We have seen several number systems:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R},$$

where the latter is an extension of the former, and $\mathbb{Q}, \mathbb{R}$ are fields.

We may do another extension to $\mathbb{R}$, and get the set of complex numbers:

$$\mathbb{R} \subset \mathbb{C}.$$

For this extension, we introduce an element i, which satisfies

$$i^2 = -1.$$

Since for every $x \in \mathbb{R}$, $x^2 \geq 0$, the i can not be a real number. The set $\mathbb{C}$ is then the collection of

$$x + iy,$$

where x and y are both real numbers, called the real part and imaginary part:

$$x = \mathrm{Re}(x + iy), \quad y = \mathrm{Im}(x + iy).$$

We know every real number corresponds to a point on a line. Every complex number $z = x + iy \in \mathbb{C}$ then corresponds to a point (x, y) in the plane. We understand $\mathbb{R}$ as a subset of $\mathbb{C}$ by

$$x = x + i0.$$

So $\mathbb{R}$ corresponds to the points on the x-axis. We can do addition and subtraction on complex numbers. The addition formula and subtraction formula are

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

This means that we operate on the real parts and the imaginary parts respectively. The multiplication formula is

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

We can derive this formula using $i^2 = -1$. We have the commutative law, associative law, and distributive law. Then $1 = 1 + i0$ and $0 = 0 + i0$ satisfy $0 + z = z$ and $1 \cdot z$ for all $z \in \mathbb{C}$. The quotient formula is a little bit complicated. For $x + iy \neq 0 + 0i$, its reciprocal is

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

It is straightforward to check that $(x + iy)(x + iy)^{-1} = 1$. Then z/w for $z, w \in \mathbb{C}$ with $w \neq 0$ is defined as zw^{-1}. The set $\mathbb{C}$ with these operations is a field.

For $z = x + iy \in \mathbb{C}$, we define its absolute value

$$|z| = \sqrt{x^2 + y^2},$$

which is the Euclidean distance from the point (x, y) from $(0, 0)$. For $z, w \in \mathbb{C}$, $|z - w|$ is the distance between z and w. For $z_0 \in \mathbb{C}$ and $r > 0$, we call $D(z_0, r)$ defined as $\{z \in \mathbb{C} : |z - z_0| < r\}$ an open disc. It is the set of all points in the plane with distance less than r from z_0. The set $D(z_0, r) \setminus \{z_0\}$ is then called a punctured disc. We say that a sequence of complex numbers (z_n) converge to a complex number z_0 if $|z_n - z_0| \to 0$. This is equivalent to that $\operatorname{Re} z_n \to \operatorname{Re} z_0$ and $\operatorname{Im} z_n \to \operatorname{Im} z_0$. A subset U of $\mathbb{C}$ is called open if for every $z_0 \in U$, there is $r > 0$ such that $D(z_0, r) \subset U$.

For $L \in \mathbb{C}$ and a function f defined on $D(z_0, r) \setminus \{z_0\}$ taking values in $\mathbb{C}$, we write

$$\lim_{z \to z_0} f(z) = L,$$

if for any sequence (z_n) in $D(z_0, r) \setminus \{z_0\}$ with $z_n \to z$, we have $f(z_n) \to L$. If in addition, f is also defined at z_0 and $\lim_{z \to z_0} f(z) = f(z_0)$, then we say that f is continuous at z_0. Suppose f is defined on $D(z_0, r)$, and there is $L \in \mathbb{C}$ such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

Then we say that f is (complex) differentiable at z_0, and write $\frac{d}{dz} f(z_0)$ or $f'(z_0)$ for the complex derivative L. If f is defined on an open set U, and is differentiable at every $z \in U$, then we say that f is differentiable on U.

We also have sum rule, product rule, quotient rule, and chain rule for derivatives, i.e.,

$$(af + bg)' = af' + bg', \quad (fg)' = f'g + fg', \quad (\frac{f}{g})' = \frac{f'g - fg'}{g^2}, \quad (g \circ f)' = g'(f)f'.$$

It is straightforward to check that $\frac{d}{dz} z = 1$. We define $z^n = \underbrace{z \cdots z}_{n}$. Using the product rule and induction, we find that $\frac{d}{dz} z^n = nz^{n-1}$ for all $n \in \mathbb{N}$. Here z^0 is constant 1.

For a sequence $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$, the series

$$\sum_{n=0}^{\infty} a_n z^n$$

is called a power series centered at 0. Let

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

If $R = 0$, the series converges only at 0; if $R = \infty$, the series converges at every $z \in \mathbb{C}$; if $0 < R < \infty$, the series converges at every $z \in D(0, R)$ and diverges at every $z \in \mathbb{C}$ with $|z| > R$.

Suppose $R > 0$, and we let f denote the sum of the series in $D(0, R)$. It turns out that f is complex differentiable, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad z \in D(z_0, R).$$

Since f' is also a sum of a power series with positive radius, we may further differentiate f' and then f'' and so on. So f is infinitely many times complex differentiable. The coefficients a_n then satisfy

$$a_n = \frac{f^{(n)}(0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

A function f defined on an open set U is called analytic if for every $z_0 \in U$, there exist $r > 0$ and $a_0, a_1, a_2, \cdots \in \mathbb{C}$ such that $D(z_0, r) \subset U$, and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0, r).$$

Then an analytic function must be infinitely times differentiable. A remarkable fact is that if f is once complex differentiable on U, then it must be analytic on U, and so is infinitely times differentiable. This is not true for real differentiable functions. We know that for a real differentiable function f on $\mathbb{R}$, the derivative function f' may not even be continuous.

Here are a few important examples of analytic functions. For $a_0, \ldots, a_n \in \mathbb{C}$, the function $P(z) = \sum_{k=0}^{n} a_k z^k$ is called a complex polynomial. This expression is just a power series centered at 0. Its derivative is $\sum_{k=1}^{n} k a_k z^{k-1}$, which is is still a polynomial. Another example is the complex exponential function. We define for $x + iy \in \mathbb{C}$,

$$\exp(x + iy) = e^{x+iy} = e^x \cos y + i e^x \sin y.$$

Note when $y = 0$, $e^{x+i0} = e^x$. So it extends the real exponential function. On can also compute directly that $e^{z_1} e^{z_2} = e^{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$. It is analytic on $\mathbb{C}$ with the power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad z \in \mathbb{C}.$$

Unlike the real exponential function, the complex exponential function is not injective because $e^{z+i2\pi} = e^z$ for any $z \in \mathbb{C}$. The logarithm function $\log z$ is defined as the inverse of e^z, which is multi-valued. The complex trigonometric functions $\cos z$ and $\sin z$ are defined using e^z by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

They are both analytic on $\mathbb{C}$. The power series expansions are

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad z \in \mathbb{C};$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad z \in \mathbb{C}.$$

The remarkable **Fundamental Theorem of Algebra** states that, for any nonconstant complex polynomial P, the equation $P(z) = 0$ has at least one complex solution. This statement is not true for real numbers. For example, the equation $x^2 + 1 = 0$ has no real solution. This is one of the reasons why complex numbers are important.