r.t. the product of its marginal measures. The property of M will be checked later.

Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Let $(\mathcal{F}_t^j)$ be the filtration generated by $B_j(t)$, $j = 1, 2$. Fix $j \ne k \in \{1, 2\}$. Suppose $\xi_j(t)$, $0 \le t < T_j$, is the maximal solution to the SDE:

$$d\xi_j(t) = \sqrt{\kappa_j} dB_j(t) + \left(\frac{-\kappa_j/2}{\xi_j(t) - \varphi_j(t_j, x_k)} + \sum_{m=1}^{N} \frac{\rho_{j,m}}{\xi_j(t) - \varphi_j(t_j, p_m)} \right) dt, \tag{4.35}$$

with $\xi_j(0) = x_j$. Then $(K_j(t), 0 \le t < T_j)$ is an $\mathrm{SLE}(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ process started from $(x_1; x_2, \vec{p})$. Since $\varphi_{k,t}(t, \cdot) = \mathrm{id}$, so at $t_j = t$ and $t_k = 0$, $A_{j,1} = 1$, $A_{j,2} = 0$, $E_{j,0} = \xi(j) - \varphi_j(t, \xi_k)$, and $E_{j,m} = \xi(j) - \varphi_j(t, p_m)$, $1 \le m \le N$. Thus

$$\left(\frac{6-\kappa_j}{2\kappa_j} \cdot \frac{A_{j,2}}{A_{j,1}} - \frac{1}{2} \cdot \frac{A_{j,1}}{E_{j,0}} + \sum_{m=1}^{N} \rho_{j,m}^* \frac{A_{j,1}}{E_{j,m}} \right) \Bigg|_{t_j = t, t_k = 0} = \frac{-1/2}{\xi_j(t) - \varphi_j(t_j, x_k)} + \sum_{m=1}^{N} \frac{\rho_{j,m}^*}{\xi_j(t) - \varphi_j(t_j, p_m)}. \tag{4.36}$$

For $j \ne k \in \{1, 2\}$ and $t_k \in [0, T_k)$, let $T_j(t_k) \in (0, T_j]$ be the largest number such that for $0 \le t < T_j(t_k)$, $\overline{K_j(t)} \cap \overline{K_k(t_k)} = \emptyset$.

Theorem 4.2 *Fix $j \ne k \in \{1, 2\}$. Let $\bar{t}_k$ be an $(\mathcal{F}_t^k)$-stopping time. Then the process $t \mapsto M|_{t_j = t, t_k = \bar{t}_k}$, $0 \le t < T_j(\bar{t}_k)$, is an $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)_{t \ge 0}$-local martingale, and*

$$\frac{\partial_j M}{M} \Bigg|_{t_j = t, t_k = \bar{t}_k} = \Bigg[\left(\frac{6-\kappa_j}{2\kappa_j} \cdot \frac{A_{j,2}}{A_{j,1}} - \frac{1}{2} \cdot \frac{A_{j,1}}{E_{j,0}} + \sum_{m=1}^{N} \rho_{j,m}^* \frac{A_{j,1}}{E_{j,m}} \right) \Bigg|_{t_j = t, t_k = \bar{t}_k} - \left(\frac{-1/2}{\xi_j(t) - \varphi_j(t, x_k)} + \sum_{m=1}^{N} \frac{\rho_{j,m}^*}{\xi_j(t) - \varphi_j(t, p_m)} \right) \Bigg] \sqrt{\kappa_j} \partial B_j(t). \tag{4.37}$$

Proof. This follows from (4.33)–(4.36), where all functions are valued at $t_j = t$ and $t_k = \bar{t}_k$, and all SDE are $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)$-adapted. □

Now we make some improvement over the above theorem. Let $\bar{t}_2$ be an $(\mathcal{F}_t^2)$-stopping time with $\bar{t}_2 < T_2$. Suppose R is an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t \ge 0}$-stopping time with $R < T_1(\bar{t}_2)$. Let $\mathcal{F}_{R, \bar{t}_2}$ denote the σ-algebra obtained from the filtration $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t \ge 0}$ and its stopping time R, i.e., $\mathcal{E} \in \mathcal{F}_{R, \bar{t}_2}$ iff for any $t \ge 0$, $\mathcal{E} \cap \{R \le t\} \in \mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2$. For every $t \ge 0$, $R + t$ is also an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t \ge 0}$-stopping time. So we have a filtration $(\mathcal{F}_{R+t, \bar{t}_2})_{t \ge 0}$.

Theorem 4.3 *Let $\bar{t}_2$ and R be as above. Let $I \in [0, \bar{t}_2]$ be $\mathcal{F}_{R, \bar{t}_2}$-measurable. Then $(M(R + t, I), 0 \le t < T_1(I) - R)$ is a continuous $(\mathcal{F}_{R+t, \bar{t}_2})_{t \ge 0}$-local martingale.*

Proof. Let $B_1^R(t) = B_1(R+t) - B_1(R)$, $0 \le t < \infty$. Since $B_1(t)$ is an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t\ge 0}$-Brownian motion, so $B_1^R(t)$ is an $(\mathcal{F}_{R+t,\bar{t}_2})_{t\ge 0}$-Brownian motion. Since $\varphi_1(R+t,\cdot)$ is $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t\ge 0}$-adapted, so $\xi_1(R+t)$ satisfies the $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t\ge 0}$-adapted SDE

$$d\xi_1(R+t) = \sqrt{\kappa} dB_1^R(t) d\xi_j(t) + \frac{-\kappa_1/2}{\xi_1(R+t) - \varphi_1(R+t, x_2)} dt + \sum_{m=1}^{N} \frac{\rho_{1,m}}{\xi_1(R+t) - \varphi_1(R+t, p_m)} dt.$$

Now we show that $\varphi_2(I,\cdot)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable. Fix $n \in \mathbb{N}$. Let $I_n = \lfloor nI \rfloor / n$. For $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{E}_n(m) = \{m/n \le I < (m+1)/n\}$. Then $\mathcal{E}_n(m)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable, and $I_n = m/n$ on $\mathcal{E}_n(m)$. Since $m/n \le \bar{t}_2$ and $I_n = m/n$ on $\mathcal{E}_n(m)$, so I_n agrees with $(m/n) \wedge \bar{t}_2$ on $\mathcal{E}_n(m)$. Now $(m/n) \wedge \bar{t}_2$ is an $(\mathcal{F}_t^2)$-stopping time, and $\mathcal{F}^2_{(m/n)\wedge \bar{t}_2} \subset \mathcal{F}^2_{\bar{t}_2} \subset \mathcal{F}_{R,\bar{t}_2}$. So $\varphi_2((m/n) \wedge \bar{t}_2, \cdot)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable. Since $\varphi_2(I_n,\cdot) = \varphi_2((m/n) \wedge \bar{t}_2, \cdot)$ on $\mathcal{E}_n(m)$, and $\mathcal{E}_n(m)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable for each $m \in \mathbb{N} \cup \{0\}$, so $\varphi_2(I_n,\cdot)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable. Since $\varphi_2(I_n,\cdot) \to \varphi_2(I,\cdot)$ as $n \to \infty$, so $\varphi_2(I,\cdot)$ is also $\mathcal{F}_{R,\bar{t}_2}$-measurable. Thus $K_2(I)$ is $\mathcal{F}_{R,\bar{t}_2}$-measurable. Hence for any $t \ge 0$, $\varphi_{K_1(R+t)\cup K_2(I)}$ is $\mathcal{F}_{R+t,\bar{t}_2}$-measurable. From (4.2), $\varphi_{1,I}(R+t,\cdot)$ and $\varphi_{2,R+t}(I,\cdot)$ are both $\mathcal{F}_{R+t,\bar{t}_2}$-measurable. If the t_j and t_k in (4.17)–(4.25) are replaced by $R+t$ and I, respectively, then all these SDEs are $\mathcal{F}_{R+t,\bar{t}_2}$-adapted. From the same computation, we conclude that $(M(R+t, I), 0 \le t < T_1(I) - R)$ is a continuous $(\mathcal{F}_{R+t,\bar{t}_2})_{t\ge 0}$-local martingale. □

Let HP denote the set of (H_1, H_2) such that H_j is a hull in $\mathbb{H}$ w.r.t. ∞ that contains some neighborhood of x_j in $\mathbb{H}$, $j = 1, 2$, $\overline{H_1} \cap \overline{H_2} = \emptyset$, and $p_m \notin \overline{H_1} \cup \overline{H_2}$, $1 \le m \le N$. For $(H_1, H_2) \in$ HP, let $T_j(H_j)$ be the first time that $\overline{K_j(t)} \cap \overline{\mathbb{H} \setminus H_j} \ne \emptyset$, $j = 1, 2$. An argument that is similar to the proof of Theorem 5.1 in [19] gives the following.

Theorem 4.4 *For any* $(H_1, H_2) \in$ HP, *there are* $C_2 > C_1 > 0$ *depending only on* H_1 *and* H_2 *such that* $M(t_1, t_2) \in [C_1, C_2]$ *for any* $(t_1, t_2) \in [0, T_1(H_1)] \times [0, T_2(H_2)]$.

Fix $(H_1, H_2) \in$ HP. Let μ denote the joint distribution of $(\xi_1(t) : 0 \le t \le T_1)$ and $(\xi_2(t) : 0 \le t \le T_2)$. From Theorem 4.2 and Theorem 4.4, we have

$$\int M(T_1(H_1), T_2(H_2)) d\mu = \mathbf{E}_\mu[M(T_1(H_1), T_2(H_2))] = M(0,0) = 1.$$

Note that $M(T_1(H_1), T_2(H_2)) > 0$. Suppose ν is a measure on $\mathcal{F}^1_{T_1(H_1)} \times \mathcal{F}^2_{T_2(H_2)}$ such that $d\nu/d\mu = M(T_1(H_1), T_2(H_2))$. Then ν is a probability measure. Now suppose the joint distribution of $(\xi_1(t), 0 \le t \le T_1(H_1))$ and $(\xi_2(t), 0 \le t \le T_2(H_2))$ is ν instead of μ. Fix an $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2$ with $\bar{t}_2 \le T_2(H_2)$. From (4.35)–(4.37) and Girsanov theorem, there is an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)$-Brownian motion $\bar{B}_1(t)$ such that $\xi_1(t)$ satisfies the $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)$-adapted SDE for $0 \le t \le T_1(H_1)$:

$$d\xi_1(t) = \sqrt{\kappa_1} d\bar{B}_1(t) + \left(\frac{6-\kappa_1}{2} \cdot \frac{A_{1,2}}{A_{1,1}} - \frac{\kappa_1}{2} \cdot \frac{A_{1,1}}{E_{1,0}} + \sum_{m=1}^{N} \rho_{1,m} \frac{A_{1,1}}{E_{1,m}}\right)\Bigg|_{(t,\bar{t}_2)} dt.$$

Let $\xi_{1,\bar{t}_2}(t) = A_{1,0}(t,\bar{t}_2) = \varphi_{2,t}(\bar{t}_2, \xi_1(t))$, $0 \le t \le T_1(H_1)$. From Ito's formula and (4.3), $\xi_{1,\bar{t}_2}(t)$ satisfies

$$d\xi_{1,\bar{t}_2}(t) = A_{1,1}(t,\bar{t}_2)\sqrt{\kappa_1} d\bar{B}_1(t) + \left(-\frac{\kappa_1}{2} \cdot \frac{A_{1,1}^2}{E_{1,0}} + \sum_{m=1}^{N} \rho_{1,m} \frac{A_{1,1}^2}{E_{1,m}}\right)\Bigg|_{(t,\bar{t}_2)} dt. \tag{4.38}$$

Since $\varphi_2(\bar{t}_2, \cdot)$ is a conformal map, and from (4.1), for $0 \le t_1 < T_1(\bar{t}_2)$,

$$K_{1,\bar{t}_2}(t) = (K_1(t) \cup K_2(\bar{t}_2))/K_2(\bar{t}_2) = \varphi_2(\bar{t}_2, K_1(t)),$$

so $(K_{1,\bar{t}_2}(t))$ is a Loewner chain. Let $v(t) = \mathrm{hcap}(K_{1,\bar{t}_2}(t))/2$. From Proposition 2.2, $v(t)$ is a continuous increasing function with $v(0) = 0$, and $(\widetilde{K}(t) = K_{1,\bar{t}_2}(v^{-1}(t)))$ are chordal Loewner hulls driven by some real continuous function, say $\widetilde{\xi}(t)$, and the chordal Loewner maps are $\widetilde{\varphi}(t,\cdot) = \varphi_{K_{1,\bar{t}_2}(v^{-1}(t))} = \varphi_{1,\bar{t}_2}(v^{-1}(t), \cdot)$. Moreover,

$$\{\widetilde{\xi}(v(t))\} = \bigcap_{\varepsilon>0} \overline{\widetilde{K}(v(t+\varepsilon))/\widetilde{K}(v(t))}; \quad \{\xi_1(t)\} = \bigcap_{\varepsilon>0} \overline{K_1(t+\varepsilon)/K_1(t)}.$$

Let $W_t = \varphi_{2,t}(\bar{t}_2, \cdot)$. From (4.2), for $\varepsilon > 0$, we have

$$\begin{aligned} &W_t(K_1(t+\varepsilon)/K_1(t)) \\ &\quad = W_t \circ \varphi_1(t,\cdot)(K_1(t+\varepsilon) \setminus K_1(t)) \\ &\quad = \varphi_{K_1(t)\cup K_2(\bar{t}_2)}(K_1(t+\varepsilon) \setminus K_1(t)) \\ &\quad = \varphi_{K_1(t)\cup K_2(\bar{t}_2)}((K_1(t+\varepsilon) \cup K_2(\bar{t}_2)) \setminus (K_1(t) \cup K_2(\bar{t}_2))) \\ &\quad = (K_1(t+\varepsilon) \cup K_2(\bar{t}_2))/(K_1(t) \cup K_2(\bar{t}_2)) \\ &\quad = [(K_1(t+\varepsilon) \cup K_2(\bar{t}_2))/K_2(\bar{t}_2)]/[(K_1(t) \cup K_2(\bar{t}_2))/K_2(\bar{t}_2)] \\ &\quad = K_{1,\bar{t}_2}(t+\varepsilon)/K_{1,\bar{t}_2}(t) = \widetilde{K}(v(t+\varepsilon))/\widetilde{K}(v(t)). \end{aligned}$$

Thus $\widetilde{\xi}(v(t)) = W_t(\xi_1(t)) = \varphi_{2,t}(\bar{t}_2, \xi_1(t)) = \xi_{1,\bar{t}_2}(t)$. Since $\mathrm{hcap}(K_1(t+\varepsilon)/K_1(t)) = 2\varepsilon$ and $\mathrm{hcap}(\widetilde{K}(v(t+\varepsilon))/\widetilde{K}(v(t))) = 2v(t+\varepsilon) - 2v(t)$, so from Proposition 2.1, $v'(t) = W_t'(\xi_1(t))^2 = \partial_z \varphi_{2,t}(\bar{t}_2, \xi_1(t))^2 = A_{1,1}(t,\bar{t}_2)^2$.

From the definitions of $E_{1,0}$ and $E_{1,m}$, and (4.2), we have

$$E_{1,0}\big(v^{-1}(t), \bar{t}_2\big) = \widetilde{\xi}(t) - \widetilde{\varphi}(t, \xi_2(\bar{t}_2)); \tag{4.39}$$

$$E_{1,m}\big(v^{-1}(t), \bar{t}_2\big) = \widetilde{\xi}(t) - \widetilde{\varphi}(t, \varphi_2(\bar{t}_2, p_m)). \tag{4.40}$$

From (4.38)–(4.40), and the properties of $v(t)$ and $\widetilde{\xi}(t)$, there is a Brownian motion $\widetilde{B}_1(t)$ such that $\widetilde{\xi}(t)$, $0 \le t < v(T_1(H_1))$, satisfies the SDE:

$$d\widetilde{\xi}(t) = \sqrt{\kappa_1} d\widetilde{B}_1(t) + \left(\frac{-\kappa_1/2}{\widetilde{\xi}(t) - \widetilde{\varphi}(t, \xi_2(\bar{t}_2))} + \sum_{m=1}^{N} \frac{\rho_{1,m}}{\widetilde{\xi}(t) - \widetilde{\varphi}(t, \varphi_2(\bar{t}_2, p_m))} \right) dt.$$

Note that $\widetilde{\xi}(0) = \xi_{1,\bar{t}_2}(0) = \varphi_2(\bar{t}_2, x_1)$. Thus conditioned on $\mathcal{F}^2_{\bar{t}_2}$, $\widetilde{K}(t)$, $0 \le t < v(T_1(H_1))$, is a partial chordal SLE$(\kappa_1; -\frac{\kappa_1}{2}, \vec{\rho}_1)$ process started from $(\varphi_2(\bar{t}_2, x_1); \xi_2(\bar{t}_2), \varphi_2(\bar{t}_2, \vec{p}))$. By symmetry, we may exchange the subscripts 1 and 2 in the above statement.

Theorem 4.5 *Suppose $n \in \mathbb{N}$ and $(H_1^m, H_2^m) \in$ HP, $1 \le m \le n$. There is a continuous function $M_*(t_1, t_2)$ defined on $[0, \infty]^2$ that satisfies the following properties:*

(i) $M_* = M$ *on* $[0, T_1(H_1^m)] \times [0, T_2(H_2^m)]$ *for* $1 \le m \le n$;
(ii) $M_*(t, 0) = M_*(0, t) = 1$ *for any* $t \ge 0$;
(iii) $M(t_1, t_2) \in [C_1, C_2]$ *for any* $t_1, t_2 \ge 0$, *where* $C_2 > C_1 > 0$ *are constants depending only on* H_j^m, $j = 1, 2$, $1 \le m \le n$;
(iv) *for any* $(\mathcal{F}_t^2)$*-stopping time* $\bar{t}_2$, $(M_*(t_1, \bar{t}_2), t_1 \ge 0)$ *is a bounded continuous* $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar{t}_2})_{t_1 \ge 0}$*-martingale; and*
(v) *for any* $(\mathcal{F}_t^1)$*-stopping time* $\bar{t}_1$, $(M_*(\bar{t}_1, t_2), t_2 \ge 0)$ *is a bounded continuous* $(\mathcal{F}^1_{\bar{t}_1} \times \mathcal{F}^2_{t_2})_{t_2 \ge 0}$*-martingale.*

Proof. This is Theorem 6.1 in [19]. For reader's convenience, we include the proof here. The first quadrant $[0, \infty]^2$ will be divided by the vertical or horizontal lines $\{x_j = T_j(H_j^m)\}$, $1 \le m \le n$, $j = 1, 2$, into small rectangles, and M_* will be piecewise defined on these rectangles. Theorem 4.4 will be used to prove the boundedness, and Theorem 4.3 will be used to prove the martingale properties.

Let $\mathbb{N}_n := \{k \in \mathbb{N} : k \le n\}$. Write T_j^k for $T_j(H_j^k)$, $k \in \mathbb{N}_n$, $j = 1, 2$. Let $S \subset \mathbb{N}_n$ be such that $\bigcup_{k \in S} [0, T_1^k] \times [0, T_2^k] = \bigcup_{k=1}^n [0, T_1^k] \times [0, T_2^k]$, and $\sum_{k \in S} k \le \sum_{k \in S'} k$ if $S' \subset \mathbb{N}_n$ also satisfies this property. Such S is a random nonempty set, and $|S| \in \mathbb{N}_n$ is a random number. Define a partial order "$\preceq$" on $[0, \infty]^2$ such that $(s_1, s_2) \preceq (t_1, t_2)$ iff $s_1 \le t_1$ and $s_2 \le t_2$. If $(s_1, s_2) \preceq (t_1, t_2)$ and $(s_t, s_2) \ne (t_1, t_2)$, we write $(s_1, s_2) \prec (t_1, t_2)$. Then for each $m \in \mathbb{N}_n$ there is $k \in S$ such that $(T_1^m, T_2^m) \preceq (T_1^k, T_2^k)$; and for each $k \in S$ there is no $m \in \mathbb{N}_n$ such that $(T_1^k, T_2^k) \prec (T_1^m, T_2^m)$.

There is a map σ from $\{1, \ldots, |S|\}$ onto S such that if $1 \le k_1 < k_2 \le |S|$, then

$$T_1^{\sigma(k_1)} < T_1^{\sigma(k_2)}, \quad T_2^{\sigma(k_1)} > T_2^{\sigma(k_2)}. \tag{4.41}$$

Define $T_1^{\sigma(0)} = T_2^{\sigma(|S|+1)} = 0$ and $T_1^{\sigma(|S|+1)} = T_2^{\sigma(0)} = \infty$. Then (4.41) still holds for $0 \le k_1 < k_2 \le |S| + 1$.

Extend the definition of M to $[0,\infty]\times\{0\}\cup\{0\}\times[0,\infty]$ such that $M(t,0)=M(0,t)=1$ for $t\ge 0$. Fix $(t_1,t_2)\in[0,\infty]^2$. There are $k_1\in\mathbb{N}_{|S|+1}$ and $k_2\in\mathbb{N}_{|S|}\cup\{0\}$ such that

$$T_1^{\sigma(k_1-1)}\le t_1\le T_1^{\sigma(k_1)},\quad T_2^{\sigma(k_2+1)}\le t_2\le T_2^{\sigma(k_2)}. \tag{4.42}$$

If $k_1\le k_2$, let

$$M_*(t_1,t_2)=M(t_1,t_2). \tag{4.43}$$

If $k_1\ge k_2+1$, let

$$M_*(t_1,t_2)=\frac{M\big(T_1^{\sigma(k_2)},t_2\big)M\big(T_1^{\sigma(k_2+1)},T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1-1)},T_2^{\sigma(k_1-1)}\big)M\big(t_1,T_2^{\sigma(k_1)}\big)}{M\big(T_1^{\sigma(k_2)},T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1-2)},T_2^{\sigma(k_1-1)}\big)M\big(T_1^{\sigma(k_1-1)},T_2^{\sigma(k_1)}\big)}. \tag{4.44}$$

In the above formula, there are k_1-k_2+1 terms in the numerator, and k_1-k_2 terms in the denominator. For example, if $k_1-k_2=1$, then

$$M_*(t_1,t_2)=M\big(T_1^{\sigma(k_2)},t_2\big)M\big(t_1,T_2^{\sigma(k_1)}\big)/M\big(T_1^{\sigma(k_2)},T_2^{\sigma(k_1)}\big).$$

We need to show that $M_*(t_1,t_2)$ is well-defined. First, we show that the $M(\cdot,\cdot)$ in (4.43) and (4.44) are defined. Note that M is defined on

$$Z:=\bigcup_{k=0}^{|S|+1}\big[0,T_1^{\sigma(k)}\big]\times\big[0,T_2^{\sigma(k)}\big].$$

If $k_1\le k_2$ then $t_1\le T_1^{\sigma(k_1)}\le T_1^{\sigma(k_2)}$ and $t_2\le T_2^{\sigma(k_2)}$, so $(t_1,t_2)\in Z$. Thus $M(t_1,t_2)$ in (4.43) is defined. Now suppose $k_1\ge k_2+1$. Since $t_2\le T_2^{\sigma(k_2)}$ and $t_1\le T_1^{\sigma(k_1)}$, so $(T_1^{\sigma(k_2)},t_2),(t_1,T_2^{\sigma(k_1)})\in Z$. It is clear that $(T_1^{\sigma(k)},T_2^{\sigma(k)})\in Z$ for $k_2+1\le k\le k_1-1$. Thus the $M(\cdot,\cdot)$ in the numerator of (4.44) are defined. For $k_2\le k\le k_1-1$, $T_1^{\sigma(k)}\le T_1^{\sigma(k+1)}$, so $(T_1^{\sigma(k)},T_2^{\sigma(k+1)})\in Z$. Thus the $M(\cdot,\cdot)$ in the denominator of (4.44) are defined.

Second, we show that the value of $M_*(t_1,t_2)$ does not depend on the choice of (k_1,k_2) that satisfies (4.42). Suppose (4.42) holds with (k_1,k_2) replaced by (k_1',k_2), and $k_1'\ne k_1$. Then $|k_1'-k_1|=1$. We may assume $k_1'=k_1+1$. Then $t_1=T_1^{\sigma(k_1)}$. Let $M_*'(t_1,t_2)$ denote the $M_*(t_1,t_2)$ defined using (k_1',k_2). There are three cases.

Case 1 $k_1<k_1'\le k_2$. Then from (4.43), $M_*'(t_1,t_2)=M(t_1,t_2)=M_*(t_1,t_2)$.

Case 2 $k_1=k_2$ and $k_1'-k_2=1$. Then $T_1^{\sigma(k_2)}=T_1^{\sigma(k_1)}=t_1$. So from (4.43) and (4.44),

$$\begin{aligned}M_*'(t_1,t_2)&=M\big(T_1^{\sigma(k_2)},t_2\big)M\big(t_1,T_2^{\sigma(k_1)}\big)/M\big(T_1^{\sigma(k_2)},T_2^{\sigma(k_1)}\big)\\&=M(t_1,t_2)=M_*(t_1,t_2).\end{aligned}$$

Case 3 $k_1' > k_1 > k_2$. From (4.44) and that $T_1^{\sigma(k_1)} = t_1$, we have

$$
\begin{aligned}
&M_*'(t_1, t_2) \\
&= \frac{M\big(T_1^{\sigma(k_2)}, t_2\big)M\big(T_1^{\sigma(k_2+1)}, T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1)}, T_2^{\sigma(k_1)}\big)M\big(t_1, T_2^{\sigma(k_1+1)}\big)}{M\big(T_1^{\sigma(k_2)}, T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_1)}\big)M\big(T_1^{\sigma(k_1)}, T_2^{\sigma(k_1+1)}\big)} \\
&= \frac{M\big(T_1^{\sigma(k_2)}, t_2\big)M\big(T_1^{\sigma(k_2+1)}, T_2^{\sigma(k_2+1)}\big)\cdots M\big(t_1, T_2^{\sigma(k_1)}\big)}{M\big(T_1^{\sigma(k_2)}, T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_1)}\big)} = M_*(t_1, t_2).
\end{aligned}
$$

Similarly, if (4.42) holds with (k_1, k_2) replaced by (k_1, k_2'), then $M_*(t_1, t_2)$ defined using (k_1, k_2') has the same value as $M(t_1, t_2)$. Thus M_* is well-defined.

From the definition, it is clear that for each $k_1 \in \mathbb{N}_{|S|+1}$ and $k_2 \in \mathbb{N}_{|S|} \cup \{0\}$, M_* is continuous on $[T_1^{\sigma(k_1-1)}, T_1^{\sigma(k_1)}] \times [T_2^{\sigma(k_2+1)}, T_1^{\sigma(k_2)}]$. Thus M_* is continuous on $[0, \infty]^2$. Let $(t_1, t_2) \in [0, \infty]^2$. Suppose $(t_1, t_2) \in [0, T_1^m] \times [0, T_2^m]$ for some $m \in \mathbb{N}_n$. There is $k \in \mathbb{N}_{|S|}$ such that $(T_1^m, T_2^m) \preceq (T_1^{\sigma(k)}, T_2^{\sigma(k)})$. Then we may choose $k_1 \le k$ and $k_2 \ge k$ such that (4.42) holds, so $M_*(t_1, t_2) = M(t_1, t_2)$. Thus (i) is satisfied. If $t_1 = 0$, we may choose $k_1 = 1$ in (4.42). Then either $k_1 \le k_2$ or $k_2 = 0$. If $k_1 \le k_2$ then $M_*(t_1, t_2) = M(t_1, t_2) = 1$ because $t_1 = 0$. If $k_2 = 0$, then

$$
M_*(t_1, t_2) = M\big(T_1^{\sigma(0)}, t_2\big)M\big(t_1, T_2^{\sigma(1)}\big)/M\big(T_1^{\sigma(0)}, T_2^{\sigma(1)}\big) = 1
$$

because $T_1^{\sigma(0)} = t_1 = 0$. Similarly, $M_*(t_1, t_2) = 0$ if $t_2 = 0$. So (ii) is also satisfied. And (iii) follows from Theorem 4.4 and the definition of M_*.

Now we prove (iv). Suppose $(t_1, t_2) \in [0, \infty]^2$ and $t_2 \ge \bigvee_{m=1}^n T_2^m = T_2^{\sigma(1)}$. Then (4.42) holds with $k_2 = 0$ and some $k_1 \in \{1, \ldots, |S| + 1\}$. So $k_1 \ge k_2 + 1$. Since $T_1^{\sigma(k_2)} = 0$ and $M(0, t) = 1$ for any $t \ge 0$, so from (4.44) we have

$$
M_*(t_1, t_2) = \frac{M\big(T_1^{\sigma(k_2+1)}, T_2^{\sigma(k_2+1)}\big)\cdots M\big(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_1-1)}\big)M\big(t_1, T_2^{\sigma(k_1)}\big)}{M\big(T_1^{\sigma(k_2+1)}, T_2^{\sigma(k_2+2)}\big)\cdots M\big(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_1)}\big)}.
$$

Since the right-hand side of the above equality has no t_2, so $M_*(t_1, t_2) = M_*(t_1, \bigvee_{m=1}^n T_2^m)$ for any $t_2 \ge \bigvee_{m=1}^n T_2^m$. Similarly, $M_*(t_1, t_2) = M_*(\bigvee_{m=1}^n T_1^m, t_2)$ for any $t_1 \ge \bigvee_{m=1}^n T_1^m$.

Fix an $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2$. Since $M_*(\,\cdot\,, \bar{t}_2) = M_*(\,\cdot\,, \bar{t}_2 \wedge (\bigvee_{m=1}^n T_2^m))$, and $\bar{t}_2 \wedge (\bigvee_{m=1}^n T_2^m)$ is also an $(\mathcal{F}_t^2)$-stopping time, so we may assume that $\bar{t}_2 \le \bigvee_{m=1}^n T_2^m$. Let $I_0 = \bar{t}_2$. For $s \in \mathbb{N} \cup \{0\}$, define

$$
\begin{aligned}
R_s &= \sup\big\{T_1^m : m \in \mathbb{N}_n, T_2^m \ge I_s\big\}; \\
I_{s+1} &= \sup\big\{T_2^m : m \in \mathbb{N}_n, T_2^m < I_s, T_1^m > R_s\big\}.
\end{aligned} \tag{4.45}
$$

Here we set $\sup(\emptyset)=0$. Then we have a non-decreasing sequence (R_s) and a non-increasing sequence (I_s). Let S and $\sigma(k)$, $0\le k\le |S|+1$, be as in the definition of M_*. From the property of S, for any $s\in\mathbb{N}\cup\{0\}$,

$$R_s=\sup\left\{T_1^k: k\in S, T_2^k\ge I_s\right\}. \tag{4.46}$$

Suppose for some $s\in\mathbb{N}\cup\{0\}$, there is $m\in\mathbb{N}_n$ that satisfies $T_2^m<I_s$ and $T_1^m>R_s$. Then there is $k\in S$ such that $T_j^k\ge T_j^m$, $j=1,2$. If $T_2^k\ge I_s$, then from (4.46) we have $R_s\ge T_1^k\ge T_1^m$, which contradicts that $T_1^m>R_s$. Thus $T_2^k<I_s$. Now $T_2^k<I_s$, $T_1^k\ge T_1^m>R_s$, and $T_2^k\ge T_2^m$. Thus for any $s\in\mathbb{N}\cup\{0\}$,

$$I_{s+1}=\sup\left\{T_2^k: k\in S, T_2^k<I_s, T_1^k>R_s\right\}. \tag{4.47}$$

First suppose $\bar t_2>0$. Since $\bar t_2\le\bigvee_{m=1}^n T_2^m=T_2^{\sigma(0)}$, so there is a unique $k_2\in\mathbb{N}_{|S|}$ such that $T_2^{\sigma(k_2)}\ge\bar t_2>T_2^{\sigma(k_2+1)}$. From (4.46) and (4.47), we have $R_s=T_1^{\sigma(k_2+s)}$ for $0\le s\le |S|-k_2$; $R_s=T_1^{\sigma(|S|)}$ for $s\ge |S|-k_2$; $I_s=T_2^{\sigma(k_2+s)}$ for $1\le s\le |S|-k_2$; and $I_s=0$ for $s\ge |S|-k_2+1$. Since $R_0=T_1^{\sigma(k_2)}$ and $\bar t_2\le T_2^{\sigma(k_2)}$, so from (i),

$$M_*(t_1,\bar t_2)=M(t_1,\bar t_2),\quad \text{for } t_1\in[0,R_0]. \tag{4.48}$$

Suppose $t_1\in[R_{s-1},R_s]$ for some $s\in\mathbb{N}_{|S|-k_2}$. Let $k_1=k_2+s$. Then $T_1^{\sigma(k_1-1)}\le t_1\le T_1^{\sigma(k_1)}$. Since $I_s=T_2^{\sigma(k_2+s)}=T_2^{\sigma(k_1)}$, so from (4.44),

$$M_*(t_1,\bar t_2)/M_*(R_{s-1},\bar t_2)=M(t_1,I_s)/M(R_{s-1},I_s),\quad \text{for } t_1\in[R_{s-1},R_s]. \tag{4.49}$$

Note that if $s\ge |S|-k_2+1$, (4.49) still holds because $R_s=R_{s-1}$. Suppose $t_1\ge R_n$. Since $n\ge |S|-k_2$, so $R_n=T_1^{\sigma(|S|)}=\bigvee_{m=1}^n T_1^m$. From the discussion at the beginning of the proof of (iv), we have

$$M_*(t_1,\bar t_2)=M_*(R_n,\bar t_2),\quad \text{for } t_1\in[R_n,\infty]. \tag{4.50}$$

If $\bar t_2=0$, (4.48)–(4.50) still hold because all $I_s=0$ and so $M_*(t_1,\bar t_2)=M(t_1,I_s)=M(t_1,0)=1$ for any $t_1\ge 0$.

Let $R_{-1}=0$. We claim that for each $s\in\mathbb{N}\cup\{0\}$, R_s is an $(\mathcal{F}_t^1\times\mathcal{F}_{\bar t_2}^2)_{t\ge0}$-stopping time, and I_s is $\mathcal{F}_{R_{s-1},\bar t_2}$-measurable. Recall that $\mathcal{F}_{R_{s-1},\bar t_2}$ is the σ-algebra obtained from the filtration $(\mathcal{F}_t^1\times\mathcal{F}_{\bar t_2}^2)_{t\ge0}$ and its stopping time R_{s-1}. It is clear that $R_{-1}=0$ is an $(\mathcal{F}_t^1\times\mathcal{F}_{\bar t_2}^2)_{t\ge0}$-stopping time, and $I_0=\bar t_2$ is $\mathcal{F}_{R_{-1},\bar t_2}$-measurable. Now suppose I_s is $\mathcal{F}_{R_{s-1},\bar t_2}$-measurable. Since $I_s\le\bar t_2$ and $R_{s-1}\le R_s$, so for any $t\ge0$, $\{R_s\le t\}=\{R_{s-1}\le t\}\cap\mathcal{E}_t$, where

$$\begin{aligned}\mathcal{E}_t&=\bigcap_{m=1}^n\left(\left\{T_2^m<I_s\right\}\cup\left\{T_1^m\le t\right\}\right)\\&=\bigcap_{m=1}^n\Big(\bigcup_{q\in\mathbb{Q}}\left(\left\{T_2^m<q\le\bar t_2\right\}\cap\{q<I_s\}\right)\cup\left\{T_1^m\le t\right\}\Big).\end{aligned}$$

Thus $\mathcal{E}_t \in \mathcal{F}_{R_{s-1},\bar{t}_2} \vee (\mathcal{F}^1_t \times \mathcal{F}^2_{\bar{t}_2})$, and so $\{R_s \le t\} \in \mathcal{F}^1_t \times \mathcal{F}^2_{\bar{t}_2}$ for any $t \ge 0$. Therefore R_s is an $(\mathcal{F}^1_t \times \mathcal{F}^2_{\bar{t}_2})_{t\ge 0}$-stopping time. Next we consider I_{s+1}. For any $h \ge 0$,

$$\begin{aligned}\{I_{s+1} > h\} &= \bigcup_{m=1}^{n} \left(\left\{h < T_2^m < I_s\right\} \cap \left\{T_1^m > R_s\right\}\right)\\ &= \bigcup_{m=1}^{n} \Big(\bigcup_{q\in\mathbb{Q}} \left(\left\{h < T_2^m < q < \bar{t}_2\right\} \cap \{q < I_s\}\right) \cap \left\{T_1^m > R_s\right\}\Big)\\ &\in \mathcal{F}_{R_s,\bar{t}_2}.\end{aligned}$$

Thus I_{s+1} is $\mathcal{F}_{R_s,\bar{t}_2}$-measurable. So the claim is proved by induction.

Since $\bar{t}_2 \le \bigvee_{m=1}^n T_2^m < T_2$, so from Theorem 4.3, for any $s \in \mathbb{N}_n$, $(M(R_{s-1}+t, I_s), 0 \le t < T_1(I_s) - R_{s-1})$ is a continuous $(\mathcal{F}_{R_{s-1}+t,\bar{t}_2})_{t\ge 0}$-local martingale. For $m \in \mathbb{N}_n$, if $T_2^m \ge I_s$, then $T_1^m < T_1(T_2^m) \le T_1(I_s)$. So from (4.45) we have $R_s < T_1(I_s)$. From (4.49), we find that $(M_*(R_{s-1}+t, \bar{t}_2), 0 \le t \le R_s - R_{s-1})$ is a continuous $(\mathcal{F}_{R_{s-1}+t,\bar{t}_2})_{t\ge 0}$-local martingale for any $s \in \mathbb{N}_n$. From Theorem 4.2 and (4.48), $(M_*(t, \bar{t}_2), 0 \le t \le R_0)$ is a continuous $(\mathcal{F}_{t,\bar{t}_2})_{t\ge 0}$-local martingale. From (4.50), $(M_*(R_n + t, \bar{t}_2), t \ge 0)$ is a continuous $(\mathcal{F}_{R_n+t,\bar{t}_2})_{t\ge 0}$-local martingale. Thus $(M_*(t, \bar{t}_2), t \ge 0)$ is a continuous $(\mathcal{F}_{t,\bar{t}_2})_{t\ge 0}$-local martingale. Since by (iii) $M_*(t_1, t_2) \in [C_1, C_2]$, so this local martingale is a bounded martingale. Thus (iv) is satisfied. Finally, (v) follows from the symmetry in the definitions (4.43) and (4.44) of M_*. □

4.3 Coupling measures. Let $\mathcal{C} := \bigcup_{T\in(0,\infty]} C([0,T))$. The map $T : \mathcal{C} \to (0,\infty]$ is such that $[0, T(\xi))$ is the definition domain of ξ. For $t \in [0,\infty)$, let $\mathcal{F}_t$ be the σ-algebra on $\mathcal{C}$ generated by $\{T > s, \xi(s) \in A\}$, where A is a Borel set on $\mathbb{R}$ and $s \in [0,t]$. Then $(\mathcal{F}_t)$ is a filtration on $\mathcal{C}$, and T is an $(\mathcal{F}_t)$-stopping time. Let $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$.

For $\xi \in \mathcal{C}$, let $K_\xi(t)$, $0 \le t < T(\xi)$, denote the chordal Loewner hulls driven by ξ. Let H be a hull in $\mathbb{H}$ w.r.t. ∞. Let $T_H(\xi) \in [0, T(\xi)]$ be the maximal number such that $K_\xi(t) \cap \overline{\mathbb{H}\setminus H} = \emptyset$ for $0 \le t < T_H$. Then T_H is an $(\mathcal{F}_t)$-stopping time. Let $\mathcal{C}_H = \{T_H > 0\}$. Then $\xi \in \mathcal{C}_H$ iff H contains some neighborhood of $\xi(0)$ in $\mathbb{H}$. Define $P_H : \mathcal{C}_H \to \mathcal{C}$ such that $P_H(\xi)$ is the restriction of ξ to $[0, T_H(\xi))$. Then $P_H(\mathcal{C}_H) = \{T_H = T\}$, and $P_H \circ P_H = P_H$. Let $\mathcal{C}_{H,\partial}$ denote the set of $\xi \in \{T_H = T\}$ such that $\overline{\bigcup_{0\le t<T(\xi)} K_\xi(t)} \cap \overline{(\mathbb{H}\setminus H)} \ne \emptyset$. If $\xi \in \mathcal{C}_H \cap \{T_H < T\}$, then $P_H(\xi) \in \mathcal{C}_{H,\partial}$. If A is a Borel set on $\mathbb{R}$ and $s \in [0,\infty)$, then

$$\begin{aligned}&P_H^{-1}(\{\xi \in \mathcal{C} : T(\xi) > s, \xi(s) \in A\})\\ &\quad = \{\xi \in \mathcal{C}_H : T_H(\xi) > s, \xi(s) \in A\} \in \mathcal{F}_{T_H^-}.\end{aligned}$$

Thus P_H is $(\mathcal{F}_{T_H^-}, \mathcal{F}_\infty)$-measurable on $\mathcal{C}_H$. On the other hand, the restriction of $\mathcal{F}_{T_H^-}$ to $\mathcal{C}_H$ is the σ-algebra generated by $\{\xi \in \mathcal{C}_H : T_H(\xi) > s, \xi(s) \in A\}$, where $s \in [0, \infty)$ and A is a Borel set on $\mathbb{R}$. Thus $P_H^{-1}(\mathcal{F}_\infty)$ agrees with the restriction of $\mathcal{F}_{T_H^-}$ to $\mathcal{C}_H$.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with spherical metric. Let $\Gamma_{\widehat{\mathbb{C}}}$ denote the space of nonempty compact subsets of $\widehat{\mathbb{C}}$ endowed with Hausdorff metric. Then $\Gamma_{\widehat{\mathbb{C}}}$ is a compact metric space. Define $G : \mathcal{C} \to \Gamma_{\widehat{\mathbb{C}}}$ such that $G(\xi)$ is the spherical closure of $\{t + i\xi(t) : 0 \le t < T(\xi)\}$. Then G is a one-to-one map. Let $I_G = G(\mathcal{C})$. Let $\mathcal{F}_{I_G}^H$ denote the σ-algebra on I_G generated by Hausdorff metric. Let

$$\mathcal{R} = \{\{z \in \mathbb{C} : a < \operatorname{Re} z < b, c < \operatorname{Im} z < d\} : a, b, c, d \in \mathbb{R}\}.$$

Then $\mathcal{F}_{I_G}^H$ agrees with the σ-algebra on I_G generated by $\{\{F \in I_G : F \cap R \neq \emptyset\} : R \in \mathcal{R}\}$. Using this result, one may check that G and G^{-1} (defined on I_G) are both measurable with respect to $\mathcal{F}_\infty$ and $\mathcal{F}_{I_G}^H$.

For $j = 1, 2$, let $\xi_j(t)$, $0 \le t < T_j$, be the maximal solution to (4.35). Then ξ_j is a $\mathcal{C}$-valued random variable, and $T(\xi_j) = T_j$. Since $B_1(t)$ and $B_2(t)$ are independent, so are $\xi_1(t)$ and $\xi_2(t)$. Now we write $K_j(t)$ for $K_{\xi_j}(t)$, $0 \le t < T_j$, $j = 1, 2$. For $j = 1, 2$, let μ_j denote the distribution of ξ_j, which is a probability measure on $\mathcal{C}$. Let $\mu = \mu_1 \times \mu_2$ be a probability measure on $\mathcal{C}^2$. Then μ is the joint distribution of ξ_1 and ξ_2. Let $(H_1, H_2) \in \mathrm{HP}$. For $j = 1, 2$, H_j contains some neighborhood of $x_j = \xi_j(0)$ in $\mathbb{H}$, so $\xi_j \in \mathcal{C}_{H_j}$. Since $\bigcup_{0 \le t < T_j} K_j(t)$ disconnects some force point from ∞, so we do not have $\bigcup_{0 \le t < T_j} K_j(t) \subset H_j$, which implies that $T_{H_j}(\xi_j) < T_j$, $j = 1, 2$. Thus $P_{H_j}(\xi_j) \in \mathcal{C}_{H_j, \partial}$, and so $(P_{H_1} \times P_{H_2})_*(\mu)$ is supported by $\mathcal{C}_{H_1, \partial} \times \mathcal{C}_{H_2, \partial}$.

Let HP_* be the set of $(H_1, H_2) \in \mathrm{HP}$ such that for $j = 1, 2$, H_j is a polygon whose vertices have rational coordinates. Then HP_* is countable. Let (H_1^m, H_2^m), $k \in \mathbb{N}$, be an enumeration of HP_*. For each $n \in \mathbb{N}$, let $M_*^n(t_1, t_2)$ be the $M_*(t_1, t_2)$ given by Theorem 4.5 for (H_1^m, H_2^m), $1 \le m \le n$, in the above enumeration. For each $n \in \mathbb{N}$ define $\nu^n = (\nu_1^n, \nu_2^n)$ such that $d\nu^n / d\mu = M_*^n(\infty, \infty)$. From Theorem 4.5, $M_*^n(\infty, \infty) > 0$ and $\int M_*^n(\infty, \infty) d\mu = \mathbf{E}_\mu[M_*^n(\infty, \infty)] = 1$, so ν^n is a probability measure on $\mathcal{C}^2$. Since $d\nu_1^n / d\mu_1 = \mathbf{E}_\mu[M_*^n(\infty, \infty) | \mathcal{F}_\infty^2] = M_*^n(\infty, 0) = 1$, so $\nu_1^n = \mu_1$. Similarly, $\nu_2^n = \mu_2$. So each ν^n is a coupling of μ_1 and μ_2.

Let $\bar{\nu}^n = (G \times G)_*(\nu^n)$ be a probability measure on $\Gamma_{\widehat{\mathbb{C}}}^2$. Since $\Gamma_{\widehat{\mathbb{C}}}^2$ is compact, so $(\bar{\nu}^n)$ has a subsequence $(\bar{\nu}^{n_k})$ that converges weakly to some probability measure $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2)$ on $\Gamma_{\widehat{\mathbb{C}}} \times \Gamma_{\widehat{\mathbb{C}}}$. Then for $j = 1, 2$, $\bar{\nu}_j^{n_k} \to \bar{\nu}_j$ weakly. For $n \in \mathbb{N}$ and $j = 1, 2$, since $\nu_j^n = \mu_j$, so $\bar{\nu}_j^n = G_*(\mu_j)$. Thus $\bar{\nu}_j = G_*(\mu_j)$, $j = 1, 2$. So $\bar{\nu}$ is supported by I_G^2. Let $\nu = (\nu_1, \nu_2) = (G^{-1} \times G^{-1})_*(\bar{\nu})$ be a probability measure on $\mathcal{C}^2$. Here we use the fact that G^{-1} is $(\mathcal{F}_{I_G}^H, \mathcal{F}_\infty^j)$-measurable. For $j = 1, 2$, we have $\nu_j = (G^{-1})_*(\bar{\nu}_j) = \mu_j$. So ν is also a coupling measure of μ_1 and μ_2.

Lemma 4.1 *For any $r \in \mathbb{N}$, the restriction of ν to $\mathcal{F}^1_{T_{H_1^r}} \times \mathcal{F}^2_{T_{H_2^r}}$ is absolutely continuous w.r.t. μ, and the Radon–Nikodym derivative is $M(T_{H_1^r}(\xi_1), T_{H_2^r}(\xi_2))$.*

Proof. We may choose $s \in \mathbb{N}$ such that $\overline{H_j^r} \cap \overline{\mathbb{H} \setminus H_j^s} = \emptyset$, $j = 1, 2$. Since M is continuous, so $M(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2))$ is $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$-measurable. Let $\nu_{(s)}$ be defined on $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$ such that $d\nu_{(s)}/d\mu = M(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2))$. From Theorems 4.2 and 4.4, $\nu_{(s)}$ is a probability measure on $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$. Let $\mathring{\nu}_{(s)} = (P_{H_1^s} \times P_{H_2^s})_*(\nu_{(s)})$. Since $P_{H_j^s}$ is $(\mathcal{F}^j_{T^-_{H_j^s}}, \mathcal{F}^j_\infty)$-measurable, $j = 1, 2$, so $\mathring{\nu}_{(s)}$ is a probability measure on $\mathcal{C}^2$, and is absolute continuous w.r.t. $(P_{H_1^s} \times P_{H_2^s})_*(\mu)$. Let $\bar{\nu}_{(s)} = (G \times G)_*(\mathring{\nu}_{(s)})$. Since G is $(\mathcal{F}^j_\infty, \mathcal{F}^H_{I_G})$-measurable, $j = 1, 2$, so $\bar{\nu}_{(s)}$ is a probability measure on I_G^2. Since $d\nu^{n_k}/d\mu = M_*^{n_k}(\infty, \infty)$, and $M_*^{n_k}(\cdot\,, \cdot)$ satisfies the martingale properties, so the Radon–Nikodym derivative of the restriction of ν^{n_k} to $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$ w.r.t. μ is $M_*^{n_k}(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2))$. If $n_k \geq s$ then $M_*^{n_k}(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2)) = M(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2))$. Thus the restriction of ν^{n_k} to $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$ equals to $\nu_{(s)}$, which implies that

$$(G \times G)_* \circ (P_{H_1^s} \times P_{H_2^s})_*(\nu^{n_k}) = (G \times G)_* \circ (P_{H_1^s} \times P_{H_2^s})_*(\nu_{(s)}) = \bar{\nu}_{(s)}.$$

For $n \in \mathbb{N}$, let a $\mathcal{C}^2$-valued random variable (ζ_1^n, ζ_2^n) have the distribution ν^n, and $\eta_j^n = P_{H_j^s}(\zeta_j^n)$, $j = 1, 2$. Let $\bar{\tau}^n_{(s)}$ denote the distribution of $(G(\zeta_1^n), G(\zeta_2^n), G(\eta_1^n), G(\eta_2^n))$. Then $\bar{\tau}^n_{(s)}$ is supported by Ξ, which is the set of $(L_1, L_2, F_1, F_2) \in \Gamma^4_{\hat{C}}$ such that $F_j \subset L_j$, $j = 1, 2$. It is easy to check that Ξ is a closed subset of $\Gamma^4_{\hat{\mathbb{C}}}$. Then (n_k) has a subsequence (n_k') such that $(\bar{\tau}^{n_k'}_{(s)})$ converges weakly to some probability measure $\bar{\tau}_{(s)}$ on Ξ. Since the marginal of $\bar{\tau}^{n_k'}_{(s)}$ at the first two variables equals to $(G \times G)_*(\nu^{n_k'}) = \bar{\nu}^{n_k'}$, and $\bar{\nu}^{n_k'} \to \bar{\nu}$ weakly, so the marginal of $\bar{\tau}_{(s)}$ at the first two variables equals to $\bar{\nu}$. Since the marginal of $\bar{\tau}^{n_k'}_{(s)}$ at the last two variables equals to $(G \times G)_* \circ (P_{H_1^s} \times P_{H_2^s})_*(\nu^{n_k'}) = \bar{\nu}_{(s)}$ when $n_k' \geq s$, so the marginal of $\bar{\tau}_{(s)}$ at the last two variables equals to $\bar{\nu}_{(s)}$.

Now $\bar{\tau}_{(s)}$ is supported by I_G^4. Let $\tau_{(s)} = (G \times G \times G \times G)_*^{-1}(\bar{\tau}_{(s)})$. Let a $\mathcal{C}^4$-valued random variable $(\zeta_1, \zeta_2, \eta_1, \eta_2)$ have distribution $\tau_{(s)}$. Since $\bar{\nu} = (G \times G)_*(\nu)$ and $\bar{\nu}_{(s)} = (G \times G)_*(\mathring{\nu}_{(s)})$, so the distribution of (ζ_1, ζ_2) is ν, and the distribution of (η_1, η_2) is $\mathring{\nu}_{(s)}$. For $j = 1, 2$, since $G(\eta_j) \subset G(\zeta_j)$, so η_j is some restriction of ζ_j. Note that for $j = 1, 2$, $K_j(t)$ does not always stay in H_j^s, so μ_j is supported by $\{T_{H_j^s} < T_j\}$, so $(P_{H_j^s})_*(\mu_j)$ is supported by $\mathcal{C}_{H_j^s, \partial}$. Thus $(P_{H_1^s} \times P_{H_2^s})_*(\mu)$ is supported by $\mathcal{C}_{H_1^s, \partial} \times \mathcal{C}_{H_2^s, \partial}$. Since $\mathring{\nu}_{(s)}$ is absolutely continuous w.r.t. $(P_{H_1^s} \times P_{H_2^s})_*(\mu)$, so $\mathring{\nu}_{(s)}$ is also supported by $\mathcal{C}_{H_1^s,\partial} \times \mathcal{C}_{H_2^s,\partial}$. Thus for $j = 1, 2$, $K_{\eta_j}(t) \cap \overline{\mathbb{H} \setminus H_j^s} = \emptyset$ for $0 \le t < T(\eta_j)$, and $\overline{\bigcup_{0\le t<T(\eta_j)} K_{\eta_j}(t)} \cap \overline{(\mathbb{H} \setminus H_j^s)} \ne \emptyset$. Since η_j is a restriction of ζ_j, so from the above observation, we have $\eta_j = P_{H_j^s}(\zeta_j)$, $j = 1, 2$. Thus $\mathring{\nu}_{(s)} = (P_{H_1^s} \times P_{H_2^s})_*(\nu)$.

We now have $(P_{H_1^s} \times P_{H_2^s})_*(\nu) = \mathring{\nu}_{(s)} = (P_{H_1^s} \times P_{H_2^s})_*(\nu_{(s)})$. So $\nu(\mathcal{E}) = \nu_{(s)}(\mathcal{E})$ for any $\mathcal{E} \in P^{-1}_{H_1^s}(\mathcal{F}^1_\infty) \times P^{-1}_{H_2^s}(F^2_\infty)$. Since $P^{-1}_{H_j^s}(\mathcal{F}^j_\infty)$ agrees with the restriction of $\mathcal{F}^j_{T^-_{H_j^s}}$ to $\mathcal{C}_{H_j^s}$, $j = 1, 2$, and both ν and $\nu_{(s)}$ are supported by $\mathcal{C}_{H_1^s} \times \mathcal{C}_{H_2^s}$, so the restriction of ν to $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$ equals to $\nu_{(s)}$. From the definition of $\nu_{(s)}$, the Radon–Nikodym derivative of the restriction of ν to $\mathcal{F}^1_{T^-_{H_1^s}} \times \mathcal{F}^2_{T^-_{H_2^s}}$ w. r.t. μ is $M(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2))$.

For $j = 1, 2$, since $\overline{H_j^r} \cap \overline{\mathbb{H} \setminus H_j^s} = \emptyset$, so μ_j and ν_j are supported by $\{T_{H_j^r} < T_{H_j^s}\}$. Since $\mathcal{F}^j_{T_{H_j^r}} \subset \mathcal{F}^j_{T^-_{H_j^s}}$ on $\{T_{H_j^r} < T_{H_j^s}\}$, $j = 1, 2$, so the restriction of ν to $\mathcal{F}^1_{T_{H_1^r}} \times \mathcal{F}^2_{T_{H_2^r}}$ is absolutely continuous w.r.t. μ, and the Radon–Nikodym derivative equals to

$$\mathbf{E}_\mu\left[M(T_{H_1^s}(\xi_1), T_{H_2^s}(\xi_2)) \;\middle|\; \mathcal{F}^1_{T_{H_1^r}} \times \mathcal{F}^2_{T_{H_2^r}}\right] = M(T_{H_1^r}(\xi_1), T_{H_2^s}(\xi_r)). \qquad \square$$

Proof of Theorem 4.1 Now let the $\mathcal{C}^2$-valued random variable (ξ_1, ξ_2) have distribution ν in the above theorem. Let $K_j(t)$ and $\varphi_j(t, \cdot)$, $0 \le t < T_j$, be the chordal Loewner hulls and maps, respectively, driven by ξ_j, $j = 1, 2$. For $j \ne k \in \{1, 2\}$, since $\nu_j = \mu_j$, so $K_j(t)$, $0 \le t < T_j$, is a chordal SLE$(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ process started from $(x_j; x_k, \vec{p})$.

Fix $j \ne k \in \{1, 2\}$. Suppose $\bar{t}_k$ is an $(\mathcal{F}^k_t)$-stopping time with $\bar{t}_k < T_k$. For $n \in \mathbb{N}$, define

$$R_n = \sup\left\{T_j\left(H_j^m\right) : 1 \le m \le n, T_k\left(H_k^m\right) \ge \bar{t}_k\right\}.$$

Here we set $\sup(\emptyset) = 0$. Then for any $t \ge 0$,

$$\{R_n \le t\} = \bigcap_{m=1}^{n} \left(\left\{\bar{t}_k > T_k\left(H_k^m\right)\right\} \cup \left\{T_j\left(H_j^m\right) \le t\right\}\right) \in \mathcal{F}^j_t \times \mathcal{F}^k_{\bar{t}_k}.$$

So R_n is an $(\mathcal{F}^j_t \times \mathcal{F}^k_{\bar{t}_k})_{t\ge 0}$-stopping time for each $n \in \mathbb{N}$. For $1 \le m \le n$, let $\bar{t}_k^m = \bar{t}_k \wedge T_k(H_2^m)$. Then $\bar{t}_k^m$ is an $(\mathcal{F}^k_t)$-stopping time, and $\bar{t}_k^m \le T_k(H_k^m)$. Let $(L(t))$ be a chordal SLE$(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ process started from $(\varphi_k(\bar{t}_2, x_j); \xi_k(\bar{t}_k), \varphi_k(\bar{t}_k, \vec{p}))$. From Lemma 4.1 and the discussion after Theorem 4.4, $\varphi_k(\bar{t}_k^m, K_j(t))$, $0 \le t \le T_j(H_j^m)$, has the distribution of a time-change of a partial $(L(t))$, i.e., $(L(t))$ stopped at some stopping time. Let $\mathcal{E}_{n,m} = \{\bar{t}_k \le T_k(H_k^m)\} \cap \{R_n = T_j(H_j^m)\}$. Since $\{R_n > 0\} = \bigcup_{m=1}^n \mathcal{E}_{n,m}$, and on each $\mathcal{E}_{n,m}$, $\bar{t}_k = \bar{t}_k^m$ and $R_n = T_j(H_j^m)$, so $\varphi_k(\bar{t}_k, K_j(t))$,

$0 \le t \le R_n$, has the distribution of a time-change of a partial $(L(t))$. Since $T_j(\bar t_k) = \sup\{T_j(H_j^m) : m \in \mathbb{N}, T_k(H_k^m) \ge \bar t_k\} = \bigvee_{n=1}^\infty R_n$, so $\varphi_k(\bar t_k, K_j(t))$, $0 \le t < T_j(\bar t_k)$, has the distribution of a time-change of a partial $(L(t))$. Thus after a time-change, $\varphi_k(\bar t_k, K_j(t))$, $0 \le t < T_j(\bar t_k)$, is a partial chordal SLE$(\kappa_j; -\frac{\kappa_j}{2}, \vec\rho_j)$ process started from $(\varphi_k(\bar t_2, x_j); \xi_k(\bar t_k), \varphi_k(\bar t_k, \vec p))$. □

4.4 Coupling in degenerate cases. Now we will prove that Theorem 4.1 still holds if one or more than one force points p_m are degenerate, i.e., p_m equals to $x_1^\pm$ or $x_2^\pm$. The results do not immediately follow from Theorem 4.1 in the generic case. However, we may modify the proof of Theorem 4.1 to deal with the degenerate cases. We need to find some suitable two-dimensional local martingales, and obtain some boundedness.

We use the following simplest example to illustrate the idea. Suppose there is only one degenerate force point, which is $p_1 = x_1^+$. Then the $(K_1(t))$ and $(K_2(t))$ in Theorem 4.1 should be understood as follows: $(K_1(t))$ is a chordal SLE$(\kappa_1; -\frac{\kappa_1}{2}, \vec\rho_1)$ process started from $(x_1; x_2, x_1^+, p_2, \dots, p_N)$, and $(K_2(t))$ is a chordal SLE$(\kappa_2; -\frac{\kappa_2}{2} + \rho_{2,1}, \rho_{2,2}, \dots, \rho_{2,N})$ process started from $(x_2; x_1, p_2, \dots, p_N)$. Here the force points x_1 and $p_1 = x_1^+$ for $(K_2(t))$ are combined to be a single force point x_1. And in Theorem 4.1, $\varphi_2(t_2, p_1) = \varphi_2(t_2, x_1^+)$ should be understood as $\varphi_2(t_2, x_1)$; and $\varphi_1(t_1, p_1) = \varphi_1(t_1, x_1^+)$ should be understood as $p_1(t_1)$, which is a component of the solution to the equation that generates $(K_1(t))$.

We want to define $M(t_1, t_2)$ by (4.34) and (4.26). However, for the case we study here, some factors in (4.26) does not make sense, and some factors become zero, which will cause trouble in (4.34). Let's check the factors in (4.26) one by one. Let $j \ne k \in \{1, 2\}$. First, $A_{j,h}(t_1, t_2) = \partial_z^h \varphi_{k,t_j}(t_k, \xi_j(t_j))$ is well defined for $h = 0, 1$, and $A_{j,1}$ is a positive number; and $E(t_1, t_2) = |A_{1,0}(t_1, t_2) - A_{2,0}(t_1, t_2)| > 0$ is well defined. Then $F(t_1, t_2) > 0$ is well defined by (4.24). Now $B_{m,0}(t_1, t_2) = \varphi_{K_1(t_1)\cup K_2(t_2)}(p_m)$ is well defined for each $1 \le m \le n$. For the degenerate force point $p_1 = x_1^+$, the formula $\varphi_{K_1(t_1)\cup K_2(t_2)}(x_1^+)$ is understood as $\varphi_{2,t_1}(t_2, \varphi_1(t_1, x_1^+)) = \varphi_{2,t_1}(p_1(t_1))$. So $E_{j,m} = A_{j,0} - B_{m,0}$ and $C_{m_1,m_2} = B_{m_1,0} - B_{m_2,0}$ are all well defined. Among these numbers, $|C_{m_1,m_2}(t_1, t_2)|$ is positive if $m_1 \ne m_2$, and $|E_{j,m}(t_1, t_2)|$ is positive except when $j = 1$, $m = 1$ and $t_1 = 0$. The factor $B_{m,1}(t_1, t_2) = \partial_z \varphi_{K_1(t_1)\cup K_2(t_2)}(p_m)$ is well defined and positive except when $m = 1$. Now for $(t_1, t_2) \in \mathcal{D}$, define

$$\widetilde N(t_1, t_2) = \frac{A_{1,1}^{\alpha_1} A_{2,1}^{\alpha_2}}{E^{1/2}} F^\lambda |E_{2,1}|^{\rho_{2,1}^*} \times \prod_{m=2}^N \left(B_{m,1}^{\gamma_m} \prod_{j=1}^2 |E_{j,m}|^{\rho_{j,m}^*} \right) \prod_{1 \le m_1 < m_2 \le N} |C_{m_1,m_2}|^{\delta_{m_1,m_2}};$$

and

$$N(t_1, t_2) = (\widetilde N(t_1, t_2) \widetilde N(0, 0))/(\widetilde N(t_1, 0) \widetilde N(0, t_2)). \tag{4.51}$$

Then in the generic case, we have $M(t_1,t_2)/N(t_1,t_2) = L_1(t_1,t_2)/L_2(t_1,t_2)$, where

$$\begin{aligned}
&L_1(t_1,t_2)\\
&\quad = \frac{\partial_z\varphi_{K_1(t_1)\cup K_2(t_2)}(p_1)^{\gamma_1}|\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|^{\rho^*_{1,1}}}{\partial_z\varphi_{K_1(t_1)\cup K_2(0)}(p_1)^{\gamma_1}|\varphi_{2,t_1}(0,\xi_1(t_1)) - \varphi_{2,t_1}(0,\varphi_1(t_1,p_1))|^{\rho^*_{1,1}}}\\
&\quad = \partial_z\varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))^{\gamma_1}\frac{|\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|^{\rho^*_{1,1}}}{|\xi_1(t_1)-\varphi_1(t_1,p_1)|^{\rho^*_{1,1}}},\\
&L_2(t_1,t_2)\\
&\quad = \frac{\partial_z\varphi_{K_1(0)\cup K_2(t_2)}(p_1)^{\gamma_1}|\varphi_{2,0}(t_2,\xi_1(0)) - \varphi_{2,0}(t_2,\varphi_1(0,p_1))|^{\rho^*_{1,1}}}{\partial_z\varphi_{K_1(0)\cup K_2(0)}(p_1)^{\gamma_1}|\varphi_{2,0}(0,\xi_1(0)) - \varphi_{2,0}(0,\varphi_1(0,p_1))|^{\rho^*_{1,1}}}\\
&\quad = \partial_z\varphi_2(t_2,p_1)^{\gamma_1}\frac{|\varphi_2(t_2,x_1)-\varphi_2(t_2,p_1)|^{\rho^*_{1,1}}}{|x_1-p_1|^{\rho^*_{1,1}}}.
\end{aligned}$$

In the above equalities, (4.2) is used. Thus in the generic case,

$$\begin{aligned}
\frac{M(t_1,t_2)}{N(t_1,t_2)} &= \left(\frac{\partial_z\varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))}{\partial_z\varphi_2(t_2,p_1)}\right)^{\gamma_1}\cdot\left(\frac{|x_1-p_1|}{|\varphi_2(t_2,x_1)-\varphi_2(t_2,p_1)|}\right)^{\rho^*_{1,1}}\\
&\quad\cdot\left(\frac{|\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|}{|\xi_1(t_1)-\varphi_1(t_1,p_1)|}\right)^{\rho^*_{1,1}}.
\end{aligned}$$

Now come back to the degenerate case $p_1 = x_1^+$ we are studying here. Then

$$\partial_z\varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1)) = \partial_z\varphi_{2,t_1}(t_2,\varphi_1(t_1,x_1^+)) \quad \text{and}$$
$$\partial_z\varphi_2(t_2,p_1) = \partial_z\varphi_2(t_2,x_1)$$

still make sense and are both positive. If $t_1 > 0$, then $|\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|$ and $|\xi_1(t_1)-\varphi_1(t_1,p_1)|$ both make sense and are positive. And we have

$$\begin{aligned}
&\lim_{t_1\to 0^+} |\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|/|\xi_1(t_1)-\varphi_1(t_1,p_1)|\\
&\quad = \partial_z\varphi_{2,t_1}(t_2,\xi_1(t_1)).
\end{aligned}$$

Since $p_1 = x_1^+$, we may view $|x_1-p_1|/|\varphi_2(t_2,x_1)-\varphi_2(t_2,p_1)|$ as

$$\lim_{p\to x_1^+} |x_1-p|/|\varphi_2(t_2,x_1)-\varphi_2(t_2,p)| = 1/\partial_z\varphi_2(t_2,x_1).$$

These observations suggest us to define $M(t_1,t_2)$ in the case $p_1 = x_1^+$ as follows. For $(t_1,t_2)\in\mathcal{D}$, define $U(t_1,t_2)$ such that $U(0,t_2) = \partial_z\varphi_{2,t_1}(t_2,\xi_1(t_1))$; and if $t_1 > 0$, then

$$U(t_1,t_2) = |\varphi_{2,t_1}(t_2,\xi_1(t_1)) - \varphi_{2,t_1}(t_2,\varphi_1(t_1,p_1))|/|\xi_1(t_1)-\varphi_1(t_1,p_1)|.$$

Then U is continuous on $\mathcal{D}$. Now for $(t_1, t_2) \in \mathcal{M}$, define

$$M(t_1, t_2) = N(t_1, t_2) \cdot \frac{U(t_1, t_2)^{\rho^*_{1,1}}}{U(0, t_2)^{\rho^*_{1,1}}} \cdot \frac{\partial_z \varphi_{2,t_1}(t_2, \varphi_1(t_1, x_1^+))^{\gamma_1}}{\partial_z \varphi_2(t_2, x_1)^{\gamma_1}}. \tag{4.52}$$

Then M is continuous on $\mathcal{D}$. It is direct to check that $M(t_1, 0) = M(0, t_2) = 1$ for any $t_1 \in [0, T_1)$ and $t_2 \in [0, T_2)$.

Suppose $(\xi_1(t), 0 \le t < T_1)$ and $(\xi_2(t), 0 \le t < T_2)$ are independent. Let μ_j denote the distribution of $(\xi_j(t))$, $j = 1, 2$, and $\mu = \mu_1 \times \mu_2$. Let $(\mathcal{F}_t^j)$ be the filtration generated by $(\xi_j(t))$, $j = 1, 2$. Let $j \ne k \in \{1, 2\}$. Then for any fixed $(\mathcal{F}_t^k)$-stopping time $\bar{t}_k$ with $\bar{t}_k < T_k$, the process $M|_{t_j = t, t_k = \bar{t}_k}$, $0 \le t < T_j(\bar{t}_k)$, is an $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)$-adapted local martingale, under the probability measure μ. The argument is similar to that used in Sect. 4.2.

Let HP denote the set of (H_1, H_2) such that H_j is a hull in $\mathbb{H}$ w.r.t. ∞ that contains some neighborhood of x_j in $\mathbb{H}$, $j = 1, 2$, $\overline{H_1} \cap \overline{H_2} = \emptyset$, and $p_m \notin \overline{H_1} \cup \overline{H_2}$, $2 \le m \le N$. Here we only require that the non-degenerate force points are bounded away from H_1 and H_2. Then Theorem 4.4 still holds here. For the proof, one may check that Theorem 4.4 holds with $M(t_1, t_2)$ replaced by $N(t_1, t_2)$, $U(t_1, t_2)$, $\partial_z \varphi_{2,t_1}(t_2, \varphi_1(t_1, x_1^+))$, and $\partial_z \varphi_2(t_2, x_1)$, respectively. So for any $(H_1, H_2) \in$ HP, $\mathbf{E}_\mu[M(T_1(H_1), T_2(H_2))] = 1$. Suppose ν is a measure on $\mathcal{F}^1_{T_1(H_1)} \times \mathcal{F}^2_{T_2(H_2)}$ such that $d\nu/d\mu = M(T_1(H_1), T_2(H_2))$. Then ν is a probability measure. Now suppose the joint distribution of $(\xi_1(t), 0 \le t \le T_1(H_1))$ and $(\xi_2(t), 0 \le t \le T_2(H_2))$ is ν instead of μ. Let $j \ne k \in \{1, 2\}$. Using Girsanov theorem, one may check that for any fixed $(\mathcal{F}_t^k)$-stopping time $\bar{t}_k$ with $\bar{t}_k \le T_k(H_k)$, conditioned on $\mathcal{F}^k_{\bar{t}_k}$, $(\varphi_k(\bar{t}_k, K_j(t)))$, $0 \le t < T_j(H_j)$, is a time-change of a partial chordal SLE$(\kappa_j; -\frac{\kappa_j}{2}, \vec{\rho}_j)$ process started from $(\varphi_k(\bar{t}_k, x_j); \xi_k(\bar{t}_k), \varphi_k(\bar{t}_k, \vec{p}))$. We now can use the argument in Sect. 4.3 to derive Theorem 4.1 in this degenerate case.

5 Applications

5.1 Duality. We say α is a crosscut in $\mathbb{H}$ on $\mathbb{R}$ if α is a simple curve that lies inside $\mathbb{H}$ except for the two ends of α, which lie on $\mathbb{R}$. If α is a crosscut, then $\mathbb{H} \setminus \alpha$ has two connected components: one is bounded, the other is unbounded. Let $D(\alpha)$ denote the bounded component. We say that such α strictly encloses some $S \subset \overline{\mathbb{H}}$ if $\overline{S} \subset \overline{D(\alpha)}$ and $\overline{S} \cap \alpha = \emptyset$.

In Theorem 4.1, let $\kappa_1 < 4 < \kappa_2$; $x_1 < x_2$; $N = 3$; $p_1 \in (-\infty, x_1)$, $p_2 \in (x_2, \infty)$, $p_3 \in (x_1, x_2)$; for $j = 1, 2$, $\rho_{j,1} = C_1(\kappa_j - 4)$, $\rho_{j,2} = C_2(\kappa_j - 4)$, and $\rho_{j,3} = \frac{1}{2}(\kappa_j - 4)$ for some $C_1 \le 1/2$ and $C_2 = 1 - C_1$. Let $K_j(t)$, $0 \le t < T_j$, $j = 1, 2$, be given by Theorem 4.1. Let $\varphi_j(t, \cdot)$ and $\beta_j(t)$, $0 \le t < T_j$, $j = 1, 2$, be the corresponding Loewner maps and traces.

Let $K_2(T_2^-) = \bigcup_{0 \le t < T_2} K_2(t)$. Since $\kappa_1 \in (0, 4)$, so $\beta_1(t)$, $0 \le t < T_j$, is a simple curve, and $\beta_1(t) \in \mathbb{H}$ for $0 < t < T_j$. From Theorem 3.6 and

Lemma 2.3, a.s. $\beta_1(T_1) := \lim_{t\to T_1}\beta_1(t)$ exists and lies on (x_2, p_2). For simplicity, we use β_1 to denote the image $\{\beta_1(t) : 0 \le t \le T_1\}$. Thus β_1 is a crosscut in $\mathbb{H}$ on $\mathbb{R}$.

Suppose $S \subset \mathbb{H}$ is bounded. Then there is a unique unbounded component of $\mathbb{H} \setminus \overline{S}$, which is denoted by D_∞. Then we call $\partial D_\infty \cap \mathbb{H}$ the outer boundary of S in $\mathbb{H}$. Let it be denoted by $\partial^{\text{out}}_{\mathbb{H}} S$.

Lemma 5.1 *Almost surely* $\beta_1 = \partial^{\text{out}}_{\mathbb{H}} K_2(T_2^-)$.

Proof. For $j = 1, 2$, let $\mathcal{P}_j$ denote the set of polygonal crosscuts in $\mathbb{H}$ on $\mathbb{R}$ whose vertices have rational coordinates, which strictly enclose x_j, and which do not contain or enclose x_{3-j} or p_m, $m = 1, 2, 3$. For each $\gamma \in \mathcal{P}_j$, let $T_j(\gamma)$ denote the first time that β_j hits γ. Then $T_j(\gamma)$ is an $(\mathcal{F}^j_t)$-stopping time, and $T_j(\gamma) < T_j$. Moreover, we have $T_j = \bigvee_{\gamma\in\mathcal{P}_j} T_j(\gamma)$. Let $\mathcal{P}^*_2$ denote the set of polygonal crosscuts in $\mathbb{H}$ on $\mathbb{R}$ whose vertices have rational coordinates, and which strictly enclose x_2.

We first show that $K_2(T_2^-) \subset D(\beta_1) \cup \beta_1$ a.s. Let $\mathcal{E}$ denote the event that β_2 intersects $\mathbb{H} \setminus (D(\beta_1) \cup \beta_1)$. We need to show that $\mathbf{P}[\mathcal{E}] = 0$. For $\alpha \in \mathcal{P}^*_2$ and $\gamma \in \mathcal{P}_2$, let $\mathcal{E}_{\alpha;\gamma}$ denote the event that α strictly encloses β_1, and β_2 hits α before γ. Then $\mathcal{E} = \bigcup_{\alpha\in\mathcal{P}^*_2;\gamma\in\mathcal{P}_2} \mathcal{E}_{\alpha;\gamma}$. Since $\mathcal{P}^*_2$ and $\mathcal{P}_2$ are countable, so we suffice to show $\mathbf{P}[\mathcal{E}_{\alpha;\gamma}] = 0$ for any $\alpha \in \mathcal{P}^*_2$ and $\gamma \in \mathcal{P}_2$.

Now fix $\alpha \in \mathcal{P}^*_2$ and $\gamma \in \mathcal{P}_2$. Let $\bar{t}_2$ denote the first time that β_2 hits $\alpha\cup\gamma$. Then $\bar{t}_2$ is an $(\mathcal{F}^2_t)$-stopping time, and $\bar{t}_2 \le T_2(\gamma) < T_2$. From Theorem 4.1, after a time-change, $\varphi_2(\bar{t}_2, \beta_1(t))$, $0 \le t < T_1(\bar{t}_2)$, has the same distribution as a full chordal SLE$(\kappa_1; -\frac{\kappa_1}{2}, C_1(\kappa_1 - 4), C_2(\kappa_1 - 4), \frac{1}{2}(\kappa_1 - 4))$ trace started from $(\varphi_2(\bar{t}_2, x_1); \xi_2(\bar{t}_2), \varphi_2(\bar{t}_2, p_1), \varphi_2(\bar{t}_2, p_2), \varphi_2(\bar{t}_2, p_3))$. Here we have

$$\varphi_2(\bar{t}_2, p_1) < \varphi_2(\bar{t}_2, x_1) < \varphi_2(\bar{t}_2, p_3) < \xi_2(\bar{t}_2) < \varphi_2(\bar{t}_2, p_2).$$

Since $C_1(\kappa_1 - 4) \ge \kappa_1/2 - 2$, $\frac{1}{2}(\kappa_1 - 4) \ge \kappa_1/2 - 2$, and

$$\left|(C_1(\kappa_1 - 4) + C_2(\kappa_1 - 4)) - \left(\frac{1}{2}(\kappa_1 - 4) + \left(-\frac{\kappa_1}{2}\right)\right)\right| = |\kappa_1 - 2| < 2,$$

so from Theorem 3.6 and Lemma 2.3, a.s. $\lim_{t\to T_1(\bar{t}_2)} \varphi_2(\bar{t}_2, \beta_1(t)) \in (\xi_2(\bar{t}_2), \varphi_2(\bar{t}_2, p_2))$. Thus a.s. $\{\varphi_2(\bar{t}_2, \beta_1(t)) : 0 \le t < T_1(\bar{t}_2)\}$ disconnects $\xi_2(\bar{t}_2)$ from ∞ in $\mathbb{H}$. So a.s. β_1 disconnects $\beta_2(\bar{t}_2)$ from ∞ in $\mathbb{H} \setminus K_2(\bar{t}_2)$.

Assume that the event $\mathcal{E}_{\alpha;\gamma}$ occurs. Since β_2 starts from x_2, which is strictly enclosed by α, so $\beta_2(t) \in \overline{D(\alpha)}$ for $0 \le t \le \bar{t}_2$, which implies that $K_2(\bar{t}_2) \subset \overline{D(\alpha)}$. On the other hand, β_1 is strictly enclosed by α, and $\beta_2(\bar{t}_2) \in \alpha$. Thus β_1 does not disconnect $\beta_2(\bar{t}_2)$ from ∞ in $\mathbb{H} \setminus K_2(\bar{t}_2)$. So we have $\mathbf{P}[\mathcal{E}_{\alpha;\gamma}] = 0$. Thus $K_2(T_2^-) \subset D(\beta_1) \cup \beta_1$ a.s.

Next we show that a.s. $\beta_1 \subset \overline{K_2(T_2^-)}$. Fix $\gamma \in \mathcal{P}_1$ and $q \in \mathbb{Q}_{\ge 0}$. Let $\bar{t}_1 = q \wedge T_1(\gamma)$. Then $\bar{t}_1$ is an $(\mathcal{F}^1_t)$-stopping time, and $\bar{t}_1 \le T_1(\gamma) < T_1$. From

Theorem 4.1, after a time-change, $\varphi_1(\bar{t}_1, \beta_2(t)), 0 \le t < T_2(\bar{t}_1)$, has the same distribution as a full chordal $\mathrm{SLE}(\kappa_2; -\frac{\kappa_2}{2}, C_1(\kappa_2-4), C_2(\kappa_2-4), \frac{1}{2}(\kappa_2-4))$ trace started from $(\varphi_1(\bar{t}_1, x_2); \xi_1(\bar{t}_1), \varphi_1(\bar{t}_1, p_1), \varphi_1(\bar{t}_1, p_2), \varphi_1(\bar{t}_1, p_3))$. Here we have

$$\varphi_1(\bar{t}_1, p_1) < \xi_1(\bar{t}_1) < \varphi_1(\bar{t}_1, p_3) < \varphi_1(\bar{t}_1, x_2) < \varphi_1(\bar{t}_1, p_2).$$

Since $\frac{1}{2}(\kappa_2 - 4) \ge \kappa_2/2 - 2$, $C_2(\kappa_2 - 4) \ge \kappa/2 - 2$, and $C_2(\kappa_2 - 4) + C_1(\kappa_2 - 4) = \kappa_2 - 4 \ge \kappa_2/2 - 2$, so from Theorem 3.1 and Lemma 2.1, a.s. $\xi_1(\bar{t}_1)$ is a subsequential limit point of $\varphi_1(\bar{t}_1, \beta_2(t))$ as $t \to T_2(\bar{t}_1)$. Thus a.s. $\beta_1(\bar{t}_1)$ is a subsequential limit point of $\beta_2(t)$ as $t \to T_2(\bar{t}_1)$. So $\beta_1(\bar{t}_1) \in \overline{K_2(T_2(\bar{t}_1)^-)} \subset \overline{K_2(T_2^-)}$ a.s. Since $\mathbb{Q}_{\ge 0}$ is countable, so a.s. $\beta_1(q \wedge T_1(\gamma)) \in \overline{K_2(T_2^-)}$ for any $q \in \mathbb{Q}_{\ge 0}$. Since $\mathbb{Q}_{\ge 0}$ is dense in $\mathbb{R}_{\ge 0}$, so a.s. $\beta_1(t) \in \overline{K_2(T_2^-)}$ for any $t \in [0, T_1(\gamma)]$. Since $\mathscr{P}_1$ is countable and $T_1 = \bigvee_{\gamma \in \mathscr{P}_1} T_1(\gamma)$, so almost surely $\beta_1(t) \in \overline{K_2(T_2^-)}$ for any $t \in [0, T_1)$, i.e., $\beta_1 \subset \overline{K_2(T_2^-)}$ a.s. Finally, $K_2(T_2^-) \subset D(\beta_1) \cup \beta_1$ and $\beta_1 \subset \overline{K_2(T_2^-)}$ imply that $\beta_1 = \partial_{\mathbb{H}}^{\mathrm{out}} K_2(T_2^-)$. □

Theorem 5.1 *Suppose $\kappa > 4$; $p_1 < x_1 < p_3 < x_2 < p_2$; $C_1 \le 1/2 \le C_2$ and $C_1 + C_2 = 1$. Let $K(t)$, $0 \le t < T$, be chordal* $\mathrm{SLE}(\kappa; -\frac{\kappa}{2}, C_1(\kappa - 4), C_2(\kappa - 4), \frac{1}{2}(\kappa - 4))$ *process started from* $(x_2; x_1, p_1, p_2, p_3)$. *Let $K(T^-) = \bigcup_{0 \le t < T} K(t)$. Then a.s. $K(T^-)$ is bounded, and $\partial_{\mathbb{H}}^{\mathrm{out}} K(T^-)$ has the distribution of the image of a chordal* $\mathrm{SLE}(\kappa'; -\frac{\kappa'}{2}, C_1(\kappa' - 4), C_2(\kappa' - 4), \frac{1}{2}(\kappa' - 4))$ *trace started from* $(x_1; x_2, p_1, p_2, p_3)$, *where $\kappa' = 16/\kappa$.*

Theorem 5.1 still holds if we let $p_1 \in (-\infty, x_1)$, or $= -\infty$, or $= x_1^-$; let $p_2 \in (x_2, \infty)$, or $= \infty$, or $= x_2^+$; and let $p_3 \in (x_1, x_2)$, or $= x_1^+$, or $= x_2^-$. In some cases we may use Theorem 3.3, Theorem 3.4, or Theorem 3.5 instead of Theorem 3.9 to prove that β_1 is a crosscut. We may derive some interesting theorems from some cases.

Theorem 5.2 *Suppose $\kappa \ge 8$, and $K(t)$, $0 \le t < \infty$, is a standard chordal* $\mathrm{SLE}(\kappa)$ *process, i.e., the chordal Loewner chain driven by $\xi(t) = \sqrt{\kappa} B(t)$. Let $x \in \mathbb{R} \setminus \{0\}$ and T_x be the first t such that $x \in \overline{K(t)}$. Then $\partial K(T_x) \cap \mathbb{H}$ has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; -\frac{\kappa'}{2}, -\frac{\kappa'}{2}, \frac{\kappa'}{2} - 2)$ *trace started from $(x; 0, x^a, x^b)$, where $\kappa' = 16/\kappa$, $a = \mathrm{sign}(x)$ and $b = \mathrm{sign}(-x)$.*

Proof. $K(t)$, $0 \le t < T_x$, is a full chordal $\mathrm{SLE}(\kappa; 0)$ process started from $(0; x)$. Since $\kappa \ge 8$, so $K(T_x) = \bigcup_{0 \le t < T_x} K_t$ and $\partial K(T_x) \cap \mathbb{H} = \partial_{\mathbb{H}}^{\mathrm{out}} K(T_x)$. If $x < 0$, this follows from Theorem 5.1 with $x_1 = x$, $x_2 = 0$, $p_1 = x_1^-$, $p_2 = \infty$, $p_3 = x_1^+$; $C_1 = 2/(\kappa - 4)$ and $C_2 = 1 - C_1$. If $x > 0$, this follows from symmetry. □

One may expect that after reasonable modifications, the above theorem also holds for $\kappa \in (4, 8)$. In this case, for the $\mathrm{SLE}(\kappa'; -\frac{\kappa'}{2}, -\frac{\kappa'}{2}, \frac{\kappa'}{2}-2)$ trace started from $(x; 0, x^a, x^b)$, the force $-\frac{\kappa'}{2}$ that corresponds to the degenerate force point x^a does not satisfy $-\frac{\kappa'}{2} \ge \kappa'/2 - 2$. So we must allow that the process continue growing after the degenerate force point is swallowed. This will make sense because $-\frac{\kappa'}{2} > -2$.

Corollary 5.1 *For $\kappa > 8$, chordal* SLE(κ) *trace is not reversible.*

Proof. Let $\beta(t), 0 \le t < \infty$, be a standard chordal SLE(κ) trace. Let $W(z) = 1/\overline{z}$ and $\gamma(t) = W(\beta(1/t))$. Suppose chordal SLE($\kappa$) trace is reversible, then after a time-change, $(\gamma(t), 0 < t < \infty)$ has the same distribution as $(\beta(t), 0 < t < \infty)$. Let T be the first time such that $1 \in \beta(t)$. Since $\kappa > 8$, 1 is visited by β exactly once a.s. Thus $1/T$ is the first time such that $1 \in \gamma(t)$. From the above theorem, $\partial(\beta((0, T])) \cap \mathbb{H}$ and $\partial(\gamma((0, 1/T])) \cap \mathbb{H}$ both have the distribution of the image of a chordal $\mathrm{SLE}(\kappa'; -\frac{\kappa'}{2}, -\frac{\kappa'}{2}, \frac{\kappa'}{2}-2)$ trace started from $(1; 0, 1^+, 1^-)$, where $\kappa' = 16/\kappa$. From Lemma 2.1 and the definition of γ, we find that $\partial(\beta([T, \infty))) \cap \mathbb{H}$ has the distribution of the image of a chordal $\mathrm{SLE}(\kappa'; \frac{3\kappa'}{2} - 4, \frac{\kappa'}{2} - 2, -\frac{\kappa'}{2})$ trace started from $(1; 0, 1^+, 1^-)$. Since $\kappa' < 2$, so $-\frac{\kappa'}{2} \ne \frac{3\kappa'}{2} - 4$. Thus $\partial(\beta((0, T])) \cap \mathbb{H}$ and $\partial(\beta([T, \infty))) \cap \mathbb{H}$ have different distributions. However, since β is space-filling and never crosses its past, the two boundary curves coincide, which gives a contradiction. □

Suppose $S \subset \mathbb{H}$ and $\overline{S} \cap [a, \infty) = \emptyset$ for some $a \in \mathbb{R}$. Then there is a unique component of $\mathbb{H} \setminus \overline{S}$, which has $[a, \infty)$ as part of its boundary. Let D_+ denote this component. Then $\partial D_+ \cap \mathbb{H}$ is called the right boundary of S in $\mathbb{H}$. Let it be denoted by $\partial^+_{\mathbb{H}} S$.

Theorem 5.3 *Let $\kappa > 4$, $C \ge 1/2$, and $K(t)$, $0 \le t < \infty$, be a chordal* $\mathrm{SLE}(\kappa; C(\kappa - 4), \frac{1}{2}(\kappa - 4))$ *process started from* $(0; 0^+, 0^-)$. *Let $K(\infty) = \bigcup_{t<\infty} K(t)$. Let $W(z) = 1/\overline{z}$. Then $W(\partial^+_{\mathbb{H}} K(\infty))$ has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; C'(\kappa' - 4))$ *trace started from* $(0; 0^+)$, *where $\kappa' = 16/\kappa$ and $C' = 1 - C$.*

Proof. Let $W_0(z) = 1/(1 - z)$. Then W_0 is a conformal automorphism of $\mathbb{H}$, and $W_0(0) = 1$, $W_0(\infty) = 0$, $W_0(0^{\pm}) = 1^{\pm}$. From Lemma 2.1, after a time-change, $(W_0(K(t)))$ has the same distribution as a chordal $\mathrm{SLE}(\kappa; C'(\kappa-4)-\frac{\kappa}{2}, C(\kappa-4), \frac{1}{2}(\kappa-4))$ process started from $(1; 0, 1^+, 1^-)$. Applying Theorem 5.1 with $x_1 = 0$, $x_2 = 1$, $p_1 = 0^-$, $p_2 = 1^+$, $p_3 = 1^-$, $C_1 = C'$ and $C_2 = C$, we find that $\partial^{\text{out}}_{\mathbb{H}} W_0(K_\infty)$ has the same distribution as the image of a chordal $\mathrm{SLE}(\kappa'; C(\kappa' - 4) - 2, C'(\kappa' - 4))$ trace started from $(0; 1, 0^-)$. Let β denote this trace. From Lemma 2.3 and Theorem 3.4, β is a crosscut in $\mathbb{H}$ from 0 to 1. Thus $\partial^+_{\mathbb{H}} K_\infty = W_0^{-1}(\beta)$, and so $W(\partial^+_{\mathbb{H}} K_\infty) = W \circ W_0^{-1}(\beta)$. Let $W_1 = W \circ W_0^{-1}$. Then $W_1(z) = \overline{z}/(\overline{z} - 1)$. So $W_1(0) = 0$,

$W_1(1) = \infty$, $W_1(0^-) = 0^+$. From Lemma 2.1, after a time-change, $W_1(\beta)$ has the same distribution as the image of a chordal SLE$(\kappa'; C'(\kappa'-4))$ trace started from $(0; 0^+)$. □

Theorem 5.4 *Let $\kappa > 4$, $C \geq 1/2$, and $K(t)$, $0 \leq t < \infty$, be a chordal* SLE$(\kappa; C(\kappa - 4))$ *process started from* $(0; 0^+)$. *Let $K(\infty) = \bigcup_{t<\infty} K(t)$. Let $W(z) = 1/\overline{z}$. Then $W(\partial_{\mathbb{H}}^+ K(\infty))$ has the same distribution as the image of a chordal* SLE$(\kappa'; C'(\kappa' - 4), \frac{1}{2}(\kappa' - 4))$ *trace started from* $(0; 0^+, 0^-)$, *where $\kappa' = 16/\kappa$ and $C' = 1 - C$.*

Proof. This proof is similar to the previous one. We use the same W_0, W_1, x_1, x_2, p_1, and p_2, except that now $p_3 = 0^+$ instead of 1^-. And the β here is a chordal SLE$(\kappa'; C(\kappa' - 4) - \frac{\kappa'}{2}, \frac{1}{2}(\kappa' - 4), C'(\kappa' - 4))$ trace started from $(0; 1, 0^+, 0^-)$. □

Corollary 5.2 *Let $\kappa > 4$, and $K(t)$, $0 \leq t < \infty$, be a chordal* SLE$(\kappa; \kappa - 4, \frac{1}{2}(\kappa - 4))$ *process started from* $(0; 0^+, 0^-)$. *Let $K(\infty) = \bigcup_{t<\infty} K(t)$. Then $\partial_{\mathbb{H}}^+ K(\infty)$ has the same distribution as the image of a standard chordal* SLE(κ') *trace, where $\kappa' = 16/\kappa$.*

Proof. This follows from Theorem 5.3 and the reversibility of chordal SLE(κ') trace when $\kappa' \in (0, 4]$ (see [19]). □

If we assume that Conjecture 1 is true, then in Theorem 5.3 we conclude that $\partial_{\mathbb{H}}^+ K(\infty)$ has the same distribution as a chordal SLE$(\kappa'; C'(\kappa'-4))$ trace started from $(0; 0^+)$; and in Theorem 5.4 we conclude that $\partial_{\mathbb{H}}^+ K(\infty)$ has the same distribution as the image of a chordal SLE$(\kappa'; C'(\kappa' - 4), \frac{1}{2}(\kappa' - 4))$ trace started from $(0; 0^+, 0^-)$, where $\kappa' = 16/\kappa$ and $C' = 1 - C$. Moreover, assuming Conjecture 1, and letting $C = 1$ in Theorem 5.4, we conclude that the right boundary of the final hull of a chordal SLE$(\kappa; \kappa - 4)$ process started from $(0; 0^+)$ has the same distribution as the image of a chordal SLE$(\kappa'; \frac{1}{2}(\kappa - 4))$ trace started from $(0; 0^-)$, which is Conjecture 2 in [4]. Moreover, we conjecture that for $C_r, C_l \geq 1/2$, if $(K(t))$ is a chordal SLE$(\kappa; C_r(\kappa-4), C_l(\kappa-4))$ started from $(0; 0^+, 0^-)$, then $\partial_{\mathbb{H}}^+ K(\infty)$ has the same distribution as the image of a chordal SLE$(\kappa'; C_r'(\kappa' - 4), C_l'(\kappa' - 4))$ trace started from $(0; 0^+, 0^-)$, where $C_r' = 1 - C_r$ and $C_l' = 1/2 - C_l$.

5.2 Reversibility

Theorem 5.5 *Let $\vec{p}_\pm = (p_{\pm 1}, \ldots, p_{\pm N_\pm})$ and $\vec{\rho}_\pm = (\rho_{\pm 1}, \ldots, \rho_{\pm N_\pm})$, where $0 < \pm p_{\pm m} < \pm p_{\pm n}$ for $1 \leq m < n \leq N_\pm$; $\sum_{m=1}^{n} \rho_{\pm m} \geq 0$ for $1 \leq n \leq N_\pm$, and $\sum_{m=1}^{N_\pm} \rho_{\pm m} = 0$. Let $\beta(t)$, $0 \leq t < \infty$, be a chordal* SLE$(4; \vec{\rho}_+, \vec{\rho}_-)$ *trace started from* $(0; \vec{p}_+, \vec{p}_-)$. *Let $W(z) = 1/\overline{z}$. Then a.s. $\lim_{t\to\infty} \beta(t) = \infty$, and after a time-change, the reversal of $(W(\beta(t)))$ has the same distribution as a chordal* SLE$(4; -\vec{\rho}_+, -\vec{\rho}_-)$ *trace started from* $(0; W(\vec{p}_+), W(\vec{p}_-))$, *where $W(\vec{p}_\pm) = (W(p_{\pm 1}), \ldots, W(p_{\pm N_\pm}))$.*

Proof. Choose $x_0 > p_{N_+}$. Let $W_0(z) = x_0/(x_0 - z)$. Then W_0 maps $\mathbb{H}$ conformally onto $\mathbb{H}$, and $W_0(0) = 1$, $W_0(\infty) = 0$. Let $q_{\pm j} = W_0(p_{\pm j})$, $1 \le j \le N_\pm$. Then $0 < q_{-N_-} < \cdots < q_{-1} < 1 < q_1 < \cdots < q_{N_+}$. Let $x_1 = 1$, $x_2 = 0$, $\vec{\rho}_{1,\pm} = \vec{\rho}_\pm$, and $\vec{\rho}_{2,\pm} = -\vec{\rho}_\pm$. From Theorem 4.1, there is a coupling of two curves $\beta_j(t)$, $0 \le t < T_j$, $j = 1, 2$, such that for fixed $j \ne k \in \{1, 2\}$, (i) $(\beta_j(t))$ is a chordal SLE$(4; -2, \vec{\rho}_{j,+}, \vec{\rho}_{j,-})$ trace started from $(x_j; x_k, \vec{p}_+, \vec{p}_-)$; and (ii) for any $(\mathcal{F}^k_t)$-stopping time $\bar{t}_k$ with $\bar{t}_k < T_k$, $\varphi_k(\bar{t}_k, \beta_j(t))$, $0 \le t < T_j(\bar{t}_k)$, has the same distribution as a chordal SLE$(4; -2, \vec{\rho}_{j,+}, \vec{\rho}_{j,-})$ trace started from $(\varphi_k(\bar{t}_k, x_j); \xi_k(\bar{t}_k), \varphi_k(\bar{t}_k, \vec{p}_+), \varphi_k(\bar{t}_k, \vec{p}_-))$, where $\varphi_j(t, \cdot)$ and $\xi_j(t)$, $0 \le t < T_j$, are chordal Loewner maps and driving function for the trace β_j, $j = 1, 2$. Note the symmetry between $\vec{\rho}_{1,\pm}$ and $\vec{\rho}_{2,\pm}$: $\sum_{m=1}^{n} \rho_{1,\pm m} \ge 0$ for all $1 \le n \le N_\pm$, and $\sum_{m=1}^{N_\pm} \rho_{1,\pm m} = 0$; $\sum_{m=n}^{N_\pm} \rho_{2,\pm m} \ge 0$ for all $1 \le n \le N_\pm$, and $\sum_{m=1}^{N_\pm} \rho_{2,\pm m} = 0$.

Fix $j \ne k \in \{1, 2\}$. From Lemma 2.1 and Theorem 3.1, we have a.s. $x_k \in \overline{\beta_j((0, T_j))}$. Now fix an $(\mathcal{F}^k_t)$-stopping time $\bar{t}_k \in (0, T_k)$. From Lemma 2.1 and Theorem 3.1, we have a.s. $\overline{\varphi_k(\bar{t}_k, \beta_j((0, T_j(\bar{t}_k))))} \cap \mathbb{R} = \{\xi_k(\bar{t}_k)\}$, which implies that $\overline{\beta_j((0, T_j(\bar{t}_k)))} \cap (\mathbb{R} \cup \beta_k((0, \bar{t}_k))) = \{\beta_k(\bar{t}_k)\}$. Since $\bar{t}_k > 0$, so $\beta_k(\bar{t}_k) \ne \beta_k(0) = x_k$. If $T_j(\bar{t}_k) = T_j$, then $x_k \in \overline{\beta_j((0, T_j(\bar{t}_k)))}$, which a.s. does not happen. Thus a.s. $T_j(\bar{t}_k) < T_j$. So we have a.s. $\beta_j(T_j(\bar{t}_k)) = \lim_{t \to T_j(\bar{t}_k)^-} \beta_j(t) \in \overline{\beta_j((0, T_j(\bar{t}_k)))}$. From the definition of $T_j(\bar{t}_k)$, we have a.s. $\beta_j(T_j(\bar{t}_k)) \in \beta_k([0, \bar{t}_k])$. Thus a.s. $\beta_j(T_j(\bar{t}_k)) = \beta_k(\bar{t}_k)$.

We may choose a sequence of $(\mathcal{F}^k_t)$-stopping times $(\bar{t}^{(n)}_k)$ on $(0, T_k)$ such that $\{\bar{t}^{(n)}_k : n \in \mathbb{N}\}$ is dense on $[0, T_k]$. Then a.s. $\beta_k(\bar{t}^{(n)}_k) = \beta_j(T_j(\bar{t}^{(n)}_k))$ for any $n \in \mathbb{N}$. From the denseness of $\{\bar{t}^{(n)}_k : n \in \mathbb{N}\}$ and the continuity of β_j and β_k, we have a.s. $\beta_k((0, T_k)) \subset \beta_j((0, T_j))$. Similarly, a.s. $\beta_j((0, T_j)) \subset \beta_k((0, T_k))$. So a.s. β_2 is a time-change of the reversal of β_1.

From Lemma 2.1, $(W_0(\beta(t)))$ has the same distribution as $(\beta_1(t))$ after a time-change. Thus the reversal of $(W(\beta(t)))$ has the same distribution as $(W \circ W_0^{-1}(\beta_2(t)))$ after a time-change. From Lemma 2.1, $(W \circ W_0^{-1}(\beta_2(t)))$ has the same distribution as a chordal SLE$(4; -\vec{\rho}_+, -\vec{\rho}_-)$ trace started from $(0; W(\vec{p}_+), W(\vec{p}_-))$. □

This theorem may also be proved using the convergence of discrete Gaussian free field on some triangle lattice with suitable boundary conditions (see [13]). It also holds in the degenerate cases, i.e., $p_1 = 0^+$ and/or $p_{-1} = 0^-$ and/or $p_{N_+} = +\infty$ and/or $p_{-N_-} = -\infty$. For example, let $\rho_+, \rho_- \ge 0$, and apply Theorem 5.5 with $N_+ = N_- = 2$, $p_1 = 0^+$, $p_{-1} = 0^-$, $p_2 = +\infty$, $p_{-2} = -\infty$, $\rho_1 = \rho_+$, $\rho_2 = -\rho_+$, $\rho_{-1} = \rho_-$, and $\rho_{-2} = -\rho_-$. Then we conclude that if $\beta(t)$, $0 \le t < \infty$, is a chordal SLE$(4; \rho_+, \rho_-)$ trace started from $(0; 0^+, 0^-)$, then after a time-change, the reversal of $(W(\beta(t)))$ has the same distribution as $(\beta(t))$. This is the case when $\kappa = 4$ in Conjecture 1 of this paper.

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